A Vélu's like formula for computing isogenies on Abelian Varieties

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Outline

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1. Abelian varieties and isogenies
2. Theta functions
3. Computing isogenies

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Discrete logarithm

Definition (DLP)

Let G be a commutative finite group, $g \in G$ and $x \in \mathbb{N}$. Let $h = x \cdot g$. The discrete logarithm $\log_g(h)$ is x.

- The DLP is hard (in a generic group) if the order of g is divisible by a large prime.
  - $\Rightarrow$ Usual tools of public key cryptography (and more!)
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An Abelian variety is a complete connected group variety over a base field k.

- Abelian variety = points on a projective space (locus of homogeneous polynomials) + an algebraic group law.
- Abelian varieties are projective, smooth, irreducible with an Abelian group law.
- *Example:* Elliptic curves, Jacobians of genus g curves...

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- Isogenies = Rational map + group morphism + finite kernel.
- Isogenies $\Leftrightarrow$ Finite subgroups.

$$(f : A \to B) \mapsto \operatorname{Ker} f$$
$$(A \to A/H) \leftarrow H$$

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Cryptographic usage of isogenies

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- Point counting algorithms (ℓ-adic or p-adic).
- Compute the class field polynomials.
- Compute the modular polynomials.
- Determine $\mathrm{End}(A)$.

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Vélu's formula

Theorem

Let $E : y^2 = f(x)$ be an elliptic curve. Let $G \subset E(k)$ be a finite subgroup. Then E/G is given by $Y^2 = g(X)$ where

$$X(P) = x(P) + \sum_{Q \in G \setminus \{0_E\}} x(P+Q) - x(Q)$$
$$Y(P) = y(P) + \sum_{Q \in G \setminus \{0_E\}} y(P+Q) - y(Q)$$

- Uses the fact that x and y are characterised in $k(E)$ by

$$\begin{aligned} v_{0_E}(x) &= -2 \qquad v_P(x) \geq 0 \quad \text{if } P \neq 0_E \\ v_{0_E}(y) &= -3 \qquad v_P(y) \geq 0 \quad \text{if } P \neq 0_E \\ y^2/x^3(O_E) &= 1 \end{aligned}$$

- No such characterisation in genus $g \geq 2$.

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The modular polynomial

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- The modular polynomial is a polynomial $\varphi_n(x, y) \in \mathbb{Z}[x, y]$ such that $\varphi_n(x, y) = 0$ iff $x = j(E)$ and $y = j(E')$ with E and E' n-isogeneous.
- If $E : y^2 = x^3 + ax + b$ is an elliptic curve, the j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

- Roots of $\varphi_n(j(E), .) \Leftrightarrow$ elliptic curves n-isogeneous to E.
- In genus 2, modular polynomials use Igusa invariants. The height explodes: $\varphi_2 = 50$ MB.

$\Rightarrow$ Use the moduli space given by theta functions.

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Complex abelian varieties

- Abelian variety over $\mathbb{C}$: $A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, where $\Omega \in \mathcal{H}_g(\mathbb{C})$ the Siegel upper half space.
- The theta functions with characteristic give a lot of analytic (quasi periodic) functions on $\mathbb{C}^g$.

$$\vartheta(z,\Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \,{}^t n \Omega n + 2\pi i \,{}^t n z}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\Omega) = e^{\pi i \,{}^t a \Omega a + 2\pi i \,{}^t a (z+b)} \vartheta(z + \Omega a + b, \Omega) \; a, b \in \mathbb{Q}^g$$

- The quasi-periodicity is given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z + m + \Omega n, \Omega) = e^{2\pi i ({}^t a m - {}^t b n) - \pi i \,{}^t n \Omega n - 2\pi i \,{}^t n z} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\Omega)$$

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Projective embeddings given by theta functions

Theorem

- *Let $\mathcal{L}_\ell$ be the space of analytic functions f satisfying:*

$$f(z+n) = f(z)$$
$$f(z+n\Omega) = \exp(-\ell \cdot \pi i n' \Omega n - \ell \cdot 2\pi i n' z) f(z)$$

- *A basis of $\mathcal{L}_\ell$ is given by*

$$\left\{ \vartheta \begin{bmatrix} 0 \\ b \end{bmatrix} (z, \Omega/\ell) \right\}_{b \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g}$$

- *Let $\mathcal{Z}_\ell = \mathbb{Z}^g/\ell\mathbb{Z}^g$. If $i \in \mathcal{Z}_\ell$ we define $\vartheta_i = \vartheta \begin{bmatrix} 0 \\ i/\ell \end{bmatrix} (., \Omega/\ell)$. If $l \geq 3$ then*

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The action of the Theta group

- The point $(a_i)_{i \in \mathcal{Z}_\ell} := (\vartheta_i(0))_{i \in \mathcal{Z}_\ell}$ is called the theta null point of level ℓ of the Abelian variety $A := \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$.
- $(a_i)_{i \in \mathcal{Z}_\ell}$ determines the equations of the projective embedding of A of level ℓ.
- The symplectic basis $\mathbb{Z}^g \oplus \Omega \mathbb{Z}^g$ induce a decomposition into isotropic subgroups for the commutator pairing:

$$\begin{aligned} A[\ell] &= A[\ell]_1 \oplus A[\ell]_2 \\ &= \frac{1}{\ell}\mathbb{Z}^g / \mathbb{Z}^g \oplus \frac{1}{\ell}\Omega \mathbb{Z}^g / \Omega \mathbb{Z}^g \end{aligned}$$

  This decomposition can be recovered by $(a_i)_{i \in \mathcal{Z}_\ell}$.
- The action by translation is given by

$$\vartheta_k \left(z - \frac{i}{\ell} - \Omega \frac{j}{\ell} \right) = e_{\mathcal{L}_\ell}(i+k, j) \vartheta_{i+k}$$

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The isogeny theorem

Theorem

- Let $\ell = n.m$, and $\varphi : \mathcal{Z}_n \to \mathcal{Z}_\ell, x \mapsto m.x$ be the canonical embedding. Let $K = A[m]_2 \subset A[\ell]_2$.
- Let $(\vartheta_i^A)_{i \in \mathcal{Z}_\ell}$ be the theta functions of level ℓ on $A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$.
- Let $(\vartheta_i^B)_{i \in \mathcal{Z}_n}$ be the theta functions of level n of $B = A/K = \mathbb{C}^g/(\mathbb{Z}^g + \frac{\Omega}{m}\mathbb{Z}^g)$.
- We have:

$$(\vartheta_i^B(x))_{i \in \mathcal{Z}_n} = (\vartheta_{\varphi(i)}^A(x))_{i \in \mathcal{Z}_n}$$

Proof.

$$\vartheta_i^B(z) = \vartheta\begin{bmatrix} 0 \\ i/n \end{bmatrix}\left(z, \frac{\Omega}{m}/n\right) = \vartheta\begin{bmatrix} 0 \\ mi/\ell \end{bmatrix}(z, \Omega/\ell) = \vartheta_{m \cdot i}^A(z)$$

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Mumford: On equations defining Abelian varieties

Theorem (car $k \nmid \ell$)

- *The theta null point of level ℓ $(a_i)_{i \in \mathcal{Z}_\ell}$ satisfy the Riemann Relations:*

$$\sum_{t \in \mathcal{Z}_2} a_{x+t} a_{y+t} \sum_{t \in \mathcal{Z}_2} a_{u+t} a_{v+t} = \sum_{t \in \mathcal{Z}_2} a_{x'+t} a_{y'+t} \sum_{t \in \mathcal{Z}_2} a_{u'+t} a_{v'+t} \quad (1)$$

  We note $\mathcal{M}_\ell$ the moduli space given by these relations together with the relations of symmetry:

$$a_x = a_{-x}$$

- *$\mathcal{M}_\ell(k)$ is the modular space of k-Abelian variety with a theta structure of level ℓ. The locus of theta null points of level ℓ is an open subset $\mathcal{M}_\ell^0(k)$ of $\mathcal{M}_\ell(k)$.*

Remark

- *Analytic action:* $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathcal{H}_g$ (and preserves the isomorphic classes).
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Summary

$$\begin{tikzcd} A_k, A_k[\ell] = A_k[\ell]_1 \oplus A_k[\ell]_2 \arrow[d, shift left, "\pi"] & (a_i)_{i \in \mathcal{Z}_\ell} \in \mathcal{M}_\ell(k) \arrow[l, dotted, "\text{determines}"'] \arrow[d] \\ B_k, B_k[n] = B_k[n]_1 \oplus B_k[n]_2 \arrow[u, shift left, "\hat{\pi}"] & (b_i)_{i \in \mathcal{Z}_n} \in \mathcal{M}_n(k) \arrow[l, dotted] \end{tikzcd}$$

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Outline

An Example with $n \wedge m = 1$

We will show an example with $g = 1$, $n = 4$ and $\ell = 12$ ($m = 3$).

- Let B be the elliptic curve $y^2 = x^3 + 23x + 3$ over $k = \mathbb{F}_{31}$. The corresponding theta null point (b_0, b_1, b_2, b_3) of level 4 is $(3 : 1 : 18 : 1) \in \mathcal{M}_4(\mathbb{F}_{31})$.
- We note $V_B(k)$ the subvariety of $\mathcal{M}_{12}(k)$ defined by

$$a_0 = b_0, a_3 = b_1, a_6 = b_2, a_9 = b_3$$

- By the isogeny theorem, to every valid theta null point $(a_i)_{i \in \mathcal{Z}_{\ell n}} \in V_B^0(k)$ corresponds a 3-isogeny $\pi : A \to B$:

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Program

3. Computing isogenies
   - Computing the contragredient isogeny
   - Vélu-like formula in dimension g

The kernel of $\hat{\pi}$

- Let $(a_i)_{i \in \mathcal{Z}_\ell}$ be a valid theta null point solution. Let ζ be a primitive m root of unity. The kernel of π is

$$\{(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}),$$
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- If $i \in \mathcal{Z}_m$ we define

$$\pi_i(x) = (x_{ni+mj})_{j \in \mathcal{Z}_n}$$

Let $R_0 := \pi_0(\widetilde{0}_{A_k}) = (a_0, a_3, a_6, a_9)$, $R_1 := \pi_1(\widetilde{0}_{A_k}) = (a_4, a_7, a_{10}, a_1)$, $R_2 := \pi_2(\widetilde{0}_{A_k}) = (a_8, a_{11}, a_2, a_5)$.

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The addition law

Theorem

$$\Big(\sum_{t\in\mathcal{Z}_2}\chi(t)\vartheta_{i+t}(x+y)\vartheta_{j+t}(x-y)\Big).\Big(\sum_{t\in\mathcal{Z}_2}\chi(t)\vartheta_{k+t}(0)\vartheta_{l+t}(0)\Big) =$$
$$\Big(\sum_{t\in\mathcal{Z}_2}\chi(t)\vartheta_{-i'+t}(y)\vartheta_{j'+t}(y)\Big).\Big(\sum_{t\in\mathcal{Z}_2}\chi(t)\vartheta_{k'+t}(x)\vartheta_{l'+t}(x)\Big).$$

$$\textit{where} \quad A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\chi \in \hat{\mathcal{Z}}_2, i, j, k, l \in \mathcal{Z}_n$$
$$(i', j', k', l') = A(i, j, k, l)$$

Addition and isogenies

Proposition

$\pi_i(x) = \pi_0(x) + R_i$ so we have:

$$\pi_{i+j}(x+y) = \pi_i(x) + \pi_j(y)$$
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- $x \in A$ is entirely determined by $\pi_0(x), \pi_1(x), \pi_2(x)$.
- $\pi_2(x) = \texttt{chaine\_add}(\pi_1(x), R_1, \pi_0(x))$

Corollary

- *x is entirely determined by*

$$\{\pi_i(x)\}_{i \in \{0, e_1, \cdots, e_g, e_1+e_2, \cdots, e_{g-1}+e_g\}}$$

- *Use $(1+g(g+1)/2)n^g$ coordinates rather than $(\ell n)^g$.*
- *The decompression use $O(m^g)$ chain additions.*
- *Can still do chain additions with this representation.*

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The contragredient isogeny

$$\begin{tikzcd} x \in A \arrow[rr, "{[m]}"] \arrow[dr, "\pi"'] & & z \in A \\ & y \in B \arrow[ur, "\hat{\pi}"'] & \end{tikzcd}$$

Let $\pi : A \to B$ be the isogeny associated to $(a_i)_{i \in Z_{\ell n}}$. Let $y \in B$ and $x \in A$ be one of the ℓ^g antecedents. Then

$$\hat{\pi}(y) = m.x$$

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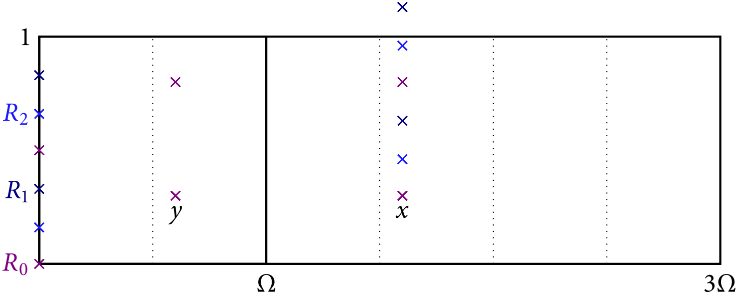

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- Let $y \in B$. We can compute $y_i = y \oplus R_i$ with a normal addition. We have $y_i = \lambda_i \pi_i(x)$.

$$\pi_i(m.x) = (m-1).y + \pi_i(x) = \lambda_i^m (m-1).y + y_i)$$
$$y = (m-1).R_i + \pi_i(x)) = \lambda_i^m (m-1)R_i + y_i$$

Corollary

We can compute $\pi_i(m.x)$ with two fast multiplications of length m. To recover the compressed coordinates of $\hat{\pi}(y)$, we need $g(g+1)/2 \cdot O(\log(m))$ additions.

The contragredient isogeny

$$\begin{tikzcd} x \in A \arrow[rr, "{[m]}"] \arrow[dr, "\pi"'] & & z \in A \\ & y \in B \arrow[ur, "\hat{\pi}"'] & \end{tikzcd}$$

Let $\pi : A \to B$ be the isogeny associated to $(a_i)_{i \in \mathcal{Z}_{\ell n}}$. Let $y \in B$ and $x \in A$ be one of the ℓ^g antecedents. Then

$$\hat{\pi}(y) = m.x$$

- Let $y \in B$. We can compute $y_i = y \oplus R_i$ with a normal addition. We have $y_i = \lambda_i \pi_i(x)$.

$$\pi_i(m.x) = (m-1).y + \pi_i(x) = \lambda_i^m (m-1).y + y_i)$$
$$y = (m-1).R_i + \pi_i(x)) = \lambda_i^m (m-1)R_i + y_i$$

Corollary

We can compute $\pi_i(m.x)$ with two fast multiplications of length m. To recover the compressed coordinates of $\hat{\pi}(y)$, we need $g(g+1)/2 \cdot O(\log(m))$ additions.

Example

Let $K = \{(3:1:18:1), (22:15:4:1), (18:29:23:1)\}$, a point solution corresponding to this kernel is given by $(3, \eta^{14233}, \eta^{2317}, 1, \eta^{1324}, \eta^{5296}, 18, \eta^{5296}, \eta^{1324}, 1, \eta^{2317}, \eta^{14233})$ where $\eta^3 + \eta + 28 = 0$. We have to compute:

$$\begin{array}{cccc} & y & & \\ R_1 & y + R_1 & y + 2R_1 & y + 3R_1 = y \\ & 2y + R_1 & & \\ & 3y + R_1 & & \end{array}$$

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$$R_1 = (\eta^{1324},\eta^{5296},\eta^{2317},\eta^{14233}) \quad y = (\eta^{19406},\eta^{19805},\eta^{10720},1)$$

$$y + R_1 = \lambda_1(\eta^{2722},\eta^{28681},\eta^{26466},\eta^{2096})$$

$$y + 2R_1 = \lambda_1^2(\eta^{28758},\eta^{11337},\eta^{27602},\eta^{22972})$$

$$y + 3R_1 = \lambda_1^3(\eta^{18374},\eta^{18773},\eta^{9688},\eta^{28758}) = y/\eta^{1032}$$

$$2y + R_1 = \lambda_1^2(\eta^{17786},\eta^{12000},\eta^{16630},\eta^{365})$$

$$3y + R_1 = \lambda_1^3(\eta^{7096},\eta^{11068},\eta^{8089},\eta^{20005}) = \eta^{5772}R_1$$

We have $\lambda_1^3 = \eta^{28758}$ and

$$\hat{\pi}(y) = (3,\eta^{21037},\eta^{15925},1,\eta^{8128},\eta^{18904},18,\eta^{12100},\eta^{14932},1,\eta^{9121},\eta^{27841})$$

Program

3. Computing isogenies
   - Computing the contragredient isogeny
   - Vélu-like formula in dimension g

The action of the symplectic group on the modular space

- Let $K \subset B[\ell]$ be an isotropic subgroup of maximal rank. Let $(a_i)_{i \in \mathcal{Z}_{\ell n}}$ be a theta null point corresponding to the isogeny $\pi : B \to B/K$.
- The actions of the symplectic group compatible with the isogeny π are generated by

$$\{R_i\}_{i \in \mathcal{Z}_\ell} \mapsto \{R_{\psi_1(i)}\}_{i \in \mathcal{Z}_\ell} \tag{2}$$

$$\{R_i\}_{i \in \mathcal{Z}_\ell} \mapsto \{e(\psi_2(i), i) R_i\}_{i \in \mathcal{Z}_\ell} \tag{3}$$

where ψ_1 is an automorphism of $\mathcal{Z}_\ell$ and ψ_2 is a symmetric endomorphism of $\mathcal{Z}_\ell$.

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- These points corresponds to the same isogeny:

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$$

$$(a_0, \zeta a_1, \zeta^{2^2} a_2, a_3, \zeta a_4, \zeta^{2^2} a_5, a_6, \zeta a_7, \zeta^{2^2} a_8, a_9, \zeta a_{10}, \zeta^{2^2} a_{11})$$

$$(a_0, \zeta^2 a_1, \zeta^2 a_2, a_3, \zeta^2 a_4, \zeta^2 a_5, a_6, \zeta^2 a_7, \zeta^2 a_8, a_9, \zeta^2 a_{10}, \zeta^2 a_{11})$$

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- In particular by action (2), if $\{T_{e_i}\}_{i \in [1..g]}$ is a basis of K, we may suppose that $R_{e_i} = \lambda_{e_i} T_{e_i}$.

Recovering the projective factors

- Since we are working with symmetric Theta structures, we have $\vartheta_i(-x) = \vartheta_{-i}(x)$.
- In particular if $m = 2m' + 1$

$$(m'+1).R_i = -m'.R_i$$
$$\lambda_i^{(m'+1)^2}(m'+1).T_i = \lambda_i^{m'^2} m'.T_i$$

So we may recover λ_i up to a ℓ-root of unity.

- But we only need to recover R_i for $i \in \{e_1, \cdots, e_{g-1} + e_g\}$ and the action (3) shows that each choice of a m-root of unity corresponds to a valid theta null point.

Corollary

We have Vélu-like formulas to recover the compressed modular point solution, by computing $g(g+1)/2$ m-roots and $g(g+1)/2 \cdot O(\log(m))$ additions. The compressed coordinates are sufficient to compute the compressed coordinates of the associated isogeny.

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Isogeny graphs

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\begin{tikzcd}
B \arrow[dd, "{[m]}"'] \arrow[dr, "\hat{\pi}"] & & \\
& A, A[m] = A[m]_1 \oplus A[m]_2 \arrow[dl, "\pi"'] \arrow[dr, "\pi_2"] & \\
B = A/A[m]_2 & & C = A/A[m]_1
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- $\pi_2 \circ \hat{\pi}$ is an m^2 isogeny between two varieties of level n.
- Each choice of the m-roots of unity in the Vélu's-like formulas give a different decomposition $A[m] = A[m]_1 \oplus K$. All the m^2-isogenies $B \to C$ containing K come from these choices.
- We know the kernel of the contragredient isogeny $C \to A$, this is helpful for computing isogeny graphs.

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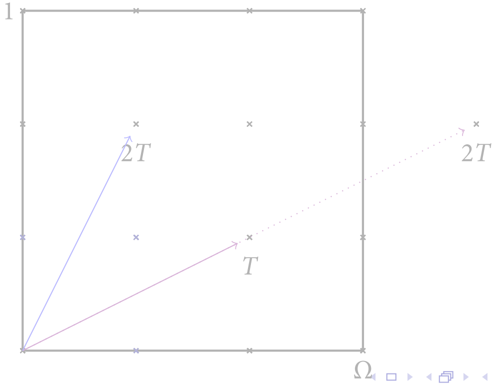
Computing all modular points

- Let $T_{e_1}, \cdots, T_{e_{2g}}$ be a basis for $B[m]$. If x,y and $x-y$ are true points of ℓ-torsion, so is $x+y := \texttt{chaine\_add}(x,y,x-y)$. This means we can compute "true" representatives of $B[m]$ with $g(2g+1)$ m-roots of unity, $g(2g-1)$ additions and m^{2g} chain additions.
- **Warning:** When applying our Vélu's formulas to an isotropic kernel, take into account the action of the commutator pairing:



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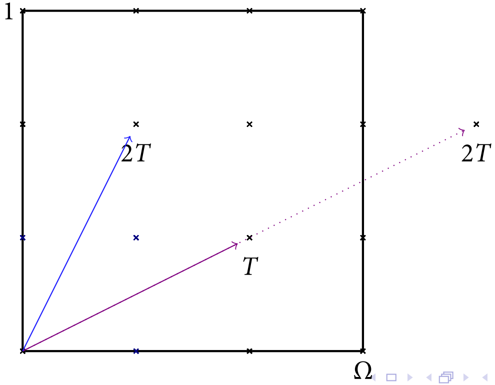

Perspectives

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