Isospectrality, regulators and torsion of Vignéras manifolds

Aurel Page
joint work with Alex Bartel

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Isospectral manifolds

A famous question

Mark Kac 1966: "Can you hear the shape of a drum?"

Vibrating frequencies $\longleftrightarrow$ eigenvalues of **Laplace operator**

$$\Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$$

A mathematical question

In this talk: manifold M = closed connected orientable Riemannian manifold.

$\rightsquigarrow$ Laplace operator Δ acting on space $\Omega^i(M)$ of i-forms, with discrete spectrum.

Definition

Two manifolds M and N are **isospectral** if for all i, the spectra of Δ on $\Omega^i(M)$ and $\Omega^i(N)$ agree with multiplicity.

Question: isospectral $\Longrightarrow$ isometric?
Answer: no in all dimensions ≥ 2 (Vignéras 1978).

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A refined question

Question: which invariants of Riemannian manifolds are isospectral invariants?

- dimension $\dim M$: yes
- volume $\mathrm{Vol}(M)$: yes
- Betti numbers $\mathrm{rk}\, H_i(M)$: yes
- ring $H^\bullet(M)$: no (Lauret–Miatello–Rossetti 2013)
- torsion homology $\#H_i(M)[p^\infty]$: no $\forall p$ (Bartel–P. 2016).

Special values of zeta functions

The spectral **zeta function**

$$\zeta_{M,i}(s) = \sum_{\lambda>0} (\dim \Omega^i(M)_{\Delta=\lambda})\lambda^{-s} \text{ for } \Re(s) \gg 0$$

has a **special value** formula (Cheeger, Müller 1978):

$$\prod_{i=0}^{\dim M} \exp(\zeta'_{M,i}(0))^{i(-1)^i} = \prod_{i=0}^{\dim M} \left(\frac{\# H_i(M)_{\mathrm{tors}}}{\mathrm{Reg}_i(M)} \right)^{(-1)^i}$$

where

$$\mathrm{Reg}_i(M) = \mathrm{Vol}\left(\frac{H_i(M,\mathbb{R})}{H_i(M)} \right).$$

Example: $\mathrm{Reg}_0(M) = \mathrm{Vol}(M)^{-1/2}$, $\mathrm{Reg}_{\dim M - i}(M) = \mathrm{Reg}_i(M)^{-1}$.

Special values of zeta functions

Example: if two 3-manifolds M and N are isospectral, then

$$\frac{\#H_1(M)_{\mathrm{tors}}}{\mathrm{Reg}_1(M)^2} = \frac{\#H_1(N)_{\mathrm{tors}}}{\mathrm{Reg}_1(N)^2},$$

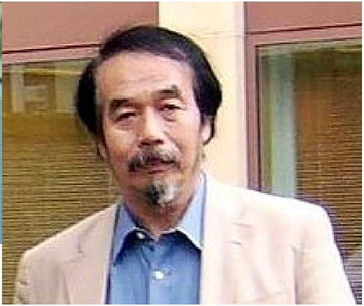
and in particular

$$\frac{\mathrm{Reg}_1(M)^2}{\mathrm{Reg}_1(N)^2} = \frac{\#H_1(M)_{\mathrm{tors}}}{\#H_1(N)_{\mathrm{tors}}} \in \mathbb{Q}^\times.$$

Questions:

- Is this rationality true more generally?
- What primes can enter these rational numbers?
- At which primes can $H_i(M)_{\mathrm{tors}}$ and $H_i(N)_{\mathrm{tors}}$ differ?

Two constructions of isospectral manifolds



Marie-France Vignéras 1978: **number theory** (arithmetic groups)

Toshikazu Sunada 1983: **group theory** (finite group G)
Bad primes = divisors of $\#G$.

Vignéras's construction

Arithmetic manifolds

$\mathrm{GL}_2(\mathbb{C})$ acts on hyperbolic 3-space $\mathcal{H}^3 = \mathrm{GL}_2(\mathbb{C})/\mathrm{U}_2(\mathbb{C})\mathbb{C}^\times$.

Let F be a field[1]. A **quaternion algebra** over F is

$$A = \left(\frac{a,b}{F}\right) = F + Fi + Fj + Fij,$$

where $i^2 = a \in F^\times$, $j^2 = b \in F^\times$ and $ij = -ji$.

Pick A/F a division quaternion algebra over a number field such that

$$\mathbb{R} \otimes A \cong M_2(\mathbb{C}) \times \left(\frac{-1,-1}{\mathbb{R}}\right)^m.$$

Let $\mathcal{O} \subset A$ be an **order** (subring with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O} \cong A$).
Then $M(\mathcal{O}) = \mathcal{O}^\times \backslash \mathcal{H}^3$ is a hyperbolic 3-manifold.

[1] of characteristic not 2

Vignéras's theorem

Maximal order: maximal for inclusion.

- Always exists.
- Not unique: $\mathcal{O} \rightsquigarrow x\mathcal{O}x^{-1}$ for $x \in A^\times$.
- Finite number up to conjugation.

Theorem (Vignéras)

*If $\mathcal{O}_1$ and $\mathcal{O}_2$ are maximal orders **and extra conditions hold**, then $M(\mathcal{O}_1)$ and $M(\mathcal{O}_2)$ are isospectral.*

Representation equivalence

The proof uses the **trace formula** and in fact proves the stronger fact that there is an isomorphism

$$L^2(\mathcal{O}_1^\times \backslash \mathrm{GL}_2(\mathbb{C})) \cong L^2(\mathcal{O}_2^\times \backslash \mathrm{GL}_2(\mathbb{C}))$$

of unitary representations of $\mathrm{GL}_2(\mathbb{C})$.

When such an isomorphism holds, we say that $M(\mathcal{O}_1)$ and $M(\mathcal{O}_2)$ are **representation-equivalent.**

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Theorem (DeTurck–Gordon 1989)

Representation-equivalent $\Longrightarrow$ *isospectral.*

Dimension 2: rigidity

Question (Pesce 1995): how much stronger is representation-equivalence compared to isospectrality?

Theorem (Doyle–Rossetti 2011)

If two hyperbolic manifolds of ***dimension*** 2 *are isospectral, then they are representation-equivalent.*

Conjecture (Doyle–Rossetti 2011)

If two hyperbolic manifolds are isospectral, then they are representation-equivalent.

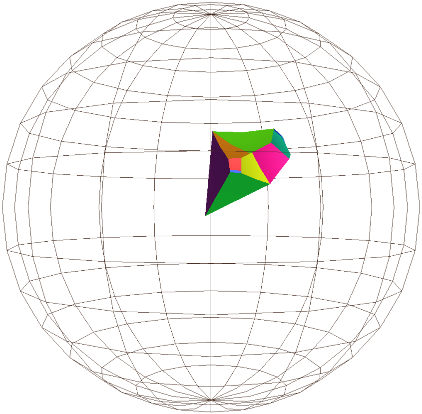
Dimension 3: an exotic pair

Theorem (Bartel–P.–PARI/GP)

There exists a pair of isospectral hyperbolic 3*-manifolds with volume* $0.251\ldots$ *that are* ***isospectral****, but* ***not representation-equivalent****.*

Vignéras's construction with $F = \mathbb{Q}(\sqrt{-10-14\sqrt{5}})$, the unique A ramified exactly at the real places, and maximal orders.

Smallest possible volume? Previous record was $2.83\ldots$ (Linowitz–Voight 2014) and Sunada's construction cannot produce smaller ones.



Isospectrality, regulators and torsion

Theorem template

Theorem$\langle * \rangle$ (Bartel–P.)

At least one of the following two statements is true:

1. *there exists a number field L in an a-priori finite list and a certain Hecke character of L;*
2. *$M(\mathcal{O}_1)$ and $M(\mathcal{O}_2)$ are $*$**-isospectral**.*

For each instance of $*$, existence can be checked using PARI/GP's new Hecke characters package!

Theorem template

Theorem$\langle * \rangle$ (Bartel–P.)

At least one of the following two statements is true:

1. *there exists a number field L in an a-priori finite list and a* $*$***-shady character*** *of L;*
2. $M(\mathcal{O}_1)$ *and* $M(\mathcal{O}_2)$ *are* $*$***-isospectral****.*

For each instance of $*$, existence can be checked using PARI/GP's new Hecke characters package!

Instantiating the template

- $* =$ representation-equivalence $\longleftrightarrow$ $*$-shady characters $=$ certain (possibly transcendental) **Hecke characters.**
- $* =$ isospectrality $\longleftrightarrow$ $*$-shady characters $=$ certain (possibly transcendental) Hecke characters of a **more restricted type.**
- $* =$ rational regulator ratios $\longleftrightarrow$ $*$-shady characters $=$ certain **algebraic Hecke characters.**
- $* =$ same regulators and torsion at p $\longleftrightarrow$ $*$-shady characters $=$ certain **mod** p **Hecke characters** (assuming a conjecture about mod p Galois representations attached to torsion homology).

Sketch of proof

- Use **Hecke operators**

$$\mathcal{F}(\mathcal{O}_1) \to \mathcal{F}(\mathcal{O}_2)$$

- They are sums of $T_{\mathfrak{p}}$ for all $\mathfrak{p}$ inert in some quadratic L/F.
- We would like an **invertible** one.
- If none is invertible, by dévissage there is an eigenvector f such that $a_{\mathfrak{p}}(f) = 0$ for all $\mathfrak{p}$ inert in L/F, i.e.

$$a_{\mathfrak{p}}(f) = \chi(\mathfrak{p}) a_{\mathfrak{p}}(f) \text{ for all } \mathfrak{p}.$$

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- This implies that f is "CM" and comes from some ψ.
- ρ irreducible 2-dimensional representation of a group G: $\rho \cong \rho \otimes \chi \iff \rho \cong \mathrm{ind}_{G/\ker\chi}\,\psi$.

Conclusion

Thanks!

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