On symmetric theta structures

Damien Robert

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1 Symmetric theta structures and the isogeny theorem

Let A be an abelian variety of dimension g defined over an algebraically closed field $\overline{k}$. Let $\mathscr{L}_0$ be a symmetric ample line bundle of degree one on A, $\mathscr{L}_0$ defines a principal polarization: $A \to \hat{A}$. If n is even $\mathscr{L} = \mathscr{L}_0^n$ is then totally symmetric, and the kernel $K(\mathscr{L})$ of the polarization associated to $\mathscr{L}$ is $A[n]$.

From now on, we assume that n is prime to the characteristic of k, so that $\mathscr{L}$ defines a separable polarisation. Since $\mathscr{L}$ is totally symmetric, there exist a symmetric theta structure on the theta group $G(\mathscr{L})$. Fixing such a structure fix a unique projective basis of theta functions [Mum66] that we call theta functions of level n. Note: the theta structure induces an isomorphism between the symplectic spaces $Z(\overline{n}) \times \hat{Z}(\overline{n})$ and $K(\mathscr{L}) = A[n]$ where $Z(\overline{n}) = (\mathbb{Z}/n\mathbb{Z})^g$ and $\hat{Z}(\overline{n})$ is the Cartier dual of $Z(\overline{n})$. We note $K(\mathscr{L}) = K_1(\mathscr{L}) \oplus K_2(\mathscr{L})$ where $K_1(\mathscr{L})$ corresponds to $Z(\overline{n})$ and $K_2(\mathscr{L})$ to $\hat{Z}(\overline{n})$. Usually the canonical basis of the theta functions of level n are indexed by $i \in Z(\overline{n})$, but in these notes we will index them by $i \in K_1(\mathscr{L})$ which permit us to not track explicitly the isomorphism between $Z(\overline{n})$ and $K_1(\mathscr{L})$.

If $n > 2$ then the theta functions of level n give a projective embedding of A into $\mathbb{P}_{\overline{k}}^{n^g-1}$, while if $n = 2$ we only get an embedding of the Kummer variety $A/\pm 1$ (the $n = 2$ case assume that A is absolutely simple, see [BL04]). Under a generic condition (the even theta null coordinates are non zero), this embedding of the Kummer variety is actually projectively normal (see [Koi76]).

Theorem 1.1 :
The symmetric theta structure on $G(\mathscr{L})$ is uniquely determined by a choice of symplectic basis $(e_1, \dots e_g, e'_1, \dots e'_g)$ on $A[n]$ and a choice of symplectic basis $(f_1, \dots f_g, f'_1, \dots f'_g)$ on $A[2n]$ such that $e_i = 2f_i, e'_i = 2f'_i$. (Here symplectic mean for the commutator pairing $e_{\mathscr{L}}$ and $e_{\mathscr{L}^2}$ respectively).

Moreover, changing these symplectic basis do not change the resulting symmetric theta structure if and only if

- *The symplectic basis of $A[n]$ is left invariant;*
- *The f_i are replaced by points $f_i + t_i$ with $t_i \in A[2]$ such that $e_{\mathscr{L}}(e_i, t_i) = 1$.*

In particular, fixing a symplectic basis of $A[n]$ and a symplectic decomposition $A[2n] = A_1[2n] \oplus A_2[2n]$ of the $2n$-torsion into a sum of maximal isotropic subspaces is enough (and even stronger) to fix the symmetric theta structure.

Proof : This is implicit in [Mum66, Section 3]. A symmetric theta structure comes from an isomorphism between the Heisenberg group and the theta group that commutes with the action of $[-1]$. It induces an isomorphism between the symplectic spaces $Z(\overline{n}) \times \hat{Z}(\overline{n})$ and $K(\mathscr{L}) = A[n]$ and hence fix a symplectic basis of the n-torsion.

Conversely, having fixed a symplectic basis of the n-torsion, since $\mathscr{L}$ is totally symmetric, there is always a symmetric theta structure respecting this symplectic basis. Such a choice of a symmetric theta structure can be seen as a choice of a symmetric element above each of the element of the basis $(e_1, \dots e'_g)$; since there is only two symmetric elements $\pm g_i$ above each e_i a symmetric theta structure above the symplectic basis can be seen as a choice of sign for each element of the basis.

If $g_i \in G(\mathscr{L}^2)$ is a symmetric element of the theta group above a point f_i such that $e_i = 2f_i$, then $(g_i)^2$ determines a symmetric element of the theta group above e_i that uniquely depends on the choice of f_i (since the other symmetric element above f_i is $-g_i$ which gives rise to $(-g_i)^2 = (g_i)^2$ above e_i. Via the transfer map δ_2 from [Mum66], we see how the choices of the f_i above the e_i are enough to determine the symmetric theta structure on $G(\mathscr{L})$.

It is a straightforward verification to see that replacing f_i by $f_i + t_i$ where t_i is a point of 2-torsion involve replacing $(g_i)^2$ by $e_{\mathscr{L}^2}(f_i, t_i)(g_i)^2$ which concludes the proof.

(One could also replace the application δ_2 by the isogeny [2] which would involve working in $G(\mathscr{L}^4)$, as in [Kem89].) ■

Corollary 1.2 :
Let $(A, \mathscr{L}_0)/\mathbb{F}_q$ be a ppav over the finite field $\mathbb{F}_q$. Assume that $\mu_n(\overline{\mathbb{F}}_q) \subset \mathbb{F}_q$ ($n = 2n_0$ even). Then there exist a rational symmetric theta structure on $\mathscr{L} = \mathscr{L}_0^n$ iff there exist a rational symplectic basis $(e_1, \dots e_g, e'_1, \dots, e'g)$ such that $e_{T,2}(n_0 e_i, e_i) = 1$; where $e_{T,2}$ denotes the 2-Tate pairing. (In other words, e_i form a symplectic basis consisting of elements whose self n-Tate pairing is not a primitive n-th root of unity).

Proof : This is clear from Theorem 1.1 and the definition of the Tate pairing as $e_{T,2}(n_0 e_i, e_i) = e_{W,2}(n_0 e_i, \pi(f_i) - f_i)$ where $2f_i = e_i$ and π is the Frobenius of $\mathbb{F}_q$. ■

Remark 1.3 :
In the case that $\mathbb{F}_q$ does not contain the n-th root of unity, a rational theta structure of level n induces an equivariant (for the Galois action) isomorphism between $A[n]$ and $Z(\overline{n}) \times \hat{Z}(\overline{n})$. In particular, this does not impose that all geometric points of $A[n]$ are rational.

Proposition 1.4 :
Let $\mathscr{L}$ be a symmetric line bundle on A, defining a polarization of type $\delta = (\delta_1, \dots, \delta_g)$. Then there exists a symmetric theta structure on $G(\mathscr{L})$ if and only if for every $x \in A[2] \bigcap K(\mathscr{L})$, we have $e_*(x) = 1$.

In this case we call $\mathscr{L}$ totally symetrisable (because a totally symmetric line bundle satisfy the condition), and the obvious generalisation of Theorem 1.1 to this case also holds.

Proof : [Kem89; Mum66]. ■

The idea is that (for instance in dimension 2), $\mathscr{L}_0^\ell$ is of type (ℓ, ℓ) and allows to compute isogenies with maximal isotropic kernels, but for a cyclic isogeny we need a polarisation of type $(1, \ell)$ (like the type of $\mathscr{L}_0^\rho$ from Section **??**).

Theorem 1.5 :
Let $f : (A, \mathscr{L}) \to (B, \mathscr{M})$ be an isogeny between pav. Then $K = \operatorname{Ker} f$ is isotropic in $K(\mathscr{L})$ for the commutator pairing $e_{\mathscr{L}}$, and $K(\mathscr{M}) \simeq K^\perp / K$.

Assume that we have a symmetric theta structure on $G(\mathscr{L})$ coming from a symplectic basis (f_i, f'_i) on $K(\mathscr{L}^2)$. Assume that K is compatible with the induced symplectic decomposition $K(\mathscr{L}) = K_1(\mathscr{L}) \oplus K_2(\mathscr{L})$ into maximal isotropic subspaces in the sense that $K = K_1 \oplus K_2$ where $K_i = K_i(\mathscr{L}) \bigcap K$. In this case $K(\mathscr{M}) \simeq K^{2,\perp}/K_1 \oplus K^{1,\perp}/K_2$ where $K^{2,\perp} = K_2^\perp \bigcap K_1(\mathscr{L})$ and $K^{1,\perp} = K_1^\perp \bigcap K_2(\mathscr{L})$

Let $\widetilde{K}$ be the level subgroup above K induced by this theta structure; the corresponding descent data give a line bundle $\mathscr{M}'$ algebraically equivalent to $\mathscr{M}$. Moreover $\mathscr{M}'$ is totally symetrisable, and we can define a symmetric theta structure on $\mathscr{M}'$ as follow: from the symplectic basis of $K(\mathscr{L}^2)$ one derives a "canonical" basis $(g_1, \dots, g'_g)$ of $[2]^{-1}K^\perp$. Pushing this basis via the isogeny f gives a symplectic basis on $K(\mathscr{M}'^2)$, which determines the symmetric theta structure on $\mathscr{M}'$. It is easy to see that by construction, it is compatible with the theta structure on $\mathscr{L}$.

We can then apply the isogeny theorem: there exist λ such that for all $i \in K_1(\mathscr{M}')$

$$\vartheta_i^{\mathscr{M}'} = \lambda \sum_{j \in K_1(\mathscr{L}) | f(j) = i} \vartheta_j^{\mathscr{L}}.$$

Proof : [Kem89; Mum66; Rob10]. ■

Corollary 1.6 :

- *If $\mathscr{M}$ is of type δ' with $2 \mid \delta'$ (meaning that $A[2] \bigcap K(\mathscr{L}) \subset K^{\perp}$), then $\mathscr{M}'$ is the unique totally symmetric line bundle in the equivalence class of $\mathscr{M}$.*
- *If $A[2] \bigcap K(\mathscr{L}) \subset K$, then every symmetric theta structure on $G(\mathscr{L})$ induces the same symmetric theta structure on $G(\mathscr{M}')$.*

Proof : See [Kem89; Rob10]. ■