# Quantum signatures of the mixed classical phase space for three interacting particles in a circular trap 

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#### Abstract

We study theoretically two consequences of the mixed classical phase space for three repulsivelyinteracting bosonic particles in a circular trap. First, we show that the energy levels of the corresponding quantum system are well described by a Berry-Robnik distribution. Second, we identify stationary quantum states whose density is enhanced along the stable classical periodic trajectories, and calculate their energies and wavefunctions using the semiclassical Einstein-Brillouin-Keller (EBK) theory. Our EBK results are in excellent agreement with our full-fledged finite-element numerics. We discuss the impact of discrete symmetries, including bosonic exchange symmetry, on these classically localized states. They are within experimental reach, and occur in the same range of energies as the quantum scar reported in our previous work [Phys. Rev. A 107, 022217 (2023)].


## I. INTRODUCTION

The suppression of ergodicity in quantum systems has long been under intense scrutiny [1, chap. 8], and atomic systems are very well suited to its investigation Refs. [2, chap. 4]. The mechanisms leading to it in many-body systems, relying on e.g. integrability 3], the presence of disorder [4], many-body scarring [5, 6], or periodic driving [7], hold promises for quantum information processing over long times, but may hinder cooling mechanisms [8].

In the case of Hamiltonian systems, comparing the quantum system to its classical analog has been very fruitful in identifying such mechanisms [9]. Most classical systems have a mixed phase space hosting both ergodic and non-ergodic trajectories. Ergodic trajectories densely cover a substantial fraction of the energy surface; non-ergodic ones wind around tori found within the Kolmogorov-Arnold-Moser (KAM) regions of phase space, well described using KAM theory [10, appendix 8]. Ergodicity in the quantum system may be suppressed in a phase space region corresponding to classical ergodic motion, e.g. by a quantum scar [11]. The quantum system is also known to exhibit regular levels reflecting the classical non-ergodic trajectories [9, Sec. 4]. These levels may be studied using the semiclassical Einstein-Brillouin-Keller (EBK) theory 12, 13]. In contrast to the semiclassical approaches applicable to the classically chaotic region, which mainly provide information concerning the density of states [14, chap. 17], EBK theory applied to the classical KAM regions yields both quantum energy eigenvalues and eigenfunctions constructed from classically non-ergodic trajectories. The full energy spectrum, including both the regular levels to which EBK theory applies and the remaining levels related to chaotic dynamics [9, Sec. 5], exhibits energy level statistics which significantly deviate [15, 16] from both the Poisson and Wigner distributions respectively associated with classical integrability and chaos [14, chap. 16].

[^0]Mixed classical phase spaces are relevant for the description of many-body systems. The many-body scar affecting the spin dynamics of a Rydberg atom chain observed in Ref. 5] provides a recent example. The classical analog system, whose construction is involved [17], exhibits mixed phase space, and KAM regions play a key role in the many-body quantum revivals [18]. Motivated by these recent developments, we introduced in our previous article [19] the system of three interacting particles in a circular trap. We analyzed this experimentally accessible system through well-established theories applied to a phase space whose dimension matches the number of independent parameters introduced in Ref. 18], and identified a quantum scar affecting the motion of the atoms.

In this paper, we analyze the role of its mixed classical phase space. First, we show that the parameters we investigated in Ref. [19] fall within a range where the quantum energy level statistics are well described by the Berry-Robnik distribution [16]. Then, we identify quantum states whose probability density is enhanced near stable classical periodic trajectories. Using EBK theory, we characterize their energy eigenvalues and explicitly construct their wavefunctions. Our results are in excellent agreement with our full-fledged numerical solution of the Schrödinger equation using the finite-element method. We highlight the role of discrete symmetries, including bosonic echange symmetry, and their observable consequence, on the energies and wavefunctions of the considered localized states.

We formulate our analysis in terms of trapped Rydberg atoms, made accessible by recent experimental advances 20, 21]. However, similar phenomena are expected to occur with systems of magnetic atoms [22] or polar molecules [23] exhibiting the same symmetries. The classically-localized states [24, chap. 22] identified in the present paper occur for the same parameters and energy range as the previously identified quantum scar [19]. One may address one effect or the other simply by changing the initial condition defining the atomic motion. Hence, the simple, well-controlled atomic system we are proposing offers an opportunity for a detailed experimental comparison of the two effects.


FIG. 1. (a) The periodic trajectories $A_{0}, A_{1}, A_{2}$ (straight blue lines) and $C_{+}, C_{-}$(the closed green trajectory is followed anticlockwise for $C_{+}$and clockwise for $C_{-}$), shown in the ( $x, y$ ) plane for the energy $\epsilon=7 C_{6} / R^{6}$. The dotted brown line shows the classically accessible region. The inset shows the considered physical system: three interacting particles in a circular trap. (b) Periodic trajectory $A_{0}$ as a function of time for $\epsilon=7 C_{6} / R^{6}$ in terms of its coordinates $x(t)$ (solid line) and $y(t)$ (dashed line). (c) Period $T_{A}(\epsilon)$ of trajectory $A$ as a function of the energy $\epsilon$. Panels (d) and (e) show the corresponding quantities for trajectory $C_{+}$. Trajectories $A$ and $C$ are stable for the considered range of energies.

The paper is organized as follows. In Sec. II, we introduce the considered system, and briefly summarize its properties described in detail in our previous article [19]. In Sec. III] we show that its quantum energy levels are well represented by the Berry-Robnik distribution. In Sec.IV, we apply EBK theory to identify the energy levels for the quantum states localized near stable periodic trajectories and construct the corresponding EBK wavefunctions, and we compare them to our finite-element numerical results. In Sec. V we discuss experimental prospects. The article ends with the conclusive Sec. VI.

## II. THE CONSIDERED SYSTEM

The system we analyze has been introduced in our previous article [19]. We briefly summarize its key features.

We consider three identical bosonic particles of mass $m$ in a circular trap of radius $R$ (Fig. [1(a),inset). We assume that the interaction $v\left(d_{i j}\right)$ between the particles $i$ and $j$ only depends on their distance $d_{i j}=2 R\left|\sin \left[\left(\theta_{i}-\theta_{j}\right) / 2\right]\right|$. For circular Rydberg atoms whose electronic angular momenta are perpendicular to the plane, $v\left(d_{i j}\right)=C_{6} / d_{i j}^{6}$ with $C_{6}>0$. We introduce the Jacobi coordinates $x=\left[\left(\theta_{1}+\theta_{2}\right) / 2-\theta_{3}+\pi\right] / \sqrt{3}, y=\left(\theta_{2}-\theta_{1}\right) / 2-\pi / 3$, $z=\left(\theta_{1}+\theta_{2}+\theta_{3}\right) / 3-2 \pi / 3$, and their conjugate momenta $p_{x}, p_{y}, p_{z}$ (which carry the unit of action). Then, the Hamiltonian reads $H=p_{z}^{2} /\left(3 m R^{2}\right)+H_{2 \mathrm{D}}$, where

$$
\begin{equation*}
H_{2 \mathrm{D}}=\frac{p_{x}^{2}+p_{y}^{2}}{4 m R^{2}}+V(x, y) . \tag{1}
\end{equation*}
$$

Here, $V(x, y)=v(x, y) C_{6} / R^{6}$, with

$$
\begin{align*}
v(x, y)= & {\left[\sin ^{-6}(\pi / 3+y)+\sin ^{-6}(\pi / 3+x \sqrt{3} / 2-y / 2)\right.} \\
& \left.+\sin ^{-6}(\pi / 3-x \sqrt{3} / 2-y / 2)\right] / 64-1 / 9, \quad(2) \tag{2}
\end{align*}
$$

energies being measured from the minimum $V(\mathbf{0})$. The Hamiltonian $H$ may be understood as describing either a
classical system or its quantum counterpart. It is invariant under the point group $C_{3 v}$, generated by the rotation of order 3 about the axis $(x=y=0)$ and the reflection in the plane $(x=0)$. The free motion of the coordinate $z$ reflects the conservation of the total angular momentum $p_{z}$. Once the latter is fixed, the system is reduced to an effective point in the two-dimensional (2D) plane $(x, y)$ within the equilateral triangle ABC of Fig. [(a), in the presence of the potential $V(x, y)$.

From the quantum point of view, we seek the 3 -atom eigenstates of $H$ in the form $\Psi_{n}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\psi_{n}(\boldsymbol{r}) e^{i n z}$, where $\boldsymbol{r}=(x, y)$, and $n$ is an integer setting the value of the quantized angular momentum $p_{z}$. The wavefunction $\psi_{n}(\boldsymbol{r})$ is fully determined by its values within the triangle $A B C$ and vanishes along $A B, B C$, and $C A$. The constraint $\Psi_{n}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\Psi_{n}\left(\theta_{3}-2 \pi, \theta_{1}, \theta_{2}\right)$, combining bosonic symmetry and angular periodicity, yields:

$$
\begin{equation*}
\psi_{n}(\mathcal{R} \boldsymbol{r})=\psi_{n}(\boldsymbol{r}) e^{2 i n \pi / 3} \tag{3}
\end{equation*}
$$

where $\mathcal{R}$ is the rotation of angle $2 \pi / 3$ about $O$ in the $(x, y)$ plane. We sort the energy levels in terms of the three irreducible representations $A_{1}, A_{2}, E$ of $C_{3 v}$. Owing to Eq. (3), wavefunctions pertaining to the onedimensional (1D) representations $A_{1}$ or $A_{2}$ have $n=0$ modulo 3 , whereas those pertaining to the 2 D representation $E$ have $n \neq 0$ modulo 3 .

As in Ref. [19], we set the ratio $\eta=\hbar R^{2} /\left(m C_{6}\right)^{1 / 2}$ to 0.01 , and we consider energies $\epsilon \sim 7 C_{6} / R^{6}$.

## III. MIXED CLASSICAL PHASE SPACE AND QUANTUM ENERGY LEVEL STATISTICS

## A. Classical periodic trajectories

We have characterized the periodic trajectories of the model of Eq. (1) using our own $\mathrm{C}++$ implementation of the numerical approach of Ref. [25]. We find three families of periodic trajectories existing for all energies


FIG. 2. Surface of section for Eq. (2), with $p_{z}=0, \epsilon=$ $7 C_{6} / R^{6}, x=0$, and $p_{x}>0$. The periodic trajectory $A_{0}$ appears as the dark blue closed boundary of the figure. All other periodic trajectories appear as fixed points, shown in dark blue for $A_{1}$ and $A_{2}$; dark red for $B_{1}, B_{2}$, and $B_{3}$; and dark green for $C_{+}$and $C_{-}$. The stable trajectories $A_{i}$ and $C_{j}$ are surrounded by (light blue and light green) tori; no tori are present near the unstable trajectories $B_{k}$. The $\approx 287000$ thin brown dots all belong to the same ergodic trajectory.
$\epsilon>0$ : we label them $A, B, C$ in analogy with those of the Hénon-Heiles potential [26]. We have analyzed the unstable trajectories of family $B$ (i.e their Lyapunov exponent $>0$ ), along with the quantum scar it yields, in our previous article [19]. By contrast, the trajectories of families $A$ and $C$ are stable for all considered energies (i.e. their Lyapunov exponents $=0$ ).

For a given energy $\epsilon$, family A contains three straightline trajectories $A_{0}, A_{1}, A_{2}$, which follow the medians of the triangular configuration space, and transform into one another under rotations of order 3 . Family C contains two trajectories $C_{+}$and $C_{-}$, which are closed loops around the center $O: C_{+}$is followed anticlockwise and $C_{-}$clockwise, and they transform into each other under reflections about any of the three medians. All five trajectories are represented in the $(x, y)$ plane on Fig. (a). The vertical trajectory $A_{0}$ and the trajectory $C_{+}$are shown as functions of time on Figs. 1 (b, d). Trajectories of a given family have the same period as a function of energy $T_{A}(\epsilon)$ and $T_{C}(\epsilon)$ : these are plotted on Figs. $1(\mathrm{c}, \mathrm{e})$ and are both of the order of $\left(m R^{8} / C_{6}\right)^{1 / 2}$ for $\epsilon \sim 7 C_{6} / R^{6}$.

The simultaneous existence of stable and unstable periodic trajectories signals that the classical system represented by $H_{2 \mathrm{D}}$ is neither integrable nor fully chaotic: its phase space is mixed. This is apparent on the surface of section of Fig. 2 [19]. There, the non-ergodic trajectories are represented by the closed blue and green curves, which are sections in the two-dimensional plane of the KAM tori [10, appendix 8] surrounding the stable trajectories $A$ and $C$. We numerically find that the fraction of the surface of section not occupied by tori is densely


FIG. 3. The histograms show the distribution of unfolded energy level spacings $s_{r, i}=\bar{N}_{r}\left(\epsilon_{i}+1\right)-\bar{N}_{r}\left(\epsilon_{i}\right)$ for states belonging to the three irreducible representations $r=A_{1}$ (top), $A_{2}$ (center), $E$ (bottom), which are analyzed separately. They differ from the Poisson (dotted golden line) and Wigner (dashed red line). They are well represented by the BerryRobnik distribution, assuming a single chaotic region in phase space, with parameter $\rho_{1}=0.6$ for all three representations.
covered by the intersections from a single ergodic trajectory, comprising the single ergodic zone visible on Fig. 2, within which lie the 3 unstable trajectories of family $B$.

## B. Quantum energy level statistics

The quantum spectra of systems with mixed classical phase space satisfy neither the Poisson nor the Wigner distribution [14, Sec. 16.8]. We now verify this for the model of Eq. (2), and show that its energy level statistics are well represented by a Berry-Robnik distribution [16].

We numerically solve the Schrödinger equation for the Hamiltonian of Eq. (11) using the finite-element software Freefem [27]. We calculate stationary states belonging to the three irreducible representations $A_{1}, A_{2}, E$ of the point group $C_{3 v}$ separately. We exploit discrete symmetries to reduce the configuration space to a triangle which slightly exceeds $1 / 6$ of the classically accessible region for a given energy: details are given in our previous paper [19, Appendix 2]. We use a triangular mesh comprising 1000 vertices along each edge. We thus numerically obtain the energies and wavefunctions for slightly more than 1200 consecutive energy levels for Representation $A_{1}, 1200$ levels for Representation $A_{2}$, and 1700 non-degenerate levels for Representation $E$, in energy windows centered on $7 C_{6} / R^{6}$.

For each irreducible representation $r=A_{1}, A_{2}$, and $E$ of $C_{3 v}$, we introduce the integrated density of states $N_{r}(\epsilon)$, which is the staircase-like function giving the number of stationary quantum states whose energies are smaller than $\epsilon$ [14, Sec. 16.2]. We describe its smooth
component $\bar{N}_{r}(\epsilon)$ through its Weyl expansion, accounting for discrete symmetries [28]. We retain the leadingorder term, proportional to $1 / \hbar^{2}$, and the first correction, proportional to $1 / \hbar$. We calculate the spacings $s_{r, i}=\bar{N}_{r}\left(\epsilon_{i}+1\right)-\bar{N}_{r}\left(\epsilon_{i}\right)$ between consecutive 'unfolded' energies $\bar{N}_{r}\left(\epsilon_{i}\right)$ [9, Sec. 5.4]. We plot their distribution on Fig. 3, where it is seen to differ from both the Poisson and the Wigner distributions [14, Secs. $16.3 \& 16.4]$, as expected for a system with mixed classical phase space.

Figure 3 shows that the distribution of unfolded energy level spacings is well represented by the Berry-Robnik distribution [16], assuming that a single chaotic region in phase space contributes to the statistics, with the same parameter $\rho_{1}=0.6$ for all three representations. Both the assumption of a single chaotic region and the value $\rho_{1}=0.6$, representing the fraction of the energy surface over which motion is regular, are compatible with the surface of section of Fig. 2, The applicability of the Berry-Robnik distribution hinges on the statistical independence of the regular and chaotic sequences of levels. Counter-examples have been identified, e.g. the hydrogen atom in a magnetic field [29], and its numerical verification with billiards requires reaching the deep semiclassical limit 30]. By contrast, our result provides a realization of the Berry-Robnik distribution in an experimentally accessible system involving smooth interatomic interactions rather than sharp billiard walls.

## IV. QUANTUM STATIONARY STATES LOCALIZED NEAR THE CLASSICALLY STABLE PERIODIC TRAJECTORIES $A$ AND $C$

For the majority of the stationary quantum states of the Hamiltonian $H_{2 \text { D }}$ that we have obtained numerically, the probability density $|\psi(x, y)|^{2}$ is not directly related to the periodic trajectories of types $A$ and $C$. Nevertheless, we find multiple eigenstates whose probability density is enhanced along one or the other of these trajectories. Figures 6( $a, b$ ) and 7(a,b) illustrate this phenomenon for trajectories $A$ and $C$, respectively: in each case, we show the probability density for the quantum states closest to the energy $\epsilon=7 C_{6} / R^{6}$. This phenomenon superficially resembles the quantum scars stemming from trajectory $B$ which we have identified in our previous article 19]. However, the quantum states we consider in the present article do not satisfy Heller's definition for a quantum scar [24, chap. 22]. Indeed, in stark contrast to the classically unstable trajectory $B$, trajectories $A$ and $C$ are both classically stable. Hence, quantum mechanics yields no qualitative change in the behavior of the system in their vicinity. In this section, we illustrate this statement with two results. First, calculating the energies of the quantum states related to trajectories $A$ and $C$ semiclassically, we justify that they obey selection rules which we entirely explain in terms of the symmetries of the classical KAM tori. Second, we construct semiclassical wavefunctions for these quantum states. Our semiclassical results for
both the energies and the wavefunctions are in excellent agreement with our full quantum calculation.

## A. Symmetries of the regular classical trajectories

We first consider the regular classical trajectories in the KAM regions of phase space surrounding the stable periodic trajectories of families $A$ and $C$. Our numerical results show that the tori lying close to the periodic trajectories inherit the discrete symmetry properties of the corresponding periodic trajectories, namely: (i) A torus $T_{A}$ near the periodic trajectory of type $A$ invariant under the reflection $\mathcal{S}$ exhibits reflection symmetry, i.e. if the point $(\boldsymbol{r}, \boldsymbol{p})$ belongs to $T_{A}$, then so does $(\mathcal{S} \boldsymbol{r}, \mathcal{S} \boldsymbol{p})$; (ii) A torus $T_{C}$ near a periodic trajectory of type $C$ is invariant under rotations $\mathcal{R}$ of order 3, i.e. if the point $(\boldsymbol{r}, \boldsymbol{p})$ belongs to $T_{C}$, then so does $(\mathcal{R} \boldsymbol{r}, \mathcal{R} \boldsymbol{p})$.

We justify properties (i) and (ii) through the following argument. We rely on an approximation introduced in Ref. [9, Sec. 4.1]: we ignore narrow instability subregions and approximate the whole KAM region by a set of concentric tori. Our numerical results for the surface of section, shown on Fig. 2, confirm that it is very well satisfied for the inner tori, close to the periodic trajectories, which are of interest in this work (it breaks down for the outer tori in the vicinity of the ergodic zone, which we do not consider). This allows for the introduction of local action-angle coordinates, valid within this region. These are defined through the consistent choice of fundamental frequencies $\boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}\right)$ [31, Sec. III.E] on each torus within the region. Then, any conditionally-periodic trajectory $(\boldsymbol{r}(t), \boldsymbol{p}(t))$ winding around one such torus may be written as a Fourier series [32, §52]:

$$
\begin{equation*}
\boldsymbol{r}(t)=\sum_{\boldsymbol{k}} \boldsymbol{r}_{\boldsymbol{k}} \exp (i \boldsymbol{k} \cdot \boldsymbol{\omega} t), \quad \boldsymbol{p}(t)=2 m R^{2} d \boldsymbol{r} / d t \tag{4}
\end{equation*}
$$

the sum being taken over all integer pairs $\boldsymbol{k}=\left(k_{1}, k_{2}\right)$. The considered torus is uniquely determined by its actions $\boldsymbol{J}=\left(J_{1}, J_{2}\right)$, which are given by 33]:

$$
\begin{equation*}
J_{\alpha}=\sum_{\alpha^{\prime}=1,2} \sum_{\boldsymbol{k}} k_{\alpha}\left|\boldsymbol{r}_{\boldsymbol{k}}\right|^{2} k_{\alpha^{\prime}} \omega_{\alpha^{\prime}} \tag{5}
\end{equation*}
$$

Let us justify statement (ii), concerning tori in the vicinity of a periodic trajectory of type $C$. We consider a point $(\boldsymbol{r}, \boldsymbol{p})$ belonging to the KAM region surrounding trajectory $C_{+}$, and the rotated point $\left(\boldsymbol{r}^{\prime}, \boldsymbol{p}^{\prime}\right)$ with $\boldsymbol{r}^{\prime}=\mathcal{R}_{2 \pi / 3} \boldsymbol{r}$ and $\boldsymbol{p}^{\prime}=\mathcal{R}_{2 \pi / 3} \boldsymbol{p}$. Trajectory $C_{+}$is invariant under rotations of order 3 , so that $\left(\boldsymbol{r}^{\prime}, \boldsymbol{p}^{\prime}\right)$ also belongs to the same KAM region. We compare the two trajectories $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ and $\left(\boldsymbol{q}^{\prime}(t), \boldsymbol{p}^{\prime}(t)\right)$ obtained from the initial conditions $(\boldsymbol{r}, \boldsymbol{p})$ and $\left(\boldsymbol{r}^{\prime}, \boldsymbol{p}^{\prime}\right)$. Their Fourier components $\boldsymbol{r}_{\boldsymbol{k}}$ and $\boldsymbol{r}_{\boldsymbol{k}}^{\prime}$, defined by Eq. (4), satisfy $\boldsymbol{r}_{\boldsymbol{k}}^{\prime}=\mathcal{R}_{2 \pi / 3} \boldsymbol{r}_{\boldsymbol{k}}$, so that $\left|\boldsymbol{r}_{\boldsymbol{k}}^{\prime}\right|=\left|\boldsymbol{r}_{\boldsymbol{k}}\right|$. According to Eq. (5), the actions $J_{\alpha}$ only depend on the modulus $\left|\boldsymbol{r}_{\boldsymbol{k}}\right|$, hence, they are the same for both trajectories. Therefore, the points $(\boldsymbol{r}, \boldsymbol{p})$ and $\left(\boldsymbol{r}^{\prime}, \boldsymbol{p}^{\prime}\right)$ belong to the same torus $T_{C_{+}}$. Statement (i) may be justified similarly.

## B. EBK quantization: energy levels

In this section, we obtain semiclassical predictions for the energies of the quantum levels related to trajectories $A$ and $C$, which are in excellent agreement with the values obtained through our numerical solution of the Schrödinger equation (see Figs. 4(b) and 5(c)). We also explain quasidegeneracies and derive selection rules, both of which are direct consequences of the discrete symmetries of the KAM tori presented in Sec. IV A above.

Our semiclassical description relies on Einstein-Brillouin-Keller (EBK) theory [12], accounting for the Maslov phase corrections [34, §7]. This theory generalizes the Wentzel-Kramers-Brillouin approach [35, §48] to the quantization of regular classical motion with more than one degree of freedom [13]. We use our own implementation as a Python script of the EBK approach, based on Refs. 31, 36], which hinges on the representation of conditionally-periodic motion in terms of the Fourier series of Eq. (4). We integrate classical trajectories over time intervals of lengths up to $t_{\max }=3700\left(m R^{8} / C_{6}\right)^{1 / 2}$ and keep up to 3200 terms in Eq. (4).

We now characterize the quantum stationary states localized near the classically stable trajectories $A$ and $C$. In Sections IV B 1 and IV B 2 below, we derive the EBK energies for these states, considered as eigenstates of $H_{2 \mathrm{D}}$, whose wavefunctions depend on $\boldsymbol{r}=(x, y)$. In Section IVB3, we analyze the role of angular momentum so as to discuss the stationary states of the three-particle Hamiltonian $H$, whose wavefunctions depend on $(x, y, z)$.

## 1. Quantum states localized near trajectory $A$

For a given energy $\epsilon$, the three periodic trajectories $A_{0}$, $A_{1}$, and $A_{2}$ (see Fig. $\mathbb{1}(\mathrm{a})$ ) and the tori surrounding them are mapped one onto the other through the rotations $\mathcal{R}$ and $\mathcal{R}^{-1}$. Hence, we focus on the vertical trajectory $A_{0}$. In Eq. (4), we choose the fundamental frequencies $\boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}\right)$ as in Ref. [31, Fig. 8(b)]. This leads to the independent circuits $\mathcal{C}_{x}$ and $\mathcal{C}_{y}$ on Fig. 4(a). Calculating their Maslov indices [13, Sec. II.C], we obtain the EBK quantization condition for the tori near trajectory $A$ :

$$
\begin{equation*}
I_{x}=\hbar\left(\nu_{x}+1 / 2\right) \text { and } I_{y}=\hbar\left(\nu_{y}+1\right) \tag{6}
\end{equation*}
$$

where $I_{x, y}$ are the action integrals for the circuits $\mathcal{C}_{x, y}$, $\hbar$ is the reduced Planck's constant, and the integers $\nu_{x, y} \geq 0$ are the EBK quantum numbers. The action $I_{x} \geq \hbar / 2$, so that the periodic trajectory $A_{0}$ itself does not satisfy Eq. (6). The tori satisfying Eq. (6) which are closest to trajectory $A_{0}$ are those with $\nu_{x}=0$ : the corresponding energies within a window centered on $\epsilon=7 C_{6} / R^{6}$ are shown on the top line of Fig. 4(b). We compare them to the energies of the stationary quantum states of $H_{2 \mathrm{D}}$ belonging to representations $A_{1}$ and $E$ localized near the trajectories $A_{0}, A_{1}$, and $A_{2}$, obtained through our finite-element calculations (see Fig. 6(a,b)).

These are shown on Fig. 4(b), middle and bottom lines, and are in excellent agreement with the EBK results.

Figure 4 (b) reveals that each EBK energy corresponds to quasidegenerate quantum states pertaining to representations $A_{1}$ and $E$. Furthermore, no quantum stationary states pertaining to representation $A_{2}$ exhibit density profiles similar to Fig. 6(a,b). Both of these properties follow from the symmetries of the regular trajectories identified in Sec. IV A above, through a mechanism identified in Refs. 37] and [9, Sec. 4.2] in the case where the discrete symmetry at play had order 2 . The system we consider provides examples of the same phenomenon involving $C_{3 v}$ symmetry, as we now show.

We consider the EBK wavefunction $\psi_{\text {EBK }}(\boldsymbol{r})$, corresponding to a torus in the vicinity of trajectory $A_{0}$, with the energy $\epsilon_{\text {EBK }}$, satisfying Eq. (6) with $\nu_{x}=0$. This torus is invariant under the reflection $\mathcal{S}$ about the vertical axis $x=0$. Therefore, as shown in [9, Sec. 4.2]:

$$
\begin{equation*}
\psi_{\mathrm{EBK}}(\mathcal{S} \boldsymbol{r})=(-1)^{\nu_{x}} \psi_{\mathrm{EBK}}(\boldsymbol{r})=\psi_{\mathrm{EBK}}(\boldsymbol{r}) \tag{7}
\end{equation*}
$$

The EBK wavefunction $\psi_{\text {EBK }}$ reflects the symmetry of the corresponding classical torus, but does not automatically satisfy the symmetry requirements of any representation. We now project it onto the irreducible representations [35, §94] $A_{1}, A_{2}$, and $E$. This yields three linearly independent wavefunctions, $\psi_{\mathrm{EBK}}^{A_{1}}$ and $\psi_{\mathrm{EBK}}^{E, \pm}$, pertaining to the representations $A_{1}$ and $E$, corresponding to the same semiclassical energy. In terms of kets $|\psi\rangle$, with $\langle\boldsymbol{r}| \mathcal{R}|\psi\rangle=\psi\left(\mathcal{R}^{-1} \boldsymbol{r}\right)$ and $\langle\boldsymbol{r}| \mathcal{S}|\psi\rangle=\psi(\mathcal{S} \boldsymbol{r})$, they read:

$$
\left\{\begin{array}{l}
\left|\psi_{\mathrm{EBK}}^{A_{1}}\right\rangle=\alpha_{A_{1}}\left(1+\mathcal{R}+\mathcal{R}^{-1}\right)\left|\psi_{\mathrm{EBK}}\right\rangle  \tag{8}\\
\left|\psi_{\mathrm{EBK}}^{E,+}\right\rangle=\alpha_{E}\left(1+j^{*} \mathcal{R}+j \mathcal{R}^{-1}\right)\left|\psi_{\mathrm{EBK}}\right\rangle \\
\left|\psi_{\mathrm{EBK}}^{E,-}\right\rangle=\alpha_{E}\left(1+j \mathcal{R}+j^{*} \mathcal{R}^{-1}\right)\left|\psi_{\mathrm{EBK}}\right\rangle
\end{array}\right.
$$

In Eq. (8), $\alpha_{A_{1}, E}$ are normalization coefficients, and $j=e^{2 i \pi / 3}$. We have used the relations $\mathcal{S R S}=$ $\mathcal{R}^{-1}$ and Eq. (7). The states $\left|\psi_{\text {EBK }}^{A_{1}}\right\rangle$ and $\left|\psi_{\text {EBK }}^{E, \pm}\right\rangle$ satisfy $\mathcal{R}\left|\psi_{\mathrm{EBK}}^{A_{1}}\right\rangle=\left|\psi_{\mathrm{EBK}}^{A_{1}}\right\rangle, \mathcal{R}\left|\psi_{\mathrm{EBK}}^{E, \pm}\right\rangle= \pm j\left|\psi_{\mathrm{EBK}}^{E, \pm}\right\rangle$ and $\left|\psi_{\mathrm{EBK}}^{E,-}\right\rangle=\mathcal{S}\left|\psi_{\mathrm{EBK}}^{E,+}\right\rangle$. The component of $\psi_{\text {EBK }}$ pertaining to representation $A_{2}$, proportional to $\left(1+\mathcal{R}+\mathcal{R}^{-1}\right)(1-$ $\mathcal{S})\left|\psi_{\mathrm{EBK}}\right\rangle$, is 0 because of Eq. (7).

## 2. Quantum states localized near trajectory $C$

We proceed as in Sec.IVB 1. For a given energy $\epsilon$, the two periodic trajectories $C_{+}$and $C_{-}$(see Fig. 1(a)) and the tori surrounding them are mapped onto each other through the reflection $\mathcal{S}$. Hence, we focus on the trajectory $C_{+}$. In Eq. (4), we choose the fundamental frequencies $\boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}\right)$ as in Ref. 31, Fig. 8(a)], leading to the independent circuits $\mathcal{C}_{r}$ and $\mathcal{C}_{l}$ on Fig. 5(a). Calculating their Maslov indices, we obtain the EBK quantization condition for the tori near trajectory $C$ :

$$
\begin{equation*}
I_{r}=\hbar\left(\nu_{r}+1 / 2\right) \text { and } I_{l}=\hbar\left(\nu_{l}+1 / 2\right) \tag{9}
\end{equation*}
$$



FIG. 4. (a) Classical trajectory $A$ (solid dark blue) for the energy $\epsilon=7 C_{6} / R^{6}$, the nearest-energy trajectory satisfying Eq. (6) for $\eta=0.01$ (densely covering the light blue area), and two independent circuits $\mathcal{C}_{x}$ (dotted purple) and $\mathcal{C}_{y}$ (dotted red) circling the torus, in terms of which the quantum numbers are $\nu_{x}=0$ and $\nu_{y}=300$. The dashed gray lines show the caustics of this trajectory. The top left inset zooms in on the narrow region near $(x=0, y=1.2)$ to reveal the self-intersection of the caustics. (b) Top panel: energies of the EBK wavefunctions for $\nu_{x}=0$ and $295 \leq \nu_{y} \leq 305$. Center and bottom panels: energies of the corresponding quasidegenerate quantum stationary states belonging to representations $A_{1}$ (center) and $E$ (bottom), obtained through our finite-element numerical calculations. Because of the torus symmetries, there are no states in representation $A_{2}$ corresponding to the EBK quantum numbers $\left(\nu_{x}=0, \nu_{y}\right)$. The integers in the center and bottom panels specify the relative state indices within each representation, $\Delta \nu^{A_{1}}$ and $\Delta \nu^{E} / 2$, with respect to the quantum state related to Trajectory $A$ whose energy is closest to $7 C_{6} / R^{6}$. (c) Small energy differences between the quasidegenerate states of representations $A_{1}$ and $E$.
where $I_{r, l}$ are the action integrals for the circuits $\mathcal{C}_{r, l}$, and the integers $\nu_{r, l} \geq 0$ are the EBK quantum numbers. The trajectory $C_{+}$does not satisfy Eq. (9). The tori satisfying it which are closest to $C_{+}$are those with $\nu_{r}=0$ : their energies are shown on the top line of Fig. 5(b). We compare them to the energies of the stationary quantum states of $H_{2 \mathrm{D}}$ belonging to representations $A_{1}, A_{2}$, and $E$ localized near the trajectories $C_{+}$and $C_{-}$, obtained through our finite-element calculations (see Fig. 7(a,b)). These are shown on the three lower lines of Fig. 5(b), and are in excellent agreement with the EBK results.

Figure (5) shows that each EBK energy with $\nu_{r}=0$ and $\nu_{l}=0$ modulo 3 corresponds to two quasidegenerate quantum states pertaining to representations $A_{1}$ and $A_{2}$. By contrast, each EBK energy with $\nu_{r}=0$ and $\nu_{l} \neq 0$ modulo 3 corresponds to two exactly degenerate quantum states spanning a representation $E$. As for the states localized near trajectory $A$ (see Sec.IV B 1above), these properties follow from the symmetries of the regular trajectories (Sec. IV A). These are different from the symmetries of the tori surrounding trajectory $A$, leading to different selection rules, which we now derive.

We consider the EBK wavefunction $\chi_{\mathrm{EBK}}(\boldsymbol{r})$, corresponding to a torus in the vicinity of trajectory $C_{+}$, with the energy $\epsilon_{\text {EBK }}$, satisfying Eq. (9) with $\nu_{r}=0$. This torus is invariant under the rotation $\mathcal{R}$. A straightforward generalization of the argument in Ref. [9, Sec. 4.2] to symmetry operations of order 3 leads to $\chi_{\mathrm{EBK}}(\mathcal{R} r)=$ $j^{\nu_{l}} \chi_{\text {EBK }}(\boldsymbol{r})$. We now project $\chi_{\text {EBK }}$ onto the irreducible representations $A_{1}, A_{2}$, and $E$. For each $\nu_{l}$, this yields two linearly independent, degenerate EBK wavefunctions. If $\nu_{l}=0$ modulo 3 , the non-vanishing wavefunctions pertain to representations $A_{1}$ and $A_{2}$ :

$$
\begin{equation*}
\left|\chi_{\mathrm{EBK}}^{A_{1}, A_{2}}\right\rangle=\beta_{A_{1}, A_{2}}(1 \pm \mathcal{S})\left|\chi_{\mathrm{EBK}}\right\rangle, \tag{10}
\end{equation*}
$$

with $\beta_{A_{1}, A_{2}}$ being two normalization factors, whereas the component along $E$ vanishes. By contrast, if $\nu_{l} \neq 0$ modulo 3 , the components along $A_{1}$ and $A_{2}$ vanish, whereas the two non-vanishing wavefunctions $\left|\chi_{\mathrm{EBK}}^{E, \pm}\right\rangle$ span a representation $E$. For $\nu_{l}=-1$ modulo $3,\left|\chi_{\mathrm{EBK}}^{E,+}\right\rangle=\left|\chi_{\mathrm{EBK}}\right\rangle$ and $\left|\chi_{\mathrm{EBK}}^{E,-}\right\rangle=\mathcal{S}\left|\chi_{\mathrm{EBK}}\right\rangle$, and the opposite assignment holds for $\nu_{l}=+1$ modulo 3 .

## 3. The role of angular momentum

To discuss the three-particle eigenstates of $H$ in terms of the eigenstates of $H_{2 \mathrm{D}}$ identified in Secs. IV B 1 and IV B 2 we now analyze the role of angular momentum.

We first consider quantum states localized near the periodic trajectories of family $A$. The two states $\psi_{\nu_{y}}^{E, \pm}(\boldsymbol{r})$ obtained for a given $\nu_{y}$, are exactly degenerate eigenstates of $H_{2 \mathrm{D}}$ which span a 2 D representation $E$. However, in terms of three-atom eigenstates of $H$, the states $\psi_{\nu_{y}}^{E, \pm}(\boldsymbol{r}) e^{i n z}$ occur if the total angular momentum $n=$干1 modulo 3 because of Eq. (3).

The states $\psi_{\nu_{y}}^{A_{1}}(\boldsymbol{r})$ and $\psi_{\nu_{y}}^{E, \pm}(\boldsymbol{r})$ obtained for a given $\nu_{y}$ belong to different representations $A_{1}$ and $E$. Their quasidegeneracy is lifted by small couplings neglected in the EBK approach [9, Sec. 4.5], and the small energy difference is resolved in our finite-element numerical results, as shown on Fig. [4(c). Because of Eq. (3), the three-atom states $\psi_{\nu_{y}}^{A_{1}}(\boldsymbol{r}) e^{i n z}$ occur if $n=0$ modulo 3, so that none of the three states $\psi_{\nu_{y}}^{A_{1}, E \pm}(\boldsymbol{r}) e^{i n z}$ may occur for the same value of $n$. They do not reduce to an EBK wavefunction corresponding to a single classical trajectory. Instead, Eq. (8) shows that they represent coherent superpositions of the three atoms undergoing motion near the trajectories $A_{0}, A_{1}$, and $A_{2}$.


FIG. 5. (a) Classical trajectory $C$ (solid dark green) for the energy $\epsilon=7 C_{6} / R^{6}$, the nearest-energy trajectory satisfying Eq. (9) for $\eta=0.01$ (densely covering the light green area), and two independent circuits $\mathcal{C}_{r}$ (purple) and $\mathcal{C}_{l}$ (red) circling the torus, in terms of which the quantum numbers are $\nu_{r}=0, \nu_{l}=267$. The dashed gray lines show the caustics of this trajectory, which self-intersect in the top left, top right, and bottom regions. (b) Top panel: energies of the EBK wavefunctions for $\nu_{r}=0$ and $262 \leq \nu_{l} \leq 271$. Three lower panels: energies of the corresponding quantum eigenstates belonging to representations $A_{1}, A_{2}$, and $E$, obtained through our finite-element numerical calculations. States in representations $A_{1}$ and $A_{2}$ exhibit quasidegeneracies and correspond to the EBK quantum numbers $\nu_{r}=0, \nu_{l}=0$ modulo 3 ; each EBK torus with quantum numbers $\nu_{r}=0, \nu_{l} \neq 0$ modulo 3 yields two degenerate states in representation $E$. The integers specify the relative state indices within each representation, $\Delta \nu^{A_{1}}, \Delta \nu^{A_{2}}$ and $\Delta \nu^{E} / 2$, with respect to the quantum state related to trajectory $C$ whose energy is closest to $7 C_{6} / R^{6}$. (c) Small energy differences between the quasidegenerate states of representations $A_{1}$ and $A_{2}$.

We now turn to quantum states localized near the periodic trajectories of family $C$. The two states $\chi_{\nu_{l}}^{E, \pm}(\boldsymbol{r})$, obtained for a given $\nu_{l} \neq 0$ modulo 3 , are exactly degenerate. The three-atom states $\chi_{\nu_{l}}^{E, \pm}(\boldsymbol{r}) e^{i n z}$ occur for $n=\mp 1$ modulo 3 , and opposite values of $n$ lead to atoms rotating along $C$ in opposite directions. The two states $\chi_{\nu_{l}}^{A_{1}, A_{2}}(\boldsymbol{r})$ obtained for a given $\nu_{l}=0$ modulo 3 belong to different representations and, hence, are quasidegenerate: their small energy difference is shown on Fig. 5(c). The three-atom states $\chi_{\nu_{l}}^{A_{1}, A_{2}}(\boldsymbol{r}) e^{i n z}$ may occur for the same value of $n=0$ modulo 3 .

## C. EBK quantization: wavefunctions

To further illustrate the applicability of the EBK approach to the quantum states localized near the stable periodic trajectories of families $A$ and $C$, we construct primitive EBK wavefunctions for these states [38]. We focus on a given KAM torus satisfying the quantization conditions of either Eq. (6) or Eq. (9), depending on whether it lies near a trajectory of family $A$ or $C$. To obtain the corresponding EBK wavefunctions $\psi_{\text {EBK }}$ and $\chi_{\text {EBK }}$ of sections IV B 1 and IV B 2 above, the key extra required step with respect to the approach of Refs. 31, 36] is to describe the torus in terms of multiple sheets on each of which the classical momentum is univalued 13, Sec. III.A]. These sheets join along the caustics of the classical trajectory in the $(x, y)$ plane, shown as the dashed gray lines on Figs. 4(a) and 5(a). The caustics self-intersect, signalling the occurrence of catastrophes [39], and the torus sheetings must be constructed accordingly. We find that 12 sheets are required to describe tori near a trajectory of family $A$ with $\nu_{x}=0$, and that 6 sheets are required to describe tori near a trajectory of
family $C$ with $\nu_{r}=0$. We then obtain the wavefunctions $\psi_{\text {EBK }}$ and $\chi_{\text {EBK }}$ from the Fourier series of Eq. (4), in terms of linear superpositions of the contribution of each sheet [13, III.C]. Finally, we project $\psi_{\text {EBK }}$ and $\chi_{\text {EBK }}$ onto the irreducible representations $A_{1}, A_{2}$, and $E$.

Figure 6( $\mathbf{6}, \mathrm{d}$ ) shows the resulting EBK wavefunctions for the quasidegenerate quantum states $\psi^{A_{1}, E}(\boldsymbol{r})$ localized near the trajectories of family $A$ whose energies are closest to $7 C_{6} / R^{6}$. We compare them to the corresponding wavefunctions obtained through our finite-element numerical calculations (Fig. 6(a,b)). We show the analogous results for the states $\chi^{A_{1}, A_{2}}(\boldsymbol{r})$, localized near the trajectories of family $C$, on figure 6. The agreement between the finite-element and EBK results is excellent, including in the catastrophe regions where the classical caustics self-intersect, shown in the upper left insets.

Primitive EBK wavefunctions do not account for the quantum penetration of the wavefunctions through the caustics. Instead, they diverge along the caustics as in the WKB approach [35, §46] and vanish outside the classical torus, as illustrated on Figs. 8and 9 in the appendix. This causes the two limitations of the EBK wavefunctions considered here. First, interference phenomena involving decaying waves outside the torus are not captured: the top left insets of Fig. 77 provide an example. Second, the divergence of the wavefunctions leads to numerical inaccuracies near the caustics which hinder their normalization. Hence, each of our EBK wavefunctions matches the finite-element wavefunction up to an overall normalization factor of order 2 . We eliminate it by scaling the EBK wavefunction so that it matches the finite-element result at one single point chosen far from the caustics. The quantum penetration through the caustics may be accounted for, and hence both limitations be overcome, using a uniform approximation to the wavefunction 40,


FIG. 6. Quantum states localized near the trajectories of family $A$. (a,b) Wavefunction densities $\left|\psi^{A_{1}}(\boldsymbol{r})\right|^{2}$ and $\left|\psi^{E}(\boldsymbol{r})\right|^{2}$ for the two quasidegenerate eigenstates of $H_{2 \mathrm{D}}$ localized near the periodic trajectories of family $A$ whose energies are closest to $C_{6} / R^{6}$, obtained through our finite-element numerical calculations. ( $c, d$ ) The corresponding squared EBK wavefunctions $\left|\psi_{\text {EBK }}^{A_{1}}(\boldsymbol{r})\right|^{2}$ and $\left|\psi_{\text {EBK }}^{E}(\boldsymbol{r})\right|^{2}$, built from the KAM torus satisfying Eq. (6) with $\nu_{x}=0, \nu_{y}=300$ (see Fig. [4(a)). On all four panels, the left inset details the region where the caustics self-intersect, and the right one shows the region near $(x=0, y=0)$.

Sec. 7.2]. This goes beyond the scope of the present work.

## V. EXPERIMENTAL PROSPECTS AND OUTLOOK

The effects considered here may be realized e.g. on the system already considered in Ref. [19]: ${ }^{87} \mathrm{Rb}$ atoms in the circular Rydberg state $50 C$, for which $C_{6} / h=$ $3 \mathrm{GHz} \mu \mathrm{m}^{6}$. Then, the value $\eta=0.01$ is achieved in a circular trap of radius $R=7 \mu \mathrm{~m}$. The energy $\epsilon=7 C_{6} / R^{6}=$ $h \times 200 \mathrm{kHz}$ is within experimental reach. For these parameters, the periodic trajectories of families $A, B$, and $C$ all have periods of the order of 1 ms . The position of the atoms may be detected at a given time by turning on a 2D optical lattice to freeze the dynamics, followed by atomic deexcitation and site-resolved ground state imaging. We focus on realizations where the total three-atom angular momentum $n$ is well defined.

A key difference between the quantum scar of Ref. 19]
and the localization near stable orbits considered here concerns the timescale over which quantum particles follow the classical periodic trajectories. For the quantum scar, the timescale over which quantum particles follow the classically unstable periodic trajectory is expected to depend on its inverse Lyapunov exponent [24, ch. 22]. No such constraint exists for the dynamics near a classically stable orbit, so that recurrences of the initial state may be sought for over the lifetime of the trapped atoms.

Next, we point out a consequence of quantum coherence. According to Sec. IV B3 the quantum states localized near the trajectories of family $A$ are equal-weight superpositions of states localized near the three periodic trajectories of family $A$ (rather than just one trajectory). This is the impact of bosonic symmetry. By contrast, motion along a single trajectory $C_{+}$or $C_{-}$may be observed.

The following point warrants further investigation. Three atoms launched with angular momentum $n=0$ modulo 3 near the periodic trajectory $C_{+}$may undergo dynamical tunneling [41] to the trajectory $C_{-}$. The ex-


FIG. 7. Quantum states localized near the trajectories of family $C$. (a,b) Wavefunction densities $\left|\chi^{A_{1}}(\boldsymbol{r})\right|^{2}$ and $\left|\chi^{A_{2}}(\boldsymbol{r})\right|^{2}$ for the two quasidegenerate eigenstates of $H_{2 \mathrm{D}}$ localized near the periodic trajectories of family $C$ whose energies are closest to $C_{6} / R^{6}$, obtained through our finite-element numerical calculations. ( $c, d$ ) The corresponding squared EBK wavefunctions $\left|\chi_{\mathrm{EBK}}^{A_{1}}(\boldsymbol{r})\right|^{2}$ and $\left|\chi_{\mathrm{EBK}}^{A_{2}}(\boldsymbol{r})\right|^{2}$, built from the KAM torus satisfying Eq. (9) with $\nu_{r}=0$ and $\nu_{l}=267$ (see Fig. 5(a)). On all four panels, the left inset details the region where the caustics self-intersect, and the right one shows the region near $(x=0, y=0.4)$.
pected oscillation period, set by $h /\left(\epsilon_{\nu_{l}, A_{2}}-\epsilon_{\nu_{l}, A_{1}}\right)$, is $\sim 25$ s for the parameters of Fig. 5(c). This very long timescale is out of reach of current setups, but should become accessible in new experiments currently under construction promising atomic lifetimes $\sim 1$ minute 42, 43]. Furthermore, the period may be minimized by varying the energy $\epsilon$ and the parameter $\eta$. Dynamical tunneling has already been observed for non-interacting, periodically-driven atoms 44, 45]. The system we consider would provide an example involving interacting atoms described by a time-independent Hamiltonian.

## VI. CONCLUSION

We have revisited the system of three interacting bosonic particles in a circular trap that we had first considered in Ref. [19]. We have illustrated the mixed nature of its classical phase space, and shown that the statistics of the quantum levels are well described by a Berry-

Robnik distribution. We have analyzed the symmetries of the quantum states localized along the classically stable periodic trajectories $A$ and $C$, calculated their energies semiclassically using EBK theory, and constructed the corresponding EBK wavefunctions. Our semiclassical EBK results, regarding both the energies and the wavefunctions, are in excellent agreement with the quantum eigenstates and energies which we have obtained through finite-element numerical calculations. Thus, the considered system hosts both a quantum scar, analyzed in Ref. [19], and classical localization near stable periodic orbits, analyzed in the present work. These phenomena, all within experimental reach, occur in the same energy range: to observe one or the other, one simply adapts the initial conditions so as to launch the three atoms along a classical periodic orbit which is either unstable or stable. Hence, the system we propose appears promising in view of a detailed experimental comparison between quantum scars and classically localized states.

## Appendix A: Comparison between Schrödinger and EBK wavefunctions

The supplementary figures 8 and 9 on the next page compare the behavior of the EBK wavefunctions to those obtained by solving the Schrödinger equation for the Hamiltonian $H_{2 \text { D }}$ through finite-element numerics along the horizontal and vertical axes. They show excellent agreement between the two approaches, and highlight the key limitation of the EBK wavefunctions: the quantum
penetration through the caustics is not accounted for, and is replaced by a divergence along the caustics.

## ACKNOWLEDGMENTS

We acknowledge stimulating discussions with M. Brune and J.M. Raimond (LKB, Collège de France), F. Dunlop (LPTM, Cergy-Pontoise), and R.J. Papoular (IRAMIS, CEA Saclay).
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FIG. 8. Quantum states localized near the trajectories of family $A$. Comparison of the EBK $(a, b)$ wavefunction $\psi_{\mathrm{EBK}}^{A_{1}}$ and ( $c, d$ ) density $\left|\psi_{\text {EBK }}^{E}\right|^{2}$ (green) with the corresponding quantities obtained through finite-element numerics (red) shown on Fig. 4 along the horizontal $(a, c)$ and vertical $(b, d)$ axes. The insets illustrate their behaviour near the caustics (vertical dashed gray lines). Each EBK wavefunction has been scaled to match the finite-element wavefunction at the point $(x=0, y=0.5)$.


FIG. 9. Quantum states localized near the trajectories of family $C$. Comparison of the EBK wavefunctions (green) ( $a, b$ ) $\chi_{\mathrm{EBK}}^{A_{1}}$ and ( $c, d$ ) $\chi_{\mathrm{EBK}}^{A_{2}}$ and the corresponding wavefunctions obtained through finite-element numerics (red) shown on Fig. 5] along the horizontal $(a, c)$ and vertical $(b, d)$ axes. The insets illustrate their behaviour near the caustics (vertical dashed gray lines). Each EBK wavefunction has been scaled to match the finite-element wavefunction at the point $(x=0.5, y=0)$.


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