Computing fundamental domains
for arithmetic Kleinian groups

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A fundamental domain for $\mathrm{PSL}_2\big(\mathbb{Z}\big[\sqrt{-5}\big]\big)$

Acknowledgements

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List of Algorithms

Introduction

In this thesis we present an algorithm which, given an arithmetic Kleinian group Γ, returns a fundamental domain and a finite presentation for Γ with a computable isomorphism. This problem lies on the boundary between hyperbolic geometry and number theory. On one side, there are many mysteries left with hyperbolic 3-manifolds; the algorithm described in this thesis might be used to experimentally investigate conjectures about them. On the other side, the units of a semisimple algebra over a number field are, in the words of Mazur, the "gems" of algebraic number theory; an application of our algorithm is to compute the structure of the unit group in a class of quaternion algebras over number fields.

In the first part, we give the theoretical material needed, omitting most of the proofs. In the second part we give a complete description of the algorithm. In the third part we give examples of the computations performed with the implementation in Magma [BCP97] of this algorithm. In the last part we give some possible applications and generalizations of this algorithm as well as open problems.

Notations and conventions

We write $\mathcal{M}_2$ for the 2×2 matrix algebra, $\mathrm{SL}_2 = \{g \in \mathcal{M}_2 \mid \det g = 1\}$ for the special linear group, and $P : \mathrm{SL}_2 \to \mathrm{PSL}_2 = \mathrm{SL}_2 /\{\pm 1\}$ for the projection onto the projective special linear group.

Let G be a group. The identity element is written 1. If $H \subset G$ is a subgroup and $g_1, g_2 \in G$, we write $g_1 \equiv g_2 \pmod{H} \Leftrightarrow g_1 g_2^{-1} \in H \Leftrightarrow H g_1 = H g_2$. If $S \subset G$ is a subset, we write $\langle S \rangle$ for the subgroup of G generated by S, and for all $g_1, g_2 \in G$, we also write $g_1 \equiv g_2 \pmod{S} \Leftrightarrow g_1 g_2^{-1} \in \langle S \rangle$. For all $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ the commutator of g and h. If G acts on a set X and $x \in X$, $G_x = \{g \in G \mid g \cdot x = x\}$ denotes the stabilizer of x in G. Note that Int will **not** denote the group of interior automorphisms.

Every *algebra* (resp. *ring*) is an associative unital algebra (resp. an associative unital ring). For R a ring, $R^\times = \{x \in R \mid \exists y \in R, \ xy = 1\}$ denotes the unit group of R.

Part I

Kleinian groups and arithmetic

1 Hyperbolic geometry and Kleinian groups

In this section hyperbolic 3-space and Kleinian groups are introduced.

1.1 The upper half-space

Definition 1.1.1. The *upper half-space* is the set $\mathcal{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$ equipped with the metric induced by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2}{t^2}$$

where $(z, t) \in \mathcal{H}^3$, $z = x + iy$ and $t > 0$. For $x, y \in \mathcal{H}^3$, $\mathrm{d}(x, y)$ is the distance between x and y induced by $\mathrm{d}s$. The set $\widehat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ is called the *sphere at infinity*, and we define the *completed upper half-space* to be $\widehat{\mathcal{H}}^3 = \mathcal{H}^3 \cup \widehat{\mathbb{C}}$.
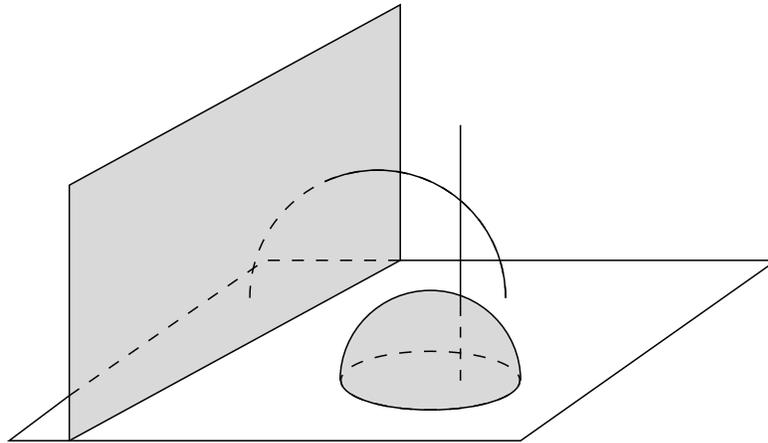
Figure 1.1: The upper half-plane, two geodesics and two geodesic planes

The space $\mathcal{H}^3$ is equipped with a volume induced by the metric, given by

$$\mathrm{d}V = \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t}{t^3}$$

where $(z, t) \in \mathcal{H}^3$, $z = x + iy$ and $t > 0$.

We recall some basic facts about $\mathcal{H}^3$. For details, the reader can refer to [Mar07, section 1]. Note that $\widehat{\mathcal{H}}^3$ can be naturally embedded into $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$ by mapping $\mathbb{C}$ to $\mathbb{C} \times \{0\}$ and ∞ to ∞, and we get a natural embedding $\mathbb{C} \hookrightarrow \widehat{\mathcal{H}}^3$. The space $\mathcal{H}^3$ has the following properties:

- The metric space $\mathcal{H}^3$ is connected, simply connected, and has constant curvature -1.

- The geodesics in $\mathcal{H}^3$ are the Euclidean hemicircles and the straight lines, orthogonal to $\mathbb{C}$.

- The geodesic planes in $\mathcal{H}^3$ are the Euclidean hemispheres and the Euclidean planes, orthogonal to $\mathbb{C}$.

- The metric space $\mathcal{H}^3$ is complete.

- The topology on $\mathcal{H}^3$ induced by its metric is the same as the topology induced by the Euclidean metric on $\mathbb{C} \times \mathbb{R}_{>0}$.

- For any distinct $x, y \in \widehat{\mathcal{H}}^3$, there is a unique geodesic passing through x and y.

Since the topology on $\mathcal{H}^3$ is the same as the Euclidean one, we can equip the set $\widehat{\mathcal{H}}^3 \setminus \{\infty\}$ with the Euclidean topology and finally define a fundamental system of neighborhoods of the point at infinity ∞ to be the sets of the form $\{(z, t) \in \mathcal{H}^3 \mid |z|^2 + t^2 > A\} \cup \{\infty\}$ for $A > 0$ to get a topology on $\widehat{\mathcal{H}}^3$.

If two geodesic planes intersect in $\mathcal{H}^3$, then there is a well-defined angle between them. When the planes are tangent on the sphere at infinity, they no longer intersect but we can still define the angle between them to be zero.

Remark 1.1.2. The metric space $\mathcal{H}^3$ is a model of the *hyperbolic 3-space* i.e. the unique connected, simply connected metric space with constant curvature -1.

Definition 1.1.3. A *segment* s in $\widehat{\mathcal{H}}^3$ is the closure in $\widehat{\mathcal{H}}^3$ of a nonempty, connected, open subset s' of a geodesic. By the last property above it is uniquely determined by the points $x, y \in \widehat{\mathcal{H}}^3$ such that $s = s' \cup \{x, y\}$ and we then write $s = [x, y]$. A subset $X \subset \widehat{\mathcal{H}}^3$ is *convex* if it contains every segment $[x, y]$ for $x, y \in X$.

1.2   The Poincaré extension

The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathcal{H}^3$ by the Poincaré extension, which we recall briefly. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{C}}$ by linear fractional transformation. An element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ can be written as a product of an even number of inversions in Euclidean circles and reflections in Euclidean lines in $\mathbb{C}$. Each such circle extends to a unique Euclidean hemisphere orthogonal to $\mathbb{C}$ and each such line extends to a unique Euclidean plane orthogonal to $\mathbb{C}$. The Poincaré extension of γ is the corresponding product of inversions in hemispheres and reflections in planes.

We recall some basic facts about this group action. For details, the reader can refer to [Mar07, section 1].

- The action of $\mathrm{PSL}_2(\mathbb{C})$ is faithful on $\mathcal{H}^3$, and transitive on the set of geodesic planes;

- Any $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ acts on $\mathcal{H}^3$ as an isometry;

- This action induces an isomorphism between the group $\mathrm{PSL}_2(\mathbb{C})$ and the group of orientation-preserving isometries of $\mathcal{H}^3$.

1.3 Classification of elements

In this section we recall geometric properties of the action of $\mathrm{PSL}_2(\mathbb{C})$; proofs can be found in [Mar07, section 1].

Let $g \in \mathrm{PSL}_2(\mathbb{C})$. The trace of g is defined up to sign, and if $g \neq \pm 1$ then the characteristic polynomials $X^2 \pm \mathrm{tr}(g)X + 1$ of the two liftings of g in $\mathrm{SL}_2(\mathbb{C})$ are also their minimal polynomials. Consequently, we have the following classification of conjugacy classes in $\mathrm{PSL}_2(\mathbb{C})$.

- If $\mathrm{tr}(g) \in \mathbb{C} \setminus [-2, 2]$, then g has two distinct fixed points in $\widehat{\mathbb{C}}$, no fixed point in $\mathcal{H}^3$ and stabilizes the geodesic between its fixed points, called its *axis*. The element g is conjugate to $\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $|\lambda| > 1$; it is *loxodromic*.

- If $\mathrm{tr}(g) \in (-2, 2)$, then g has two distinct fixed points in $\widehat{\mathbb{C}}$, and fixes every point in the geodesic between these two fixed points. The elements g is conjugate to $\pm \begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{pmatrix}$ with $\theta \in \mathbb{R} \setminus (\pi + 2\pi\mathbb{Z})$; it is *elliptic*.

- If $\mathrm{tr}(g) = \pm 2$, then g has one fixed point in $\widehat{\mathbb{C}}$ and no fixed point in $\mathcal{H}^3$. It is conjugate to $\pm \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with $\beta \in \mathbb{C}$; it is *parabolic*.

We will not define precisely what the following remark means, but it helps understanding the action of $\mathrm{PSL}_2(\mathbb{C})$.

Remark 1.3.1. A loxodromic element has two fixed points in $\widehat{\mathbb{C}}$, one is attracting and the other repelling. An elliptic element g also has two fixed points in $\widehat{\mathbb{C}}$, and the derivative of g has absolute value 1 at each of them. A parabolic element g has one fixed point in $\widehat{\mathbb{C}}$, and the derivative of g has absolute value 1 at this fixed point.

We recall the description of some standard stabilizers.

- The stabilizer of the point $(0, 1) \in \mathbb{C} \times \mathbb{R}_{>0}$ in $\mathrm{PSL}_2(\mathbb{C})$ is the subgroup $\mathrm{PSU}_2(\mathbb{C})$;

- The stabilizer of the point at infinity $\infty \in \widehat{\mathbb{C}}$ in $\mathrm{PSL}_2(\mathbb{C})$ is the subgroup $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}^\times,\ \beta \in \mathbb{C} \right\}$;

- Consider the vertical geodesic $L = \{0\} \times \mathbb{R}_{>0} \subset \mathcal{H}^3$. Then the subgroup $G = \{g \in \Gamma \mid g(L) \subset L\}$ of the elements stabilizing L decomposes as $G = T \sqcup ST$ where $T = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}^\times \right\}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- Consider the vertical geodesic plane $\mathcal{H}^2 = \mathbb{R} \times \mathbb{R}_{>0} \subset \mathcal{H}^3$. Then the subgroup $G = \{g \in \Gamma \mid g(\mathcal{H}^2) \subset \mathcal{H}^2\}$ of the elements stabilizing $\mathcal{H}^2$ decomposes as $G = \mathrm{PSL}_2(\mathbb{R}) \sqcup S\,\mathrm{PSL}_2(\mathbb{R})$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the restricted action induces an isomorphism between G and the group of isometries of $\mathcal{H}^2$.

1.4 Kleinian groups

The interesting subgroups of a real vector space for the purpose of arithmetic are lattices. In the same way, the subgroups of $\mathrm{PSL}_2(\mathbb{C})$ of interest for arithmetic are discrete.

Definition 1.4.1. Let X be a locally compact metric space.

- A family $(M_i)_{i \in I}$ of subsets of X is *locally finite* if for every compact subset K of X, the set of indices $\{i \in I \mid M_i \cap K \neq \emptyset\}$ is finite.

- Let Γ be a group acting by isometries on X. We say Γ acts *discontinuously* on X if for all $x \in X$, the Γ-orbit $(\{\gamma \cdot x\})_{\gamma \in \Gamma}$ of x is locally finite.

Remark 1.4.2. There can be repetitions in the family (M_i), but if a family is locally finite, then a nonempty subset can be repeated only finitely many times. In the same way, a group acts discontinuously if and only if every stabilizer is finite and every orbit (as a set) meets finitely many times any compact.

Definition 1.4.3. A subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a *Kleinian group* if it acts discontinuously on $\mathcal{H}^3$.

Proposition 1.4.4. *Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The following are equivalent:*

(i) *The group Γ is a Kleinian group;*

(ii) *The group Γ is discrete as a subset of $\mathrm{PSL}_2(\mathbb{C})$ equipped with the topology induced by $\mathrm{SL}_2(\mathbb{C}) \subset \mathcal{M}_2(\mathbb{C})$.*

Proof. A proof can be found in [Mar07, section 2.2]. $\qquad\square$

Remarks 1.4.5. If Γ is a Kleinian group, then the stabilizer Γ_x of any point $x \in \mathcal{H}^3$ is conjugate in $\mathrm{PSL}_2(\mathbb{C})$ to a subgroup of the stabilizer of the point $(0,1) \in \mathbb{C} \times \mathbb{R}_{>0}$ in $\mathrm{PSL}_2(\mathbb{C})$, which is equal to the compact group $\mathrm{PSU}_2(\mathbb{C})$. Therefore the subgroup Γ_x is finite as it is also discrete. Note that we know a simple classification of finite subgroups of $\mathrm{PSU}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{R})$: such a subgroup can be isomorphic to any cyclic group, any dihedral group, the alternate group on 4 elements, the symmetric group on 4 elements, or the alternate group on 5 elements. Moreover, the set of points in $\mathcal{H}^3$ that have a non-trivial stabilizer in Γ has measure 0, is closed and has empty interior: it is a countable, locally finite union of geodesics, one for each elliptic element in Γ.

1.5 Fundamental domains

A Kleinian groups admits a set of representatives that has nice topological properties.

Definition 1.5.1. Let X be a locally compact metric space X equipped with a Radon measure Vol. Let Γ be a subgroup of the isometries of X. A *fundamental domain* for Γ is an open connected subset $\mathcal{F}$ of X such that

(i) $\bigcup_{\gamma \in \Gamma} \gamma \cdot \overline{\mathcal{F}} = X$;

(ii) For all $\gamma \in \Gamma \setminus \{1\}$, $\mathcal{F} \cap \gamma \cdot \mathcal{F} = \emptyset$;

(iii) $\mathrm{Vol}(\partial \mathcal{F}) = 0$.

If the quotient space $\Gamma \backslash X$ is Hausdorff and compact, then we say Γ is *cocompact*.

Remark 1.5.2. The condition (iii) may seem strange, but it is needed to ensure that two fundamental domain have the same volume (Lemma 1.5.15).

Proposition 1.5.3. *Let Γ be a Kleinian group. The following are equivalent*

1. *The group Γ acting on $\mathrm{PSL}_2(\mathbb{C})$ by left multiplication is cocompact;*

2. *The group Γ acting on $\mathcal{H}^3$ is cocompact;*

3. *The group Γ admits a fundamental domain with compact closure in $\mathcal{H}^3$.*

Proof. It is a direct consequence of [Mar07, Proposition 3.5.1 (vii)]. $\square$

We consider fundamental domains that have a particularly nice boundary: they are polyhedra.

Definitions 1.5.4.

- A *polygon* $P \subset \mathcal{H}^3$ is a nonempty closed connected subset of a geodesic plane of $\mathcal{H}^3$ whose boundary (for the induced topology on the plane) is a countable union of sets of the form $s \cap \mathcal{H}^3$ where s is a segment in $\widehat{\mathcal{H}}^3$, called its *edges*, such that the family of the edges is locally finite. The finite endpoints of the edges of P are called *vertices*.

- A *polyhedron* is a nonempty open connected subset of $\mathcal{H}^3$ whose boundary is a countable union of polygons, called its *faces*, such that the intersection of two faces is contained in a geodesic and such that the family of the faces is locally finite.

- A fundamental domain for a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that is also a polyhedron is a *fundamental polyhedron*.

- A polyhedron is *finite* if it has only finitely many faces.

Remark 1.5.5. A polyhedron need not be convex.

A polyhedron which is a fundamental polyhedron for a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ carries structure.

Definitions 1.5.6. Let $\mathcal{F}$ be a polyhedron, and F the set of faces of $\mathcal{F}$. A *face pairing* (of $\mathcal{F}$) is a map $\cdot^* \times g : F \to F \times \mathrm{PSL}_2(\mathbb{C})$ which assigns to every face f a face f^* and a transformation $g(f) \in \mathrm{PSL}_2(\mathbb{C})$, such that

(a) $g(f) \cdot f = f^*$;

(b) $\cdot^* : F \to F$ is an involution;

(c) Every face f admits a neighborhood V such that $(g(f) \cdot (V \cap \mathcal{F})) \cap \mathcal{F} = \emptyset$.

The elements $g(f)$ where f is a face of $\mathcal{F}$ are called *pairing transformations*. If there is a face f such that $f^* = f$, then the pairing transformation satisfies the *reflection relation* $g(f)^2 = 1$.

Now assume that $\mathcal{F}$ has a face pairing. There is a natural equivalence relation $\sim$ on $\overline{\mathcal{F}}$ generated by the relations $x \sim y$ if $g \cdot x = y$ for some pairing transformation g. The *identified polyhedron* $\mathcal{F}^* = \overline{\mathcal{F}}/\sim$ comes with the canonical projection $\pi : \overline{\mathcal{F}} \to \mathcal{F}^*$. For $x, y \in \mathcal{F}^*$, let $\mathrm{d}^*(x, y) = \inf \sum_{i=1}^n \mathrm{d}(z_i, w_i)$ where the infimum is taken over every $2n$-uples of points $(z_i, w_i)_i$ of $\overline{\mathcal{F}}$ such that $\pi(z_1) = x$, $z_{i+1} \sim w_i$ and $\pi(w_n) = y$ (see Figure 1.2). We say the polyhedron is *complete* if

(d) for every $x \in \overline{\mathcal{F}}$, $\pi^{-1}(x)$ is finite, in which case d^* is a metric on $\mathcal{F}^*$, and

(e) $\mathcal{F}^*$ is complete for this metric.
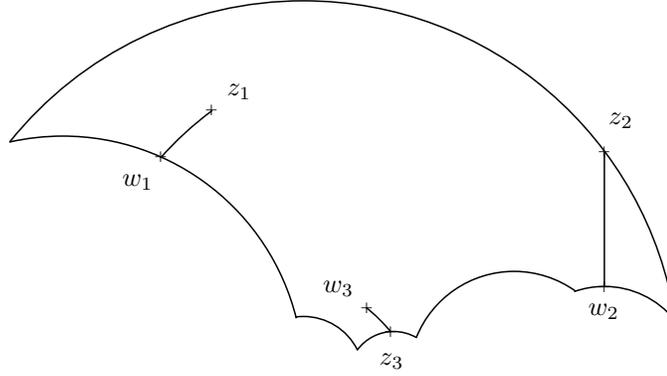
Figure 1.2: The distance d* in a planar cut

Now assume in addition that $\mathcal{F}$ is complete. Let e_1 be an edge of $\mathcal{F}$. The edge e_1 is the intersection of two faces: choose one and call it f_1. There is then a corresponding face f_1^* and a pairing transformation $g_1 = g(f_1)$. Now construct three sequences $(e_i), (f_i), (g_i)$ by induction as follows: letting $e_{i+1} = g_i \cdot e_i$, there is a unique $f_{i+1} \neq f_i^*$ such that $e_{i+1} = f_{i+1} \cap f_i^*$, and let $g_{i+1} = g(f_{i+1})$ (see Figure 1.3). Now because of condition (d), the sequence (e_i) is periodic; let m be its period. The sequence of edges $C = (e_1, \ldots, e_m)$ is called a *cycle* of edges. The *cycle transformation* at e_1 is $h = g_m g_{m-1} \ldots g_1$. At every edge $e = f_i \cap f_j$, the faces f_i and f_j make an interior angle $\alpha(e)$ inside $\mathcal{F}$. The *cycle angle* of $C = (e_1, \ldots, e_m)$ is $\alpha(C) = \sum_{i=1}^{m} \alpha(e_i)$. We say that $\mathcal{F}$ satisfies the *cycle condition* if:

(f) for every cycle C, there is $\nu \in \mathbb{Z}_{>0}$ such that $\alpha(C) = \dfrac{2\pi}{\nu}$, and

(g) if the edge e_1 is a geodesic (not only a segment) then the cycle transformation at e_1 is the identity on e_1.

If $\mathcal{F}$ satisfies the cycle condition, then for every cycle we have the *cycle relation* $h^\nu = 1$. An cycle is *elliptic* if the cycle transformation is nontrivial.

Remark 1.5.7. Some parts of these definitions require a proof; the reader should refer to [Mas71] or [Mas88, Section IV.H] for details.

One natural way to construct a set of representatives for a Kleinian group is to choose one distinguished point in the space, and then in each orbit choose "the closest point to the distinguished one": in this way, we pick generically one element in each orbit. The following proposition describes precisely how good this construction is.

Proposition 1.5.8. *Let Γ be a Kleinian group. Let $p \in \mathcal{H}^3$ be a point with trivial stabilizer in Γ. Then the set*

$$D_p(\Gamma) = \{x \in \mathcal{H}^3 \mid \text{for all } \gamma \in \Gamma \setminus \{1\}, \ \mathrm{d}(x,p) < \mathrm{d}(\gamma \cdot x, p)\}$$
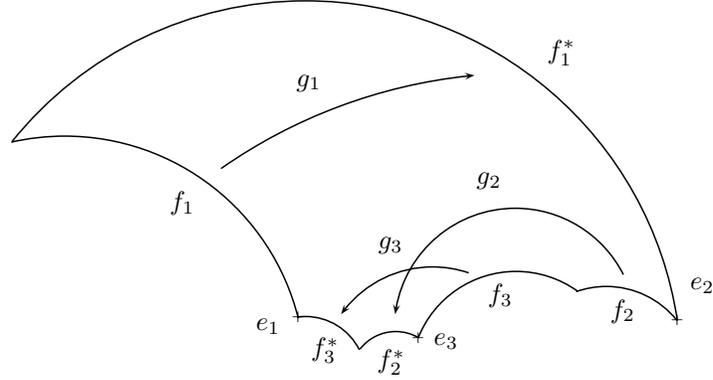
Figure 1.3: A cycle in a planar cut

is a convex fundamental polyhedron for Γ. Furthermore $D_p(\Gamma)$ admits a face pairing, is complete and satisfies the cycle condition.

Proof. This proposition can be found in [Mar07, Proposition 3.5.1] $\qquad\square$

Definition 1.5.9. Let Γ be a Kleinian group, and $p \in \mathcal{H}^3$ be a point with trivial stabilizer in Γ. Then the domain $D_p(\Gamma)$ is a *Dirichlet domain* for Γ.

Theorem 1.5.10 (Poincaré)**.** *Let $\mathcal{F}$ be a polyhedron with a face pairing. Suppose $\mathcal{F}$ is complete and satisfies the cycle condition. Let Γ be the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ generated by the face pairing transformations. Then Γ is a Kleinian group, $\mathcal{F}$ is a fundamental polyhedron for Γ, and the reflection relations together with the cycle relations form a complete set of relations for Γ.*

Proof. This version of Poincaré's theorem is due to Maskit. A proof can be found in the original article [Mas71] or in his book [Mas88, Theorem H.11] $\quad\square$

Definition 1.5.11. A Kleinian group is *geometrically finite* if it admits a finite Dirichlet domain.

Proposition 1.5.12. *Let Γ be a Kleinian group. If Γ is geometrically finite, then every Dirichlet domain for Γ is finite.*

Proof. This result can be found in [Mas88, Corollary of Proposition C.2 and Theorem C.4], [Mar07, Theorem 3.6.1] or [Rat06, Theorem 12.4.6] $\qquad\square$

Corollary 1.5.13. *Let Γ be a Kleinian group. If Γ is geometrically finite, then Γ is finitely presented.*

Proof. If Γ is a geometrically finite Kleinian group, then it admits a finite Dirichlet domain, so by Proposition 1.5.8 and Theorem 1.5.10 it is finitely presented. $\qquad\square$

Remark 1.5.14. There are finitely generated Kleinian groups that are not geometrically finite. See [Rat06, section 12.4, Example 3] or [Mas88, section IX.G] for examples.

Lemma 1.5.15. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fundamental domains for a group Γ. Assume that $\mathrm{Vol}(\mathcal{F}_1)$ is finite. Then $\mathrm{Vol}(\mathcal{F}_2)$ is also finite and*

$$\mathrm{Vol}(\mathcal{F}_1) = \mathrm{Vol}(\mathcal{F}_2).$$

Proof. The same proof as in [Kat92, Theorem 3.1.1] applies. $\square$

Definitions 1.5.16. If Γ has a fundamental domain $\mathcal{F}$ with finite volume, we say that Γ has *finite covolume* and define

$$\mathrm{Covol}(\Gamma) = \mathrm{Vol}(\mathcal{F})$$

which is well-defined by the lemma above.

Proposition 1.5.17. *Let Γ be a Kleinian group. If Γ has finite covolume, then Γ is geometrically finite.*

Proof. This result can be found in [Mar07, Lemma 3.6.4]. $\square$

Remark 1.5.18. This is easy to prove when Γ is cocompact, since a Dirichlet domain is locally finite.

Corollary 1.5.19. *Let Γ be a Kleinian group. If Γ has finite covolume, then Γ is finitely presented.*

Proof. Apply Proposition 1.5.17 and Corollary 1.5.13. $\square$

2 Quaternion algebras and Kleinian groups

In this section we describe how to construct Kleinian groups from quaternion algebras.

2.1 Quaternion algebras

In this section we recall the construction and basic properties of quaternion algebras. Every proof can be found in [MR03, Chap. 2] or in [Vig80, Chap. I]. Throughout the section, F is a field with char $F \neq 2$.

Definition 2.1.1. Let $a, b \in F^{\times}$. An F-algebra admitting a presentation of the form

$$\langle\, i, j \mid i^2 = a,\ j^2 = b,\ ij = -ji \,\rangle$$

is called a *quaternion algebra* over F, we write $\left(\frac{a,b}{F}\right)$ for such an algebra.

Examples 2.1.2.

- The matrix algebra is $\mathcal{M}_2(F) \cong \left(\frac{1,1}{F}\right)$ via the algebra homomorphism

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto i \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto j.$$

- The ring $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ of Hamiltonians is a quaternion algebra over the reals. We have an embedding $\mathbb{C} \hookrightarrow \mathbb{H}$ given by $i \mapsto i$.

Proposition 2.1.3. *Let $B = \left(\frac{a,b}{F}\right)$ be a quaternion algebra over F. Then*

(i) *The algebra B has dimension 4 over F and $\{1, i, j, ij\}$ is a basis;*

(ii) *If $F \subset K$ is a field extension, then*

$$\left(\frac{a,b}{F}\right) \otimes_F K \cong \left(\frac{a,b}{K}\right);$$

(iii) *For all $u \in F^\times$, we have*

$$\left(\frac{a,b}{F}\right) \cong \left(\frac{b,a}{F}\right) \cong \left(\frac{a,-ab}{F}\right) \cong \left(\frac{u^2 a, u^2 b}{F}\right).$$

Example 2.1.4. There are only two nonisomorphic quaternion algebras over $\mathbb{R}$: the matrix ring $\mathcal{M}_2(\mathbb{R})$ and the Hamiltonians $\mathbb{H}$.

Definitions 2.1.5.

- Let B be an F-algebra and $x \in B$. The *trace* (resp. the *norm*) $\operatorname{tr}_{B/F}(x)$ (resp. $\operatorname{N}_{B/F}(x)$) of x is the trace (resp. the determinant) of the linear endomorphism of $B : y \mapsto xy$.

- Let B be a quaternion algebra over F. Then the F-linear map $\bar{\cdot} : B \to B$ sending $1, i, j, ij$ respectively to $1, -i, -j, -ij$ is called *conjugation*. The *reduced trace* and *reduced norm* are defined to be respectively $\operatorname{trd}(x) = x + \overline{x}$ and $\operatorname{nrd}(x) = x\overline{x}$. The reduced norm is a quadratic form on B.

Proposition 2.1.6. *Let B be a quaternion algebra over F. Then for all $x \in B$, we have :*

(i) $\overline{\overline{x}} = x$;

(ii) $\overline{xy} = \overline{y}\,\overline{x}$;

(iii) $\overline{1} = 1$;

(iv) $\operatorname{nrd}(x) = x\overline{x} \in F$ *and* $\operatorname{N}_{B/F}(x) = \operatorname{nrd}(x)^2$;

(v) $\operatorname{trd}(x) \in F$ *and* $\operatorname{tr}_{B/F}(x) = 2\operatorname{trd}(x)$*;*

(vi) $x^2 - \operatorname{trd}(x)x + \operatorname{nrd}(x) = 0$*;*

(vii) $x \in B^{\times}$ *if and only if* $\operatorname{nrd}(x) \neq 0$*;*

(viii) *If* $x \in B^{\times}$*, then* $x^{-1} = \overline{x}/\operatorname{nrd}x$*.*

Examples 2.1.7.

- In $\mathcal{M}_2(F) = \left(\frac{1,1}{F}\right)$, the conjugation is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and the reduced norm is the determinant.

- In $\mathbb{H} \cong \mathbb{R}^4$, the reduced norm is the square of the usual L^2 norm, which we also note $|\cdot|^2 = \operatorname{nrd}$ as it extend the absolute value on $\mathbb{C}$.

Remark 2.1.8. If B is an F-algebra and $\sigma : F \hookrightarrow K$ is a field embedding, we write $B_{\sigma} = B \otimes_{\sigma} K = B \otimes_F K$, where K is the F-vector space induced by σ. If F is a number field, we write $B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R}$.

2.2 Splitting

Proposition 2.2.1. *Let* F *be a field with* $\operatorname{char} F \neq 2$ *and* $B = \left(\frac{a,b}{F}\right)$ *be a quaternion algebra over* F*. The following are equivalent:*

(i) *The quaternion algebra is isomorphic to the matrix ring:* $B \cong \mathcal{M}_2(F)$*;*

(ii) *The quaternion algebra* B *is not a division ring;*

(iii) *The quadratic form* nrd *is isotropic;*

(iv) *The binary form* $ax^2 + by^2$ *represents* 1*.*

Proof. This classical proposition can be found in [MR03, Theorem 2.3.1] or in [Vig80, Corollaire 3.2]. $\square$

Definitions 2.2.2. If the equivalent conditions of proposition 2.2.1 above hold, we say that B *splits*. A field K containing F is a *splitting field* for B if $B \otimes_F K$ splits.

Examples 2.2.3.

(i) Any quaternion algebra over an algebraically closed field is split;

(ii) For any quaternion algebra, there is a quadratic extension of its base field that is a splitting field.

Definition 2.2.4. Let F be a number field and B a quaternion algebra over F. If $\sigma : F \hookrightarrow \mathbb{R}$ a real embedding of F, we say that σ is *split* (resp. *ramified*) if $B_\sigma \cong \mathcal{M}_2(\mathbb{R})$ (resp. $B_\sigma \cong \mathbb{H}$). If $\mathfrak{p}$ is a prime of F and $F_\mathfrak{p}$ the $\mathfrak{p}$-adic completion of F, we say $\mathfrak{p}$ is *split* if $B_\mathfrak{p} = B \otimes_F F_\mathfrak{p} \cong \mathcal{M}_2(F_\mathfrak{p})$, and $\mathfrak{p}$ is *ramified* otherwise. The *discriminant* of B is the product of the ramified primes.

Example 2.2.5. Let $F = \mathbb{Q}(\sqrt{2})$ and $B = \left(\frac{-1, \sqrt{2}}{F} \right)$. Let σ_1 (resp. σ_2) be the real embedding of F sending $\sqrt{2}$ to $\sqrt{2}$ (resp. to $-\sqrt{2}$). Then $B_{\sigma_1} \cong \left(\frac{-1, \sqrt{2}}{\mathbb{R}} \right) \cong \left(\frac{-1, 1}{\mathbb{R}} \right) \cong \mathcal{M}_2(\mathbb{R})$ so σ_1 is split, and $B_{\sigma_2} \cong \left(\frac{-1, -\sqrt{2}}{\mathbb{R}} \right) \cong \left(\frac{-1, -1}{\mathbb{R}} \right) \cong \mathbb{H}$ so σ_2 is ramified. Let $\mathfrak{p}_2$ be the unique prime ideal of F above 2. One can show that $\mathfrak{p}_2$ is ramified, and that every other prime is split, so the discriminant of B is $\Delta_B = \mathfrak{p}_2$.

The following classification theorem is interesting, although it is not needed for computing with Kleinian groups.

Theorem 2.2.6. *Let F be a number field. For any quaternion algebra B over F, the number of ramified places is finite and even. Moreover, for any finite subset of the non-complex places of F of even cardinality, there is a unique quaternion algebra over F ramified exactly at these places.*

Proof. This fundamental theorem involves a lot of machinery. The finiteness of the number of ramified places comes from the theory of discriminants; the fact that it is even is a consequence of class field theory (when $F = \mathbb{Q}$ it follows from quadratic reciprocity); the uniqueness comes from the relationship between quadratic forms and quaternion algebras and the Hasse-Minkowski principle for quadratic forms; the existence comes from the study of splitting fields and the approximation theorem for F. The proof can be found in [MR03, Theorem 7.3.6] or in [Vig80, Théorème 3.1]. □

Example 2.2.7. The map given by the discriminant

$$\left\{ \begin{array}{c} \text{Isomorphism classes} \\ \text{of quaternion algebras over } \mathbb{Q} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Squarefree} \\ \text{positive integers} \end{array} \right\}$$

is well-defined and bijective. Indeed the ramification at the infinite place is given by the number of prime factors of the discriminant, so Theorem 2.2.6 gives the result.

2.3 Orders

An integral structure in quaternion algebras is needed to study the arithmetic. These structures are called orders.

Definition 2.3.1. Let F be a number field and $\mathbb{Z}_F$ its ring of integers. Let B be an F-algebra of finite dimension. An *order* in B (or $\mathbb{Z}_F$-*order* when there is

ambiguity) is a finitely generated $\mathbb{Z}_F$-submodule $\mathcal{O} \subset B$ with $F\mathcal{O} = B$ which is also a subring.

Example 2.3.2. Let F and $\mathbb{Z}_F$ be as above, $a, b \in \mathbb{Z}_F \setminus \{0\}$ and $B = \left(\frac{a,b}{F}\right)$. Then the $\mathbb{Z}_F$-module $\mathcal{O} = \mathbb{Z}_F + \mathbb{Z}_F i + \mathbb{Z}_F j + \mathbb{Z}_F ij$ is an order in B.

Definition 2.3.3. Let F be a number field, B a quaternion algebra over F and $\mathcal{O} \subset B$ an order. We write $\mathcal{O}_1^\times = \{x \in \mathcal{O}^\times \mid \mathrm{nrd}(x) = 1\}$. The reduced norm on B extends to a unique multiplicative map $\mathrm{nrd} : B_\mathbb{R} \longrightarrow F_\mathbb{R}$ such that $\mathrm{nrd}(1 \otimes \lambda) = 1 \otimes \lambda^2$ for all $\lambda \in \mathbb{R}$, and we write $B_{\mathbb{R},1}^\times = \{x \in B_\mathbb{R}^\times \mid \mathrm{nrd}(x) = 1\}$.

Proposition 2.3.4. *Let F be a number field of degree n, B a quaternion algebra over F and $\mathcal{O}$ an order in B. Then there is an isomorphism of $\mathbb{R}$-algebras*

$$B_\mathbb{R} \cong \mathcal{M}_2(\mathbb{R})^{s_1} \times \mathbb{H}^{r_1 - s_1} \times \mathcal{M}_2(\mathbb{C})^{r_2}$$

where s_1 is the number of split real places, and an isomorphism of topological groups

$$B_{\mathbb{R},1}^\times \cong \mathrm{SL}_2(\mathbb{R})^{s_1} \times (\mathbb{H}_1^\times)^{r_1 - s_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}.$$

The embeddings $\mathcal{O} \hookrightarrow B_\mathbb{R}$ and $\mathcal{O}_1^\times \hookrightarrow B_{\mathbb{R},1}^\times$ are discrete.

Proof. We have $F_\mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as $\mathbb{R}$-algebras, so $B_\mathbb{R} = B \otimes_F \mathbb{R} = B \otimes_F F_\mathbb{R} = \prod_{i=1}^{r_2} B_{\sigma_i}$ and $B_{\mathbb{R},1}^\times = \prod_{i=1}^{r_2} (B_{\sigma_i})_1^\times$. Then three cases have to be studied. First, if σ is a complex embedding of F then $B_\sigma \cong \mathcal{M}_2(\mathbb{C})$ and we get $(B_\sigma)_1^\times \cong \mathrm{SL}_2(\mathbb{C})$. Secondly, if σ is a split real embedding of F, then $B_\sigma \cong \mathcal{M}_2(\mathbb{R})$ and we have $(B_\sigma)_1^\times \cong \mathrm{SL}_2(\mathbb{R})$. Finally, if σ is a ramified real embedding of F, then we have $B_\sigma \cong \mathbb{H}$ and we get the isomorphism $(B_\sigma)_1^\times \cong \mathbb{H}_1^\times$. Putting these together give

$$B_\mathbb{R} \cong \mathcal{M}_2(\mathbb{R})^{s_1} \times \mathbb{H}^{r_1 - s_1} \times \mathcal{M}_2(\mathbb{C})^{r_2}$$

and

$$B_{\mathbb{R},1}^\times \cong \mathrm{SL}_2(\mathbb{R})^{s_1} \times (\mathbb{H}_1^\times)^{r_1 - s_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}.$$

By definition of an order we have $\mathcal{O} \otimes_\mathbb{Z} F = B$, so the image L of the embedding $\mathcal{O} \hookrightarrow B_\mathbb{R}$ is a finitely generated subgroup such that $L \otimes_\mathbb{Z} \mathbb{R} = B_\mathbb{R}$, so L is a full lattice : L is discrete. By restriction, the map $\mathcal{O}_1^\times \hookrightarrow B_{\mathbb{R},1}^\times$ is also discrete. □

Theorem 2.3.5. *Let F be a number field of degree n, B a quaternion algebra over F and $\mathcal{O}$ an order in B. Then the embedding*

$$\mathcal{O}_1^\times \hookrightarrow B_{\mathbb{R},1}^\times$$

is cocompact if and only if B is a division algebra.

Proof. Consider first the direction : if B is a division algebra, then the embedding $\mathcal{O}_1^\times \hookrightarrow B_{\mathbb{R},1}^\times$ is cocompact. It is a consequence of Minkowski's convex body theorem; for a proof, the reader should refer to [Vig80, Théorème 1.1]; this proof uses idèles, but there exists also an elementary proof using only Minkowski's theorem.

The other direction is actually more difficult in the general case. We will give a direct proof in the case of quaternion algebras. Suppose B is not a division algebra. Then we have $B \cong \mathcal{M}_2(F)$, so we may assume $B = \mathcal{M}_2(F)$, and up to conjugacy, $\mathcal{O} \subset \mathcal{M}_2(\mathbb{Z}_F)$ where $\mathbb{Z}_F$ is the ring of integers of F so we may assume $\mathcal{O} = \mathcal{M}_2(\mathbb{Z}_F)$. Suppose $\mathcal{O}_1^\times = \mathrm{SL}_2(\mathbb{Z}_F)$ is cocompact, and consider the matrix $x_n = \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \in B_{\mathbb{R},1}^\times$, then there exists a bound $A \in \mathbb{R}$ and an element $\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_F)$ for all n, such that for every infinite place σ, the inequality $\|\gamma_n x_n\|_\sigma \leq A$ holds. But we have $\gamma_n x_n = \begin{pmatrix} a_n n & b_n n^{-1} \\ c_n n & d_n n^{-1} \end{pmatrix}$, so we have both $|a_n n|_\sigma \leq A$ and $|c_n n|_\sigma \leq A$ for every infinite place σ. Hence for n large enough, $a_n = c_n = 0$ so $\det \gamma_n = 0$, which is a contradiction. □

Definition 2.3.6. Let F be a number field. We say F is *quasi totally real* or *QTR* if F has exactly one complex place. A *Kleinian quaternion algebra* is a quaternion algebra over a QTR number field, ramified at every real place.

Example 2.3.7. A quadratic imaginary field is a QTR number field. For any positive cubefree integer $d \neq 1$, $\mathbb{Q}(\sqrt[3]{d})$ is a QTR number field.

Remark 2.3.8. By Theorem 2.2.6, a Kleinian quaternion algebra is uniquely determined by its discriminant.

Corollary 2.3.9. *Let B be a Kleinian quaternion algebra and $\mathcal{O}$ an order in B. Then there is a discrete embedding*

$$\mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$$

which is cocompact if and only if B is a division algebra. Furthermore, if B is not a division algebra then the base field F of B is a quadratic imaginary field and $B \cong \mathcal{M}_2(F)$.

Proof. We always have an embedding $\rho : B_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$: take for example the restriction of the algebra homomorphism

$$i \mapsto \begin{pmatrix} \sqrt{\sigma(a)} & 0 \\ 0 & -\sqrt{\sigma(a)} \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \sigma(b) & 0 \end{pmatrix}$$

where $B = \left(\frac{a,b}{F}\right)$, σ is the unique (up to complex conjugation) complex embedding of F and $\sqrt{\sigma(a)}$ is a chosen complex square root of $\sigma(a)$. Applying the previous theorem, the embedding $\mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C}) \times (\mathbb{H}_1^\times)^{n-1}$ is discrete and is cocompact if and only if B is a division algebra. But $(\mathbb{H}_1^\times)^{n-1}$ is compact, and ρ is the composition of $\mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C}) \times (\mathbb{H}_1^\times)^{n-1} \longrightarrow \mathrm{SL}_2(\mathbb{C})$, so it is also discrete, and cocompact if and only if B is a division algebra.

If B is not a division algebra, then it cannot be ramified at any place; but it is ramified at every real place, so F has no real places and one complex place, and then F is a quadratic imaginary field. Since B is not a division algebra, we get $B \cong \mathcal{M}_2(F)$. $\qquad\square$

2.4   Arithmetic Kleinian groups and covolumes

Definitions 2.4.1.

- Let Γ_1 and Γ_2 be subgroups of $\mathrm{PSL}_2(\mathbb{C})$. We say that Γ_1 and Γ_2 are *directly commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. We say that Γ_1 and Γ_2 are *commensurable* if Γ_1 and a conjugate of Γ_2 in $\mathrm{PSL}_2(\mathbb{C})$ are directly commensurable;

- Let Γ be a Kleinian group. We say that Γ is *arithmetic* if it is commensurable with some $P\rho(\mathcal{O}_1^\times)$ where $\mathcal{O}$ is an order in a quaternion algebra over F, ramified at every real places of a QTR number field F, and ρ is a discrete embedding $\rho : \mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$.

Theorem 2.4.2. *Let F be a QTR number field of degree n, B a Kleinian quaternion algebra over F, and $\mathcal{O}$ an order in B. Let $\Gamma = P\rho(\mathcal{O}_1^\times)$ where ρ is a discrete embedding $\rho : \mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. Then Γ has finite covolume. Furthermore, if $\mathcal{O}$ is maximal, then we have*

$$\mathrm{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2}\zeta_F(2)\prod_{\mathfrak{p}|\Delta_B}(N(\mathfrak{p})-1)}{(4\pi^2)^{n-1}}$$

where Δ_F is the discriminant of F, ζ_F is the Dedekind zeta function of F and Δ_B is the discriminant of B.

Proof. This result can be found in [MR03, Theorem 11.1.3], but the proof omits an important computation (the "Tamagawa number"). A complete proof is given in [Vig80, Corollaire 1.8 and Corollaire 3.8]. $\qquad\square$

Remark 2.4.3. In the nonsplit case, we already knew that Γ had finite covolume since it was cocompact. However, the exact formula for the covolume will be crucial for computations.

Part II

Algorithms for Kleinian groups

3   Algorithms for hyperbolic geometry

In this section we provide formulas for computing in the hyperbolic 3-space, and describe how to perform geometric computations, including how to compute with finite polyhedra.

3.1   The unit ball model and explicit formulas

In order to be able to do some computations, formulas for the action of $\mathrm{SL}_2(\mathbb{C})$ on the hyperbolic 3-space are needed.

Proposition 3.1.1. *Identify $\mathcal{H}^3$ with the subset $\mathbb{C} + \mathbb{R}_{>0}j \subset \mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$. Then for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $z \in \mathcal{H}^3$, we have*

$$g \cdot z = (az + b)(cz + d)^{-1} = (zc + d)^{-1}(za + b). \tag{1}$$

Proof. We will proceed as follows. First we will derive the generic formula for a reflection in a plane and an inversion in a sphere. The formulas will be given in $\mathcal{H}^3$ but by continuity they will still be valid on $\widehat{\mathcal{H}}^3$ with the obvious image and preimage for the point at infinity. Then we will prove the Formula (1) for some standard matrices: we will decompose Formula (1) for these matrices as a product of reflections and inversions; since Formula (1) is the action by linear fractional transformations on $\mathbb{C}$ this will prove that it is the Poincaré extension of these matrices. Finally we will prove that the action on $\mathbb{C}$ and by Formula (1) of any element in $\mathrm{SL}_2(\mathbb{C})$ can be written as the same product of the action of the standard matrices.

Let $a, u \in \mathbb{C}$ be such that $|u| = 1$, and let P be the plane containing a, orthogonal to u. Let r be the map given by

$$\text{for all } z \in \mathcal{H}^3, \ r(z) = \mathrm{trd}(a\bar{u})u - u\bar{z}u = a + u\bar{a}u - u\bar{z}u.$$

We claim that $r(\mathcal{H}^3) \subset \mathcal{H}^3$ and that r is the reflection in the plane P. Note first that r is $\mathbb{R}$-affine and fixes a. Let $z \in P$, so that we have $0 = \mathrm{trd}((z - a)\bar{u}) = z\bar{u} + u\bar{z} - \mathrm{trd}(a\bar{u})$ i.e. $\mathrm{trd}(a\bar{u}) = z\bar{u} + u\bar{z}$. This gives $r(z) = (z\bar{u} + u\bar{z})u - u\bar{z}u = z|u|^2 = z$. Now we compute $r(a + u) = a + u\bar{a}u - u\overline{(a + u)}u = a - u|u|^2 = a - u$. This proves the claim, and the general form of a reflection in a plane is thus $z \mapsto \lambda u - u\bar{z}u$ with $\lambda \in \mathbb{R}, u \in \mathbb{C}$ and $|u| = 1$.

Let $a \in \mathbb{C}, R \in \mathbb{R}_{>0}$. Let $\mathcal{S}$ be the sphere of center a and radius R. Let ι be the map given by

$$\text{for all } z \in \mathcal{H}^3, \ \iota(z) = a + \frac{R^2}{\bar{z} - \bar{a}}.$$

We claim that $\iota(\mathcal{H}^3) \subset \mathcal{H}^3$ and that ι is the inversion in $\mathcal{S}$. Indeed we have for all $z \in \mathcal{H}^3$, $\iota(z) - a = \frac{R^2}{|z-a|^2}(z - a)$.

We now prove Formula (1) for the following standard matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ D_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ for } a \in \mathbb{C}^\times, \text{ and } T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for } b \in \mathbb{C}.$$

Let $z \in \mathcal{H}^3$. For the matrix S, Formula (1) becomes $-z^{-1} = -\overline{(\bar{z}^{-1})}$ which is the composition of an inversion and a reflection. Let $R \in \mathbb{R}_{>0}$. Then Formula (1) for the matrix D_R becomes $RzR = R^2z = \frac{R^2}{\overline{(1/\bar{z})}}$ which is the composition of two inversions. Let $u \in \mathbb{C}$ with $|u| = 1$. Then Formula (1) for the matrix D_u becomes $uzu = -u\overline{(-\bar{z})}u$ which is the composition of two reflections. Noting that $D_R D_u = D_{Ru}$ gives the formula for D_a for all $a \in \mathbb{C}^\times$. Let $b \in \mathbb{R}$. Then Formula (1) for the matrix T_b becomes $z+b = b-\overline{(-\bar{z})}$ which is the composition of two reflections. Noting that $D_a T_b D_{a^{-1}} = T_{a^2 b}$ gives the formula for all $b \in \mathbb{C}$.

We now express every element of $\mathrm{SL}_2(\mathbb{C})$ as a product of the standard matrices. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $z \in \mathcal{H}^3$. If $c = 0$, then $d = a^{-1}$ and Formula (1) becomes $(az+b)d^{-1} = d^{-1}(za+b) = aza+ab = T_{ab}D_a \cdot z$. If $c \neq 0$ then Formula (1) becomes $(az+b)(cz+d)^{-1}$ and $(zc+d)^{-1}(za+b)$. We give only the computation for the first expression, the same method gives the result for the second one. We can rewrite $az+b = (a/c)(cz+d)+b-(ad/c) = (a/c)(cz+d)-1/c$ and $cz+d = c(z+d/c)$. This gives $(az+b)(cz+d)^{-1} = a/c-c^{-1}(c(z+d/c))^{-1} = a/c - (c(z+d/c)c)^{-1} = T_{a/c}SD_c T_{d/c} \cdot z$.

$\square$

Actually another model of the hyperbolic 3-space will be used. The reason will appear when isometric spheres will be introduced: they will provide a nice way to express a Dirichlet domain.

Definition 3.1.2. The *unit ball* $\mathcal{B}$ is the open ball of center 0 and radius 1 in $\mathbb{R}^3 \cong \mathbb{C} + \mathbb{R}j \subset \mathbb{H}$, equipped with the metric

$$ds^2 = \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2)}{(1 - |w|^2)^2}$$

where $w = (z,t) \in \mathcal{B}$, $z = x + iy$ and $|w|^2 = x^2 + y^2 + t^2 \leq 1$. The *sphere at infinity* $\partial\mathcal{B}$ is the sphere of center 0 and radius 1. Let $\widehat{\mathcal{B}} = \mathcal{B} \cup \partial\mathcal{B}$ be the closed ball of radius 1. We equip $\widehat{\mathcal{B}}$ with the Euclidean topology.

Proposition 3.1.3 (formulas for the unit ball model)**.**

- *The map*

$$\eta\colon \begin{cases} \widehat{\mathcal{H}}^3 \longrightarrow \widehat{\mathcal{B}} \\ z \longmapsto (z-j)(1-jz)^{-1} = (1-zj)^{-1}(z-j) \\ \infty \longmapsto j \end{cases}$$

is a continuous bijection, with continuous inverse

$$\eta^{-1}\colon \begin{cases} \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{H}}^3 \\ w \longmapsto (w+j)(1+jw)^{-1} = (1+wj)^{-1}(w+j) \\ j \longmapsto \infty \end{cases}$$

and the restrictions $\eta : \mathcal{H}^3 \longrightarrow \mathcal{B}$ and $\eta^{-1} : \mathcal{B} \longrightarrow \mathcal{H}^3$ are well-defined, bijective isometries;

- *For all $w, z \in \mathcal{B}$, $\mathrm{d}(w, z) = \cosh^{-1}\left(1 + 2\dfrac{|w-z|^2}{(1-|w|^2)(1-|z|^2)}\right)$;*

- *For all $w \in \mathcal{B}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, transport the action on the upper half-space to the unit ball by setting*

$$g \cdot w = \eta(g \cdot \eta^{-1}(w));$$

we then have

$$g \cdot w = (Aw + B)(Cw + D)^{-1}$$

where

$$A = a + \bar{d} + (b - \bar{c})j, \ \ B = b + \bar{c} + (a - \bar{d})j,$$

$$C = c + \bar{b} + (d - \bar{a})j, \ \ D = d + \bar{a} + (c - \bar{b})j.$$

We also have $|A|^2 = |D|^2 = \|g\|^2 + 2$ and $|B|^2 = |C|^2 = \|g\|^2 - 2$, where $\|\cdot\|$ is the usual L^2 norm on $\mathcal{M}_2(\mathbb{C})$.

Proof. For the first two points the reader should refer to [Rat06, Theorem 4.5.1]. We only prove the third one. Let $w \in \mathcal{B}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. We

let $x = w + j$ and $y = 1 + jw$ so that $\eta^{-1}(w) = xy^{-1}$

$$
\begin{aligned}
g \cdot \eta^{-1}(w) &= (axy^{-1} + b)(cxy^{-1} + d)^{-1} \\
&= (ax + by)y^{-1}((cx + dy)y^{-1})^{-1} \\
&= (ax + by)(cx + dy)^{-1} \\
&= ((a + bj)w + (aj + b))((c + dj)w + (cj + d))^{-1} \\
&= XY^{-1}
\end{aligned}
$$

where
$$
X = (a + bj)w + (aj + b), \ \ Y = (c + dj)w + (cj + d).
$$

This gives

$$
\begin{aligned}
\eta(g \cdot \eta^{-1}(w)) &= (XY^{-1} - j)(1 - jXY^{-1})^{-1} \\
&= (X - jY)(Y - jX)^{-1}
\end{aligned}
$$

so we can compute

$$
\begin{aligned}
X - jY &= (a + bj)w + (aj + b) - j((c + dj)w + (cj + d)) \\
&= (a + bj - jc - jdj)w + (aj + b - jcj - jd) \\
&= (a + bj - \bar{c}j + \bar{d})w + (aj + b + \bar{c} - \bar{d}j) \\
&= Aw + B
\end{aligned}
$$

and

$$
\begin{aligned}
Y - jX &= (c + dj)w + (cj + d) - j((a + bj)w + (aj + b)) \\
&= (c + dj - ja - jbj)w + (cj + d - jaj - jb) \\
&= (c + dj - \bar{a}j + \bar{b})w + (cj + d + \bar{a} - \bar{b}j) \\
&= Cw + D.
\end{aligned}
$$

Finally we can compute $|A|^2, |B|^2, |C|^2$ and $|D|^2$. We give the calculation for $|A|^2$, the others being similar.

$$
\begin{aligned}
|A|^2 &= |a + \bar{d} + (b - \bar{c})j|^2 \\
&= |a + \bar{d}|^2 + |b - \bar{c}|^2 \\
&= |a|^2 + |d|^2 + 2\Re\left(a\bar{\bar{d}}\right) + |b|^2 + |c|^2 - 2\Re\left(b\bar{c}\right) \\
&= \|g\|^2 + 2\Re\left(\det g\right) \\
&= \|g\|^2 + 2.
\end{aligned}
$$

$\square$

Remark 3.1.4. It is tempting to believe that $\overline{A} = D$ and $\overline{B} = C$. Actually we have $\overline{A} = \bar{a} + d + \bar{j}(\bar{b} - c) = \bar{a} + d + (\bar{c} - b)j \neq D$; in the same way we can compute $\overline{B} = \bar{b} + c + (\bar{d} - a)j \neq C$.

Because of the isometry η, we can transport every structure from $\mathcal{H}^3$ to $\mathcal{B}$. The geodesics (resp. the geodesic planes) in $\mathcal{B}$ are the intersections between $\mathcal{B}$ and the Euclidean circles and straight lines (resp. the Euclidean spheres and planes), orthogonal to $\partial\mathcal{B}$. We apply to $\mathcal{B}$ the same definitions of a segment, convexity, a polyhedron and the related properties, and a Dirichlet domain.

Definitions 3.1.5. Suppose $g \in \mathrm{SL}_2(\mathbb{C})$ does not fix 0 in $\mathcal{B}$. Then let

- $\mathrm{I}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) = \mathrm{d}(g \cdot w, 0)\}$;

- $\mathrm{Ext}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) < \mathrm{d}(g \cdot w, 0)\}$;

- $\mathrm{Int}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) > \mathrm{d}(g \cdot w, 0)\}$;

We call $\mathrm{I}(g)$ the *isometric sphere* of g. For a subset $S \subset \mathrm{SL}_2(\mathbb{C})$ such that no element of S fixes 0, the *exterior domain* of S is $\mathrm{Ext}(S) = \bigcap_{g \in S} \mathrm{Ext}(g)$. The set S is a *boundary* for $\mathrm{Ext}(S)$. A *normalized boundary* for $\mathrm{Ext}(S)$ is a subset $S' \subset S$ such that $\mathrm{Ext}(S') = \mathrm{Ext}(S)$ and for all $g \in S'$, the geodesic plane $\mathrm{I}(g)$ contains a face of $\mathrm{Ext}(S)$ (i.e. it is a minimal boundary). For $\mathcal{S}$ a Euclidean sphere, also define $\mathrm{Ext}(\mathcal{S})$ (resp. $\mathrm{Int}(\mathcal{S})$) to be the intersection of $\mathcal{B}$ and the exterior (resp. the interior) of the sphere.

Proposition 3.1.6. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ *and* A, B, C, D *as in Proposition 3.1.3. Then* $g{\cdot}0 = 0$ *if and only if* $C = 0$. *If* g *does not fix* 0*, then* $\mathrm{I}(g)$ *is the intersection of* $\mathcal{B}$ *and the Euclidean sphere of center* $-C^{-1}D$ *and radius* $2/|C|$*, and we have* $\mathrm{Int}(g) = \mathrm{Int}(\mathrm{I}(g))$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and A, B, C, D as in Proposition 3.1.3. We first claim that for all $w \in \mathcal{B}$, we have $\mathrm{trd}(\overline{B}Aw) = \mathrm{trd}(\overline{D}Cw)$. Indeed since the action of $\mathrm{SL}_2(\mathbb{C})$ preserves $\widehat{\mathbb{C}}$, it also preserves $\partial\mathcal{B}$. Let $w \in \partial\mathcal{B}$, so that $|w| = 1$. We then have $|g \cdot w| = 1$, i.e. $|Aw + B|^2 = |Cw + D|^2$. We can rewrite

$$|Aw + B|^2 = |A|^2|w|^2 + |B|^2 + \mathrm{trd}(\overline{B}Aw)$$

and

$$|Cw + D|^2 = |C|^2|w|^2 + |D|^2 + \mathrm{trd}(\overline{D}Cw).$$

Since $|A| = |D|$, $|B| = |C|$ and $|w| = 1$ this gives $\mathrm{trd}(\overline{B}Aw) = \mathrm{trd}(\overline{D}Cw)$. Now $\partial\mathcal{B}$ generates $\mathbb{C} + \mathbb{R}j$ as a real vector space, so the last equality is still true for all $w \in \mathbb{C} + \mathbb{R}j$ and in particular for all $w \in \mathcal{B}$.

Now we turn to the proof of the proposition. By Proposition 3.1.3, we have $g \cdot 0 = BD^{-1}$, and $g \cdot 0 = 0$ if and only if $B = 0$ if and only if $C = 0$. Let $w \in \mathcal{B}$ and let

$$\alpha(w) = \mathrm{d}(w, 0) - \mathrm{d}(g \cdot w, 0).$$

Since the map $x \mapsto \cosh^{-1}(1 + 2x)$ is strictly increasing on $\mathbb{R}_{\geq 0}$, $\alpha(w)$ has the same sign as

$$\beta(w) = \frac{|w - 0|^2}{(1 - |w|^2)(1 - |0|^2)} - \frac{|g \cdot w - 0|^2}{(1 - |g \cdot w|^2)(1 - |0|^2)} = \frac{|w|^2}{1 - |w|^2} - \frac{|g \cdot w|^2}{1 - |g \cdot w|^2}.$$

Now since $x \mapsto \frac{x}{1-x} = \frac{1}{1-u} - 1$ is strictly increasing on $[0, 1)$, $\beta(w)$ has the same sign as $\delta(w) = |w|^2 - |g \cdot w|^2$. We compute

$$\begin{aligned}
\delta(w) &= |w|^2 - |g \cdot w|^2 \\
&= |w|^2 - |(Aw + B)(Cw + D)^{-1}|^2 \\
&= |Cw + D|^{-2}\big(|w|^2|Cw + D|^2 - |Aw + B|^2\big).
\end{aligned}$$

Letting $\epsilon(w) = |Cw + D|^2 \delta(w)$ we obtain

$$\begin{aligned}
\epsilon(w) &= |w|^2|Cw + D|^2 - |Aw + B|^2 \\
&= |C|^2|w|^4 + |D|^2|w|^2 + \mathrm{trd}(\overline{D}Cw)|w|^2 - \big(|A|^2|w|^2 + |B|^2 + \mathrm{trd}(\overline{B}Aw)\big) \\
&= |C|^2|w|^4 + |D|^2|w|^2 + \mathrm{trd}(\overline{D}Cw)|w|^2 - |D|^2|w|^2 - |C|^2 - \mathrm{trd}(\overline{D}Cw) \\
&= |C|^2(|w|^4 - 1) + \mathrm{trd}(\overline{D}Cw)(|w|^2 - 1) \\
&= \big(|C|^2(|w|^2 + 1) + \mathrm{trd}(\overline{D}Cw)\big)(|w|^2 - 1) \\
&= \big(|C|^2|w|^2 + |D|^2 - 4 + \mathrm{trd}(\overline{D}Cw)\big)(|w|^2 - 1) \\
&= \big(|Cw + D|^2 - 4\big)(|w|^2 - 1).
\end{aligned}$$

Finally $\alpha(w)$ has the same sign as $2 - |Cw + D|$ so we have

- $x \in \mathrm{I}(g) \Leftrightarrow \alpha(w) = 0 \Leftrightarrow |Cw + D| = 2 \Leftrightarrow |w - (-C^{-1}D)| = 2/|C|$, and

- $x \in \mathrm{Int}(g) \Leftrightarrow \alpha(w) > 0 \Leftrightarrow |Cw + D| < 2 \Leftrightarrow |w - (-C^{-1}D)| < 2/|C|$

as claimed. $\qquad \square$

Lemma 3.1.7. *Let Γ be a Kleinian group such that 0 has a trivial stabilizer in Γ and let $g, h \in \Gamma$. Then we have $\mathrm{I}(g) = \mathrm{I}(h)$ if and only if $g = h$.*

Proof. For all $g \in \Gamma$, we have $\mathrm{I}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) = \mathrm{d}(g \cdot w, 0)\} = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) = \mathrm{d}(w, g^{-1} \cdot 0)\}$ so $\mathrm{I}(g)$ is the perpendicular bisector of $[0, g^{-1} \cdot 0]$. Now suppose $\mathrm{I}(g) = \mathrm{I}(h)$, then the segments $[0, g^{-1} \cdot 0]$ and $[0, h^{-1} \cdot 0]$ have a common endpoint and the same perpendicular bisector, so they are equal and we have $g^{-1} \cdot 0 = h^{-1} \cdot 0$. Since 0 has trivial stabilizer in Γ this proves $g = h$. $\quad \square$

Remark 3.1.8. For every Kleinian group Γ such that $0 \in \mathcal{B}$ has trivial stabilizer, we have $D_0(\Gamma) = \mathrm{Ext}(\Gamma \setminus \{1\})$, so now the goal will be to compute the normalized boundary of such an exterior domain.

3.2 Geometric computations

In this section, every computation will be done in the unit ball model. The following geometrical objects have to be represented in bits:

- Reals, complex numbers, and Hamiltonians are represented using exact real arithmetic (see [PER89] and [Wei00] for theoretical foundations and for example [Boe05] or [GL01] for a discussion on practical implementations). Alternatively, we can use fixed and sufficiently large precision; we cannot predict in advance the required precision, but in practice it is not likely that an error due to round-off will occur (see also section 11.4);

- A point in $\mathcal{B}$ is represented as a vector of norm less than 1 in $\mathbb{C} + \mathbb{R}j \subset \mathbb{H}$;

- A geodesic plane not containing 0 is the intersection of a Euclidean sphere and $\mathcal{B}$, so it is represented by a couple (c, r) where c is its Euclidean center and r is its Euclidean radius;

- A geodesic not containing 0 is the intersection of a Euclidean circle with $\mathcal{B}$, and a Euclidean circle is the intersection of a Euclidean sphere and a unique Euclidean plane containing the center of the sphere, so it is represented by a 5-uple (c, r, e_1, e_2, e_3) where c, r are the center and the radius of the sphere, and (e_1, e_2, e_3) is an orthonormal basis of the ambiant Euclidian 3-space such that (e_1, e_2) is a basis of the plane;

- A segment not collinear with 0 is an arc of a geodesic; it is represented by a couple (ℓ, I) where ℓ is the geodesic, and I is a segment of $\mathbb{R}/2\pi\mathbb{Z}$, which is represented by a union of at most two segments of $[0, 2\pi]$;

- A finite convex polyhedron P containing 0 is represented in bits by a triple (F, FE, IE), where F is the set of the geodesic planes containing the faces of P (this already uniquely determines it), FE is the set of the edges of P, an edge e is represented by a couple $(s, \{f_1, f_2\})$ where s is a segment and f_1, f_2 are faces such that $e = f_1 \cap f_2$, and IE is the set of its infinite edges (the intersection of the sphere at infinity with the Euclidean closure of the faces) represented by arcs of Euclidian circles.

The following basic geometric computations are required and can be computed by explicit formulas:

- The intersection of two Euclidean spheres can be in general a Euclidean circle or the empty set, otherwise it is a single point; this gives the intersection of two geodesic planes;

- The intersection of two Euclidean planes is in general a Euclidean line, otherwise it can be the empty set or a Euclidean plane;

- The intersection of a Euclidean sphere and a Euclidean circle can be in general two points or the empty set, otherwise it can be a Euclidean circle or a single point; this gives the intersection of a geodesic plane and a geodesic.

3.3   Computing an exterior domain

A natural way to compute an exterior domain $\text{Ext}(S)$ is to iteratively compute $\text{Ext}(\{s_1, \ldots, s_t\})$ for $t = 1, \ldots, \#S$. Here are the elementary operations needed to do this computation:

(i) given a Euclidean circle or a union of segments of that circle, and an exterior domain, compute the part of that circle sitting inside (resp. outside) of that exterior domain: for every face of the exterior domain, compute the intersection points with the circle, compute the corresponding angles and the corresponding interval, then compute the whole intersection;

(ii) given a finite convex polyhedron and a Euclidean sphere, remove from its edges and infinite edges the part sitting inside this sphere: for every edge and infinite edge, use (i) with the exterior domain consisting of the interior of the single sphere and replace the edge by the result; this operation is called *reducing the edges* of the polyedron by the sphere;

(iii) given a finite convex polyhedron P and a Euclidean sphere S, compute the edges of $P \cap \text{Ext}(S)$ having a nonempty intersection with S: for every face of the polyhedron, compute its intersection with S, if it is not empty, use (i) with the resulting circle and P; this operation is called *computing the new edges of P associated to S*;

(iv) given a finite convex polyhedron P and a Euclidean sphere S, compute the infinite edges of $P \cap \text{Ext}(S)$ having a nonempty intersection with S: compute the intersection of S and the sphere at infinity, then use (i) with the resulting circle and P; this operation is called *computing the new infinite edges of P associated to S*;

(v) given a finite convex polyhedron P, a Euclidean sphere $\mathcal{S}$ and a point x on that sphere, compute what angle fraction of $\mathcal{S}$, locally around x, is in the boundary of P: take a Euclidean sphere s centered at x of small enough radius (smaller than half the minimum Euclidean distance between x and any vertex of P), compute the intersection $c = s \cap \mathcal{S}$, and use (i) with this circle and P, then add up the length of the angle intervals; this operation is called *computing the angle of $\mathcal{S}$ around x*.

Definition 3.3.1. A subset $X \subset \mathcal{B}$ *intersects trivially* a polyhedron P if the intersection $X \cap P$ is contained in a union of edges of P.

This leads to Algorithm 1 below: given a finite convex polyhedron P and a geodesic plane $\mathcal{S}$ not containing 0, it computes $P' = P \cap \text{Ext}(\mathcal{S})$.

Algorithm 1 Exterior domain algorithm

Input: A finite convex polyhedron P, a geodesic plane $\mathcal{S}$ not containing 0
 1: $P' \leftarrow P$
 2: Reduce the edges of P' by $\mathcal{S}$
 3: Remove the edges and infinite edges of P' reduced to a finite set of points
 4: Compute the new edges and the new infinite edges of P' associated to $\mathcal{S}$, add them to the edges of P'
 5: Add $\mathcal{S}$ to the faces of P'
 6: For every face f of P' and every edge and infinite edge e of f, compute the angle of f around a point in e to remove edges intersecting P' trivially
 7: Remove faces having no egde and no infinite edge
Output: $P' = P \cap \text{Ext}(\mathcal{S})$

Proof of Algorithm 1. In order to simplify the notations, in this proof "edge" means finite or infinite edges, and otherwise "finite" or "infinite" is specified. Any face of P' is either contained in $\mathcal{S}$ or a face of P; hence after step 5, the stored faces contain every possible face for P'. Now a face f is the convex hull of its edges: let $x \in f$; take any geodesic λ passing through x and contained in a geodesic plane containing f, and consider the geodesic segment $[y, z] = \lambda \cap f$. Then y, z are in the edges of f, and x is in the convex hull $[y, z]$ of $\{y, z\}$. Hence a face is empty if and only if it has no edge, and a face f intersects P trivially if and only if every edge of f intersects P trivially. After step 6, the only remaining faces are the ones intersecting nontrivially P, so after step 7, the remaining faces are exactly the faces of P'. Hence we need to compute the edges of P': they are either edges of P, or new edges of P associated to $\mathcal{S}$. Hence after steps 2 and 3, the remaining edges are the edges of P' contained in some edge of P, and after step 4, the remaining edges are exactly the edges of P'. $\square$

Remark 3.3.2. Actually there is a much simpler way to compute an exterior domain: first compute every intersection of three planes, which gives a set of points, then keep only the points inside the polyhedron (these are the vertices), and finally keep only the planes containing a point (these are the faces). However this method is much slower since it takes $\mathcal{O}(n^4)$ elementary geometric operations where n is the number of planes, and in practice it seems to be numerically less stable.

3.4 Computing the volume of a convex finite polyhedron

In this section a description of how to compute the volume of a convex finite polyhedron will be given. First, in order to get formulas for hyperbolic volumes, the Lobachevsky function has to be introduced.

Proposition 3.4.1. *The integral*

$$-\int_0^\theta \ln|2\sin u|\,\mathrm{d}u$$

converges for $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$ and admits a continuous extension to $\mathbb{R}$, which is odd and periodic with period π.

Proof. [Rat06, Theorem 10.4.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.4.2. This extension is called the *Lobachevsky function $\mathcal{L}(\theta)$*.

Proposition 3.4.3. *The Lobachevsky function admits a power series expansion:*

$$\mathcal{L}(\theta) = \theta\left(1 - \ln(2\theta) + \sum_{n=1}^\infty \frac{2^{2n}|B_{2n}|}{2n(2n+1)!}\theta^{2n}\right)$$

where the B_n are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}.$$

Proof. In [Rat06, paragraph 10.4], the identity

$$\mathcal{L}''(\theta) = -\cot(\theta)$$

is proved. Integrating twice the classical power series of the cotangent ([Ser73, Chap. VII, 4.1, Proposition 7]) gives the result. $\qquad\qquad\square$

   This power series expansion can be used to compute approximate values of the Lobachevsky function as follows. Using oddness and π-periodicity we can reduce to $\theta \in [0, \frac{\pi}{2}]$. Then we use the classical expression of the Bernoulli numbers (see [Ser73, Chap. VII, 4.1, Proposition 7])

$$B_{2n} = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n) \text{ for all } n \in \mathbb{Z}_{>0}$$

where $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the Riemann zeta function. Finally for all $n \in \mathbb{Z}_{>0}$

and $\theta \in [0, \frac{\pi}{2}]$ we compute

$$\frac{2^{2n}|B_{2n}|}{2n(2n+1)!}\theta^{2n} = \frac{2^{2n}2(2n)!}{2n(2n+1)!(2\pi)^{2n}}\zeta(2n)\theta^{2n}$$
$$= \frac{1}{(2n+1)\pi^{2n}}\zeta(2n)\theta^{2n}$$
$$< \left(\frac{\theta}{\pi}\right)^{2n} < 4^{-n}.$$

This gives a bound for the tail of the series

$$\sum_{n>m}\frac{2^{2n}|B_{2n}|}{2n(2n+1)!}\theta^{2n} < \sum_{n>m}4^{-n} = \frac{4^{-m}}{3}.$$

From this bound we know in advance the number of terms to add to reach some given precision.

With this a formula for the volume of a certain standard tetrahedron can be derived. It will be used to compute the volume of any finite convex polyhedron.

Proposition 3.4.4. *Let $T_{\alpha,\gamma}$ be the tetrahedron in $\mathcal{H}^3$ with one vertex at ∞ and the other vertices A, B, C on the unit hemisphere such that they project vertically onto A, B', C' in $\mathbb{C}$ with $A' = 0$ to form a Euclidean triangle, with angles $\frac{\pi}{2}$ at B' with and α at A', and such that the angle along BC is γ (see Figure 3.1). Then $T_{\alpha,\gamma}$ is unique up to isometry and*

$$\mathrm{Vol}(T_{\alpha,\gamma}) = \frac{1}{4}\left[\mathcal{L}(\alpha + \gamma) + \mathcal{L}(\alpha - \gamma) + 2\mathcal{L}\left(\frac{\pi}{2} - \alpha\right)\right].$$
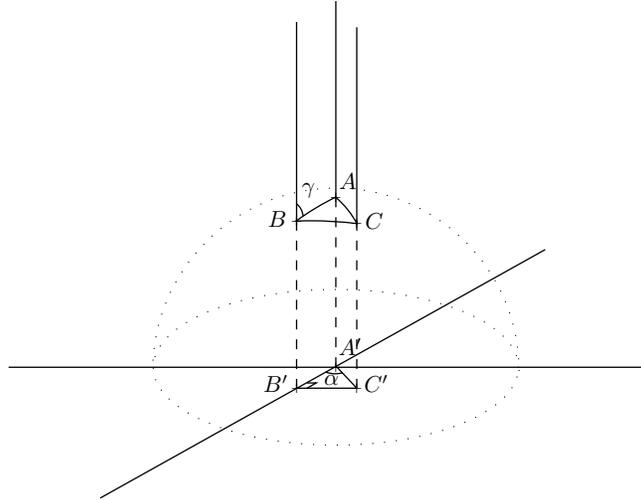
Figure 3.1: The standard tetrahedron $T_{\alpha,\gamma}$

Proof. This formula can be found in [MR03, paragraph 1.7]. □

Using this, Algorithm 2 computes the volume of a finite convex polyhedron.

Algorithm 2 Volume of a finite convex polyhedron

Input: A convex finite polyhedron P
 1: Split every face of P into triangles
 2: Split P into tetrahedra
 3: Express every tetrahedron as a difference of two tetrahedra, each having a vertex in the sphere at infinity
 4: For every tetrahedron having a vertex in the sphere at infinity, apply an isometry to map it to a tetrahedron with one vertex at ∞ and the other vertices on the unit hemisphere
 5: Express every such tetrahedron as an algebraic sum of tetrahedra of the same type having one vertex at j
 6: Express every such tetrahedron as an algebraic sum of tetrahedra of the same type such that the projected Euclidean triangle has a right angle not at 0
 7: For every such tetrahedron, compute the angles α and γ and use Proposition 3.4.4 to compute the volume
 8: $\mathrm{Vol}(P) \leftarrow$ sum of every contribution
Output: $\mathrm{Vol}(P)$

Remarks 3.4.5.

- For step 1, choose a vertex of the face and link it to every other vertex;

- For step 2, choose a vertex of P and link it to every computed triangle;

- For step 3, choose an edge and extend it into a geodesic, choose an endpoint of that geodesic and then the tetrahedron appears as the difference between two tetrahedra, each having the geodesic as an edge and a face of the initial tetrahedron as a base (see Figure 3.2);

- In step 5, the signs that appear in the sum are the signs of certain determinants.

4 The reduction algorithm

In this section we describe the reduction algorithm, and an algorithm that uses reduction to compute a fundamental domain for a Kleinian group.

4.1 Reduction

When we have a fundamental domain, it is natural to try to compute for any point in the hyperbolic 3-space an equivalent point in the fundamental domain
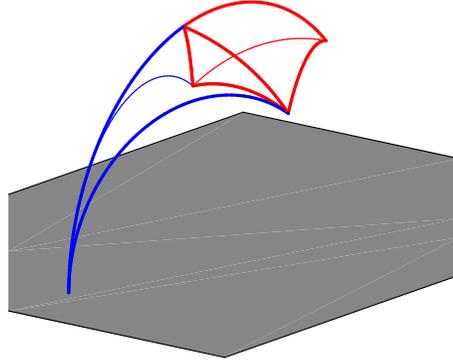
Figure 3.2: Reducing to tetrahedra with a vertex at infinity

and elements of the considered Kleinian group sending them to each other. This is called reduction.

Definition 4.1.1. Let Γ be a Kleinian group and $S \subset \Gamma$. A point $z \in \mathcal{B}$ is *S-reduced* if for all $g \in S$, we have $\mathrm{d}(z, 0) \leq \mathrm{d}(g \cdot z, 0)$, i.e. if $z \in \overline{\mathrm{Ext}(S)}$.

Given S a finite subset of $\mathrm{PSL}_2(\mathbb{C})$ and a point $w \in \mathcal{B}$, Algorithm 3 below returns a point z and $\delta \in \langle S \rangle$ such that z is S-reduced and $z = \delta \cdot w$.

Algorithm 3 Reduction algorithm

Input: A point $w \in \mathcal{B}$, a finite ordered subset $S \subset \mathrm{PSL}_2(\mathbb{C})$
 1: $z \leftarrow w$, $\delta \leftarrow 1$
 2: $g \leftarrow 1$
 3: **repeat**
 4:    $z \leftarrow g \cdot z$, $\delta \leftarrow g\delta$
 5:    $g \leftarrow$ the first $g \in S$ such that $\mathrm{d}(g \cdot z, 0)$ is minimal
 6: **until** $\mathrm{d}(g \cdot z, 0) \geq \mathrm{d}(z, 0)$
Output: z, $\delta \in \langle S \rangle$ s.t. z is S-reduced and $z = \delta \cdot w$

Proof of Algorithm 3. After step 4, we have $z = \delta \cdot w$ and $\delta \in \langle S \rangle$: after the initialization, we have $z = w$ and $\delta = 1$, and writing z', δ' the values after the step, we also get $z' = g \cdot z = g \cdot (\delta \cdot w)$ by induction, so $z' = (g\delta) \cdot w = \delta' \cdot w$ and $\delta' = g\delta \in \langle S \rangle$ since $g \in S$ and $\delta \in \langle S \rangle$ by induction. At the end of the loop, the algorithm terminates if $\mathrm{d}(g \cdot z, 0) \geq \mathrm{d}(z, 0)$. Hence while the algorithm runs, the distance $\mathrm{d}(z, 0)$ is decreasing. But z stays in the Γ-orbit of w and this orbit is discrete, so the algorithm terminates, and when this happens g is an element in S such that $\mathrm{d}(g \cdot z, 0)$ is minimal and $\mathrm{d}(g \cdot z, 0) \geq \mathrm{d}(z, 0)$, so z is S-reduced. $\qquad\square$

Remarks 4.1.2.

- The element δ can alternatively be computed as a word in the elements of S instead of being evaluated;

- At step 5, the g achieving the minimal $d(g \cdot z, 0)$ may not be unique. We can then pick any of these elements. Ordering S gives us a canonical choice.

Reducing points can give interesting information about the elements of the group, because if w has a trivial stabilizer, then the orbit map $\gamma \mapsto \gamma \cdot w$ is a bijection. This is the reason for introducing the following definition:

Definition 4.1.3. Let Γ be a Kleinian group, $S \subset \Gamma$ and $w \in \mathcal{B}$. An element $\gamma \in \Gamma$ is (S, w)-*reduced* if $\gamma \cdot w$ is S-reduced, i.e. if $\gamma \cdot w \in \overline{\mathrm{Ext}(S)}$.

Given S, w and Γ, an (S, w)-reduced element $\bar{\gamma}$ such that $\bar{\gamma} \equiv \gamma \pmod{S}$ can now be computed as follows: reduce $\gamma \cdot w$ with respect to S; if $\delta \in \langle S \rangle$ is such that $\delta \cdot (\gamma \cdot w)$ is S-reduced, then $\bar{\gamma} = \delta\gamma$ is (S, w)-reduced. We also write the reduced element $\bar{\gamma} = \mathrm{Red}_S(\gamma; w)$ and simply $\mathrm{Red}_S(\gamma) = \mathrm{Red}_S(\gamma; 0)$.

Proposition 4.1.4. *Suppose that* $\mathrm{Ext}(S)$ *is a fundamental domain for* $\langle S \rangle$. *Then for almost every* $w \in \mathcal{B}$ *the following holds: for every* $\gamma \in \Gamma$, *there exists a unique* (S, w)-*reduced* $\bar{\gamma} \equiv \gamma \pmod{S}$. *If* $w \in \mathrm{Ext}(S)$ *then* $\bar{\gamma} = 1$ *if and only if* $\gamma \in \langle S \rangle$.

Remark 4.1.5. Almost every means outside of a zero measure, closed subset of $\mathcal{H}^3$ with empty interior.

Proof. Let w be in the orbit of $\mathrm{Ext}(S)$. The existence follows from Algorithm 3. For uniqueness, suppose $\bar{\gamma}$ and $\bar{\gamma}'$ are (S, w)-reduced and $\bar{\gamma} \equiv \bar{\gamma}' \equiv \gamma \pmod{S}$. Then $\bar{\gamma} \cdot w, \bar{\gamma}' \cdot w \in \overline{\mathrm{Ext}(S)}$, and since w is in the orbit of $\mathrm{Ext}(S)$, we have in fact $\bar{\gamma} \cdot w, \bar{\gamma}' \cdot w \in \mathrm{Ext}(S)$. But $\bar{\gamma} \cdot w$ and $\bar{\gamma}' \cdot w$ are in the same $\langle S \rangle$-orbit, so $\bar{\gamma} = \bar{\gamma}'$. Now assume $w \in \mathrm{Ext}(S)$. If $\bar{\gamma} = 1$ then $\gamma \equiv \bar{\gamma} \equiv 1 \pmod{S}$, i.e. $\gamma \in \langle S \rangle$. If $\gamma \in \langle S \rangle$ then $\gamma \equiv 1 \pmod{S}$ and 1 is (S, w)-reduced since $1 \cdot w = w \in \mathrm{Ext}(S)$ so by uniqueness $\bar{\gamma} = 1$. $\qquad\square$

Since this means that elements in $\langle S \rangle$ can be explicitly written down as words in the elements of S and that computation in $\langle S \rangle \backslash \Gamma$ can be performed (with explicit unique representatives), this particular kind of generating set deserves a name.

Definition 4.1.6. A subset S of a Kleinian group Γ is a *basis* if $\mathrm{Ext}(S)$ is a fundamental domain for $\langle S \rangle = \Gamma$. If S is also a normalized boundary for $\mathrm{Ext}(S)$, it is called a *normalized basis* for Γ.

4.2 Computing a normalized basis

In this section we describe an algorithm that computes a normalized basis for a geometrically finite Kleinian group, using the reduction algorithm. We begin by two lemmas. The first one describes the face pairing in terms of the boundary of an exterior domain; the second one will allow us to make some assumptions on the Kleinian group given as an input.

Lemma 4.2.1. *Let $g \in \Gamma$ and $\mathcal{F} = \mathrm{Ext}(\Gamma \setminus \{1\})$. Then $g \cdot \mathrm{I}(g) = \mathrm{I}(g^{-1})$, and $\mathrm{I}(g)$ contributes to the boundary of $\mathcal{F}$ if and only if $\mathrm{I}(g^{-1})$ does.*

Proof. Let $y \in \mathcal{B}$. Then

$$
\begin{aligned}
y \in g \cdot \mathrm{I}(g) &\Leftrightarrow x \in \mathrm{I}(g) \text{ where } x = g^{-1} \cdot y \\
&\Leftrightarrow \mathrm{d}(g \cdot x, 0) = \mathrm{d}(x, 0) \\
&\Leftrightarrow \mathrm{d}(g \cdot (g^{-1} \cdot y), 0) = \mathrm{d}(g^{-1} \cdot y, 0) \\
&\Leftrightarrow \mathrm{d}(y, 0) = \mathrm{d}(g^{-1} \cdot y, 0) \\
&\Leftrightarrow y \in \mathrm{I}(g^{-1}).
\end{aligned}
$$

Note first that a point $z \in \mathcal{B}$ contributes to the boundary of $\mathcal{F}$ if and only if $z \in \overline{\mathcal{F}} = \overline{\mathrm{Ext}(\Gamma \setminus \{1\})}$, and let $z \in \mathrm{I}(g)$. Then

$$
\begin{aligned}
g \cdot z \in \overline{\mathrm{Ext}(\Gamma \setminus \{1\})} &\Leftrightarrow \forall \gamma \in \Gamma, \ \mathrm{d}(\gamma \cdot (g \cdot z), 0) \geq \mathrm{d}(g \cdot z, 0) \\
&\Leftrightarrow \forall \gamma \in \Gamma, \ \mathrm{d}(\gamma g \cdot z, 0) \geq \mathrm{d}(z, 0) \text{ since } \gamma \in \mathrm{I}(g) \\
&\Leftrightarrow \forall \gamma \in \Gamma, \ \mathrm{d}(\gamma \cdot z, 0) \geq \mathrm{d}(z, 0) \\
&\Leftrightarrow z \in \overline{\mathrm{Ext}(\Gamma \setminus \{1\})}.
\end{aligned}
$$

$\square$

Remark 4.2.2. This proves that the pairing transformations of such a Dirichlet domain are exactly the elements of a normalized boundary.

Lemma 4.2.3. *Let Γ be a Kleinian group. Let $p \in \mathcal{B}$, and let $h \in \mathcal{B}$ be such that $h \cdot 0 = p$ and $\Gamma' = h^{-1} \Gamma h$. Then for almost all choices of p, the stabilizer of 0 in Γ' is trivial and every elliptic cycle in $D_0(\Gamma)$ has length 1.*

Remark 4.2.4. By almost all we mean outside of a zero measure, closed subset of $\mathcal{H}^3$ with empty interior.

Proof. Let $z \in \mathcal{B}$ and $g \in \Gamma$. Then $h^{-1} g h \cdot z = z \Leftrightarrow g h \cdot z = h \cdot z \Leftrightarrow g \cdot p = p$: for the first property we can take p outside of the set of elliptic fixed points of Γ.

We now turn to the second property. If there is an elliptic cycle in $D_0(\Gamma)$ with length greater than 1, then there is a geodesic L', an elliptic element $h^{-1} g' h \in \Gamma'$ fixing every point in L' and a pairing transformation $h^{-1} g h \in \Gamma'$ such

that $h^{-1}gh \cdot L'$ is also the axis of an elliptic element of Γ'. By Lemma 4.2.1, we have $L \subset \mathrm{I}(h^{-1}gh)$. Now we conjugate back to Γ: we have $L = h \cdot L'$ is the axis of g', $g \cdot L$ is the axis of an elliptic element of Γ and for all $x \in L$, we have $h^{-1} \cdot x \in \mathrm{I}(h^{-1}gh)$, i.e. $\mathrm{d}(h^{-1} \cdot x, 0) = \mathrm{d}(h^{-1}g \cdot x, 0)$ which is equivalent to $\mathrm{d}(x, h \cdot 0) = \mathrm{d}(g \cdot x, h \cdot 0)$: the center p is in the intersection of the perpendicular bisectors of $[x, g \cdot x]$ for all $x \in L$. Thus we can take p outside of every such intersection of perpendicular bisectors corresponding to a couple of elliptic elements of Γ, and this is a zero measure, closed subset of $\mathcal{H}^3$ with empty interior since it is a countable, locally finite union of such sets. $\square$

Thanks to Lemma 4.2.3 we can always assume that 0 is not a fixed point in Γ and that every elliptic cycle has length 1. We can now describe an algorithm to compute a normalized basis. It uses two subalgorithms, Enumerate and IsSubgroup. The first subalgorithm Enumerate takes as an input a positive integer n and returns elements in Γ. The second subalgorithm IsSubgroup takes as an input a normalized boundary S for a subgroup $\langle S \rangle \subset \Gamma$ and returns **true** or **false** according to whether $\langle S \rangle \neq \Gamma$. We describe first a completely naive algorithm for computing a normalized basis for Γ.

Algorithm 4 Naive normalized basis algorithm

Input: A Kleinian group Γ
1: $S \leftarrow \emptyset$, $n \leftarrow 0$
2: **repeat**
3:     **repeat**
4:         $n \leftarrow n + 1$
5:         $S \leftarrow S \cup \mathrm{Enumerate}(\Gamma, n)$
6:         $S \leftarrow$ normalized boundary of $\mathrm{Ext}(S)$
7:     **until** every edge in $\mathrm{Ext}(S)$ is paired **and** $\mathrm{Ext}(S)$ is complete **and** the cycle condition holds
8: **until not** $\mathrm{IsSubgroup}(\Gamma, S)$
Output: A normalized basis S for Γ

Remark 4.2.5. Methods for checking whether every egde is paired and whether the polyhedron $\mathrm{Ext}(S)$ is complete are described respectively in Lemma 4.2.1 and Lemma 4.2.13.

Definition 4.2.6. Let X be a set, and A is an algorithm that takes as an input a positive integer and returns elements in X. The algorithm A is a *complete enumeration* of X if $X = \bigcup_{n>0} A(n)$.

Proposition 4.2.7. *If Γ is geometrically finite and Enumerate is a complete enumeration of Γ, then Algorithm 4 terminates after a finite number of steps and the output S is a normalized basis for Γ.*

Proof. Since Enumerate is a complete enumeration, a boundary for the Dirichlet domain centered at 0 will be enumerated after a finite number of steps.

By Proposition 1.5.8, the algorithm will then terminate. The output will then be a normalized basis for Γ by Step 6.   □

We will now use the reduction algorithm to improve tremendously Algorithm 4. For clarity Algorithm 9 will be divided into four routines.

Algorithm 5 returns a subset $S' \subset \langle S \rangle$ such that the following properties hold: $1 \notin S'$, $\text{Ext}(S') \subset \text{Ext}(S)$, $\langle S' \rangle = \langle S \rangle$ and $S' = U \cup T$ where U is the normalized boundary of $\text{Ext}(S')$ and every $g \in T$ is U-reduced.

Algorithm 5 KeepSameGroup

Input: A finite subset $S \subset \text{PSL}_2(\mathbb{C})$
  1: $U' \leftarrow$ normalized boundary of $\text{Ext}(S)$
  2: **repeat**
  3:    $U \leftarrow U'$
  4:    $S' \leftarrow U$
  5:    **for all** $g \in S$ **do**
  6:       $\bar{g} \leftarrow \text{Red}_{U \setminus \{g^{-1}\}}(g)$
  7:       **if** $\bar{g} \neq 1$ **then**
  8:          $S' \leftarrow S' \cup \{\bar{g}\}$
  9:       **end if**
10:    **end for**
11:    $U' \leftarrow$ normalized boundary of $\text{Ext}(S')$
12:    $S \leftarrow S'$
13: **until** $U = U'$
Output: S'

Proof of Algorithm 5. Let S_0 be the value of S in the input. We claim that after Step 10, we have

  (i) $S' \subset \langle S_0 \rangle$ with $1 \notin S'$;

  (ii) $S' = U \cup T$ where every $g \in T$ is U-reduced;

 (iii) $\text{Ext}(S') \subset \text{Ext}(S_0)$; and

 (iv) $\langle S' \rangle = \langle S \rangle = \langle S_0 \rangle$.

First we prove (i). We have $U \subset S$ because of Steps 1 and 3 at the first iteration and because of Steps 11 and 12 at the other iterations. But after Step 10, S' contains only elements of U and nontrivial reductions by a subset of U of elements of S. So by induction we have $S' \subset \langle S \rangle \subset \langle S_0 \rangle$ and $1 \notin S'$.

Next we turn to (ii). Let $T = S' \setminus U$, so that $S' = U \cup T$. Then every element $g \in T$ is added to S' at Step 8, and $U \setminus \{g^{-1}\} = U$, so g is U-reduced.

Now we prove (iii). After the first execution of Step 4 we have $\text{Ext}(S') = \text{Ext}(S) = \text{Ext}(S_0)$. Next, every time S' decreases it is because we compute

its normalized boundary, so $\text{Ext}(S')$ remains the same, and when S' increases then $\text{Ext}(S')$ decreases.

Finally we prove (iv). After Step 10, $S' = U \cup T$. We must prove that $S \subset \langle S' \rangle$. Let $g \in S$. If $\bar{g} = 1$, then $g \in \langle U \setminus \{g^{-1}\} \rangle \subset \langle S' \rangle$. If $\bar{g} \neq 1$, then $g = \delta \bar{g}$ with $\delta \in \langle U \rangle \subset \langle S' \rangle$ and $\bar{g} \in S'$ so $g \in \langle S' \rangle$. By induction $\langle S' \rangle = \langle S_0 \rangle$.

Let's prove that Algorithm 5 terminates after a finite number of steps, with the previous claim it will prove correctness. Let $A = \max\{d(g \cdot 0, 0) : g \in S_0\}$. Let $X_0 = \{g \in \langle S_0 \rangle : d(g \cdot 0, 0) \leq A\}$. The set X_0 is finite since $\langle S \rangle$ is a Kleinian group, and we have $S_0 \subset X_0$. Define after the n-th execution of Step 11 $X_n = X_{n-1} \setminus (U \setminus U')$.

We claim that after Step 11, we have $U' \subset X_n$. Indeed the only elements that can be added come from Step 8, and they are reductions of elements of S, but $d(g \cdot 0, 0)$ decreases as we reduce an element g, so by induction we have $S, S' \subset X_0$. Now $U' \subset S'$, so $U' \subset X_0$. But every $g \in X_0 \setminus X_n$ has $\text{I}(g)$ contained in the complement of $\text{Ext}(S')$, so $g \notin U'$ and $U' \subset X_n$. Finally after every iteration, if $U' = U$ then the algorithm terminates, and otherwise $X_n \setminus U'$ decreases. Since X_0 is finite, the algorithm terminates after finitely many steps. □

Algorithm 6 adds to S elements of $\langle S \rangle$ such that if not every edge of $\text{Ext}(S)$ is paired, then $\text{Ext}(S)$ is strictly smaller after adding these elements.

Algorithm 6 CheckPairing

Input: A finite subset $S \subset \text{PSL}_2(\mathbb{C})$
1: **for all** $g \in S$ with e edge in $\text{I}(g)$ not paired **do**
2:     $x \leftarrow x \in e$ such that $g \cdot x \notin \overline{\text{Ext}(S)}$
3:     $\bar{g} \leftarrow \text{Red}_S(g; x)$
4:     $S \leftarrow S \cup \{\bar{g}, \bar{g}^{-1}\}$
5: **end for**
Output: S

Remark 4.2.8. In Step 1, we test whether $\text{I}(g)$ is paired by using Lemma 4.2.1.

Proof of Algorithm 6. By Lemma 4.2.1, an edge $e \subset \text{I}(g)$ is not paired if and only if $g \cdot e$ is not an edge of $\text{Ext}(S)$, if and only if there is $x \in e$ such that $g \cdot x \notin \overline{\text{Ext}(S)}$. Now if there is a nonpaired edge, at Step 4, since $x \in \text{I}(g)$ we have $d(g \cdot x, 0) = d(x, 0)$ and since $g \cdot x \notin \overline{\text{Ext}(S)}$ we have $d(g \cdot x, 0) > d(\bar{g} \cdot x, 0)$. Putting these two together gives $d(\bar{g} \cdot x, 0) < d(x, 0)$, i.e. $x \in \text{Int}(\bar{g})$ so finally we have $\text{Ext}(S \cup \{\bar{g}\}) \subsetneq \text{Ext}(S)$. □

Lemma 4.2.9. *Let $g \in \text{SL}_2(\mathbb{C})$ be loxodromic. Then $\text{Int}(g)$ contains a fixed point of g in $\partial \mathcal{B}$.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ be loxodromic. The fixed points of g in $\partial\mathcal{B}$ are the images under η of the fixed points of g in $\mathbb{P}^1(\mathbb{C})$. Replacing g by a close element in $\mathrm{SL}_2(\mathbb{C})$ we may assume that the fixed points of g are in $\mathbb{C}$. Let $z_1, z_2 \in \mathbb{C}$ denote the fixed points of g and $z \in \mathbb{C}$ denotes any fixed point of g. Then $\eta(z) = xy^{-1}$ where $x = z - j$, $y = 1 - jz$. With notations of Proposition 3.1.3, by Proposition 3.1.6 we have

$$\begin{aligned} \eta(z) \in \mathrm{Int}(g) &\Leftrightarrow |C\eta(z) + D| < 2 \\ &\Leftrightarrow |Cxy^{-1} + D| < 2 \\ &\Leftrightarrow |Cx + Dy| < 2|y|. \end{aligned}$$

We compute

$$Cx = (c + \bar{b} + (d - \bar{a})j)(z - j) = (c + \bar{b})z + d - \bar{a} + ((d - \bar{a})\bar{z} - c - \bar{b})j,$$
$$Dy = (d + \bar{a} + (c - \bar{b})j)(1 - jz) = (c - \bar{b})z + d + \bar{a} + (c - \bar{b} - (d + \bar{a})\bar{z})j, \text{ and}$$
$$Cx + Dy = 2(d + cz) + 2(\overline{b + az})j.$$

We then get

$$\begin{aligned} \eta(z) \in \mathrm{Int}(g) &\Leftrightarrow |Cx + Dy|^2 < 4|y|^2 \\ &\Leftrightarrow |az + b|^2 + |cz + d|^2 < 1 + |z|^2 \\ &\Leftrightarrow |z|^2|cz + d|^2 + |cz + d|^2 < 1 + |z|^2 \text{ since } g \cdot z = z \\ &\Leftrightarrow |cz + d|^2 < 1. \end{aligned}$$

We need to prove that one of z_1, z_2 satisfies $|cz + d| < 1$. To prove this we need some information on z_1, z_2. We have

$$cz^2 + (d - a)z - b = 0$$

so letting $\Delta = (d - a)^2 - 4bc = \mathrm{tr}(g)^2 - 4 = \delta^2$ we have

$$z_1 z_2 = -\frac{b}{c},$$
$$z_1 + z_2 = \frac{a - d}{c}, \text{ and}$$
$$z = \frac{a - d \pm \delta}{2c}.$$

We compute

$$\begin{aligned} (cz_1 + d)(cz_2 + d) &= c^2 z_1 z_2 + cd(z_1 + z_2) + d^2 \\ &= -bc + d(a - d) + d^2 = 1. \end{aligned}$$

This gives $|cz_1 + d||cz_2 + d| = 1$ so we have to prove that $|cz_1 + d|$, $|cz_2 + d|$ cannot be both equal to 1, which is now equivalent to $|cz_1 + d|^2 + |cz_2 + d|^2 \neq 2$ since the product is 1. So we compute

$$
\begin{aligned}
|cz_1 + d|^2 + |cz_2 + d|^2 &= |c|^2(|z_1|^2 + |z_2|^2) + 2|d|^2 + 2\Re\left(\bar{d}c(z_1 + z_2)\right) \\
&= |c|^2(|z_1 + z_2|^2 - 2\Re\left(z_1\overline{z_2}\right)) + 2|d|^2 + 2\Re\left(\bar{d}(a - d)\right) \\
&= |a - d|^2 - 2|c|^2\Re\left(z_1\overline{z_2}\right) + 2\Re\left(\bar{d}a\right).
\end{aligned}
$$

Making a choice for z_1, z_2 gives

$$
\begin{aligned}
z_1\overline{z_2} &= \frac{a - d + \delta}{2c} \frac{\overline{(a - d)} - \bar{\delta}}{2\bar{c}} \\
&= \frac{|a - d|^2 - |\delta|^2 + 2i\Im\left((\overline{a - d})\delta\right)}{4|c|^2}, \text{ so} \\
\Re\left(z_1\overline{z_2}\right) &= \frac{|a - d|^2 - |\delta|^2}{4|c|^2},
\end{aligned}
$$

and we can go on with

$$
\begin{aligned}
|cz_1 + d|^2 + |cz_2 + d|^2 &= \frac{1}{2}|a - d|^2 + \frac{1}{2}|\delta|^2 + 2\Re\left(\bar{d}a\right) \\
&= \frac{1}{2}\left(\left|\text{tr}(g)^2\right| + \left|\text{tr}(g)^2 - 4\right|\right)
\end{aligned}
$$

but this last quantity is equal to 2 if and only if $\text{tr}(g)^2 \in [0, 4]$, if and only if $\text{tr}(g) \in [-2, 2]$ which is not since g is loxodromic. $\square$

Lemma 4.2.10. *Suppose $S \subset \Gamma$ be a subset of a Kleinian group Γ such that 0 has a trivial stabilizer in Γ, and suppose there are elements $g \in S$, $h \in \Gamma$ and $x \in \text{I}(g) \cap \overline{\text{Ext}(S)}$ such that $g \neq h$, $h \cdot x \in \overline{\text{Ext}(S)}$ and x does not lie in an edge of $\text{Ext}(S)$. Then $\text{Ext}(S \cup \{h, h^{-1}\}) \subsetneq \text{Ext}(S)$.*

Proof. We consider three cases. First suppose $\text{d}(x, 0) > \text{d}(h \cdot x, 0)$. Then we have $x \in \text{Int}(h) \cap \overline{\text{Ext}(S)}$, so we get $\text{Ext}(S \cup \{h\}) \subsetneq \text{Ext}(S)$. Now suppose $\text{d}(x, 0) < \text{d}(h \cdot x, 0)$. Then letting $y = h \cdot 0$ this gives $\text{d}(h^{-1}y, 0) < \text{d}(y, 0)$ so $y \in \text{Int}(h) \cap \overline{\text{Ext}(S)}$, and we have $\text{Ext}(S \cup \{h^{-1}\}) \subsetneq \text{Ext}(S)$. Finally suppose $\text{d}(x, 0) = \text{d}(h \cdot x, 0)$. Then $x \in \text{I}(g) \cap \text{I}(h)$ but x does not lie in an edge of $\text{Ext}(S)$ and since $h \neq g$ we have $\text{I}(h) \neq \text{I}(g)$ by Lemma 3.1.7, so $\text{I}(g) \cap \text{I}(h)$ is a geodesic not containing any edge of $\text{Ext}(S)$: we get $\text{Ext}(S \cup \{h\}) \subsetneq \text{Ext}(S)$. $\square$

Algorithm 7 adds to S elements of $\langle S \rangle$ such that if condition (g) of the cycle condition does not hold for the already existing edge cycles in $\text{Ext}(S)$ or if some cycle angle for a non-elliptic cycle is larger than 2π, then $\text{Ext}(S)$ is strictly smaller after adding these elements.

Algorithm 7 CheckCycleCondition

Input: A finite subset $S \subset \mathrm{PSL}_2(\mathbb{C})$
 1: Compute every edge cycle
 2: **for all** g cycle transformation for the edge e **do**
 3:    **if** $g \neq 1$ fixes at most one point in e **then**
 4:      $S \leftarrow S \cup \{g, g^{-1}\}$
 5:    **else if** $g \neq 1$ fixes every point in e **then**
 6:      $S \leftarrow S \cup \langle g \rangle$
 7:    **else**
 8:      $m \leftarrow$ period of the cycle
 9:      **for all** $0 < i < m$ **do**
10:        $x \leftarrow$ point in f_{i+1} close to e_i
11:        $h \leftarrow g_i \ldots g_1$
12:        **if** $h^{-1} \cdot x \in \overline{\mathrm{Ext}(S)}$ **then**
13:          $S \leftarrow S \cup \{h, h^{-1}\}$
14:        **end if**
15:      **end for**
16:    **end if**
17: **end for**
Output: S

Remark 4.2.11. At Step 6, g is elliptic so $\langle g \rangle$ is finite and we can compute it by computing the successive powers of g.

Proof of Algorithm 7. Suppose there is an edge cycle for an edge e equal to a geodesic and such that condition (g) is not satisfied, and let g be the corresponding cycle transformation. From the description of the stabilizer of a geodesic, the transformation g is either loxodromic, or elliptic of order 2 with exactly one fixed point in e. In both cases, Step 4 is executed. In the first case, by Lemma 4.2.9 above $\mathrm{Ext}(\{g, g^{-1}\}) \cap e \subsetneq e$ so $\mathrm{Ext}(S \cup \{g, g^{-1}\}) \subsetneq \mathrm{Ext}(S)$. In the second case, the edge e contains exactly one fixed point of g in $\mathcal{H}^3$, so we again have $\mathrm{Ext}(\{g\}) \cap e \subsetneq e$ and we get $\mathrm{Ext}(S \cup \{g, g^{-1}\}) \subsetneq \mathrm{Ext}(S)$.

Now suppose some cycle angle for a non-elliptic cycle is larger than 2π. Then considering the images $P, g_1^{-1} \cdot P, \ldots, (g_i \ldots g_1)^{-1} \cdot P$ of $P = \mathrm{Ext}(S)$ that glue one after another around e, there is an overlap which will be detected at Step 12. But then after Step 13 we have $\mathrm{Ext}(S \cup \{h, h^{-1}\}) \subsetneq \mathrm{Ext}(S)$ by Lemma 4.2.10. $\square$

In order to have the conditions of Theorem 1.5.10, we need to check completeness. We give here a simple criterion.

Definition 4.2.12. Let P be a finite polyhedron. A point $z \in \partial \mathcal{B}$ is a *tangency vertex* if it is a point of tangency $z = f \cap f'$ of two faces f, f' of P. Suppose P has a face pairing. If $z_1 = f_0 \cap f_1$ is a tangency vertex, then we define a sequence by letting $z_{i+1} = g(f_i) \cdot z_i = f_i^* \cap f_{i+1}$ while z_{i+1} is a tangency vertex

(otherwise the sequence ends at z_i). If such a sequence (z_i) is infinite, let m be its period, then $(z_1, \ldots, z_m)$ is a *tangency vertex cycle* and the *tangency vertex transformation* is $h = g_m g_{m-1} \ldots g_1$.

Lemma 4.2.13. *Let P be a finite polyhedron with a face pairing. Then every tangency vertex transformation in P is either loxodromic or parabolic, and P is complete if and only if every tangency vertex transformation is parabolic.*

Proof. A proof can be found in [Mas71] or [Mas88, Proposition I.6]   □

Algorithm 8 adds to S elements of $\langle S \rangle$ such that if the polyhedron $\mathrm{Ext}(S)$ is not complete, then $\mathrm{Ext}(S)$ is strictly smaller after adding these elements.

Algorithm 8 CheckComplete

Input: A finite subset $S \subset \mathrm{PSL}_2(\mathbb{C})$
 1: Compute every tangency vertex cycle
 2: **for all** g tangency vertex transformation **do**
 3:     **if** $g \neq 1$ is loxodromic **then**
 4:         $S \leftarrow S \cup \{g, g^{-1}\}$
 5:     **end if**
 6: **end for**
Output: S

Proof of Algorithm 8. Suppose the polyhedron $\mathrm{Ext}(S)$ has a face pairing and is not complete. By Lemma 4.2.13 above, there is a non-parabolic tangency vertex transformation g for a tangency vertex z. Since the transformation g is loxodromic, Step 4 is executed. But then by Lemma 4.2.9 and since $g \cdot z = z$, we get $\mathrm{Ext}(S \cup \{g, g^{-1}\}) \subsetneq \mathrm{Ext}(S)$.   □

Algorithm 9 takes as an input a geometrically finite Kleinian group Γ, and returns a normalized basis for Γ. The behavior of Algorithm 9 depends on these subalgorithms and is described in Proposition 4.2.14.

Proposition 4.2.14. *Let Γ be a Kleinian group. The following holds for Algorithm 9 applied to Γ:*

(i) *Suppose the inner loop terminates. Then after Step 10 the set S is a normalized basis for a geometrically finite subgroup $\langle S \rangle \subset \Gamma$, and $\langle S \rangle = \langle T \rangle$ where T is the union of the outputs of Enumerate up to that point.*

(ii) *Suppose that the algorithm terminates. Then Γ is geometrically finite and S is a normalized basis for Γ.*

(iii) *Suppose that at some point in the execution $\overline{\mathrm{Ext}(S)}$ is compact. Then the inner loop terminates.*

(iv) Suppose that Enumerate is a complete enumeration of Γ. Then the algorithm terminates.

Algorithm 9 Normalized basis algorithm

Input: A Kleinian group Γ
 1: $S \leftarrow \emptyset, \ n \leftarrow 0$
 2: **repeat**
 3:    **repeat**
 4:      $n \leftarrow n + 1$
 5:      $S \leftarrow S \cup \mathrm{Enumerate}(\Gamma, n), \ S \leftarrow S \cup S^{-1}$
 6:      $S \leftarrow \mathrm{KeepSameGroup}(S)$
 7:      $S \leftarrow \mathrm{CheckPairing}(S)$
 8:      $S \leftarrow \mathrm{CheckCycleCondition}(S)$
 9:      $S \leftarrow \mathrm{CheckComplete}(S)$
10:    **until** every edge in $\mathrm{Ext}(S)$ is paired **and** $\mathrm{Ext}(S)$ is complete **and** the cycle condition holds
11: **until not** $\mathrm{IsSubgroup}(\Gamma, S)$
Output: A normalized basis S for Γ

Remark 4.2.15. It would be interesting to prove that the algorithm terminates without any condition. It seems true in practise, but we have no proof for this result; one difficulty is that there are finitely generated Kleinian groups that are not geometrically finite. The improvement of Algorithm 9 over the naive Algorithm 4 would then appear clearly: instead of having to completely enumerate Γ we would only need generators as an input to Algorithm 9 that uses reduction to compute exactly the elements needed.

Proof of Proposition 4.2.14. We first prove (i). Suppose the inner loop terminates. Then after Step 10, $P = \mathrm{Ext}(S)$ is a finite, convex polyhedron. Then P has a face pairing because every edge is paired. Indeed a face f is the convex hull of the edges of f, so the face f is paired if and only if the edges of f are paired. The polyhedron P is complete since it is finite and every tangency vertex is parabolic (Lemma 4.2.13). We need to check the cycle condition. Because of Lemma 4.2.3, we may assume that every elliptic cycle has length 1; let g be the cycle transformation. Such a cycle automatically satisfies the cycle condition since it is a cycle in the Dirichlet domain of $\langle g \rangle$ because of Step 6 of Algorithm 7. If a cycle $C = (e_1, \ldots, e_m)$ is not elliptic, then condition (g) holds for C because of Algorithm 7, so the cycle transformation is the identity. Consider the images $P, g_1^{-1} \cdot P, \ldots, (g_m \ldots g_1)^{-1} \cdot P$ around the edge e: they glue one after another around e. Since the cycle transformation is the identity, their union covers a neighborhood of e and the cycle angle is an integer multiple of 2π. But because of Algorithm 7, this angle is at most 2π, so the cycle condition holds. By Theorem 1.5.10, the polyhedron P is a fundamental domain for the group $\langle S \rangle$. But at every step, $\langle S \rangle$ remains the same, so $\langle S \rangle = \langle T \rangle$ and

because of Algorithm 5, S is a normalized basis.

Now we turn to (ii). If the algorithm terminates, then by (i) the output S is a normalized basis for the subgroup $\langle S \rangle \subset \Gamma$, and because of Step 11 we have $\langle S \rangle = \Gamma$.

Next we prove (iii). Suppose that at some point in the execution $\overline{\text{Ext}(S)}$ is compact, then only finitely many I(g) for a $g \in \Gamma$ intersect $\overline{\text{Ext}(S)}$. While the inner loop runs, one of the conditions of Step 10 does not hold, so by the properties of Algorithms 6, 7 and 8, Ext(S) is strictly decreasing. So the inner loop terminates after a finite number of steps.

Finally for (iv), the same proof as for Proposition 4.2.7 applies. $\qquad\square$

Example 4.2.16. If Γ is a geometrically finite (hence finitely generated by Corollary 1.5.13) Kleinian group given by a finite set of generators in $\text{SL}_2(\mathbb{C})$, then we can take for Enumerate the algorithm that writes every word of length n in the generators, and by Proposition 4.1.4 we can take for SubGroup the algorithm that reduces every generator with respect to the given normalized basis and returns whether there is a nontrivial reduction.

5 Element enumeration in arithmetic Kleinian groups

This section focuses on ways of enumerating elements in a Kleinian group associated to an order $\mathcal{O}$ in a Kleinian quaternion algebra B with base field F.

5.1 Lattice enumeration

Recall from section 2.3 that the natural map $\rho : \mathcal{O} \hookrightarrow B \hookrightarrow B_\mathbb{R}$ is a discrete embedding. Now suppose $B_\mathbb{R}$ is equipped with a positive definite quadratic form $Q : B_\mathbb{R} \to \mathbb{R}$. Then the order $\mathcal{O}$ becomes a full lattice in a real vector space of dimension $4n$. Hence we can use the Fincke-Pohst algorithm (see [FP85] for details) based on LLL reduction to enumerate short vectors in such a lattice, and then pick up elements having reduced norm 1, that contribute to the Kleinian group associated to $\mathcal{O}$.

Remark 5.1.1. One can prove (by the Skolem Noether Theorem, see [MR03, Theorem 9.2.8]) that an injective algebra homomorphism $\rho : B \hookrightarrow \mathcal{M}_2(\mathbb{C})$ is unique up to conjugation by an element in $\text{GL}_2(\mathbb{C})$. Thus we only need to describe one such embedding, for example if $B = \left(\frac{a,b}{F} \right)$, then we can take

$$x + yi + zj + tij \mapsto \begin{pmatrix} x + y\sqrt{\sigma(a)} & z - t\sqrt{\sigma(a)} \\ z\sigma(b) - t\sigma(b)\sqrt{\sigma(a)} & x - y\sqrt{\sigma(a)} \end{pmatrix}$$

where σ is a complex embedding of F.

5.2   The absolute reduced norm

Definition 5.2.1. Suppose $\rho : B \hookrightarrow \mathcal{M}_2(\mathbb{C})$ is an injective algebra homomorphism such that $\mathcal{O}_1^\times$ embeds discretely into $\mathrm{SL}_2(\mathbb{C})$. Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$ and define $\mathrm{invrad}(m) = \left| (c + \bar{b}) + (d - \bar{a})j \right|^2$. For $g \in \mathrm{SL}_2(\mathbb{C})$ not fixing 0 in $\mathcal{B}$, let $\mathrm{rad}(g)$ be the Euclidean radius of $\mathrm{I}(g)$.

Proposition 5.2.2. *The quadratic form $Q : B \to \mathbb{R}$ defined by*

$$Q(x) = \mathrm{invrad}(\rho(x)) + \mathrm{tr}_{F/\mathbb{Q}}(\mathrm{nrd}(x)) \text{ for all } x \in B$$

gives $\mathcal{O}$ the structure of a lattice, and we have

$$\text{for all } x \in \mathcal{O}_1^\times, \ Q(x) = \frac{4}{\mathrm{rad}(\rho(x))^2} + n.$$

Remark 5.2.3. If $\rho(x)$ fixes 0 then $\mathrm{rad}(\rho(x))$ is not defined, but $Q(x) = n$ (infinite radius).

Proof. Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$. Then we have

$$
\begin{aligned}
\mathrm{invrad}(m) &= |c + \bar{b}|^2 + |d - \bar{a}|^2 \\
&= |c|^2 + |b|^2 + 2\,\Re(c\bar{\bar{b}}) + |d|^2 + |a|^2 - 2\,\Re(d\bar{a}) \\
&= |a|^2 + |b|^2 + |c|^2 + |d|^2 - 2\,\Re(\det m) \\
&= \|m\|^2 - 2\,\Re(\det m)
\end{aligned}
$$

where $\|\cdot\|$ is the usual L^2 norm on $\mathcal{M}_2(\mathbb{C})$, so that $\|\cdot\|^2$ is a positive definite quadratic form on $\mathcal{M}_2(\mathbb{C})$. Since nrd is a positive definite quadratic form on $\mathbb{H}$ and we have the decomposition $B_\mathbb{R} \cong \mathcal{M}_2(\mathbb{C}) \oplus \mathbb{H}^{n-2}$, a positive definite quadratic form on $B_\mathbb{R}$ can be constructed by letting for all $x \in B_\mathbb{R}$

$$Q(x) = \|m\|^2 + \mathrm{nrd}(h_1) + \cdots + \mathrm{nrd}(h_{n-2}) = \mathrm{invrad}(m) + \mathrm{tr}_{F_\mathbb{R}/\mathbb{R}}(\mathrm{nrd}(x))$$

where

$$x = m + h_1 + \cdots + h_{n-2} \in \mathcal{M}_2(\mathbb{C}) \oplus \mathbb{H}^{n-2},$$

since $2\,\Re(\det m) + \mathrm{nrd}(h_1) + \cdots + \mathrm{nrd}(h_{n-2}) = \mathrm{tr}_{F_\mathbb{R}/\mathbb{R}}(\mathrm{nrd}(x))$. This gives the first statement of the proposition.

For the second statement, note that according to Proposition 3.1.6,

$$\text{invrad}(g) = \left|(c + \bar{b}) + (d - \bar{a})j\right|^2 = \frac{4}{\text{rad}(g)^2}$$

for $g \in \text{SL}_2(\mathbb{C})$ not fixing 0 in $\mathcal{B}$, and if g fixes 0 then $\text{invrad}(g) = 0$. $\qquad\square$

Definition 5.2.4. The quadratic form Q is the *absolute reduced norm.*

Remark 5.2.5. The last statement of Proposition 5.2.2 says that elements of $\rho(\mathcal{O}_1^\times)$ with a small absolute reduced norm have a large radius. Since a Dirichlet domain for $\rho(\mathcal{O}_1^\times)$ has finitely many faces, the radii of the isometric spheres containing these faces are bounded by below, so we can expect the boundary elements of a Dirichlet domain to have a large radius, and it is relevant to enumerate $\mathcal{O}_1^\times$ by increasing absolute reduced norm. But it is not the only interest of this particular quadratic form: it also enables to geometrically detect some boundary element by moving the center: the propositions below make this precise.

Lemma 5.2.6. *Fix a matrix $m \in \mathcal{M}_2(\mathbb{C})$. Let $h \in \text{SL}_2(\mathbb{C})$ and $z = h \cdot 0$. Then the quantity $\|h^{-1}mh\|$ only depends on z.*

Proof. Every $h' \in \text{SL}_2(\mathbb{C})$ such that $h' \cdot 0 = z$ is equal to h up to right multiplication by an element of the stabilizer of 0, which is $\text{SU}_2(\mathbb{C})$. $\qquad\square$

Definition 5.2.7. Let $m \in \mathcal{M}_2(\mathbb{C})$, $z \in \mathcal{B}$ and let $h \in \text{SL}_2(\mathbb{C})$ be an isometry such that $h \cdot 0 = z$. We define the *norm centered at z* of m to be

$$\|m\|_z = \|h^{-1}mh\|$$

which depends only on z by the lemma above.

Proposition 5.2.8. *Let $g \in \text{SL}_2(\mathbb{C})$ and $z \in \mathcal{B}$. Then we have*

$$\|g\|_z^2 = 2\cosh(\text{d}(g \cdot z, z)).$$

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C})$ and $L = \frac{1}{2}(\cosh(\text{d}(g \cdot 0, 0)) - 1)$. Because of Proposition 3.1.3 we have

$$\begin{aligned} L &= \frac{|g \cdot 0|^2}{1 - |g \cdot 0|^2} \\ &= \frac{|BD^{-1}|^2}{1 - |BD^{-1}|^2} \end{aligned}$$

where

$$B = b + \bar{c} + (a - \bar{d})j \text{ and } D = d + \bar{a} + (c - \bar{b})j,$$

so we get

$$
\begin{aligned}
L &= \frac{|B|^2}{|D|^2 - |B|^2} \\
&= \frac{\|g\|^2 - 2}{(\|g\|^2 + 2) - (\|g\|^2 - 2)} \\
&= \frac{\|g\|^2 - 2}{4}
\end{aligned}
$$

so that

$$
\|g\|^2 = 4L + 2 = 2\cosh(\mathrm{d}(g \cdot 0, 0)).
$$

Now let $h \in \mathrm{SL}_2(\mathbb{C})$ such that $h \cdot 0 = z$. Then

$$
\begin{aligned}
\|h^{-1}gh\|^2 &= 2\cosh(\mathrm{d}(h^{-1}gh \cdot 0, 0)) \\
&= 2\cosh(\mathrm{d}(g \cdot z, h \cdot 0)) \\
&= 2\cosh(\mathrm{d}(g \cdot z, z))
\end{aligned}
$$

is the result as claimed. $\qquad\square$

Remark 5.2.9. We already knew that $\|g\|^2 \geq 2$ since the Cauchy-Schwarz inequality already gives $2 = |2\det g| = |\langle X, Y \rangle| \leq \|g\|^2$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product, $X = (a, d, b, c)$, $Y = (d, a, -c, -b)$ and $\|X\| = \|Y\| = \|g\|$.

The previous proposition suggests that the norm of an element decreases as we move the center toward a fixed point. Let's examine precisely this statement.

Definition 5.2.10. Let $z \in \partial \mathcal{B}$. A *horosphere* H at z is a Euclidean sphere such that $H \setminus \{z\}$ is contained in $\mathcal{B}$ and such that H is tangent to $\partial \mathcal{B}$ at z.

Remark 5.2.11. A horosphere H at z, containing a point $w \in \mathcal{B}$, is the limit of hyperbolic spheres containing w and whose center tends to z along the segment $[w, z]$. Hence a hyperbolic isometry sends horospheres to horospheres.

Proposition 5.2.12. *Let $g \in \mathrm{SL}_2(\mathbb{C})$.*

(i) *Suppose g is loxodromic or elliptic. Then the map $z \mapsto \|g\|_z$ has a minimum on the axis of g and we have*

$$
\|g\|_z^2 - 2 = (|\lambda| - |\lambda^{-1}|)^2 \text{ for all } z \text{ on the axis}
$$

where λ, λ^{-1} are the eigenvalues of g;

(ii) *Suppose g is parabolic, let w be the fixed point of g and let H_r be the horosphere at w of Euclidean radius r. Then we have*

$$
\|g\|_z^2 - 2 = |\beta|^2 \left(\frac{r}{1-r}\right)^2 \text{ for all } z \in H_r
$$

where g is conjugate in $\mathrm{SL}_2(\mathbb{C})$ *to* $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$.

Proof. Let $z \in \mathcal{B}$ and let

$$h = \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} -x & t+1 \\ t-1 & \bar{x} \end{pmatrix}$$

where $z = x + tj$ and $x \in \mathbb{C}$, $t \in \mathbb{R}$. Note that $h \in \mathrm{SL}_2(\mathbb{C})$ since the determinant of h is $\det h = \frac{1}{1-|z|^2}(-|x|^2 - (t^2 - 1)) = 1$. We claim that $h \cdot 0 = z$. Using the notations of Proposition 3.1.3, we have

$$\left(\sqrt{1 - |z|^2}\right) B = t + 1 + t - 1 + (-x - x)j = 2(t - xj)$$

and

$$\left(\sqrt{1 - |z|^2}\right) D = \bar{x} - \bar{x} + (t - 1 - t - 1)j = -2j$$

so

$$h \cdot 0 = BD^{-1} = 2(t - xj)(2j^{-1})^{-1} = (t - xj)j = x + tj = z.$$

First we prove (i). Let $g \in \mathrm{SL}_2(\mathbb{C})$ be loxodromic or elliptic. Without loss of generality, we may conjugate g so that $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Then the axis of g is the vertical j-line geodesic. We compute

$$h^{-1}gh = \frac{1}{1 - |z|^2} \begin{pmatrix} \lambda|x|^2 + (t^2 - 1)\lambda^{-1} & \bar{x}(t+1)(\lambda^{-1} - \lambda) \\ -x(t-1)(\lambda^{-1} - \lambda) & \lambda^{-1}|x|^2 + (t^2 - 1)\lambda \end{pmatrix}.$$

The point z is in the axis of g if and only if $x = 0$. When t is fixed and as $|x|$ decreases, $1 - |z|^2$ increases so all four coefficients of $h^{-1}gh$ decrease in norm. And finally for z in the axis, the expression becomes simply

$$h^{-1}gh = \frac{1}{1 - t^2} \begin{pmatrix} (t^2 - 1)\lambda^{-1} & 0 \\ 0 & (t^2 - 1)\lambda \end{pmatrix} = -\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$$

so $\|g\|_z^2 - 2 = \|h^{-1}gh\|^2 - 2 = |\lambda|^2 + |\lambda^{-1}|^2 - 2 = (|\lambda| - |\lambda^{-1}|)^2$ as claimed.

Next we turn to (ii). Let $g \in \mathrm{SL}_2(\mathbb{C})$ be parabolic. We again conjugate so that $g = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with fixed point j. We then compute

$$h^{-1}gh = \frac{1}{1 - |z|^2} \begin{pmatrix} |z|^2 - 1 - \beta\bar{x}(t-1) & -\beta\bar{x}^2 \\ \beta(t-1)^2 & |z|^2 - 1 + \beta\bar{x}(t-1) \end{pmatrix}$$

$$= -\begin{pmatrix} 1 + \beta\frac{\bar{x}(t-1)}{1-|z|^2} & \beta\frac{\bar{x}^2}{1-|z|^2} \\ -\beta\frac{(t-1)^2}{1-|z|^2} & 1 - \beta\frac{\bar{x}(t-1)}{1-|z|^2} \end{pmatrix}$$

so

$$\|g\|_z^2 = \|h^{-1}gh\|^2$$

$$= 2 + |\beta|^2 \frac{2|x|^2(t-1)^2}{(1-|z|^2)^2} + |\beta|^2 \frac{|x|^4 + (t-1)^4}{(1-|z|^2)^2}$$

$$= 2 + |\beta|^2 \frac{(|x|^2 + (t-1)^2)^2}{(1-|z|^2)^2}$$

$$= 2 + |\beta|^2 \left(\frac{1 + |z|^2 - 2t}{1 - |z|^2} \right)^2$$

$$= 2 + |\beta|^2 \left(1 - 2\frac{1-t}{1-|z|^2} \right)^2 .$$

Now let $u = 2(1-t)(1-|z|^2)^{-1} - 1$. Then

$$u = 2\frac{1-t}{1-|z|^2} - 1 \Leftrightarrow (1-|z|^2)(u+1) = 2(1-t)$$

$$\Leftrightarrow 1 - |z|^2 = 2\frac{1-t}{u+1}$$

$$\Leftrightarrow |z|^2 - 1 = \frac{2t}{u+1} - \frac{2}{u+1}$$

$$\Leftrightarrow |x|^2 + \left(t - \frac{1}{u+1} \right)^2 = \left(1 - \frac{1}{u+1} \right)^2$$

$$\Leftrightarrow \left| z - \frac{j}{u+1} \right|^2 = \left| j - \frac{j}{u+1} \right|^2 .$$

This is the equation of the horosphere at j of radius $r = \frac{u}{u+1}$, so $u = \frac{r}{1-r}$, which gives the result. $\qquad\square$

Remark 5.2.13. If g is elliptic or parabolic, then $\|g\|_z^2 - 2$ tends to 0 as z moves toward a fixed point of g, so there is a good chance that we find them by moving the center and searching in the lattice. But if g is loxodromic, then $\|g\|_z^2 - 2 \geq (|\lambda| - |\lambda^{-1}|)^2 > 0$ so if the eigenvalues are too large, we will have to enumerate much more lattice vectors to find this element by such a method.

6   Computing a presentation

Once we have computed a normalized basis for a geometrically finite Kleinian group Γ, we need to compute the reflection relations and the cycle relations to form a presentation for Γ. The reflection relations easy: just check whether we have $g^2 = 1$ for every pairing transformation g. For the cycle relations, there is a simple description since we know that we compute with an exterior domain instead of a general polyhedron. Let e_1 be an edge of P. Choose g_1, g such that $e_1 \subset \mathrm{I}(g_1) \cap \mathrm{I}(g)$; we then compute $e_{i+1} = g_i \cdot e_i$, and we let g_{i+1}

be the unique transformation such that $e_{i+1} \subset \mathrm{I}(g_i^{-1}) \cap \mathrm{I}(g_{i+1})$. The sequence (e_i) is periodic of period m, and the corresponding cycle transformation is $h = g_m g_{m-1} \ldots g_1$. The cycle transformation h is either the identity and the cycle relation is $h = 1$, or h is elliptic and we can compute its order ν, and the relation $h^\nu = 1$ is the cycle relation.

7 Summary

As a conclusion to this part, let us sum up the global algorithm for computing a fundamental domain and a presentation for a Kleinian group associated to a maximal order. Given a Kleinian quaternion algebra B, Algorithm 10 returns a normalized basis and a presentation for the Kleinian group associated to a maximal order in B.

Algorithm 10 Presentation algorithm

Input: A Kleinian quaternion algebra B
 1: Compute a maximal order $\mathcal{O} \subset B$
 2: Choose $\rho : B \hookrightarrow \mathcal{M}_2(\mathbb{C})$ s.t. $\Gamma = P\rho(\mathcal{O}_1^\times)$ has $\Gamma_0 = \{1\}$
 3: $Q \leftarrow$ the quadratic form given in Proposition 5.2.2
 4: Compute $\mathrm{Covol}(\Gamma)$ by using the formula of Theorem 2.4.2
 5: $\mathrm{Enumerate}(\Gamma) \leftarrow$ (Fincke-Pohst algorithm applied to Q in $\mathcal{O}$) $\cap \mathcal{O}_1^\times$
 6: $\mathrm{IsSubgroup}(\Gamma) \leftarrow$ Test whether $\mathrm{Vol}(\mathrm{Ext}(S)) > \mathrm{Covol}(\Gamma)$ using Algorithm 2
 7: $S \leftarrow \mathrm{NormalizedBasis}(\Gamma)$
 8: $R \leftarrow$ cycle relations using the method of Section 6
Output: A maximal order $\mathcal{O}$, a discrete embedding in the complex matrix ring ρ, a normalized basis S and a complete set of relations R

Remarks 7.0.14.

- For step 1, algorithms for computing a maximal order can be found in [Voi10];

- For step 2, we can choose any embedding ρ, then choose a point $w \in \mathcal{B}$ such that $\Gamma_w = \{1\}$ and a matrix $h \in \mathrm{PSL}_2(\mathbb{C})$ such that $h \cdot 0 = w$, then $\Gamma' = h^{-1}\Gamma h$ has $\Gamma'_w = \{1\}$;

- We can refine step 5 by using the methods described in Section 5.2.

Part III

Examples

The author has implemented the algorithms of Part II in Magma [BCP97]. In this section we present some examples of computations performed with this implementation and some numerical data obtained.

8 Bianchi groups

A *Bianchi group* is a Kleinian group $\mathrm{PSL}_2(\mathbb{Z}_F)$ where F is a quadratic imaginary field. Methods for computing with Bianchi groups have already been studied by Swan [Swa71] and more recently Yasaki [Yas09]. The methods of Part II are not the best ones for Bianchi groups, but we still present computational results.

8.1 Examples of computations

The complexity of the computation increases a lot with the class number of F, so we have first computed Bianchi groups for quadratic imaginary fields with trivial class group. We have computed a fundamental domain and a presentation for every such field, and we present the result for the largest discriminant quadratic imaginary field with class number 1. The presentation obtained from the face pairing has been simplified with Magma functions.
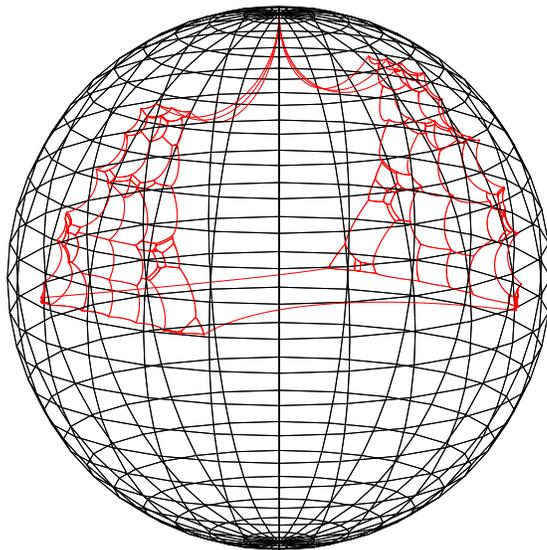
Figure 8.1: A fundamental domain for $\mathrm{PSL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-163}}{2}\right]\right)$

Proposition 8.1.1. *Let* $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-163}}{2}])$. *Then the group* Γ *has covolume* $\mathrm{Covol}(\Gamma) \approx 57.435648$, *and* Γ *is generated by the matrices*

$$\gamma_1 = \begin{pmatrix} -\frac{15+\sqrt{-163}}{2} & 14 - \sqrt{-163} \\ \frac{5-\sqrt{-163}}{2} & \frac{23+\sqrt{-163}}{2} \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 1 & -\frac{1+\sqrt{-163}}{2} \\ 0 & 1 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \gamma_4 = \begin{pmatrix} \frac{3+\sqrt{-163}}{2} & -7 \\ -6 & \frac{3-\sqrt{-163}}{2} \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} \frac{-3+\sqrt{-163}}{2} & -11 \\ 4 & \frac{3+\sqrt{-163}}{2} \end{pmatrix}, \ \gamma_6 = \begin{pmatrix} 4 + \sqrt{-163} & -18 \\ 10 & -4 + \sqrt{-163} \end{pmatrix}$$

$$\gamma_7 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \gamma_8 = \begin{pmatrix} \frac{-1+\sqrt{-163}}{2} & 7 \\ -6 & \frac{1+\sqrt{-163}}{2} \end{pmatrix}$$

$$\gamma_9 = \begin{pmatrix} \frac{-5+\sqrt{-163}}{2} & -8 \\ 6 & \frac{5+\sqrt{-163}}{2} \end{pmatrix}, \ \gamma_{10} = \begin{pmatrix} \frac{-1+\sqrt{-163}}{2} & 14 \\ -3 & \frac{1+\sqrt{-163}}{2} \end{pmatrix}$$

and the relations

$$[\gamma_2, \gamma_7] = 1, \ \gamma_3^2 = 1, \ (\gamma_3\gamma_7^{-1})^3 = 1, \ (\gamma_4^{-1}\gamma_3)^3 = 1, \ (\gamma_3\gamma_8^{-1}\gamma_4^{-1}\gamma_8)^3 = 1,$$

$$(\gamma_7^{-1}\gamma_{10}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_8\gamma_6\gamma_1)^2 = 1,$$

$$(\gamma_{10}^{-1}\gamma_1^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_3\gamma_9\gamma_7^{-1}\gamma_1\gamma_{10}\gamma_7^{-1})^3 = 1,$$

$$(\gamma_5\gamma_7^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_3\gamma_7^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_5^{-1}\gamma_7^{-1}\gamma_8^{-1}\gamma_3\gamma_4\gamma_8)^2 = 1,$$

$$\gamma_7\gamma_9\gamma_4\gamma_7\gamma_9^{-1}\gamma_8^{-1}\gamma_4^{-1}\gamma_8 = 1,$$

$$\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_8\gamma_3\gamma_8^{-1}\gamma_4^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}\gamma_9^{-1}$$
$$\gamma_7^{-1}\gamma_6\gamma_1\gamma_8\gamma_3\gamma_8^{-1}\gamma_4^{-1} = 1,$$

$$\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_3\gamma_9^{-1}\gamma_8^{-1}\gamma_4^{-1}\gamma_3\gamma_8\gamma_6\gamma_1\gamma_{10}\gamma_7\gamma_{10}^{-1}\gamma_1^{-1}\gamma_7\gamma_5^{-1}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3$$
$$\gamma_8\gamma_6\gamma_1\gamma_{10}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1 = 1,$$

$$\gamma_3\gamma_8\gamma_6\gamma_1\gamma_{10}\gamma_7\gamma_{10}^{-1}\gamma_1^{-1}\gamma_7\gamma_9^{-1}\gamma_3\gamma_7\gamma_9\gamma_4\gamma_6^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}$$
$$\gamma_7^{-1}\gamma_6\gamma_1\gamma_{10}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_6\gamma_1\gamma_5^{-1}\gamma_8^{-1}\gamma_4^{-1} = 1,$$

$$\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_7\gamma_5^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}$$
$$\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_7\gamma_5^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_7\gamma_5^{-1} = 1,$$

$$\gamma_6^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_7\gamma_5^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}$$
$$\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_3\gamma_7^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_5^{-1}\gamma_7^{-1}\gamma_6\gamma_4^{-1}\gamma_9^{-1}\gamma_3\gamma_7\gamma_9\gamma_4 = 1,$$

$$\gamma_{10}\gamma_7\gamma_5\gamma_7^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_3\gamma_7^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_5^{-1}\gamma_7^{-1}\gamma_6\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_3\gamma_9\gamma_4\gamma_8$$
$$\gamma_3\gamma_8^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_3\gamma_7\gamma_9\gamma_4\gamma_6^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_7\gamma_5^{-1}\gamma_{10}^{-1}$$
$$\gamma_7\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_8\gamma_6\gamma_1 = 1,$$

$$\gamma_3\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_4^{-1}\gamma_9^{-1}\gamma_3\gamma_7\gamma_9\gamma_4\gamma_6^{-1}\gamma_7\gamma_5\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_7\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_7$$

$$\gamma_5^{-1}\gamma_{10}^{-1}\gamma_2\gamma_{10}^{-1}\gamma_1^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_3\gamma_9\gamma_7^{-1}\gamma_1\gamma_{10}\gamma_7^{-1}\gamma_2^{-1}\gamma_{10}\gamma_5\gamma_7^{-1}\gamma_6^{-1}\gamma_7\gamma_9\gamma_4\gamma_3$$

$$\gamma_7^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_5^{-1}\gamma_7^{-1}\gamma_6\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_3\gamma_9\gamma_4\gamma_6^{-1}\gamma_7\gamma_9\gamma_4 = 1,$$

$$\gamma_8\gamma_3\gamma_8^{-1}\gamma_4^{-1}\gamma_5^{-1}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_8\gamma_6\gamma_1\gamma_{10}\gamma_7\gamma_5\gamma_4\gamma_8\gamma_3\gamma_8^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_7\gamma_9$$

$$\gamma_4\gamma_3\gamma_7^{-1}\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6\gamma_1\gamma_7^{-1}\gamma_1\gamma_{10}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}$$

$$\gamma_6\gamma_8\gamma_3\gamma_8^{-1}\gamma_4^{-1}\gamma_7^{-1}\gamma_1\gamma_{10}\gamma_7^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_6^{-1}\gamma_8^{-1}\gamma_3\gamma_4\gamma_8\gamma_9\gamma_3\gamma_4^{-1}\gamma_9^{-1}\gamma_7^{-1}\gamma_6 = 1$$

form a complete set of relations for Γ.

The fundamental polyhedron that was computed has 111 faces and 306 edges (Figure 8.1). In the lattice, 70 millions of vectors were enumerated, and 8500 of them had norm 1.

We now present an example with nontrivial class group: the field $\mathbb{Q}(\sqrt{-14})$ has class number 4; the presentation obtained from the face pairing has also been simplified.

Proposition 8.1.2. *Let* $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-14}])$. *Then the group* Γ *has covolume* $\mathrm{Covol}(\Gamma) \approx 20.351341$, *and* Γ *is generated by the matrices*

$$\gamma_1 = \begin{pmatrix} 1+\sqrt{-14} & 7 \\ 2 & 1-\sqrt{-14} \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 3+\sqrt{-14} & 6-\sqrt{-14} \\ 2-\sqrt{-14} & -5-\sqrt{-14} \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 7+\sqrt{-14} & -2+3\sqrt{-14} \\ 4-\sqrt{-14} & 7+\sqrt{-14} \end{pmatrix}, \ \gamma_4 = \begin{pmatrix} \sqrt{-14} & 5 \\ -3 & \sqrt{-14} \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 1 & -\sqrt{-14} \\ 0 & 1 \end{pmatrix}, \ \gamma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \gamma_7 = \begin{pmatrix} 3\sqrt{-14} & 19 \\ 11 & -5\sqrt{-14} \end{pmatrix}$$

and the relations

$$\gamma_6^2 = 1, \ (\gamma_6\gamma_7\gamma_1^{-1}\gamma_7^{-1})^3 = 1, \ (\gamma_4^{-1}\gamma_6\gamma_4\gamma_1)^3 = 1,$$
$$(\gamma_1\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_6\gamma_4\gamma_2^{-1}\gamma_6)^2 = 1, \ \gamma_6\gamma_7\gamma_4^{-1}\gamma_6\gamma_4\gamma_7^{-1} = 1,$$
$$\gamma_5^{-1}\gamma_4\gamma_7\gamma_1\gamma_7^{-1}\gamma_4^{-1}\gamma_5\gamma_1^{-1} = 1,$$
$$\gamma_4^{-1}\gamma_6\gamma_4\gamma_2^{-1}\gamma_6\gamma_3\gamma_4\gamma_6\gamma_2\gamma_4^{-1}\gamma_6\gamma_3^{-1} = 1,$$
$$\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_6\gamma_4\gamma_2^{-1}\gamma_6\gamma_5^{-1}\gamma_6\gamma_2\gamma_4^{-1}\gamma_6\gamma_4\gamma_1\gamma_2^{-1}\gamma_5 = 1,$$
$$\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_6\gamma_4\gamma_2^{-1}\gamma_6\gamma_4\gamma_2^{-1}\gamma_7\gamma_1\gamma_7^{-1}\gamma_6\gamma_2\gamma_4^{-1}\gamma_6 = 1,$$
$$\gamma_6\gamma_1\gamma_5^{-1}\gamma_3^{-1}\gamma_1\gamma_2\gamma_1^{-1}\gamma_3\gamma_6\gamma_4\gamma_2^{-1}\gamma_6\gamma_4^{-1}\gamma_5\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_6\gamma_4\gamma_2^{-1} = 1$$

form a complete set of relations for Γ.

The fundamental polyhedron that was computed has 48 faces and 132 edges (Figure 8.2). In the lattice, 30000 vectors were enumerated, and 300 of them had norm 1.
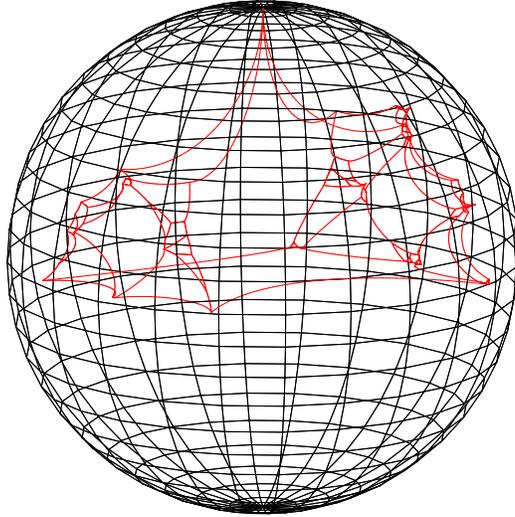
Figure 8.2: A fundamental domain for $\mathrm{PSL}_2\big(\mathbb{Z}\big[\sqrt{-14}\big]\big)$

8.2   Numerical data

We present here some numerical data obtained from the computation of Bianchi groups. In Table 8.1 below, F is a quadratic imaginary field with discriminant Δ_F and class number h_F, then we indicate the number of faces and edges of the fundamental polyhedron, the number of generators g, elliptic relations e, commutator relations c, and other relations o in the simplified presentation obtained, and finally the maximum absolute reduced norm $\max Q$ of the elements in the boundary of the fundamental polyhedron.

Δ_F	h_F	faces	edges	g	e	c	o	$\max Q$
-3	1	5	8	5	6	0	0	10
-4	1	5	8	4	6	0	0	21
-7	1	9	17	3	3	1	0	20
-8	1	6	11	4	4	1	0	20
-11	1	9	17	3	3	0	1	35
-19	1	11	22	4	6	1	0	160
-43	1	19	44	4	3	0	4	236
-67	1	35	88	6	6	1	2	749
-163	1	111	306	10	7	1	9	4536
-15	2	13	26	4	2	1	3	222
-20	2	15	36	5	3	1	1	420
-24	2	13	31	5	5	1	1	494
-35	2	21	50	4	2	0	3	1100
-40	2	23	61	6	4	1	3	1743
-23	3	27	62	5	2	2	1	333
-56	4	48	132	7	4	0	6	3504

Table 8.1: Computational data for Bianchi groups

9 Cocompact groups

By Corollary 2.3.9, an arithmetic Kleinian group is cocompact if and only if it is not commensurable with a Bianchi group. An example of computation of a fundamental domain for a cocompact arithmetic Kleinian group has been performed by Corrales, Jespers, Leal and del Río [CJLdR04]. In this section we present our computational results for such groups. Troughout the section, when F is a number field, a prime ideal of $\mathbb{Z}_F$ is denoted by $\mathfrak{p}_p$ whenever $\mathfrak{p}_p | p$ (although it does not determine it uniquely).

9.1 Examples of computations

The complexity of the computations increases a lot with the degree of the base field, simply because of it takes much time to enumerate elements in the associated Kleinian group. So we have first computed examples over quadratic fields. In opposition to Bianchi groups, the complexity does not seem to depend much on the class number. We present here the result of a computation over a field with class number 8.

Proposition 9.1.1. *Let* $F = \mathbb{Q}(\sqrt{-95})$, $\alpha = \sqrt{-95}$, $B = \left(\frac{\frac{1}{2}(13-\alpha), 653}{F} \right)$, $\mathcal{O}$ *the maximal order in* B *with* $\mathbb{Z}$-*basis* $\{1, \frac{1}{2}(\alpha-1), i, \frac{1}{2}(\alpha-1)i, \frac{1}{4}(\alpha+1) + \frac{1}{4}(\alpha-1)i + \frac{1}{2}j, -12 + \frac{1}{4}(-\alpha-47)i + \frac{1}{4}(\alpha-1)j, \frac{1}{4}(\alpha-13) + \frac{1}{4}(\alpha-3)i + \frac{1}{653}(34\alpha-1300)j + \frac{1}{1306}ij, -3 + \frac{1}{44}(5\alpha-67)i + \frac{1}{653}(-39\alpha-915)j + \frac{1}{28732}(\alpha+13)ij\}$, *and* $\Gamma = \mathcal{O}_1^\times / \pm 1$. *The quaternion algebra* B *has discriminant* $\mathfrak{p}_2\mathfrak{p}_3$ *where* $\mathfrak{p}_2 | 2$, $N(\mathfrak{p}_2) = 2$, $\mathfrak{p}_3 | 3$ *and* $N(\mathfrak{p}_3) = 3$. *Then the group* Γ *has covolume* $\mathrm{Covol}(\Gamma) \approx 114.113817$, *and* Γ *is generated by the elements*

$$\gamma_1 = -\frac{1}{4}(\alpha+5) + \frac{1}{22}(2\alpha-7)i + \frac{1}{1306}(622-201\alpha)j + \frac{1}{28732}(9367-1317\alpha)ij$$

$$\gamma_2 = \frac{1}{4}(7-\alpha) + \frac{1}{22}(\alpha-20)i + \frac{1}{1306}(1625-172\alpha)j + \frac{1}{28732}(15997-851\alpha)ij$$

$$\gamma_3 = -3/2 + \frac{1}{2612}(3\alpha-221)j - \frac{1}{2612}(\alpha+67)ij$$

$$\gamma_4 = -5/2 + \frac{1}{44}(31-\alpha)i + \frac{1}{2612}(215\alpha-2609)j + \frac{1}{14366}(232\alpha-6059)ij$$

$$\gamma_5 = -3/2 + \frac{1}{22}(\alpha-9)i + \frac{1}{2612}(331\alpha+559)j + \frac{1}{28732}(1283\alpha-1911)ij$$

$$\gamma_6 = \frac{1}{44}(9-\alpha)i + \frac{1}{2612}(79\alpha+181)j + \frac{1}{14366}(155\alpha-141)ij$$

$$\gamma_7 = \frac{1}{4}(\alpha+1) + \frac{1}{44}(1-5\alpha)i - \frac{1}{1306}(22\alpha+419)j - \frac{1}{7183}(69\alpha+633)ij$$

$$\gamma_8 = -2 + \frac{1}{2}i - \frac{1}{2612}(499\alpha+5705)j - \frac{1}{2612}(231\alpha+1359)ij$$

$$\gamma_9 = \frac{1}{4}(\alpha+5) + i - \frac{1}{2612}(539\alpha+99)j + \frac{1}{653}(138-45\alpha)ij$$

$$\gamma_{10} = \frac{1}{4}(15 - \alpha) + \frac{1}{44}(75 - \alpha)i + \frac{1}{2612}(19\alpha - 3781)j - \frac{1}{28732}(405\alpha + 14109)ij$$

and Γ admits a complete set of relations with 4 elliptic relations and 9 other relations. The elliptic relations are

$$\gamma_6^2 = 1, \ (\gamma_1^{-1}\gamma_2)^3 = 1,$$
$$(\gamma_9^{-1}\gamma_7^{-1}\gamma_3^{-1}\gamma_7\gamma_3^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_2\gamma_{10}\gamma_3^2\gamma_7\gamma_5\gamma_9\gamma_4^{-1}\gamma_2\gamma_5\gamma_4^{-1}\gamma_6\gamma_3\gamma_6\gamma_1^{-1})^2 = 1,$$
$$(\gamma_{10}\gamma_3\gamma_7\gamma_5\gamma_3^{-1}\gamma_{10}^{-1}\gamma_1^{-1}\gamma_2\gamma_{10}\gamma_3\gamma_5^{-1}\gamma_6\gamma_5\gamma_9^{-1}\gamma_7^{-1}\gamma_3^{-1}\gamma_6\gamma_4)^3 = 1$$

and the other relations have length $\{83, 98, 111, 114, 180, 194, 200, 218, 227\}$.

The fundamental polyhedron that was computed has 376 faces and 1106 edges (Figure 9.1). In the lattice, 850 000 vectors were enumerated, and 500 of them had norm 1.
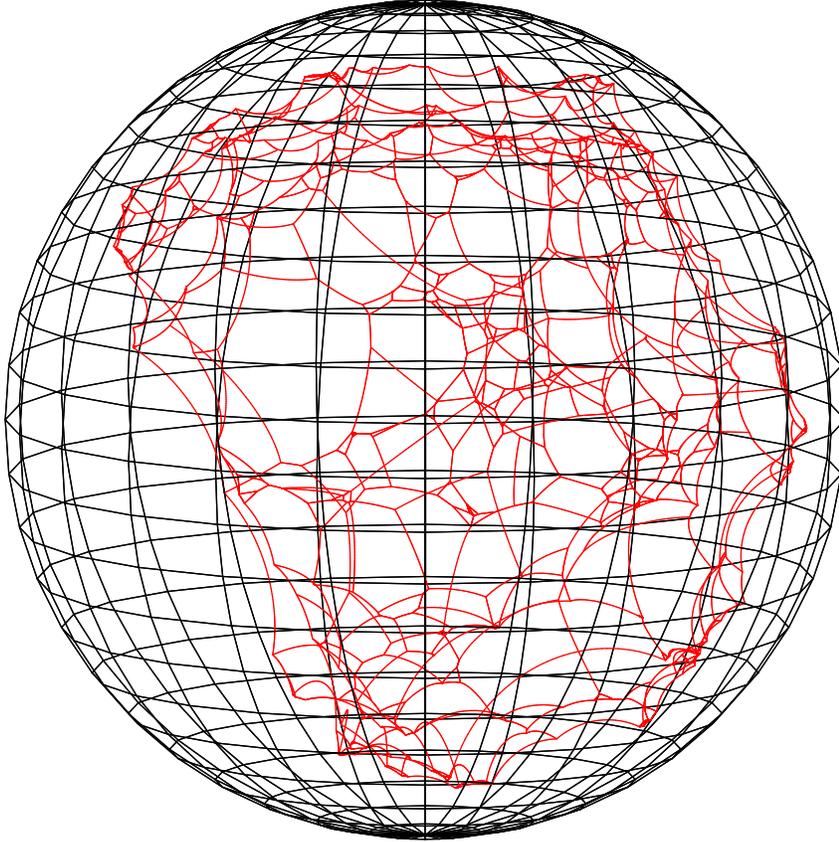
Figure 9.1: A fundamental domain for a cocompact arithmetic Kleinian group over the quadratic field $\mathbb{Q}(\sqrt{-95})$

We now present an example over a cubic field.

Proposition 9.1.2. *Let $F = \mathbb{Q}(\sqrt[3]{11})$ with discriminant -3267 and class number 2, $\alpha = \sqrt[3]{11}$, $B = \left(\frac{-2,-4\alpha^2-\alpha-2}{F}\right)$, $\mathcal{O}$ the maximal order in B with $\mathbb{Z}$-basis $\{1, \alpha, \alpha^2, i, \alpha i, \alpha^2 i, \frac{1}{2}\alpha^2 + \frac{1}{2}i + \frac{1}{2}j, 1\frac{1}{2} + \frac{1}{2}\alpha i + \frac{1}{2}\alpha j, 1\frac{1}{2}\alpha + \frac{1}{2}\alpha^2 i + \frac{1}{2}\alpha^2 j, (\alpha + 4310) + \frac{1}{2}(\alpha^2 + 2\alpha + 44)i + 1521j + \frac{1}{2}ij, \frac{1}{2}(\alpha^2 + 4311\alpha + 4310) + \frac{1}{4}(3\alpha^2 + 46\alpha + 55)i + \frac{1}{2}(1521\alpha + 1521)j + \frac{1}{4}(\alpha + 1)ij, \frac{1}{2}(\alpha^2 + 1834\alpha + 4929) + \frac{1}{4}(2\alpha^2 + 21\alpha + 55)i + \frac{1}{14998}(1521\alpha^2 + 4850469\alpha + 13044096)j + \frac{1}{29996}(\alpha^2 + 3189\alpha + 8576)ij\}$, and $\Gamma = \mathcal{O}_1^\times / \pm 1$. The quaternion algebra B has discriminant $\mathfrak{p}_2$ where $\mathfrak{p}_2 | 2$ and $N(\mathfrak{p}_2) = 2$. Then the group Γ has covolume $\mathrm{Covol}(\Gamma) \approx 206.391784$, and Γ is generated by the elements*

$$\gamma_1 = \frac{1}{2}(\alpha^2 - \alpha - 2) + \frac{1}{2}i + \frac{1}{14998}(-4391\alpha^2 + 12732\alpha - 4737)j$$
$$+ \frac{1}{14998}(-2898\alpha^2 - 2954\alpha + 20935)ij$$

$$\gamma_2 = -\frac{1}{2} + \frac{1}{4}(-\alpha^2 + 3)i + \frac{1}{14998}(349\alpha^2 - 4390\alpha + 8422)j$$
$$+ \frac{1}{29996}(3200\alpha^2 - 1339\alpha - 10639)ij$$

$$\gamma_3 = -\frac{1}{2} + \frac{1}{14998}(54\alpha^2 - 271\alpha - 1834)j + \frac{1}{14998}(355\alpha^2 - 254\alpha - 114)ij$$

$$\gamma_4 = -\frac{1}{2} + \frac{1}{4}(\alpha - 1)i + \frac{1}{14998}(161\alpha^2 + 3497\alpha - 6579)j$$
$$+ \frac{1}{29996}(-2327\alpha^2 + 3207\alpha + 5986)ij$$

$$\gamma_5 = \frac{1}{2}(-\alpha^2 + 6) + \frac{1}{4}(\alpha^2 - 2\alpha + 1)i + \frac{1}{14998}(-507\alpha^2 + 2961\alpha - 6111)j$$
$$+ \frac{1}{29996}(-2500\alpha^2 - 1063\alpha + 14639)ij$$

$$\gamma_6 = \frac{1}{2}(\alpha - 2) + \frac{1}{4}(-\alpha^2 + 2\alpha - 1)i + \frac{1}{14998}(1064\alpha^2 - 3951\alpha + 6080)j$$
$$+ \frac{1}{29996}(2880\alpha^2 - 1955\alpha - 10325)ij$$

$$\gamma_7 = \frac{1}{2}(\alpha^2 + \alpha - 7) + \frac{1}{4}(-\alpha^2 - 2\alpha + 11)i + \frac{1}{14998}(3218\alpha^2 - 11428\alpha + 8747)j$$
$$+ \frac{1}{29996}(4538\alpha^2 + 6111\alpha - 39417)ij$$

$$\gamma_8 = \frac{1}{2}(-3\alpha + 6) + \frac{1}{4}(2\alpha^2 - 3\alpha - 3)i + \frac{1}{14998}(-3586\alpha^2 + 7720\alpha - 137)j$$
$$+ \frac{1}{29996}(-2433\alpha^2 - 4871\alpha + 26806)ij$$

$$\gamma_9 = (\alpha - 3) + \frac{1}{2}(\alpha^2 - 2\alpha)i + \frac{1}{7499}(3037\alpha^2 - 3715\alpha - 6214)j$$
$$+ \frac{1}{14998}(214\alpha^2 + 7536\alpha - 16989)ij$$

$$\gamma_{10} = \frac{1}{2}(-2\alpha^2 + \alpha + 7) + \frac{1}{4}(\alpha^2 + 5\alpha - 14)i$$
$$+ \frac{1}{14998}(-1979\alpha^2 + 25624\alpha - 46661)j$$
$$+ \frac{1}{29996}(-16577\alpha^2 + 11396\alpha + 54183)ij$$

$$\gamma_{11} = \frac{1}{2}(-\alpha + 3) + \frac{1}{4}(-\alpha^2 + 2\alpha + 1)i + \frac{1}{7499}(-2128\alpha^2 + 403\alpha + 10337)j$$
$$+ \frac{1}{29996}(3478\alpha^2 - 14677\alpha + 18803)ij$$

$$\gamma_{12} = \frac{1}{2}(\alpha - 3) + \frac{1}{4}(\alpha - 3)i + \frac{1}{14998}(-594\alpha^2 + 2981\alpha - 2323)j$$
$$+ \frac{1}{29996}(-311\alpha^2 - 1911\alpha + 2508)ij$$

$$\gamma_{13} = \frac{1}{2}(-\alpha + 4) + \frac{1}{4}(-\alpha^2 + 2\alpha + 1)i + \frac{1}{14998}(-1164\alpha^2 + 9\alpha + 6204)j$$
$$+ \frac{1}{29996}(1360\alpha^2 - 4881\alpha + 2415)ij$$

$$\gamma_{14} = \frac{1}{2}(\alpha^2 - \alpha - 2) - \frac{1}{2}i + \frac{1}{14998}(-4391\alpha^2 + 12732\alpha - 4737)j$$
$$+ \frac{1}{14998}(-2898\alpha^2 - 2954\alpha + 20935)ij$$

$$\gamma_{15} = \frac{1}{2}(\alpha^2 + \alpha - 7) + \frac{1}{14998}(6991\alpha^2 - 7727\alpha - 14687)j$$
$$+ \frac{1}{14998}(271\alpha^2 + 9333\alpha - 23091)ij$$

$$\gamma_{16} = \frac{1}{2}(\alpha^2 - 2\alpha + 1) + \frac{1}{4}(\alpha^2 + 2\alpha - 11)i$$
$$+ \frac{1}{14998}(-1655\alpha^2 + 9000\alpha - 12671)j$$
$$+ \frac{1}{29996}(-4818\alpha^2 + 849\alpha + 22819)ij$$

$$\gamma_{17} = \frac{1}{2}(7 - \alpha^2 - \alpha) + \frac{1}{4}(-\alpha^2 + 2\alpha - 1)i + \frac{1}{14998}(606\alpha^2 - 2208\alpha + 249)j$$
$$+ \frac{1}{29996}(1302\alpha^2 + 5131\alpha - 15057)ij$$

and Γ admits a complete set of relations with 11 elliptic relations and 21 other relations. The elliptic relations are

$$\gamma_2^3 = 1, \ \gamma_4^3 = 1, \ \gamma_3^3 = 1, \ (\gamma_{14}^{-1}\gamma_1)^2 = 1,$$
$$(\gamma_3\gamma_{12}^{-1})^3 = 1, \ (\gamma_{14}^{-1}\gamma_{15}^{-1})^3 = 1, \ (\gamma_{15}\gamma_1)^3 = 1,$$
$$(\gamma_9\gamma_{11}^{-1}\gamma_{15}\gamma_6\gamma_8^{-1}\gamma_{14}\gamma_2\gamma_7^{-1}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_{12}\gamma_3^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_{17})^3 = 1,$$
$$(\gamma_6\gamma_8^{-1}\gamma_{14}\gamma_2\gamma_7^{-1}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_{12}\gamma_3^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_{17}\gamma_{16}^{-1}\gamma_9^{-1}\gamma_4^{-1}$$
$$\gamma_{15}\gamma_1\gamma_4\gamma_{12}^{-1}\gamma_2^{-1}\gamma_{11}\gamma_{13}^{-1}\gamma_3^{-1}\gamma_{12}\gamma_9\gamma_{11}^{-1}\gamma_{15})^2 = 1,$$

$$(\gamma_{12}\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_9\gamma_{11}^{-1}\gamma_{15}\gamma_1\gamma_4\gamma_3^{-1}\gamma_5^{-1}\gamma_{17}^{-1}\gamma_{15}^{-1}\gamma_{11}\gamma_9^{-1}\gamma_{12}^{-1}\gamma_3\gamma_{13}\gamma_{11}^{-1}\gamma_2\gamma_{12}$$
$$\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_9\gamma_{16}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_3\gamma_{12}^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_{17}\gamma_7\gamma_2^{-1}$$
$$\gamma_{14}^{-1}\gamma_8\gamma_6^{-1}\gamma_3^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{15}\gamma_6\gamma_3\gamma_{12}^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_{17}\gamma_7\gamma_2^{-1}\gamma_{14}^{-1}\gamma_8\gamma_6^{-1}$$
$$\gamma_{15}^{-1}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_3\gamma_4^{-1}\gamma_{16})^2 = 1,$$
$$(\gamma_8^{-1}\gamma_4\gamma_1\gamma_{16}\gamma_{13}\gamma_2^{-1}\gamma_{13}^{-1}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_3\gamma_{12}^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_{17}\gamma_7$$
$$\gamma_2^{-1}\gamma_{14}^{-1}\gamma_8\gamma_6^{-1}\gamma_{15}^{-1}\gamma_{11}\gamma_9^{-1}\gamma_{12}^{-1}\gamma_3\gamma_{13}\gamma_{11}^{-1}\gamma_2\gamma_{12}\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_9\gamma_{16}\gamma_{12}^{-1}\gamma_3\gamma_{13}$$
$$\gamma_{11}^{-1}\gamma_2\gamma_{12}\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_{10}\gamma_{13}^{-1}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_3\gamma_{12}^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5$$
$$\gamma_{17}\gamma_7\gamma_2^{-1}\gamma_{14}^{-1}\gamma_8\gamma_6^{-1}\gamma_{15}^{-1}\gamma_{11}\gamma_9^{-1}\gamma_{12}^{-1}\gamma_3\gamma_{13}\gamma_{11}^{-1}\gamma_2\gamma_{12}\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_9\gamma_{16}\gamma_{13}$$
$$\gamma_2\gamma_{13}^{-1}\gamma_{16}^{-1}\gamma_1^{-1}\gamma_4^{-1}\gamma_{11}^{-1}\gamma_7\gamma_{13}^{-1}\gamma_3\gamma_{13}\gamma_{11}^{-1}\gamma_2\gamma_{16}^{-1}\gamma_4\gamma_3^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{15}\gamma_6\gamma_8^{-1}\gamma_{14}$$
$$\gamma_2\gamma_7^{-1}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_{12}\gamma_3^{-1}\gamma_6^{-1}\gamma_{15}^{-1}\gamma_{11}\gamma_{10}^{-1}\gamma_5\gamma_3\gamma_{15}^{-1}\gamma_{11}\gamma_9^{-1}\gamma_{12}^{-1}\gamma_3\gamma_{13}$$
$$\gamma_{11}^{-1}\gamma_2\gamma_{12}\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_4\gamma_9\gamma_{16}\gamma_{17}^{-1}\gamma_5^{-1}\gamma_{10}\gamma_{11}^{-1}\gamma_{14}^{-1}\gamma_6\gamma_3\gamma_{12}^{-1}\gamma_6^{-1}\gamma_{14}\gamma_{11}\gamma_{10}^{-1}\gamma_5$$
$$\gamma_{17}\gamma_7\gamma_2^{-1}\gamma_{14}^{-1}\gamma_8\gamma_6^{-1}\gamma_{17}\gamma_5\gamma_3\gamma_4^{-1}\gamma_1^{-1}\gamma_{15}^{-1}\gamma_{11}\gamma_9^{-1}\gamma_4^{-1}\gamma_{15}\gamma_1\gamma_4\gamma_{12}^{-1}\gamma_2^{-1}\gamma_{11}\gamma_{13}^{-1}$$
$$\gamma_3^{-1}\gamma_{13}\gamma_7^{-1}\gamma_{11}\gamma_{13}^{-1}\gamma_3^{-1}\gamma_{13}\gamma_6\gamma_{12}\gamma_3^{-1})^3 = 1$$

and the other relations have length $\{76, 379, 414, 463, 499, 520, 567, 666, 688, 707,$
$810, 894, 942, 960, 960, 1039, 1181, 1196, 1497, 1602, 2057\}.$

The fundamental polyhedron that was computed has 647 faces and 1877 edges (Figure 9.2), and the maximum absolute reduced norm of the elements in the boundary of this polyhedron is 5802. In the lattice, 80 millions of vectors were enumerated, and 300 of them had norm 1.
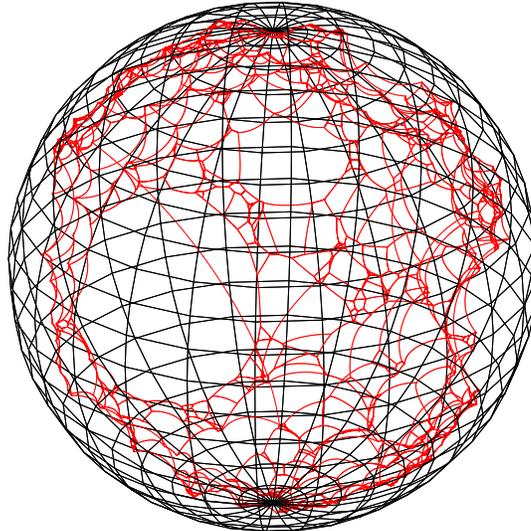
Figure 9.2: A fundamental domain for a cocompact arithmetic Kleinian group over the cubic QTR field $\mathbb{Q}(\sqrt[3]{11})$

9.2 Numerical data

We now present some numerical data obtained from the computation of cocompact arithmetic groups over quadratic imaginary fields. In Table 9.1 below, F is a quadratic imaginary field with discriminant Δ_F and class number h_F, then we indicate the discriminant Δ_B of the quaternion algebra B, the number of faces and edges of the fundamental polyhedron, the number of generators g, elliptic relations e and other relations o in the simplified presentation obtained, and finally the maximum absolute reduced norm $\max Q$ of the elements in the boundary of the fundamental polyhedron.

Δ_F	h_F	Δ_B	faces	edges	g	e	o	$\max Q$
-3	1	$\mathfrak{p}_2\mathfrak{p}_3$	29	81	3	2	2	24
-4	1	$\mathfrak{p}_2\mathfrak{p}_3$	32	86	3	2	2	47
-7	1	$\mathfrak{p}_2\mathfrak{p}_3$	46	128	3	1	3	128
-8	1	$\mathfrak{p}_2\mathfrak{p}_3$	31	79	3	3	3	41
-11	1	$\mathfrak{p}_2\mathfrak{p}_3$	53	153	3	1	3	100
-15	2	$\mathfrak{p}_2\mathfrak{p}_3$	44	120	4	3	3	69
-20	2	$\mathfrak{p}_2\mathfrak{p}_3$	50	134	3	3	2	189
-23	3	$\mathfrak{p}_2\mathfrak{p}_3$	69	193	5	3	2	222
-24	2	$\mathfrak{p}_2\mathfrak{p}_3$	56	156	4	2	3	110
-39	4	$\mathfrak{p}_2\mathfrak{p}_3$	121	357	5	2	4	374
-40	2	$\mathfrak{p}_2\mathfrak{p}_3$	262	762	9	4	9	948
-47	5	$\mathfrak{p}_2\mathfrak{p}_3$	148	426	6	3	7	572
-95	8	$\mathfrak{p}_2\mathfrak{p}_3$	376	1106	10	4	9	2365

Table 9.1: Computational data for cocompact arithmetic Kleinian groups over quadratic imaginary fields

Part IV

Prospect

10 Applications

In this section we present three possible applications of the algorithms of Part II.

10.1 Computing the unit group of quaternion algebras

We claimed in the introduction that one could compute the unit group of a quaternion algebra by using arithmetic Kleinian groups. We will now describe precisely how this can be performed. In this section $\mathcal{O}$ is a maximal order in a Kleinian quaternion algebra B over a QTR number field F.

Definition 10.1.1. The group of *positive integers of F relative to B* is

$$\mathbb{Z}_{F,(+)}^{\times} = \{x \in \mathbb{Z}_F \mid \sigma(x) > 0 \text{ for all real embeddings } \sigma \text{ ramified in } B\}.$$

Theorem 10.1.2. *We have*

$$\mathrm{nrd}(\mathcal{O}^{\times}) = \mathbb{Z}_{F,(+)}^{\times}.$$

Proof. A proof can be found in [Vig80, Théorème 4.1 and Corollaire 4.2]. $\qquad\square$

This theorem gives the short exact sequence

$$1 \longrightarrow \mathcal{O}_1^{\times} \longrightarrow \mathcal{O}^{\times} \xrightarrow{\mathrm{nrd}} \mathbb{Z}_{F,(+)}^{\times} \longrightarrow 1$$

which we can mod out by the center to give the exact sequence

$$1 \longrightarrow \mathcal{O}_1^{\times}/\{\pm 1\} \longrightarrow \mathcal{O}^{\times}/\mathbb{Z}_F^{\times} \xrightarrow{\mathrm{nrd}} \mathbb{Z}_{F,(+)}^{\times}/\mathbb{Z}_F^{\times 2} \longrightarrow 1.$$

We also have the exact sequence

$$1 \longrightarrow \mathbb{Z}_F^{\times} \longrightarrow \mathcal{O}^{\times} \longrightarrow \mathcal{O}^{\times}/\mathbb{Z}_F^{\times} \longrightarrow 1.$$

We can compute the structure of $\mathcal{O}^{\times}$ with these two sequences and the following proposition.

Definition 10.1.3. Let S be a set, we write $F(S)$ the free group generated by S. For all subsets $R \subset F(S)$ we write $\mathrm{N}(R)$ the normal subgroup generated by R. Suppose we have an exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

and suppose that for every $K \in \{H, G/H\}$, there exists a finite presentation $K \cong F(S(K))/\operatorname{N}(R(K))$ where $S(K) \subset K$ and $R(K) \subset F(S(K))$. We write the projection $\pi : G \twoheadrightarrow G/H$. Let $\lambda : S(G/H) \hookrightarrow G$ be a lifting, so that for all $x \in S(G/H)$, $\pi(\lambda(x)) = x$, which extends to a unique group homomorphism $\lambda : F(S(G/H)) \to G$. We write the projection $e : F(S(H)) \twoheadrightarrow H$. Let $\mu : H \hookrightarrow F(S(H))$ be a lifting, so that for all $h \in H$, $e(\mu(h)) = h$. The set of *lifted generators* is the set $S(G) = \lambda(S(G/H))$; the set of *lifted relations* is the set $R(G) = \{\mu(\lambda(r))\lambda(r)^{-1} : r \in R(G/H)\} \subset F(S(H) \cup S(G))$; and the set of *conjugation relations* is the set $R_C = \{xhx^{-1}\mu(xhx^{-1})^{-1} : x \in S(G), h \in S(H)\}$.

Proposition 10.1.4. *Suppose we have an exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

and suppose that for every $K \in \{H, G/H\}$, there exists a finite presentation $K \cong F(S(K))/\operatorname{N}(R(K))$ where $S(K) \subset K$ and $R(K) \subset F(S(K))$. Then the set $S(H) \cup S(G)$ is a set of generators for G and we have a finite presentation $G \cong F(S(H) \cup S(G))/\operatorname{N}(R(H) \cup R(G) \cup R_C)$.

Proof. The set $S(H) \cup S(G)$ generates G since every element $g \in G$ is in some coset in G/H, so this coset is a product of elements in $S(G/H)$, and g is a product of elements in $S(G)$ times an element in H, which is a product of elements in $S(H)$.

Now we prove that the lifted relations are satisfied. Let $r \in R(G/H)$, then $\pi(\lambda(r)) = e'(r) = 1$ where $e' : F(S(G/H)) \twoheadrightarrow G/H$ is the projection. So $\lambda(r) \in H$ so $\mu(\lambda(r))$ is well-defined, and $e''(\mu(\lambda(r))\lambda(r)^{-1}) = e(\mu(\lambda(r)))\lambda(r)^{-1} = \lambda(r)\lambda(r)^{-1} = 1$ where $e'' : F(S(H) \cup S(G)) \twoheadrightarrow G$ is the projection.

Next we prove that the conjugation relations are statisfied. Let $x \in S(G)$ and $h \in S(H)$. Since H is normal, we have $xhx^{-1} \in H$ so the conjugation relation $xhx^{-1}\mu(xhx^{-1})^{-1}$ is well-defined and $e''(xhx^{-1}\mu(xhx^{-1})^{-1}) = xhx^{-1}e(\mu(xhx^{-1})^{-1}) = 1$.

Finally we prove that the set $R(H) \cup R(G)$ is a complete set of relations for G. Let $w \in F(S(H) \cup S(G))$ be such that $e''(w) = 1$. Using the conjugation relations we may assume that $w = xh$ with $x \in F(S(G))$ and $h \in F(S(H))$. Then the word $x = wh^{-1}$ projects to 1 in G/H, so it is a product of the relations $R(G/H)$ and using the conjugation relations again, the word w is a product of the lifted relations times a word in $F(S(H))$, which is a product of the relations $R(H)$. $\square$

To treat the exact sequence

$$1 \longrightarrow \mathcal{O}_1^{\times}/\{\pm 1\} \longrightarrow \mathcal{O}^{\times}/\mathbb{Z}_F^{\times} \xrightarrow{\mathrm{nrd}} \mathbb{Z}_{F,(+)}^{\times}/\mathbb{Z}_F^{\times 2} \longrightarrow 1$$

a presentation for $\mathbb{Z}^{\times}_{F,(+)}/\mathbb{Z}^{\times 2}_{F}$ can be obtained easily since it is finite abelian, and μ is given by the reduction algorithm.

10.2   Computing the cohomology of arithmetic Kleinian groups and Hecke operators

In this section we describe the first cohomology of an arithmetic Kleinian group and the Hecke operators acting on this module. There are general constructions from homology theory, but we give here a very explicit description to emphasize the fact that they are computable.

Let Γ be a group and M a $\mathbb{Z}\Gamma$-module. A *cocycle* is a map $f : \Gamma \to M$ such that for all $g, h \in \Gamma$ we have

$$f(gh) = g \cdot f(h) + f(g).$$

The module of all cocycles is written $Z^1(\Gamma, M)$. For all $m \in M$, the map $f_m : \Gamma \to M$ defined by

$$f_m(g) = g \cdot m - m \text{ for all } g \in \Gamma$$

is a cocycle. The submodule of *coboundaries* is the module $B^1(\Gamma, M) = \{f_m : m \in M\} \subset Z^1(\Gamma, M)$. The *cohomology module* is

$$H^1(\Gamma, M) = Z^1(\Gamma, M)/B^1(\Gamma, M).$$

From a finite presentation for Γ we can compute the cohomology module. But the most interesting structure comes with Hecke operators.

Now suppose that we also have a subgroup $H \subset \Gamma$ with finite index $m = [\Gamma : H]$. The inclusion $H \hookrightarrow \Gamma$ induces the *restriction map* res $: H^1(\Gamma, M) \to H^1(H, M)$. Let $\gamma_1, \dots, \gamma_m$ be representatives for Γ/H. Then for all $\gamma \in \Gamma$, there is a unique permutation $\gamma^* \in \mathcal{S}_m$ and for all $i \in \{1, \dots, m\}$, there is a unique element $h_i(\gamma) \in H$ such that

$$\gamma\gamma_i = \gamma_{\gamma^*(i)}h_i(\gamma).$$

The *transfer map* is the map tr $: H^1(H, M) \to H^1(\Gamma, M)$ defined by

$$\mathrm{tr}(f)(\gamma) = \sum_{i=1}^{m} \gamma_{\gamma^*(i)}f(h_i(\gamma)).$$

One can prove that the transfer map is well-defined, and does not depend on the choice of representatives $\gamma_1, \dots, \gamma_m$, and that the composition tr $\circ$ res is the multiplication by $[\Gamma : H]$ on $H^1(\Gamma, M)$. We prove only the last result, the others are similar computations.

Proof. For all $f \in Z^1(\Gamma, M)$ and for all $\gamma \in \Gamma$ we have

$$\mathrm{tr}(\mathrm{res}(f))(\gamma) = \sum_{i=1}^{m} \gamma_{\gamma^*(i)} f(h_i(\gamma)).$$

For all $i \in \{1, \ldots, m\}$ we have $\gamma\gamma_i = \gamma_{\gamma^*(i)} h_i(\gamma)$ and since f is a cocycle we have

$$\begin{aligned}
\gamma \cdot f(\gamma_i) + f(\gamma) &= f(\gamma\gamma_i) \\
&= f(\gamma_{\gamma^*(i)h_i(\gamma)} \\
&= \gamma_{\gamma^*(i)} \cdot f(h_i(\gamma)) + f(\gamma_{\gamma^*(i)}).
\end{aligned}$$

This gives

$$\begin{aligned}
\mathrm{tr}(\mathrm{res}(f))(\gamma) &= \sum_{i=1}^{m}(\gamma \cdot f(\gamma_i) + f(\gamma) - f(\gamma_{\gamma^*(i)})) \\
&= \sum_{i=1}^{m} f(\gamma) + \gamma \cdot \sum_{i=1}^{m} f(\gamma_i) - \sum_{i=1}^{m} f(\gamma_{\gamma^*(i)}) \\
&= mf(\gamma) + \gamma \cdot \sum_{i=1}^{m} f(\gamma_i) - \sum_{i=1}^{m} f(\gamma_i) \\
&\in mf(\gamma) + B^1(\Gamma, M).
\end{aligned}$$

$$\square$$

Now suppose $\mathcal{O}$ is an order in a Kleinian quaternion algebra B and consider a discrete embedding $\rho : B \hookrightarrow M_2(\mathbb{C})$. Let $\Gamma = P\rho(\mathcal{O}_1^\times)$ and let $\delta \in B^\times$. Then $\mathcal{O} \cap \delta\mathcal{O}\delta^{-1}$ and $\delta\mathcal{O}\delta^{-1} \cap \mathcal{O}$ are orders in B. Thus by Theorem 2.4.2, the subgroups $\Gamma \cap \delta\Gamma\delta^{-1}$ and $\delta\Gamma\delta^{-1} \cap \Gamma$ have finite index in Γ. Furthermore, conjugation by δ induces an isomorphism

$$\tilde{\delta} : H^1(\Gamma \cap \delta\Gamma\delta^{-1}, M) \to H^1(\delta^{-1}\Gamma\delta \cap \Gamma, M).$$

The *Hecke operator* T_δ associated to δ is then defined by the commutative diagram:

$$\begin{array}{ccc}
H^1(\Gamma, M) & \xrightarrow{\ T_\delta\ } & H^1(\Gamma, M) \\
{\scriptstyle \mathrm{res}}\downarrow & & \uparrow{\scriptstyle \mathrm{tr}} \\
H^1(\Gamma \cap \delta\Gamma\delta^{-1}, M) & \xrightarrow[\ \tilde{\delta}\ ]{} & H^1(\delta^{-1}\Gamma\delta \cap \Gamma, M)
\end{array}$$

The cohomology modules of arithmetic groups for suitably chosen modules M and the action of Hecke operators are the objects of many studies, and an implementation of their computation using the algorithms of Part II would be very interesting.

10.3   Studying a large class of hyperbolic 3-manifolds

Compact hyperbolic 3-manifolds, i.e. quotients of $\mathcal{H}^3$ by a cocompact torsion-free Kleinian group, form a large class of compact 3-manifolds. However they are still not as well-understood as other compact 3-manifolds, and remain full of open problems. The algorithms described in Part II provide a way of computing with arithmetic hyperbolic 3-manifolds by producing a fundamental polyhedron with the gluing of the faces (the face pairing) that gives the quotient manifold. It would be interesting to see whether such algorithms enable to experimentally investigate these manifolds.

We give an example: the virtual positive Betti number conjecture. Let Γ be a torsion-free Kleinian group. The *first Betti number* of Γ is the integer $\beta_1(\Gamma) = \dim_{\mathbb{R}} H^1(\Gamma, \mathbb{R})$ where Γ acts trivially on $\mathbb{R}$. Note that $\beta_1(\Gamma)$ is the rank of the abelianization of Γ. We say that $M = \mathcal{H}^3/\Gamma$ is *irreducible* if for all embeddings $f : \mathbb{S}^2 \to M$ where $\mathbb{S}^2$ denotes the 2-sphere, $f(\mathbb{S}^2)$ bounds a 3-ball in M. The virtual positive Betti number conjecture is:

Conjecture 10.3.1. *Let Γ be an infinite torsion-free Kleinian group, and suppose that the hyperbolic 3-manifold $M = \mathcal{H}^3/\Gamma$ is irreducible. Then there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that $\beta_1(\Gamma') > 0$.*

It would be interesting to see whether it is valuable to experimentally test this conjecture for arithmetic Kleinian groups with the algorithms of Part II.

11   Generalizations and open problems

In this section we describe how the algorithms of Part II could be generalized, and list some open problems about these algorithms.

11.1   Computing with smaller orders

In Section 7 and in the implementation so far, we have assumed that the quaternionic order $\mathcal{O}$ was maximal. Actually this is an unnecessary restriction; all we need is a volume formula like Theorem 2.4.2. But if $\mathcal{O}$ is a maximal order and $\mathcal{O}' \subset \mathcal{O}$ is another order, then $\mathrm{Covol}(\mathcal{O}') = \mathrm{Covol}(\mathcal{O})[\mathcal{O}_1^{\times} : \mathcal{O}_1'^{\times}]$ where $[\mathcal{O}_1^{\times} : \mathcal{O}_1'^{\times}]$ denotes the index of $\mathcal{O}_1'^{\times}$ in $\mathcal{O}_1^{\times}$. For some orders we may have an explicit formula for this index : for example Eichler orders (see [MR03, Definition 6.1.1 and Section 11.2.2]). For a non-Eichler order $\mathcal{O}'$, we may compute first a fundamental domain and a presentation for the group associated to a maximal order $\mathcal{O} \supset \mathcal{O}'$, and then while enumerating $\mathcal{O}_1'^{\times}$, locating the elements in the Cayley graph of $\mathcal{O}_1^{\times}$ by using reduction: this way we would know when we have enumerated a set of generators for $\mathcal{O}_1'^{\times}$, and then we would have also

computed the index $[\mathcal{O}_1^\times : \mathcal{O}_1'^\times]$. An implementation of these methods would be interesting.

11.2   Improving the lattice enumeration

The lattice enumeration techniques (Section 5.2) seem to be a critical part of the algorithm. There are at least two possible ideas to improve this part. First, we could find new geometrically parametrized quadratic forms to detect the boundary elements of the exterior domain. For example, we have seen in Proposition 5.2.12 that loxodromic elements are difficult to detect with the absolute reduced norm; maybe a quadratic form detecting fixed points on the sphere at infinity could help detect these elements. The other idea would be to improve the lattice enumeration algorithms: Schnorr, Euchner and Hörner ([SE94], [SH95]) introduced techniques called "pruning" which enumerate almost all short vectors in much shorter time than classical algorithms; Fieker and Stehlé [FS06] described an algorithm to perform LLL-reduction in $\mathbb{Z}_F$-modules (where F is a number field), such as an order in a quaternion algebra. It would be interesting to test whether these algorithms can provide significant improvements.

11.3   Allowing more split places

As we have seen in Proposition 2.3.4 and Theorem 2.3.5, if we consider an order $\mathcal{O}$ in quaternion algebra B with no restriction on the ramification at infinity, then the space on which $\mathcal{O}_1^\times$ naturally acts is $X = (\mathcal{H}^2)^{s_1} \times (\mathcal{H}^3)^{r_2}$ where s_1 is the number of split real places and r_2 is the number of complex places. Suppose we can do geometric computations in this space X, then we can apply the methods of Part II to get a fundamental domain and a presentation for $\mathcal{O}_1^\times$. The simplest case not yet studied is probably when $s_1 = 2$ and $r_2 = 0$ so that $X = \mathcal{H}^2 \times \mathcal{H}^2$, and having this case solved might naturally lead to the general case.

11.4   Getting rid of approximation

As we have seen in Section 3.2, the geometric computations can be performed by using exact real arithmetic, but in practise it is simpler to use sufficiently large, fixed precision. Thus it would be interesting to have a way, either to predict the required precision, or to prove after the computation that the computed polyhedron is indeed the correct one. The first problem is closely related to the one of having a bound on the size of the coefficients of the elements used in the computation, and thus to Section 11.6. The second problem can be seen as partially solved as there are already many constraints on the polyhedron computed: it has a side pairing, it has the correct volume up to some large

precision, it satisfies the cycle condition; furthermore Riley [Ril83] has studied this problem.

11.5   The choice of the center

Once we have a discrete embedding $\rho : \mathcal{O}_1^\times \hookrightarrow \mathrm{SL}_2(\mathbb{C})$, we can still conjugate the group obtained before computing with it. Actually, it is natural to consider the element conjugating up to right multiplication by an element of $\mathrm{PSU}_2(\mathbb{C})$, since these act as Euclidean rotations and preserve the absolute definite norm: they will not change the computations. Thus noting that $\mathrm{PSU}_2(\mathbb{C})$ is the stabilizer of 0, what we have to choose is an element in $\mathcal{B}$: the center. This choice may affect the computation, and it would be interesting to describe how the combinatorial and geometric structure of the fundamental domain changes with the center, and to predict which center would give the most efficient computations. We have made some small observations in this direction: Propositions 5.2.8 and 5.2.12 try to describe the behaviour of the radii of the individual elements as the center moves, but what would be interesting is the behaviour of the radii of the elements of the group as a whole, as the center moves (again, this is closely related to Section 11.6); Lemma 4.2.3 gives a result about the combinatorial behaviour of the exterior domain as the center varies, but a global description is also missing.

11.6   Estimating the complexity

So far we have only proved that the algorithm involved terminate. It would be interesting, although it seems quite difficult, to give an estimate of the number of operations required to perform the computation. An idea would be to estimate the minimum radius of the boundary elements of an exterior domain. This would give an upper bound, which would probably be pessimistic as it would not take into account the work performed by the reduction algorithm. Another idea would be to estimate a bound A such that the set $\{x \in \mathcal{O}_1^\times \mid Q(x) < A\}$ generates the group $\mathcal{O}_1^\times$, and then to prove that Algorithm 9 terminates after a finite number of steps when it is given a set of generators, and to estimate the number of steps needed. This would also give an upper bound, although the finest approach would probably be a mix of the two ideas.

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