Torsion homology and regulators of Vignéras isospectral manifolds

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Isospectral manifolds

Isospectral manifolds

Mark Kac '60: "Can you hear the shape of a drum?"

Formally:
M compact Riemannian orientable manifold
$\rightsquigarrow$ Laplace operator Δ acting on k-forms $\Omega^k(M)$ with discrete spectrum.

M and N are **isospectral** if for all k, the Laplace spectrum with multiplicity is the same on $\Omega^k(M)$ and $\Omega^k(N)$.

Question: isospectral $\implies$ isometric?

Answer: **no** in all dimensions ≥ 2 (Vignéras '70, Sunada '80).

Isospectral invariants

More general question: which invariants of Riemannian manifolds are **isospectral invariants**?

- dimension $d = \dim(M)$;
- volume $\mathrm{Vol}(M)$;
- Betti numbers $b_k = \mathrm{rk}\, H_k(M)$;
- spectral zeta function

$$\zeta_{M,k}(s) = \sum_{\lambda>0} (\dim \Omega^k(M)_{\Delta=\lambda})\lambda^{-s}$$

for $\Re(s) \gg 0$.

The Cheeger–Müeller formula

We have

$$\prod_{k=0}^{d} \exp(\zeta'_{M,k}(0))^{k(-1)^k} = \prod_{k=0}^{d} \left(\frac{|H_k(M)_{\mathrm{tors}}|}{\mathrm{Reg}_k(M)} \right)^{(-1)^k}$$

where

$$\mathrm{Reg}_k(M) = \mathrm{Vol}\left(\frac{H_k(M,\mathbb{R})}{H_k(M)} \right).$$

Example: $\mathrm{Reg}_0(M) = \mathrm{Vol}(M)^{-1/2}$, $\mathrm{Reg}_{d-i}(M) = \mathrm{Reg}_i(M)^{-1}$.

Torsion and regulators

If M and N are isospectral, then

- $\operatorname{Reg}_k(M) = \operatorname{Reg}_k(N)$? No.
- $H_k(M)_{\mathrm{tors}} \cong H_k(N)_{\mathrm{tors}}$? No.

$$\prod_{k=0}^{d} \left(\frac{\operatorname{Reg}_k(M)}{\operatorname{Reg}_k(N)} \right)^{(-1)^k} = \prod_{k=0}^{d} \left(\frac{|H_k(M)_{\mathrm{tors}}|}{|H_k(N)_{\mathrm{tors}}|} \right)^{(-1)^k} \in \mathbb{Q}^\times .$$

- $\operatorname{Reg}_k(M) / \operatorname{Reg}_k(N) \in \mathbb{Q}^\times$?
- At which primes can torsion or regulators differ?

Constructions of Sunada and Vignéras

Can we find a finite set of **bad primes**, outside of which torsion and regulators must be the same?

Sunada's construction involves a finite group G:
bad primes $\{p \text{ dividing } |G|\}$.

Vignéras's construction is arithmetic: bad primes?

Vignéras's construction

Quaternion algebras

Let K be a number field. A **quaternion algebra** over K is

$$A = K + Ki + Kj + Kij,$$

where $i^2 = a \in K^\times$, $j^2 = b \in K^\times$ and $ij = -ji$.

- For $\sigma : K \hookrightarrow \mathbb{C}$, we have $A \otimes_{K,\sigma} \mathbb{C} \cong M_2(\mathbb{C})$.
- For $\sigma : K \hookrightarrow \mathbb{R}$, say σ is **unramified** if $A \otimes_{K,\sigma} \mathbb{R} \cong M_2(\mathbb{R})$ and **ramified** otherwise.
- For $\mathfrak{p}$ a prime of K, say $\mathfrak{p}$ is **unramified** if $A \otimes_K K_\mathfrak{p} \cong M_2(K_\mathfrak{p})$ and **ramified** otherwise.

Assume (for simplicity) that K has exactly one nonreal complex embedding and that all real embeddings of K are ramified in A.

Maximal orders

Order $\mathcal{O}$ in A: subring, finitely rank over $\mathbb{Z}$, $K\mathcal{O} = A$.

Example: $\mathcal{O} = \mathbb{Z}_K + \mathbb{Z}_K i + \mathbb{Z}_K j + \mathbb{Z}_K ij$ if $a, b \in \mathbb{Z}_K$.

Maximal order: maximal for inclusion.

- Not unique: $\mathcal{O} \rightsquigarrow x\mathcal{O}x^{-1}$ for $x \in A^\times$.
- Finite number up to conjugation.
- Conjugacy classes of maximal orders are in bijection with

$$C = \mathrm{Cl}_K(\infty)/\langle \mathfrak{a}^2, \mathfrak{p} \text{ ramified} \rangle.$$

Assume (for simplicity) that $|C| = 2$: corresponds to a quadratic extension L/K.

Vignéras's theorem

$\mathrm{PGL}_2(\mathbb{C})$ acts on hyperbolic 3-space $\mathcal{H}^3$.

Let $\Gamma(\mathcal{O})$ be the image of $\mathcal{O}^\times$ in $\mathrm{PGL}_2(\mathbb{C})$, and

$$M(\mathcal{O}) = \Gamma(\mathcal{O})\backslash\mathcal{H}^3$$

of finite volume, compact iff $A \ncong M_2(K)$.

Theorem (Vignéras)

If $\mathcal{O}$ and $\mathcal{O}'$ are maximal orders, then $M(\mathcal{O})$ and $M(\mathcal{O}')$ are isospectral.

Vignéras's theorem

$\mathrm{PGL}_2(\mathbb{C})$ acts on hyperbolic 3-space $\mathcal{H}^3$.

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of finite volume, compact iff $A \not\cong M_2(K)$.

Theorem (Vignéras)

*If $\mathcal{O}$ and $\mathcal{O}'$ are **non-selective** maximal orders, then $M(\mathcal{O})$ and $M(\mathcal{O}')$ are isospectral.*

Bad primes, an example

Does the action of Hecke operators control the bad primes?

Field: $K = \mathbb{Q}(\alpha)$ where $\alpha^5 - 2\alpha^4 - 4\alpha^3 + 8\alpha^2 + 3\alpha - 5 = 0$.
3 real embeddings, 1 pair of conjugate embeddings.

Algebra: $a = -4$, $b = -24\alpha^4 - 12\alpha^3 + 80\alpha^2 + 24\alpha - 71$.
A ramified at $\mathfrak{p}_7 = 7\mathbb{Z}_K + (\alpha + 4)\mathbb{Z}_K$ and all real embeddings.
$|C| = 2$, consider $\mathcal{O}$ and $\mathcal{O}'$ non-conjugate.

$\mathrm{Vol}(M(\mathcal{O})) = \mathrm{Vol}(M(\mathcal{O}')) \approx 22.876$.
$H_1(M(\mathcal{O})) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z} \oplus \mathbb{Z}$.
$H_1(M(\mathcal{O}')) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}$.
Field of Hecke eigenvalues: $\mathbb{Q}(\sqrt{11})$.

Rationality of regulators and bad primes

Hecke characters

Special case: characters of ray class groups (finite order)

$$\psi\colon \mathrm{Cl}_K(\mathfrak{M}) \to \mathbb{C}^\times.$$

More generally, idèle class group characters (finite or infinite order)

$$\psi\colon \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times.$$

They have a conductor (ideal of K) and parameters $m_\sigma \in \mathbb{Z}$ and $\varphi_\sigma \in \mathbb{R}$ for $\sigma\colon K \hookrightarrow \mathbb{C}$.

Isospectrality condition

Theorem (Bartel–P.)

$M = M(\mathcal{O})$ and $M' = M(\mathcal{O}')$ are isospectral, except possibly if no prime ideal of K ramifies in A and L/K is totally complex.

In addition, $\Omega^k(M)_{\Delta=\lambda} \cong \Omega^k(M')_{\Delta=\lambda}$, except possibly if there exists a Hecke character ψ of L satisfying a condition depending on k and λ.

A similar condition appears in work of Rajan, using the Labesse–Langlands multiplicity formula.

Regulator rationality condition

Theorem (Bartel–P.)

$$\left(\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')}\right)^2 \in \mathbb{Q}^\times,$$

except possibly if there exists a Hecke character ψ of L of conductor 1 such that

- $m_\sigma = \pm 1$ *and* $\varphi_\sigma = 0$ *for* σ *real on* K, *and*
- $m_\sigma = -m_{\sigma'} = \pm 1$ *and* $\varphi_\sigma = \varphi_{\sigma'} = 0$ *for* $\sigma \neq \sigma'$ *restricting to the complex embedding of* K.

This condition is weaker than the one for isospectrality!

Bad primes

Theorem (Bartel–P.)

Let $p > 2$ be a prime. Assume a "standard" conjecture on Galois representations attached to mod p cohomology classes.

$$v_p\left(\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')}\right) = 0 \text{ and } H_k(M)[p^\infty] \cong H_k(M')[p^\infty]$$

except possibly if there exists a mod p Hecke character

$$\psi\colon \mathrm{Cl}_L(p\mathbb{Z}_L) \to \overline{\mathbb{F}}_p^\times$$

satisfying certain conditions.

Refinement: Hecke action

There is a Hecke algebra acting on $H_k(M)$ and $H_k(M')$.

Consider **maximal ideals** $\mathfrak{m}$ of the Hecke algebra (essentially, the data of an eigenvalue in $\overline{\mathbb{F}}_p$ for each Hecke operator).

We have a decomposition

$$H_k(M, \mathbb{Z}_p) \cong \bigoplus_{\mathfrak{m}} H_k(M)_{\mathfrak{m}}.$$

We would like to understand the situation separately in each summand.

Calegari–Venkatesh

Calegari and Venkatesh studied an analogous situation, **Jacquet–Langlands pairs** of manifolds: not isospectral, but closely related spectrum.

Conjecture (Calegari–Venkatesh)

If (M, M') is a Jacquet–Langlands pair and $\mathfrak{m}$ is "non-Eisenstein", then

$$|H_k(M)_{\mathfrak{m}}^{\mathrm{new}}| = |H_k(M')_{\mathfrak{m}}^{\mathrm{new}}|.$$

Proved some versions without the Hecke action.

Bad primes with Hecke action

We can define a notion of $\left(\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')}\right)_{\mathfrak{m}} \in \mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2$, such that

$$\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')} = \prod_{\mathfrak{m}} \left(\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')}\right)_{\mathfrak{m}} \bmod (\mathbb{Z}_p^\times)^2.$$

Theorem (Bartel–P.)

Let $p > 2$ be a prime. If $\mathfrak{m}$ is "non-CM", then

$$H_k(M)_{\mathfrak{m}} \cong H_k(M')_{\mathfrak{m}} \text{ and } \left(\frac{\mathrm{Reg}_k(M)}{\mathrm{Reg}_k(M')}\right)_{\mathfrak{m}} \in \mathbb{Z}_p^\times.$$

Ideas of proof

Hecke algebras

Hecke operators are indexed by primes of K.

We have two kinds of Hecke operators

- $T_{\mathfrak{p}}$ for $[\mathfrak{p}] = 0$ in C, acting on M and on M', generating a **small Hecke algebra** $\mathbb{T}_0$.
- $T_{\mathfrak{q}}$ for $[\mathfrak{q}] \neq 0$ in C, swapping M and M', generating a $\mathbb{T}_0$-module $\mathbb{T}_1$.

The **big Hecke algebra** $\mathbb{T} = \mathbb{T}_0 \oplus \mathbb{T}_1$ acts on $M \sqcup M'$.

All the isomorphisms we prove come from the action of an operator $T \in \mathbb{T}_1$ acting invertibly.

Regulator quotients: algebraic interpretation

From a diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ \big\downarrow{\scriptstyle \langle\cdot,\cdot\rangle} & & \big\downarrow{\scriptstyle \langle\cdot,\cdot\rangle'} \\ V^* & \xrightarrow[T^*]{} & (V')^* \end{array}$$

we get

$$\left(\frac{\mathrm{Reg}'}{\mathrm{Reg}}\right)^2 = \frac{\det T^*}{\det T}.$$

Invertibility of Hecke operators

Invertible operator?

- By dévissage, we may assume that $\mathbb{T}_0$ is a field.
- $T \in \mathbb{T}_1 \Rightarrow T^2 \in \mathbb{T}_0$. When $\mathbb{T}$ is reduced, this implies $\mathbb{T}_1 = 0$ or there exists $T \in \mathbb{T}_1$ acting invertibly.
- $\mathbb{T}_1 = 0 \iff a_\mathfrak{p} = 0$ for all $[\mathfrak{p}] \neq 0$ in $C \iff a_\mathfrak{p} = a_\mathfrak{p}\chi(\mathfrak{p})$ for all $\mathfrak{p}$, where $\chi \colon C \to \{\pm 1\}$ is nontrivial.
- ρ irreducible 2-dimensional representation of a group G: $\rho \cong \rho \otimes \chi \iff \rho \cong \mathrm{ind}_{G/\ker\chi}\,\psi$.

Questions?

Thank you!