VIGNÉRAS ORBIFOLDS: ISOSPECTRALITY, REGULATORS, AND TORSION HOMOLOGY

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Dedicated to the memory of Nicolas Bergeron

ABSTRACT. We develop a new approach to the isospectrality of the orbifolds constructed by Vignéras. We give fine sufficient criteria for i-isospectrality in given degree i and for representation equivalence. These allow us to produce very small exotic examples of isospectral orbifolds: hyperbolic 3-orbifolds that are i-isospectral for all i but not representation equivalent, hyperbolic 3-orbifolds that are 0-isospectral but not 1-isospectral, and others. Using the same method, we also give sufficient criteria for rationality of regulator quotients $\mathrm{Reg}_i(Y_1)^2/\mathrm{Reg}_i(Y_2)^2$ for Vignéras orbifolds Y_1, Y_2, sometimes even when they are not isospectral. Moreover, we establish a link between the primes that enter in these regulator quotients and at which torsion homology of Y_1 and Y_2 can differ, and Galois representations.

CONTENTS

1. Introduction

1.1. **Isospectral orbifolds.** This paper is devoted to the venerable question [34] of which geometric and topological properties of a closed Riemannian manifold or orbifold are encoded in the spectrum of the Laplace-de Rham operator Δ on the space of differential i-forms for various $i \in \mathbb{Z}_{\geq 0}$.

Given $i \in \mathbb{Z}_{\geq 0}$, two closed Riemannian orbifolds are said to be i-*isospectral* if the multisets of eigenvalues of Δ on the spaces of differential i-forms of these two orbifolds coincide. We abbreviate "0-isospectral" to just *isospectral*, and "i-isospectral for all i" to $\Omega^\bullet$-*isospectral*. Sunada [62] and Vignéras [66] each proposed a general construction for pairs of orbifolds that are $\Omega^\bullet$-isospectral. Vignéras's method is the focus of this paper.

We briefly sketch the setup of Vignéras in a special case. In Section 4 we describe the construction in detail and in much greater generality. Let F be a number field with exactly one complex place, let D be a division quaternion algebra over F that is ramified at all real places, and let Γ_1 and Γ_2 be unit groups of two maximal orders in D. A complex embedding of F identifies them with two discrete subgroups of $\mathrm{GL}_2(\mathbb{C})$. For the rest of the introduction we fix two such groups. They act on hyperbolic 3-space $X = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \mathrm{U}_2(\mathbb{C})$, and under some additional conditions the closed orbifolds $\Gamma_1 \backslash X$ and $\Gamma_2 \backslash X$ are isospectral. A lot of literature is devoted to the exploration of such conditions, see [44] and extensive references therein, but all known conditions actually imply the stronger relationship that the L^2-spaces $\mathrm{L}^2(\Gamma_1 \backslash \mathrm{PGL}_2(\mathbb{C}))$ and $\mathrm{L}^2(\Gamma_2 \backslash \mathrm{PGL}_2(\mathbb{C}))$ are isomorphic as unitary representations of $\mathrm{GL}_2(\mathbb{C})$. When that relationship holds, we say that Γ_1 and Γ_2 are *representation equivalent*. It is a theorem of DeTurck and Gordon [20, Theorem 1.16, Remark 1.18] that the quotients by representation equivalent groups are in fact $\Omega^\bullet$-isospectral. In particular, as far as we are aware, the smallest currently known pair of connected isospectral hyperbolic 3-orbifolds comes from representation equivalent groups, and was found by Linowitz–Voight [44]. Their orbifolds have (necessarily equal) volume $2.83366\ldots$. The converse, whether $\Omega^\bullet$-isospectrality of two hyperbolic 3-orbifolds implies representation equivalence of the corresponding groups, has until now been an open question [53], [56, §4.1], [44, Remark 2.6], [40, Question 8.11]. The converse does hold for hyperbolic 2-orbifolds [22] and it has been conjectured to hold in dimension 3, too [22, §12].

In this paper we develop much finer criteria for different kinds of isospectrality in Vignéras's construction. Before describing the nature of these criteria, we state some of their applications. The following result will be proven in Example 7.4.

Theorem A. *There exists a pair of closed connected orientable arithmetic hyperbolic 3-orbifolds with volume $0.251\ldots$ that are i-isospectral for all i, but not representation equivalent, only one of which has a cyclic isotropy group of order 10.*

This might well be the smallest $\Omega^\bullet$-isospectral pair of connected hyperbolic 3-orbifolds. We make two remarks on this. Firstly, there exist only finitely many hyperbolic 3-orbifolds of volume less than that in Theorem A, see [1]. As far as we are aware, no isospectral pair with this property was known until now. Secondly, we will see in Section 1.5 that Sunada's method can never produce a smaller pair.

It is known that there exist pairs of orbifolds that are i-isospectral for some i but not all [29]. However, no such examples have been known that are 3-dimensional or hyperbolic [40, Question 8.10]. In [41] it is predicted that such examples should exist, but that they must be difficult to construct. We have the following application of our criteria, which will be proven in Example 7.6.

Theorem B. *There exists a pair of closed connected orientable arithmetic hyperbolic 3-orbifolds with volume $0.246\ldots$ that are isospectral, but not 1-isospectral.*

Again, one may wonder whether this is the smallest pair of connected hyperbolic isospectral 3-orbifolds.

Our method also shows that, without any further hypotheses, the groups Γ_1 and Γ_2 are always "close to" representation equivalent, in the following sense. Let $j \in \{1, 2\}$. As a representation of $U_2(\mathbb{C})$ the space $L^2(\Gamma_j \backslash PGL_2(\mathbb{C}))$ decomposes as a direct sum $\bigoplus_V V^{m_{V,j}}$ over the irreducible representations V of $U_2(\mathbb{C})$, where $m_{V,j} \in \mathbb{Z}_{\geq 0}$. Each of the isotypical subspaces $\Omega(V, \Gamma_j) = V^{m_{V,j}}$ decomposes into a direct sum of eigenspaces $\Omega(V, \Gamma_j)_{\Delta = \lambda}$ under the Casimir operator, which is the appropriate generalisation of the Laplace–de Rham operator. Weyl's law implies that as $T \to \infty$, one has $\sum_{\lambda \leq T} \dim \Omega(V, \Gamma_j)_{\Delta = \lambda} \sim cT^{3/2}$ for some constant $c > 0$. The following result, which is a special case of Theorem 5.45, therefore implies that the difference in the spectra for Γ_1 and Γ_2 is always vanishingly small, in the limit, even when no criteria for isospectrality apply.

Theorem C. *For every irreducible representation V of $U_2(\mathbb{C})$ there exists a constant $c_V \geq 0$ such that for all $T > 0$ one has*

$$\sum_{\lambda \leq T} \left| \dim \Omega(V, \Gamma_1)_{\Delta = \lambda} - \dim \Omega(V, \Gamma_2)_{\Delta = \lambda} \right| \leq c_V T^{1/2}.$$

This result defies some expert expectations. Indeed, Kelmer proves [35, Theorem 1] that, in the opposite direction, if the groups Γ_1 and Γ_2 are close to being representation equivalent, then in fact they are representation equivalent. He shows that if for all irreducible $U_2(\mathbb{C})$-representations V the quotient between the left hand side in Theorem C and $T^{1/2}$ tends to 0 as $T \to \infty$, then Γ_1 and Γ_2 are representation equivalent. He then speculates that his bound might be far from optimal, and that perhaps this type of repulsion already happens close to the asymptotic of Weyl's law. Theorem C demonstrates that in fact Kelmer's result is sharp, since there do exist groups Γ_1 and Γ_2 as above that are not representation equivalent.

1.2. **Hierarchy of isospectralities.** Suppose that Y_1 and Y_2 are closed orientable Riemannian manifolds of a common dimension d. If they are $\Omega^\bullet$-isospectral, then the Cheeger–Müller Theorem [18, 49, 50] implies that one has

$$(1.1) \qquad \prod_{i=0}^{d} \left(\frac{\mathrm{Reg}_i(Y_1)}{\# H_i(Y_1, \mathbb{Z})_{\mathrm{tors}}} \right)^{(-1)^i} = \prod_{i=0}^{d} \left(\frac{\mathrm{Reg}_i(Y_2)}{\# H_i(Y_2, \mathbb{Z})_{\mathrm{tors}}} \right)^{(-1)^i},$$

where, for a Riemannian manifold Y, the real number $\mathrm{Reg}_i(Y)$ is the covolume of the lattice $H_i(Y, \mathbb{Z})/H_i(Y, \mathbb{Z})_{\mathrm{tors}}$ in the inner product space $H_i(Y, \mathbb{R})$ [54, §1.6].

There has been a lot of recent interest in understanding regulators and torsion homology of arithmetic manifolds – see for instance [6, 7, 12, 14, 54]. In light of equation (1.1), the following vague question seems natural.

Question 1.2. What can be said about $\mathrm{Reg}_i(Y_1)/\mathrm{Reg}_i(Y_2)$ if Y_1 and Y_2 are i-isospectral manifolds?

Poincaré duality implies that for a closed orientable d-manifold Y and for all i, one has $\mathrm{Reg}_i(Y) = \mathrm{Reg}_{d-i}(Y)^{-1}$. Thus, if d is even, then equation (1.1) says nothing about $\mathrm{Reg}_i(Y_1)/\mathrm{Reg}_i(Y_2)$; while if d is odd, and Y_1 and Y_2 are $\Omega^\bullet$-isospectral, equation (1.1) implies that one has

$$(1.3) \qquad \prod_{i=0}^{(d-1)/2} \left(\frac{\mathrm{Reg}_i(Y_1)^2}{\mathrm{Reg}_i(Y_2)^2} \right)^{(-1)^i} \in \mathbb{Q}^\times.$$

There is no analogue of the Cheeger–Müller formula for a single degree i. Nevertheless, it seems natural to wonder: if Y_1 and Y_2 are merely assumed to be i-isospectral for some i, then does it follow that $\frac{\mathrm{Reg}_i(Y_1)^2}{\mathrm{Reg}_i(Y_2)^2}$ is a rational number? And if it is, then

what primes can enter? Is there a connection between that and the rational number $\frac{\#H_i(Y_1,\mathbb{Z})_{\mathrm{tors}}}{\#H_i(Y_2,\mathbb{Z})_{\mathrm{tors}}}$? We do not know the answers to these general questions, but we prove strong results in the case when $Y_j = \Gamma_j \backslash X$ arise from the Vignéras construction.

We think of different "kinds of isospectrality" between Γ_1 and Γ_2 as sitting in a kind of approximate hierarchy: representation equivalence[1], which we sometimes also call L^2-isospectrality, $\Omega^\bullet$-isospectrality, isospectrality, which we also refer to as Ω^0-isospectrality, rationality of $\frac{\mathrm{Reg}_i(Y_1)^2}{\mathrm{Reg}_i(Y_2)^2}$ for all i, which we refer to as $\mathcal{H}^\bullet$-isospectrality, and finally, for a prime number p, the conditions

$$\mathrm{ord}_p\left(\frac{\mathrm{Reg}_i(Y_1)^2}{\mathrm{Reg}_i(Y_2)^2}\right) = 0 \quad \text{and} \quad \mathrm{ord}_p\left(\frac{\#H_i(Y_1,\mathbb{Z})_{\mathrm{tors}}}{\#H_i(Y_2,\mathbb{Z})_{\mathrm{tors}}}\right) = 0$$

for all i, where ord_p denotes p-adic valuation, which we call $\mathbb{Z}_p$-isospectrality. Of course that last condition only makes sense if the regulator quotients are rational in the first place, i.e. under the assumption of $\mathcal{H}^\bullet$-isospectrality. For each of these kinds of isospectralities, we prove sufficient criteria. We do not state them here precisely, but we give an impressionistic sketch of the general shape of our results.

For each kind $*$ of isospectrality as above, we identify certain types of Hecke characters such that if there are no Hecke characters of that type, then Γ_1 and Γ_2 are $*$-isospectral, but we do not expect the converse to hold. For that reason we do not call the characters an obstruction, but instead call them $*$-*shady characters*: their presence merely stops our method from proving $*$-isospectrality. In summary, our main results take the following shape.

Theorem D. *At least one of the following two statements is true:*
  (i) *there exists a number field L in an a-priori finite list and a $*$-shady character of L;*
  (ii) *the groups Γ_1 and Γ_2 are $*$-isospectral.*

For L^2-, $\Omega^\bullet$-, Ω^0-, and $\mathcal{H}^\bullet$-isospectrality, we define the respective shady characters and prove Theorem D as Theorem 5.15, Corollaries 5.29 and 5.31, and Theorem 5.34, respectively. Our theorem for $\mathbb{Z}_p$-isospectrality is proven as Theorem 6.10, subject to a widely believed conjecture on Galois representations, Conjecture 6.5.

The conditions of being $*$-shady are completely explicit. Presence or absence of $*$-shady can be checked efficiently using existing algorithms for computations of Hecke characters [48]. Moreover, we have the following implications between existence of different shady characters:
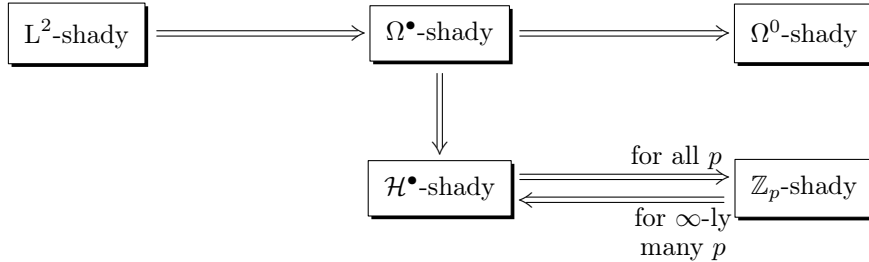
It is worth noting, in particular, that our criteria for rationality of regulator quotients are *weaker* than those for i-isospectrality, and in fact in Example 7.8 we exhibit the following peculiar phenomenon.

Theorem E. *There exists a pair of closed connected orientable hyperbolic 3-orbifolds Y_1, Y_2 with volume $5.902\ldots$ that are not isospectral, nor 1-isospectral, and for which $\dim H_1(Y_1,\mathbb{R}) = \dim H_1(Y_2,\mathbb{R}) = 1$, yet $\mathrm{Reg}_1(Y_1)^2/\mathrm{Reg}_1(Y_2)^2$ is rational.*

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[1]For hyperbolic manifolds, representation equivalence of the groups is equivalent to so-called strong isospectrality of the manifolds [53], but in general it is stronger.

1.3. **Homology and regulators: theorems and musings.** From a number theoretic point of view, it is natural to consider the homology of arithmetic orbifolds together with their action of Hecke operators. Taking this additional structure into account, we considerably strengthen our results on the homology of the Y_j sketched in Theorem D.

The "right" setup for Vignéras's construction is an adélic one, in which $Y_1 = \Gamma_1 \backslash X$ and $Y_2 = \Gamma_2 \backslash X$ are two connected components of one orbifold $\mathcal{Y}$. For now we continue with a special case. Let F be a number field, let $\mathbb{Z}_F$ be its ring of integers, and let D be a division quaternion algebra over F such that at least one infinite place of F splits in D. There is a ray class group C that parametrises left ideal classes of maximal orders of D, and thereby connected components of $\mathcal{Y}$; for simplicity of the exposition, assume that C has order 2, corresponding to a quadratic extension L of F, so that one has $\mathcal{Y} = Y_1 \sqcup Y_2$. Let $\mathbb{T}_1$ denote the Hecke algebra generated by Hecke operators $T_{\mathfrak{a}}$ for all ideals $\mathfrak{a}$ of $\mathbb{Z}_F$ whose class in C is trivial. Then $\mathbb{T}_1$ naturally acts on each of the homology groups $H_i(Y_j, \mathbb{Z})$ for $j \in \{1, 2\}$, and for every prime number p we have a decomposition of $\mathbb{Z}_p \otimes \mathbb{T}_1$-modules

$$H_i(Y_j, \mathbb{Z}_p) \cong \bigoplus_{\mathfrak{n}} H_i(Y_j, \mathbb{Z})_{\mathfrak{n}},$$

where the sum ranges over maximal ideals $\mathfrak{n}$ of $\mathbb{T}_1$ of residue characteristic p. We say that such a maximal ideal $\mathfrak{n}$ *corresponds to* a $\mathbb{Z}_p$-shady character Ψ of L if for every prime ideal $\mathfrak{q}$ of $\mathbb{Z}_F$ that splits in L, we have $T_{\mathfrak{q}} \bmod \mathfrak{n} = \Psi(\mathfrak{Q}) + \Psi(\mathfrak{Q}')$, where $\mathfrak{q}\mathbb{Z}_L = \mathfrak{Q}\mathfrak{Q}'$. We prove the following result as Theorem 6.12.

Theorem F. *Assume Conjecture 6.5. Let p be an prime number, and let $\mathfrak{n}$ be a maximal ideal of $\mathbb{T}_1$ of residue characteristic p that does not correspond to a $\mathbb{Z}_p$-shady character. Then for every i, there is an isomorphism of $\mathbb{Z}_p \otimes \mathbb{T}_1$-modules*

$$H_i(Y_1, \mathbb{Z})_{\mathfrak{n}} \cong H_i(Y_2, \mathbb{Z})_{\mathfrak{n}}.$$

We could also prove an unconditional version, at the cost of replacing the condition on $\mathfrak{n}$ with a less useful one, namely $T_{\mathfrak{q}^2} \equiv -\operatorname{Nm}(\mathfrak{q}) \bmod \mathfrak{n}$ for all $\mathfrak{q}$ inert in L, which can be shown to be equivalent to the previous one under Conjecture 6.5.

It is instructive to compare Theorem F with work of Calegari and Venkatesh. In [14], they study the torsion homology and regulators in the more difficult setting of Jacquet–Langlands pairs (Y_1', Y_2') of orbifolds, which share some similarities with Vignéras orbifolds. The orbifolds Y_1' and Y_2' are not isospectral, but their spectra are closely related. In this setting there is also a Hecke algebra $\mathbb{T}'$ generated by Hecke operators $T_{\mathfrak{p}}$ for maximal ideals $\mathfrak{p}$ of $\mathbb{Z}_F$, and acting on all $H_i(Y_j', \mathbb{Z})$. We say that a maximal ideal $\mathfrak{m}$ of $\mathbb{T}'$ is *Eisenstein* if there exist Hecke characters Ψ, Ψ' of F such that $T_{\mathfrak{p}} \bmod \mathfrak{m} = \Psi(\mathfrak{p}) + \Psi'(\mathfrak{p})$ for all but finitely many $\mathfrak{p}$. Calegari and Venkatesh define a certain quotient $H_i(Y_j', \mathbb{Z})^{\text{new}}$ of $H_i(Y_j', \mathbb{Z})$ and formulate the following conjecture [14, Conjecture 2.2.8].

Conjecture 1.4 (Calegari–Venkatesh)**.** *Let (Y_1', Y_2') be a Jacquet–Langlands pair of 3-orbifolds. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}'$. Then we have*

$$|H_1(Y_1', \mathbb{Z})_{\mathfrak{m}}^{\text{new}}| = |H_1(Y_2', \mathbb{Z})_{\mathfrak{m}}^{\text{new}}|.$$

In addition, they prove several partial results [14, Section 6.8] towards their conjecture, but only when averaging over all maximal ideals $\mathfrak{m}$, hence forgetting the Hecke action. The Calegari–Venkatesh conjecture is an analogue for torsion homology of a famous theorem from the theory of automorphic representations, the Jacquet–Langlands correspondence (see Theorem 5.8). We now explain how Theorem F can be seen as a torsion analogue of another automorphic phenomenon, studied by Labesse and Langlands [36]. The relationship between Vignéras's

construction of isospectral orbifolds and the work of Labesse–Langlands was first exposed by Rajan [55]. The Langlands programme postulates the existence of a compact group L_F and conjectures that, to each cuspidal automorphic representation Π of $D^\times$, one should be able to attach a continuous irreducible representation, called a Langlands parameter,

$$\varphi = \varphi_\Pi \colon L_F \to \mathrm{GL}_2(\mathbb{C}),$$

such that $\varphi_\Pi \cong \varphi_{\Pi'}$ if and only if $\Pi \cong \Pi'$. Moreover, each such Π appears with multiplicity 1 in the space of automorphic forms (see Theorem 5.6). In contrast, to each cuspidal automorphic representation Π of the kernel D^1 of the reduced norm $D^\times \to F^\times$, one should be able to attach an irreducible Langlands parameter

$$\varphi \colon L_F \to \mathrm{PGL}_2(\mathbb{C}),$$

but several non-isomorphic Π can have isomorphic parameters φ: they are said to form an L-packet. While the existence of the group L_F and the parameters φ are still conjectural, Labesse and Langlands gave an ad-hoc definition of L-packets for D^1, realised that sometimes not all admissible Π in the same L-packet are automorphic, and proved a formula for the multiplicity of each Π in the space of automorphic forms. This manifests concretely as follows. To every $\Pi = \Pi_\infty \otimes \Pi_f$ corresponds a Casimir eigenvalue (coming from Π_∞) and system of eigenvalues for the Hecke operators of D^1 (coming from Π_f), and these eigenvalues are the same for every Π in an L-packet. However, given a compact open subgroup K_f of the points of D^1 over the finite adeles, the dimension of the space $\Pi_f^{K_f}$ of fixed points under K_f may depend on Π, even when Π varies inside a fixed L-packet. This may result in the system of eigenvalues attached to a given L-packet appearing with different multiplicities in the L^2-space attached to Γ for the various arithmetic groups $\Gamma = K_f \cap D^1$. If the distributions of the automorphic multiplicities in L-packets were completely random, Vignéras pairs of orbifolds would never be isospectral. However, as Labesse and Langlands proved [36, Proposition 7.2] (see also [55, Theorem 4]), this is far from true.

Theorem 1.5 (Labesse–Langlands)**.** *Suppose $\varphi \colon L_F \to \mathrm{PGL}_2(\mathbb{C})$ is not induced from a character of an index 2 subgroup. Then the automorphic multiplicity of every Π in the L-packet of φ is the same.*

This strong restriction allowed Rajan to prove [55, Theorem 2] the representation equivalence of many orbifolds generalising Vignéras's construction. Returning to torsion homology, let $p > 2$ be a prime number, and let G_F be the absolute Galois group of F. For every maximal ideal $\mathfrak{n}$ of $\mathbb{T}_1$ with residue field $\mathbb{F}$ of characteristic p, there should conjecturally exist a continuous semisimple Galois representation

$$\rho \colon G_F \to \mathrm{GL}_2(\mathbb{F})$$

such that one has $\mathrm{Tr}\, \rho(\mathrm{Frob}_\mathfrak{q}) \equiv T_\mathfrak{q} \bmod \mathfrak{n}$ for almost all $\mathfrak{q}$ whose class in C is trivial, see Conjecture 6.5. In fact, the maximal ideal $\mathfrak{n}$ does not completely determine ρ but it completely determines its projectivisation

$$P\rho \colon G_F \to \mathrm{PGL}_2(\mathbb{F}).$$

One may think of $P\rho$ as the "Langlands parameter" of the "L-packet" corresponding to $\mathfrak{n}$. In addition, the condition that $\mathfrak{n}$ correspond to a $\mathbb{Z}_p$-shady character Ψ of L is equivalent to $\rho \cong \mathrm{Ind}_{G_F/G_L} \psi$, where ψ corresponds to Ψ via class field theory. This makes Theorem F a torsion analogue of Theorem 1.5. More generally, Labesse and Langlands prove a formula for the automorphic multiplicity of every Π. We think that this naturally leads to a very interesting question.

Question 1.6. Can one formulate a torsion analogue of the full Labesse–Langlands multiplicity formula? Can one prove it?

We also obtain a Hecke-equivariant refinement of our results on the p-adic valuation of regulator ratios. It may seem difficult to make sense of it, as regulators are only numbers and therefore cannot afford an action of the Hecke algebra. One might hope to have a factorisation of the form $\mathrm{Reg}_i(Y_j) = \prod_{\mathfrak{n}} \mathrm{Reg}_i(Y_j)_{\mathfrak{n}}$, but this is unlikely to be possible, as regulators come from constructions over $\mathbb{R}$, while decompositions according to maximal ideals of $\mathbb{T}_1$ are p-adic in nature. However, we show that such a decomposition exists for the ratios of regulators, under conditions ensuring that they are rational numbers. More precisely, to certain modules M we attach a number $\mathcal{C}(M)$ that we call a *regulator constant* by analogy with [21], satisfying

$$\mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})) \in \mathbb{Q}^{\times},$$

and for every maximal ideal $\mathfrak{n}$ of $\mathbb{T}_1$ of residue characteristic p,

$$\mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}}) \in \mathbb{Q}_p^{\times}/(\mathbb{Z}_p^{\times})^2.$$

We will provide more details on our construction of regulator constants in Section 1.6. The following theorem is a combination of Lemma 4.13, Theorem 5.34, and Theorem 6.12.

Theorem G. *Assume that there is no $\mathcal{H}^{\bullet}$-shady character of L. Then the following assertions hold.*

(1) *For all i we have*

$$\frac{\mathrm{Reg}_i(Y_1)^2}{\mathrm{Reg}_i(Y_2)^2} = \mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})).$$

(2) *Let p be a prime number. Then for all i we have*

$$\mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})) \equiv \prod_{\mathfrak{n}} \mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}}) \bmod (\mathbb{Z}_p^{\times})^2,$$

where the product ranges over maximal ideals $\mathfrak{n}$ of $\mathbb{T}_1$ of residue characteristic p.

(3) *Let $p > 2$ be a prime number. Assume, in addition, Conjecture 6.5. Let $\mathfrak{n}$ be a maximal ideal of $\mathbb{T}_1$ of residue characteristic p that does not correspond to a $\mathbb{Z}_p$-shady character. Then for all i we have*

$$\mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}}) \in \mathbb{Z}_p^{\times}.$$

The definition of $\mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}})$ involves the choice of arbitrary non-degenerate $\mathbb{Q}_p$-valued pairings on the free parts of $H_i(Y_j, \mathbb{Z})_{\mathfrak{n}}$ satisfying certain properties. We speculate that there should exist a canonical construction of such pairings, leading to a notion of p-adic regulators $\mathrm{Reg}_{i,p}(Y_j) \in \mathbb{Q}_p^{\times}/(\mathbb{Z}_p^{\times})^2$, compatible with localisation at maximal ideals of $\mathbb{T}_1$ and such that one has

$$\frac{\mathrm{Reg}_{i,p}(Y_1)^2}{\mathrm{Reg}_{i,p}(Y_2)^2} \equiv \mathcal{C}(H_i(\mathcal{Y}, \mathbb{Z})) \bmod (\mathbb{Z}_p^{\times})^2.$$

Question 1.7. Can one define a notion of p-adic regulator in this context? Of p-adic analytic torsion? Is there a p-adic Cheeger–Müller formula?

Finally, note that the Cheeger–Müller formula merely provides a motivation for our results on torsion homology and regulators, we do not use this formula in our proofs. In particular our methods are very different from those of [14]. What makes the Vignéras case more accessible than the Jacquet–Langlands case is that the orbifolds Y_1 and Y_2 are commensurable, unlike Jacquet–Langlands pairs. The next section explains how we exploit this commensurability.

1.4. **Our approach to isospectrality.** The basic idea of our approach is applicable much more widely than in the setting sketched in the previous subsections. In Section 4, where we explain the general setup, we take D to be any division algebra over an arbitrary number field F. An adélic construction then yields a closed orbifold $\mathcal{Y}$, and the only restriction on D is that $\mathcal{Y}$ be positive-dimensional. One has a Hecke algebra $\mathbb{T}$ acting on, say, the space $\Omega^i(\mathcal{Y})$ of differential i-forms on $\mathcal{Y}$, and commuting with the action of the Laplace–de Rham operator Δ. That space decomposes as

$$\Omega^i(\mathcal{Y}) = \bigoplus_{c \in C} \Omega^i(Y_c)$$

with the direct summands corresponding to the connected components Y_c of $\mathcal{Y}$, indexed by a certain class group C. The Hecke algebra is generated by elements $T_{\mathfrak{p}}$ indexed by maximal ideals $\mathfrak{p}$ of $\mathbb{Z}_F$. The operators $T_{\mathfrak{p}}$ permute the direct summands $\Omega^i(Y_c)$ of $\Omega^i(\mathcal{Y})$, sending $\Omega^i(Y_c)$ to $\Omega^i(Y_{[\mathfrak{p}]c})$ where $[\mathfrak{p}]$ denotes the class of $\mathfrak{p}$ in C. In particular, the algebra $\mathbb{T}_1$ introduced in Subsection 1.3 is the subalgebra of $\mathbb{T}$ consisting of those Hecke operators that preserve all direct summands in the above decomposition.

Suppose once again, for simplicity, that we have $\mathcal{Y} = Y_1 \sqcup Y_2$, and let L/F be the quadratic extension corresponding to C, so that $T_{\mathfrak{p}}$ preserves the direct summands $\Omega^i(Y_j)$ if and only if $\mathfrak{p}$ splits in L/F. Fix $\lambda \in \mathbb{R}$. The algebra $\mathbb{T}$ acts on the λ-eigenspace $\Omega^i(\mathcal{Y})_{\Delta=\lambda}$. If there exists a Hecke operator $T \in \mathbb{T}$ inducing a map

$$T \colon \Omega^i(Y_1)_{\Delta=\lambda} \to \Omega^i(Y_2)_{\Delta=\lambda}$$

and acting invertibly on $\Omega^i(\mathcal{Y})_{\Delta=\lambda}$, then that Hecke operator witnesses equal multiplicity of λ in the spectra of Y_1 and Y_2 by defining an explicit isomorphism between the respective Δ-eigenspaces. A priori one might imagine that such an invertible Hecke operator can easily fail to exist in such a way that for every $\mathfrak{p}$ that is inert in L/F, there is some line in $\Omega^i(\mathcal{Y})_{\Delta=\lambda}$ that is annihilated by $T_{\mathfrak{p}}$. If these different lines had nothing to do with each other, then it would be completely unclear how to control when this happens. However, we show in Proposition 3.6 that if no Hecke operator $T \in \mathbb{T}$ that sends $\Omega^i(Y_1)_{\Delta=\lambda}$ to $\Omega^i(Y_2)_{\Delta=\lambda}$ acts invertibly, then in fact there is a *single* line in $\Omega^i(\mathcal{Y})_{\Delta=\lambda}$ that is annihilated by $T_{\mathfrak{p}}$ for every prime $\mathfrak{p}$ that is inert in L/F. This is the first main result of Section 3.

We deduce that if no Hecke operator that maps $\Omega^i(Y_1)_{\Delta=\lambda}$ to $\Omega^i(Y_2)_{\Delta=\lambda}$ acts invertibly, then there is a simultaneous eigenvector in $\Omega^i(\mathcal{Y})_{\Delta=\lambda}$ under all the Hecke operators such that "half" of the Hecke operators $T_{\mathfrak{p}}$ have eigenvalue $a_{\mathfrak{p}} = 0$. In other words, we obtain a Hecke eigenvalue system $(a_{\mathfrak{p}})_{\mathfrak{p}}$ such that for all $\mathfrak{p}$ one has

$$a_{\mathfrak{p}} = \chi(\mathfrak{p}) a_{\mathfrak{p}},$$

where $\chi \colon C \to \mathbb{C}^\times$ is the quadratic Hecke character attached by class field theory to the unique non-trivial character of the Galois group $\mathrm{Gal}(L/F)$. This *self-twist condition* might remind the reader of coefficients a_p attached to an elliptic curve with complex multiplication. The associated Galois representation is then induced from a linear character of the absolute Galois group of the quadratic extension L.

It is at this point in our analysis that we specialise to the case of quaternion algebras. In this case, as the example of elliptic curves suggests, the automorphic representation of GL_2 over F attached to the Hecke eigenvalue system $(a_{\mathfrak{p}})_{\mathfrak{p}}$ as above is the automorphic induction of a Hecke character from the quadratic extension L/F. In Section 5 we use the explicit classification of representations of GL_2 to arrive at precise conditions on this Hecke character, and thus at our definitions of $*$-shady characters, and to prove various instances of Theorem D.

Technical discussion: algebraic axiomatisation. In Section 3 we work in a completely abstract algebraic setting: the class group parametrising the connected components of $\mathcal{Y}$ is replaced by an arbitrary finite abelian group C, the Hecke algebra $R \otimes_{\mathbb{Z}} \mathbb{T}$ for a suitable ring R of scalars becomes a commutative R-algebra $\mathbf{T}$ graded by C, the space that the Hecke algebra acts on, such as $\Omega^i(\mathcal{Y})$ or $H_i(\mathcal{Y})$, is just a graded $\mathbf{T}$-module. Also in Section 3 the reader will encounter a certain group homomorphism $\nu \colon W \to C$, which could benefit from some motivating comments. Some of the components of $\mathcal{Y}$ correspond to orders in the division algebra that are conjugate to each other, so that the components are isometric to each other. To obtain the strongest results, we do not limit ourselves to Hecke operators that map $\Omega^i(Y_1)$ to $\Omega^i(Y_2)$, but consider all those mapping $\Omega^i(Y_1)$ to $\Omega^i(Y)$ for any component Y isometric to Y_2. Formalising this is not just a simple matter of passing to the quotient by this equivalence relation, since such a quotient does not carry a well-defined $\mathbf{T}$-action. The purpose of $C_{\mathrm{iso}} = C/\nu(W)$ is to parametrise connected components up to this equivalence. If $\widehat{C}_{\mathrm{iso}}$ were replaced by $\widehat{C}$ in all these results, then we would only pick out the elements of $\mathbf{T}$ that map $\Omega^i(Y_1)$ to $\Omega^i(Y_2)$ and would get more restrictive sufficient conditions for isospectrality.

1.5. Comparison with the Sunada method.

Sunada [62] constructs pairs of $\Omega^\bullet$-isospectral orbifolds as follows: let M be a Riemannian orbifold, let G be a finite group acting on M by isometries, and let $H, H' \leq G$ be two subgroups such that the linear representations $\mathbb{C}[H \backslash G]$ and $\mathbb{C}[H' \backslash G]$ of G are isomorphic. Then the quotients $H \backslash M$ and $H' \backslash M$ are $\Omega^\bullet$-isospectral.

The analogy between the permutation representations $\mathbb{C}[H \backslash G]$ and the unitary representations $\mathrm{L}^2(\Gamma \backslash \mathrm{PGL}_2(\mathbb{C}))$ has led many authors to view Vignéras's method as a special case of an extension of Sunada's. However, as Theorems A and B illustrate, this framing does not do justice to the flexibility of Vignéras's construction, which is capable of producing isospectral orbifolds without isomorphisms of L^2-spaces. Instead, in this paper we make the case that the parallel between the two methods is of a different nature: as we explained in Section 1.4, our conditions ensure that the isospectralities in Vignéras's construction are realised by Hecke operators, which one can view as so-called *transplantation* maps. It is well known that Sunada's construction can be formulated in such a way that the same is true there [30, §2.2].

We claimed earlier that Sunada's construction cannot yield connected hyperbolic 3-orbifolds of smaller volume than that appearing in Theorem A. We are now in a position to explain this. The orbifolds $H \backslash M$ and $H' \backslash M$ as above cover a common orbifold, namely $G \backslash M$, with degree $[G : H] = [G : H']$. The smallest hyperbolic 3-orbifold has volume $0.039\ldots$ [45]. Moreover, it is known that among all finite groups G and non-conjugate subgroups H, H' such that $\mathbb{C}[H \backslash G]$ and $\mathbb{C}[H' \backslash G]$ are isomorphic G-representations, the smallest index $[G : H]$ is 7 (realised by two subgroups of $G = \mathrm{GL}_3(\mathbb{F}_2)$) [52, Theorem 3]. Thus the smallest orbifolds that Sunada's method could produce have volume at least $7 \cdot 0.039\ldots = 0.273\ldots$.

In [4] we showed that the regulator quotients of Sunada-isospectral manifolds $H \backslash M$, $H' \backslash M$ are rational numbers, and that for all $i \in \mathbb{Z}_{\geq 0}$ and all prime numbers $p \nmid \#G$ we have

$$\mathrm{ord}_p\left(\frac{\mathrm{Reg}_i(H \backslash M)^2}{\mathrm{Reg}_i(H' \backslash M)^2}\right) = 0 \text{ and } \mathrm{ord}_p\left(\frac{\#H_i(H \backslash M)_{\mathrm{tors}}}{\#H_i(H' \backslash M)_{\mathrm{tors}}}\right) = 0.$$

If Y_1, Y_2 are orbifolds that come out of Vignéras's construction, then they typically do not sit in a common finite covering, so there is no obvious analogue of the condition $p \nmid \#G$ that would exclude all but finitely many primes from contributing to the regulator quotients.

In Theorem 6.22 we find an analogue of "the set of prime divisors of $\#G$" in the Vignéras setting.

Theorem H. *Assume Conjecture 6.5. Then there is an explicit set S of prime numbers such that:*

- *every prime number p for which there exists a $\mathbb{Z}_p$-shady character is contained in S, and*
- *the set S is infinite if and only if there exists a $\mathcal{H}^\bullet$-shady character.*

1.6. **Our approach to rationality of regulator quotients.** As we explained in Section 1.4, we obtain conditions under which the spaces $H_i(Y_1, \mathbb{R})$ and $H_i(Y_2, \mathbb{R})$ are not merely abstractly isomorphic, but such that there are isomorphisms given by Hecke operators. In particular, they preserve the rational structure. That by itself is still not enough to show that $\mathrm{Reg}_i(Y_1)^2 / \mathrm{Reg}_i(Y_2)^2$ is rational. The crucial additional property is that the Hecke algebra is stable under adjoints with respect to the harmonic forms pairing, which is used to define the regulators. This is sufficient to imply rationality of regulator quotients and allows us to analyse them.

In order to show this, we once again axiomatise the situation purely algebraically. In Section 2 we first describe the very general setup. As a by-product, we reprove in a Hecke-algebraic way a foundational result of the Dokchitsers on so-called regulator constants, Theorem 2.5, which we used in [4, 5] to analyse the regulator quotients of Sunada-isospectral manifolds.

Then in Section 3.2 we prove that the existence of a suitable pairing on a graded module M with the adjointness property described above is equivalent to the existence of such a pairing that takes values in $\mathbb{Q}$. The crucial ingredient in this proof is a nice criterion, Proposition 3.13, for existence of pairings with the required properties. The following is a special case. We refer to Section 3 for the definitions of the standard algebraic terms used here.

Proposition I. *Let C be a finite abelian group, let $\mathbf{A} = \bigoplus_{c \in C} \mathbf{A}_c$ be a C-graded commutative reduced finite-dimensional $\mathbb{Q}$-algebra, and let $M = \bigoplus_{c \in C} M_c$ be a finitely generated graded $\mathbf{A}$-module. Let L be a field of characteristic 0. Then the following are equivalent:*

- *there exists a non-degenerate L-valued bilinear pairing on M with respect to which all $T \in \mathbf{A}$ are self-adjoint and such that for all $c \neq c' \in C$ the homogeneous components M_c and $M_{c'}$ are orthogonal to each other;*
- *for every $c \in C$ the $(L \otimes \mathbf{A}_1)$-module $L \otimes M_c$ is self-dual.*

The descent of pairings from $\mathbb{C}$ to $\mathbb{Q}$ easily follows from this and the Deuring–Noether Theorem, see Proposition 3.16.

In Section 3.3 we define regulator quotients $\mathcal{C}(M)$ in this abstract setting, a priori with respect to a given pairing, but then we use the formalism of Section 2 to show that in fact the values of such regulator quotients do not depend on the pairing. In conjunction with the previously mentioned existence of $\mathbb{Q}$-valued pairings this proves rationality of regulator quotients of Vignéras pairs. Moreover, this formalism is very flexible, and allows us to prove Theorem G.

As a by-product of our approach, we have stumbled upon a curious connection between fields generated by Hecke eigenvalues and regulator quotients. We do not pursue this connection seriously in this paper, but Proposition 3.24 is an example of this phenomenon. The following is a special case.

Proposition J. *Suppose that $\mathcal{Y}$ has exactly two connected components, Y_1 and Y_2, suppose that $\dim H_1(Y_1, \mathbb{C}) = \dim H_1(Y_2, \mathbb{C}) = 1$, and suppose that there is no $\mathcal{H}^\bullet$-shady character. Then the field generated by the Hecke eigenvalues of $\mathbb{T}$ acting on $H_1(\mathcal{Y}, \mathbb{C})$ is $\mathbb{Q}(\mathrm{Reg}_1(Y_1)/\mathrm{Reg}_1(Y_2))$.*

It was the observation of this numerical coincidence that suggested to us that it should be possible to expand the Dokchitsers' formalism of regulator constants to Hecke algebras, and led us to our Hecke operator approach to isospectrality.

1.7. **Comparison with other approaches to Vignéras pairs.** In most of the existing work devoted to Vignéras's construction [66], the isospectrality is derived from the trace formula via so-called *selectivity* conditions [44, 43], from which one determines which Γ_i intersect a given conjugacy class. The trace formula suggests a kind of duality between this approach and ours: in the selectivity method one studies the geometric side, while we study the spectral side. In Corollary 5.21, we recover a sufficient criterion for representation equivalence that had been obtained from the selectivity method. We believe that one might be able to obtain results on finer notions of isospectrality, e.g. $\Omega^\bullet$- and Ω^0-isospectrality, from the selectivity method and a delicate analysis of the trace formula. It would be interesting to work out such criteria explicitly. With our approach, working out conditions for different notions of isospectrality amounts to different but routine local computations. The input from the trace formula required for our method is packaged in Langlands's automorphic induction theorem.

An alternative approach, due to Rajan [55], is to use the Labesse–Langlands multiplicity formula. In Corollary 5.23, we recover a variant of a sufficient criterion for representation equivalence that had been obtained by Rajan. Although the Labesse–Langlands work has so far only been applied to representation equivalence, it is likely that one could recover the results of Section 5 on finer notions of isospectrality with this approach.

Our approach has two advantages: firstly, it allows us to realise the isospectralities through direct identification of eigenfunctions rather than just a numerical comparison of multiplicities; and secondly and most importantly for our purposes, it applies to the torsion setting thanks to the Hecke action on $H_i(\mathcal{Y}, \mathbb{Z}_p)$.

Finally, we expect that one might be able to extract some information on the algebraicity or even rationality of the regulator quotients by relating them to special values of L-functions. For an analogous observation in the Jacquet–Langlands setting see [14, §6.5]. However, it seems hard to obtain precise information on the primes that enter or Hecke equivariance properties along the lines of Theorem G.

1.8. **Notation and conventions.** All our modules will be left modules. A module that is a direct sum of a number of copies of a simple module is called *isotypical*.

If B is a ring, $n \in \mathbb{Z}_{\geq 1}$, and M is a B-module, then $\mathrm{MaxSpec}(B)$ denotes the set of maximal ideals of B, the ring of $n \times n$ matrices over B is denoted by $\mathrm{M}_n(B)$, and $\mathrm{Ann}_B(M)$ denotes the annihilator of M in B. If Q is a field and A is a Q-algebra, then $\mathrm{Nm}_{A/Q}$ denotes the Q-algebra norm. The sign $\otimes$ denotes tensor product over $\mathbb{Z}$. If R is a ring, M is an R-module, and L an R-algebra, then we abbreviate the L-module $L \otimes_R M$ to M_L. By an *ideal* without further specification we mean a two-sided ideal. The *radical* $\mathrm{J}(R)$ of a ring R is defined as the intersection of its maximal left ideals. It is a two-sided ideal. If p is a prime number, then $\mathbb{Z}_{(p)} = \{a/b : a, b \in \mathbb{Z}, b \notin p\mathbb{Z}\}$ denotes the localisation of $\mathbb{Z}$ at p, and $\mathbb{Z}_p$ denotes the ring of p-adic integers.

If C is a finite abelian group, then $\widehat{C} = \mathrm{Hom}(C, \mathbb{C}^\times)$ denotes its group of complex characters. If X is a set, σ is an automorphism of X, and f is a function defined on X, then f^σ denotes the function $x \mapsto f(\sigma x)$. We will also use this notation when X is a normal subgroup of a group containing σ, in which case the automorphism of X is $x \mapsto \sigma x \sigma^{-1}$. We denote the centre of a group G by $\mathrm{Z}(G)$.

When talking about homology, differential forms, etc. on orbifolds, we always intend the meaning of these words in the orbifold sense, rather than just in reference

to the underlying topological space. We refer to [31, 16] for the definitions and many additional details. Note that all our orbifolds are "good" in the sense of ibid.: they have (finite) coverings by manifolds.

2. Regulator constants via Hecke algebras

In this section we start by describing a very general setting in which one can define algebraic invariants that specialise to regulator quotients in certain number theoretic and geometric situations. We then explain how the regulator constants of Dokchitser–Dokchitser attached to Brauer relations of finite groups [21] can be viewed as a special case of this construction, and we use this description to give a new basis-free proof of their main algebraic theorem, [21, Theorem 2.17]. That last application will not be needed in the remainder of the paper.

2.1. **Adjoint pairs.** Throughout this section, let R be a commutative domain, and let Q be its field of fractions. An *isogeny* of R-modules is a homomorphism of R-modules with R-torsion kernel and cokernel, equivalently a homomorphism ϕ of R-modules such that $Q \otimes_R \phi$ is an isomorphism of Q-vector spaces. Let N_1 and N_2 be free finite rank R-modules, and let $\phi \colon N_1 \to N_2$, $\phi^* \colon N_2 \to N_1$ be isogenies. If B_i is an R-basis for N_i for $i = 1, 2$, then we define

$$\mathcal{D}_{B_1,B_2,\phi,\phi^*} = \frac{\det_{B_1,B_2} \phi}{\det_{B_2,B_1} \phi^*} \in Q^\times,$$

where the determinants are computed with respect to the bases B_1 and B_2. If B_1' and B_2' are other R-bases of N_1, respectively N_2, then

$$\mathcal{D}_{B_1,B_2,\phi,\phi^*} \equiv \mathcal{D}_{B_1',B_2',\phi,\phi^*} \mod (R^\times)^2,$$

so we have a well-defined invariant

$$\mathcal{D}_{\phi,\phi^*} = \mathcal{D}_{B_1,B_2,\phi,\phi^*}(R^\times)^2 \in Q^\times/(R^\times)^2$$

for any choice of bases B_1, B_2 as above, which does not depend on these choices. Now let L be a field containing R, and suppose that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are non-degenerate L-valued R-bilinear pairings on N_1, respectively N_2, satisfying

$$(2.1) \qquad\qquad \langle\phi n_1, n_2\rangle_2 = \langle n_1, \phi^* n_2\rangle_1$$

for all $n_1 \in N_1$ and $n_2 \in N_2$, in other words making the diagram

$$
\begin{array}{ccc}
L \otimes_R N_1 & \xrightarrow{\;\;\phi\;\;} & L \otimes_R N_2 \\
{\scriptstyle\langle\cdot,\cdot\rangle_1}\Big\downarrow & & \Big\downarrow{\scriptstyle\langle\cdot,\cdot\rangle_2} \\
\operatorname{Hom}(N_1, L) & \xrightarrow[\;\;\phi^*\;\;]{} & \operatorname{Hom}(N_2, L),
\end{array}
$$

commute. Then one has

$$(2.2) \qquad \mathcal{D}_{B_1,B_2,\phi,\phi^*} = \frac{\det_{B_1}\langle\cdot,\cdot\rangle_1}{\det_{B_2}\langle\cdot,\cdot\rangle_2}.$$

This basic observation has the following two important consequences.

(O1) The right hand side of equation (2.2) is independent of the pairings, and determines a well-defined invariant in $Q^\times/(R^\times)^2$ (rather than just in $L^\times/(R^\times)^2$), when the determinants are evaluated with respect to any bases on N_1 and N_2. More precisely, if $\langle\cdot,\cdot\rangle_1'$ and $\langle\cdot,\cdot\rangle_2'$ are also non-degenerate R-bilinear pairings on N_1, respectively N_2, with values in a field containing R, with respect to which the maps ϕ and ϕ^* are adjoint as in equation (2.1), then we have

$$\left(\frac{\det_{B_1}\langle\cdot,\cdot\rangle_1}{\det_{B_2}\langle\cdot,\cdot\rangle_2}\right)(R^\times)^2 = \left(\frac{\det_{B_1}\langle\cdot,\cdot\rangle_1'}{\det_{B_2}\langle\cdot,\cdot\rangle_2'}\right)(R^\times)^2 \in Q^\times/(R^\times)^2.$$

(O2) The expression $\mathcal{D}_{\phi,\phi^*}$ is independent of the pair ϕ, ϕ^* of adjoint maps with respect to the pairings $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. More precisely, if $\psi\colon N_1 \to N_2$ and $\psi^*\colon N_2 \to N_1$ are R-linear embeddings satisfying adjointness property (2.1) with ϕ replaced by ψ, then $\mathcal{D}_{\phi,\phi^*} \equiv \mathcal{D}_{\psi,\psi^*} \mod (R^\times)^2$.

These consequences are particularly useful in situations in which one has a supply of pairs of maps as above that are adjoint with respect to fixed pairings and/or a supply of pairings that make a given pair of maps adjoint. One such situation is provided by the formalism of Brauer relations and regulator constants. The rest of the section is devoted to recalling that formalism, reinterpreting it in Hecke algebra terms, and applying the above observations in that context.

2.2. **Brauer relations and regulator constants.** Let G be a finite group. If H is a subgroup and V is an $R[G]$-module, then we identify $\mathrm{Hom}_{R[G]}(R[G/H], V)$ with the group V^H of H-fixed points in V via $f \mapsto f(1 \cdot H)$. In particular if H and H' are two subgroups of G, then one has isomorphisms

$$\mathrm{Hom}_{R[G]}(R[G/H'], R[G/H]) \cong R[G/H]^{H'} \cong R[H'\backslash G/H],$$

where the inverses of the last and the first isomorphism are given by

$$(2.3) \qquad H'gH \mapsto \sum_{u \in H'/(H'\cap {}^gH)} ugH \quad \text{and} \quad gH \mapsto (\alpha\colon hH' \mapsto hgH),$$

respectively. For $g \in G$, we will denote the element of $\mathrm{Hom}_{R[G]}(R[G/H'], R[G/H])$ corresponding to the double coset $H'gH$ by $T_{H'gH}$. Composition of homomorphisms defines a product on $R[H\backslash G/H]$. This R-algebra is called a *Hecke algebra*, and its elements are called *Hecke operators*. There is an R-linear map

$$\begin{aligned} R[H'\backslash G/H] &\to R[H\backslash G/H'], \\ T &\mapsto T^*, \end{aligned}$$

defined by $(T_{H'gH})^* = T_{Hg^{-1}H'}$. Moreover, for every module V over the group ring $R[G]$, there is a map

$$(2.4) \qquad \begin{aligned} \mathrm{Hom}_{R[G]}(R[G/H'], R[G/H]) &\to \mathrm{Hom}_R(V^H, V^{H'}) \\ \alpha &\mapsto \phi_\alpha = (f \mapsto f \circ \alpha). \end{aligned}$$

Slightly abusing notation, we will sometimes denote the image of an element T of $\mathrm{Hom}_{R[G]}(R[G/H'], R[G/H])$ under this map by $T\colon V^H \to V^{H'}$.

Now let V be a finitely generated $R[G]$-module that is free over R, and let $S_1 = \bigsqcup_i G/H_i$ and $S_2 = \bigsqcup_j G/H_j'$ be finite G-sets, where H_i and H_j' are subgroups

of G. Define

$$V^{S_1} = \mathrm{Hom}_{R[G]}(R[S_1], V) \cong \bigoplus_i V^{H_i}, \quad V^{S_2} = \mathrm{Hom}_{R[G]}(R[S_2], V) \cong \bigoplus_j V^{H'_j},$$

where $R[S_i]$ are the R-linear permutation modules attached to S_i for $i \in \{1, 2\}$. Then similarly to (2.4), an element $T \in \mathrm{Hom}_{R[G]}(R[S_2], R[S_1])$ induces an R-homomorphism $T \colon V^{S_1} \to V^{S_2}$.

We say that the pair $\Theta = (S_1, S_2)$ is a *Brauer relation* if there is an isogeny of R-modules in $\mathrm{Hom}_{R[G]}(R[S_2], R[S_1])$.

Assume that $\Theta = (S_1, S_2)$ is a Brauer relation. If $\langle \cdot, \cdot \rangle$ is a non-degenerate R-bilinear G-invariant pairing on V with values in a field L containing R, and U is a subgroup of G, then let $\langle \cdot, \cdot \rangle_U$ be the pairing on V^U defined by $\langle v_1, v_2 \rangle_U = \frac{1}{\#U}\langle v_1, v_2 \rangle$ for $v_1, v_2 \in V^U$. It is not hard to see that this pairing is non-degenerate on V^U. Thus, such a pairing $\langle \cdot, \cdot \rangle$ induces an R-bilinear non-degenerate pairing $\langle \cdot, \cdot \rangle_1$ on V^{S_1} whose restriction to each direct summand V^{H_i} is given by $\langle \cdot, \cdot \rangle_{H_i}$ and that makes the distinct direct summands orthogonal, and analogously there is an induced non-degenerate pairing $\langle \cdot, \cdot \rangle_2$ on V^{S_2}. The *regulator constant* of V with respect to the Brauer relation Θ is defined as

$$\mathcal{C}_\Theta(V) = \frac{\det(\langle \cdot, \cdot \rangle_1 | V^{S_1})}{\det(\langle \cdot, \cdot \rangle_2 | V^{S_2})} \in L^\times / (R^\times)^2.$$

Theorem 2.5 ([21, Theorem 2.17])**.** *The value of $\mathcal{C}_\Theta(V)$ does not depend on $\langle \cdot, \cdot \rangle$, i.e. if $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are non-degenerate G-invariant R-bilinear pairings on V, inducing pairings $\langle \cdot, \cdot \rangle_j$ and $\langle \cdot, \cdot \rangle'_j$, respectively, on V^{S_j} for $j \in \{1, 2\}$ as above, then*

$$\frac{\det(\langle \cdot, \cdot \rangle'_1 | V^{S_1})}{\det(\langle \cdot, \cdot \rangle'_2 | V^{S_2})} \equiv \frac{\det(\langle \cdot, \cdot \rangle_1 | V^{S_1})}{\det(\langle \cdot, \cdot \rangle_2 | V^{S_2})} \mod (R^\times)^2.$$

We now give a new proof of Theorem 2.5, using the observations of Section 2.1.

Proof of Theorem 2.5. We claim that if H and H' are subgroups of G, and $\langle \cdot, \cdot \rangle$ is a non-degenerate G-invariant R-bilinear pairing on V, then the adjoint of

$$T_{H'gH} \colon V^H \to V^{H'}$$

with respect to the induced pairings $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H'}$ is $T_{Hg^{-1}H'} \colon V^{H'} \to V^H$. Indeed, an elementary calculation shows that there is a well-defined function $H'gH \to H'/(H' \cap {}^g H)$ satisfying $ugh \mapsto u(H' \cap {}^g H)$ for all $u \in H'$ and $h \in H$, all of whose fibres have cardinality $\#H$, whence we deduce that for every $v \in V^H$ and $v' \in V^{H'}$ we have

$$
\begin{aligned}
\frac{1}{\#H'}\langle T_{H'gH}v, v' \rangle &= \frac{1}{\#H'} \sum_{u \in H'/(H' \cap {}^g H)} \langle ugv, v' \rangle \\
&= \frac{1}{\#H' \cdot \#H} \sum_{x \in H'gH} \langle xv, v' \rangle \\
&= \frac{1}{\#H' \cdot \#H} \sum_{y \in Hg^{-1}H'} \langle v, yv' \rangle \\
&= \frac{1}{\#H} \sum_{h \in H/(H \cap {}^{g^{-1}} H')} \langle v, hg^{-1}v' \rangle \\
&= \frac{1}{\#H} \langle v, T_{Hg^{-1}H'}v' \rangle,
\end{aligned}
$$

as claimed.

In particular, it follows that if $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are non-degenerate G-invariant R-bilinear pairings on V, and $T \in \mathrm{Hom}_{R[G]}(R[S_2], R[S_1])$ is an isogeny, then the adjoint of the induced isogeny $T \colon V^{S_1} \to V^{S_2}$ with respect to $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ is the same as with respect to $\langle \cdot, \cdot \rangle_1'$ and $\langle \cdot, \cdot \rangle_2'$, namely equal to T^* in both cases. Thus, $\mathcal{C}_\Theta(V) = \mathcal{D}_{T, T^*}$, and in particular is the same when computed with respect to $\langle \cdot, \cdot \rangle$ as with respect to $\langle \cdot, \cdot \rangle'$.   □

3. Algebras graded by finite abelian groups

In this section we continue working in a purely algebraic setting, investigating graded modules over certain rings graded by a finite abelian group.

The main result of Section 3.1 is Proposition 3.6, in which we give a sufficient criterion for existence of an isomorphism between different homogeneous components of such a graded module. In the geometric setting this isomorphism will realise an isospectrality. To prove Proposition 3.6 we first construct paths of isomorphisms for the simple subquotients, using Lemma 3.5, and then piece them together using dévissage, Lemma 3.3.

In Section 3.2 we investigate certain pairings on our graded modules, which we call polarisations. The main result is Proposition 3.13, a necessary and sufficient criterion for existence of polarisations, which in particular implies Proposition I from the Introduction. An important consequence of this criterion is Proposition 3.16, which says that the existence of a polarisation with values in a field extension implies the existence of one with values in the base field. This is an important ingredient in our proof of rationality of quotients of regulators in the geometric setting.

In Section 3.3 we apply the formalism of adjoint pairs from Section 2.1 to define our notion of regulator constants and to prove in Proposition 3.19 that it is independent of any choices involved. These invariants become quotients of regulators in the geometric setting. They enjoy some natural properties, such as rationality (Corollary 3.21), triviality (Lemma 3.22), good behaviour with respect to extension of scalars (Lemma 3.23), and with respect to various decompositions (Lemma 3.25, Proposition 3.26, and Proposition 3.27).

The following notation and assumptions will remain in place for the rest of the section. Let R be a commmutative Noetherian domain and let Q be its field of fractions. An R-*algebra* is a ring $\mathbf{A}$ equipped with a ring homomorphism from R to the centre of $\mathbf{A}$. If C is a group and $\mathbf{A}$ is an R-algebra, then a *grading of $\mathbf{A}$ by C* is a collection of R-submodules $\mathbf{A}_c \subset \mathbf{A}$ indexed by $c \in C$ such that there is a direct sum decomposition of R-modules $\mathbf{A} = \bigoplus_{c \in C} \mathbf{A}_c$ and such that for all $c, c' \in C$ one has $\mathbf{A}_c \mathbf{A}_{c'} \subset \mathbf{A}_{cc'}$. An R-algebra that is equipped with a grading by a group C is said to be *graded by C*, and the R-submodules $\mathbf{A}_c$ of such an R-algebra $\mathbf{A}$ are referred to as the *homogeneous components* of $\mathbf{A}$.

For the rest of the section, let $\mathbf{T}$ be a commutative R-algebra that is graded by a finite abelian group C. The trivial homogeneous component $\mathbf{T}_1$ is an R-subalgebra of $\mathbf{T}$, and for every $c \in C$, the R-submodule $\mathbf{T}_c$ of $\mathbf{T}$ is an $\mathbf{T}_1$-module. Next, let W be a group equipped with a group homomorphism $\nu \colon W \to C$. By setting $W_c = \nu^{-1}(c)$ for $c \in C$, we obtain a C-grading on the group algebra $\mathbf{T}[W]$. Let $C_{\mathrm{iso}} = C/\nu(W)$. Thus, $\widehat{C}_{\mathrm{iso}}$ is canonically identified with the group of all $\chi \in \widehat{C}$ that are trivial on $\nu(W)$. Let $\mathbf{A}$ be a quotient of $\mathbf{T}[W]$ by a homogeneous ideal, equivalently a C-graded R-algebra together with a grading-preserving surjection from $\mathbf{T}[W]$, and assume that $\mathbf{A}$ is finitely generated as an R-module. In particular, every $\mathbf{A}$-module is also a $\mathbf{T}[W]$-module.

For us, a *graded $\mathbf{A}$-module* is a finitely generated $\mathbf{A}$-module M equipped with a collection of R-submodules $M_c \subset M$ indexed by $c \in C$ such that one has a direct

sum decomposition of $\mathbf{T}_1$-modules $M = \bigoplus_{c \in C} M_c$ and such that for all $c, c' \in C$ one has $\mathbf{A}_c M_{c'} \subset M_{cc'}$.

If an element of a ring is invertible modulo the radical, then it is invertible. A ring is called *semilocal* if its quotient by the radical is a semisimple ring. A commutative ring is semilocal if and only if it has only finitely many maximal ideals. We refer to [37] for these definitions and basic facts from ring theory.

3.1. Linked modules.

Definition 3.1. Let M be a graded $\mathbf{A}$-module, and let $c, c' \in C$. If L is a ring containing R, we say that M_c is *L-linked* to $M_{c'}$ by $T \in \mathbf{A}_{c'c^{-1}}$ if the induced map $T \colon (M_c)_L \to (M_{c'})_L$ is an isomorphism. We say that M_c is *L-linked* to $M_{c'}$ if it is L-linked to it by some $T \in \mathbf{A}_{c'c^{-1}}$. We say that M_c is *linked* to $M_{c'}$ if for some ring L containing R, the module M_c is L-linked to $M_{c'}$.

Lemma 3.2. *Suppose that $\mathbf{T}_1$ is a field. Let $c \in C$, and let M be a non-zero $\mathbf{A}$-module. Then the following are equivalent:*

(1) there exists an element in $\mathbf{A}_c$ that is invertible in $\mathbf{A}$;
(2) there exists a simple subquotient N of M and an element of $\mathbf{A}_c$ that does not annihilate N.

If, moreover, the image of $\mathbf{T}$ in $\mathbf{A}$ is reduced, then the conditions are equivalent to the following:

(3) the homogeneous component $\mathbf{A}_c$ does not annihilate M.

Proof. The forward implications are trivial, without the reduced assumption.

We now prove that 3 implies 2. The homogeneous component $\mathbf{A}_c$ is generated by the images of $\mathbf{T}_d \cdot W_{d^{-1}c}$, as d runs over C. The hypothesis therefore implies that there exist $d \in C$, $T \in \mathbf{T}_d$, and $w \in W_{d^{-1}c}$ such that Tw does not annihilate M. Since w is invertible, this is equivalent to T not annihilating M. Since the image of $\mathbf{T}$ in $\mathbf{A}$ is reduced and finite-dimensional over the field $\mathbf{T}_1$, it is a product of fields, so its quotient $\mathbf{T}/\operatorname{Ann}_{\mathbf{T}} M$ is reduced. The element T is therefore not nilpotent in $\operatorname{End}_{\mathbf{T}_1}(M)$. Since M has finite length, this implies that T cannot annihilate every simple subquotient of M, and therefore the same is true for $Tw \in \mathbf{A}_c$.

Finally we prove that 2 implies 1. By the same argument as in the previous paragraph, the assumption implies that there exist $d \in C$, $T \in \mathbf{T}_d$, and $w \in W_{d^{-1}c}$ such that Tw does not annihilate a subquotient N, equivalently that T does not annihilate N, equivalently that T acts invertibly on N. Let n be the order of $d \in C$. Then $T^n \in \mathbf{T}_1$ also acts invertibly on N. Since $\mathbf{T}_1$ is a field, this implies that T^n is invertible, therefore so is Tw, whose image in $\mathbf{A}$ belongs to $\mathbf{A}_c$. $\square$

Lemma 3.3. *Suppose that R is semilocal. Let $c \in C$, and let M be an $\mathbf{A}$-module such that for every simple subquotient N of M, there exists an element of $\mathbf{A}_c$ that acts non-trivially (equivalently invertibly) on N. Then there exists an element of $\mathbf{A}_c$ that acts invertibly on M.*

Proof. By replacing $\mathbf{A}$ by $\mathbf{A}/(\operatorname{Ann}_{\mathbf{T}_1} M) \cdot \mathbf{A}$, we may assume that $\mathbf{T}_1$ acts faithfully on M. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}_1$. Then by faithfulness, it is the annihilator of a simple $\mathbf{T}_1$-subquotient M' of M. By Lemma 3.2, applied to $\mathbf{T}_1/\mathfrak{m}$ in place of $\mathbf{T}_1$ and to M' in place of M, there exists $u_{\mathfrak{m}} \in \mathbf{A}_c$ that acts invertibly on $M/\mathfrak{m}M$. By the Chinese Remainder Theorem, there exists $u \in \mathbf{A}_c$ that acts invertibly on $M/\operatorname{J}(\mathbf{T}_1)M$, where recall that $\operatorname{J}(\mathbf{T}_1) = \prod_{\mathfrak{m} \in \operatorname{MaxSpec}(\mathbf{T}_1)} \mathfrak{m}$ denotes the radical of $\mathbf{T}_1$. By [37, Corollary 5.9], $\operatorname{J}(\mathbf{T}_1) \cdot \mathbf{A}$ is contained in the radical of $\mathbf{A}$, so u acts invertibly on M. $\square$

Lemma 3.4. *Suppose that R is semilocal. Let M be a $\mathbf{T}$-module. Then for every simple $\mathbf{T}$-subquotient E of M there exists $\mathfrak{p} \in \mathrm{MaxSpec}(R)$ such that E is isomorphic to a $\mathbf{T}$-submodule of $R/\mathfrak{p} \otimes_R M$.*

Proof. By replacing $\mathbf{T}$ by $\mathbf{T}/\mathrm{Ann}_{\mathbf{T}} M \subset \mathrm{End}\, M$, we may assume that $\mathbf{T}$ is finitely generated over R. Let E be a simple subquotient of M. Then its annihilator is a maximal ideal $\mathfrak{P}$ of $\mathbf{T}$. By [23, Corollary 4.17] the intersection $\mathfrak{p} = \mathfrak{P} \cap R$ is a maximal ideal of R, and E is a subquotient of the localisation $(R/\mathfrak{p} \otimes_R M)_{\mathfrak{P}}$. Now, the quotient $R/\mathfrak{p}$ is a field, so that $\mathbf{T}/\mathfrak{p}\mathbf{T}$ is Artinian. By [23, Corollary 2.16], the localisation $(\mathbf{T}/\mathfrak{p}\mathbf{T})_{\mathfrak{P}}$ is local and is a direct factor of $\mathbf{T}/\mathfrak{p}\mathbf{T}$, so that $(R/\mathfrak{p} \otimes_R M)_{\mathfrak{P}}$ is a direct $\mathbf{T}$-summand of $R/\mathfrak{p} \otimes_R M$. Since this localisation is Artinian, it contains a simple submodule. But since $(\mathbf{T}/\mathfrak{p}\mathbf{T})_{\mathfrak{P}}$ is local, all simple $(\mathbf{T}/\mathfrak{p}\mathbf{T})_{\mathfrak{P}}$-modules are pairwise isomorphic, and in particular E is isomorphic to a simple $\mathbf{T}$-submodule of $(R/\mathfrak{p} \otimes_R M)_{\mathfrak{P}} \subset R/\mathfrak{p} \otimes_R M$, which proves the result. $\qquad\square$

Lemma 3.5. *Let $c \in C$ be non-trivial, and for every $\chi \in \widehat{C}$ with $\chi(c) \neq 1$, let $c_\chi \in C$ be such that $\chi(c_\chi) \neq 1$. Then c is contained in the group generated by the elements c_χ.*

Proof. Suppose that X is the set of all $\chi \in \widehat{C}$ satisfying $\chi(c) \neq 1$. Let U be the subgroup of C generated by c_χ for all $\chi \in X$. If c is not in U, then there exists $\chi' \in \widehat{C}$ such that $\chi'(c) \neq 1$ and $\chi'|_U = 1$, in particular such that we have $\chi' \notin X$. Therefore, c is in U, as claimed. $\qquad\square$

Proposition 3.6. *Suppose that R is semilocal. Let $c \in C$, and let M be a graded $\mathbf{A}$-module. Suppose that for all the finitely many $\mathfrak{p} \in \mathrm{MaxSpec}(R)$, for all simple (ungraded) $\mathbf{T}$-submodules E of $R/\mathfrak{p} \otimes_R M$, and for all $\chi \in \widehat{C}_{\mathrm{iso}}$ with $\chi(c) \neq 1$, there exist $c_{\chi,E} \in C$ with $\chi(c_{\chi,E}) \neq 1$ and $T_{\chi,E} \in \mathbf{T}_{c_{\chi,E}}$ that does not annihilate E. Then there exists an element of $\mathbf{A}_c$ that acts invertibly on M.*

Proof. Let N be a simple $\mathbf{A}$-subquotient of M. We claim that the hypothesis implies that for all $\chi \in \widehat{C}$ satisfying $\chi(c) \neq 1$ there exist $c_{\chi,N} \in C$ with $\chi(c_{\chi,N}) \neq 1$ and $S_{\chi,N} \in \mathbf{A}_{c_{\chi,N}}$ that does not annihilate N. Indeed, for $\chi \in \widehat{C}$ that do not vanish on $\nu(W)$, we may take $S_{\chi,N}$ to be the image of any $w \in W$ for which $\chi(\nu(w)) \neq 1$. Suppose instead that $\chi \in \widehat{C}_{\mathrm{iso}}$ satisfies $\chi(c) \neq 1$. Since the image of $\mathbf{T}$ in $\mathbf{A}$ is central and N is finite over R, the $\mathbf{T}$-module N is isomorphic to E^r for a simple $\mathbf{T}$-module E and some $r \in \mathbb{Z}_{>0}$. By Lemma 3.4, the module E is isomorphic to a simple submodule of $R/\mathfrak{p} \otimes_R M$ for some $\mathfrak{p} \in \mathrm{MaxSpec}(R)$. We may then take $S_{\chi,N} = T_{\chi,E}$. By Lemma 3.5, there exist $\alpha_{\chi,N} \in \mathbb{Z}$ such that $c = \prod_\chi c_{\chi,N}^{\alpha_{\chi,N}}$, so that $\prod_\chi S_{\chi,N}^{\alpha_{\chi,N}}$ belongs to $\mathbf{A}_c$. This product clearly acts invertibly on N. The conclusion follows from Lemma 3.3. $\qquad\square$

Corollary 3.7. *Let $c \in C$ and let M be a graded $\mathbf{A}$-module. Suppose that for all simple (ungraded) $\mathbf{T}$-submodules E of M_Q, and for all $\chi \in \widehat{C}_{\mathrm{iso}}$ with $\chi(c) \neq 1$, there exist $c_{\chi,E} \in C$ with $\chi(c_{\chi,E}) \neq 1$ and $T_{\chi,E} \in \mathbf{T}_{c_{\chi,E}}$ that does not annihilate E. Then for every $b \in C$, the homogeneous component M_b is linked to M_{cb}.*

Proof. Apply Proposition 3.6 with $R = Q$. $\qquad\square$

3.2. **Polarisations.**

Definition 3.8. An *involution* on $\mathbf{T}$ is an R-module homomorphism $\iota\colon \mathbf{T} \to \mathbf{T}$ satisfying the following properties:

- $\iota^2 = \mathrm{id}$,
- for all $x, y \in \mathbf{T}$ one has $\iota(xy) = \iota(y)\iota(x)$,
- for all $c \in C$ one has $\iota(\mathbf{T}_c) = \mathbf{T}_{c^{-1}}$.

In particular, every involution on $\mathbf{T}$ stabilises $\mathbf{T}_1$. Extend any involution ι on $\mathbf{T}$ to an involution on $\mathbf{T}[W]$ by defining $\iota(w) = w^{-1}$ for all $w \in W$, and assume that this involution descends to $\mathbf{A}$. Whenever M is an $\mathbf{A}$-module (resp. $\mathbf{T}_1$-module), we give $M^* = \mathrm{Hom}_R(M, R)$ the structure of an $\mathbf{A}$-module (resp. $\mathbf{T}_1$-module), called the *dual of M*, via ι, i.e. $(T\phi)(m) = \phi(\iota(T)m)$ for $\phi \in M^*, m \in M$ and $T \in \mathbf{A}$ (resp. $T \in \mathbf{T}_1$). We say that an $\mathbf{A}$-module (resp. $\mathbf{T}_1$-module) M is *self-dual* if M^* is isomorphic to M.

Assumption 3.9. For the rest of the section, assume that the image of $\mathbf{T}$ in $\mathbf{A}$ is reduced. Further, assume that $\mathbf{T}$ is equipped with an involution ι, and that ι, when extended to $\mathbf{T}W$ as above, descends to $\mathbf{A}$.

Definition 3.10. Let M be a graded $\mathbf{A}$-module that is R-torsion-free. If L is a field containing Q, then a *polarisation* on M over L is a non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle \colon M \otimes_R M \to L$ such that for $c \neq c' \in C$, the module M_c is orthogonal to $M_{c'}$, and such that for every $T \in \mathbf{A}$, the element T is adjoint to $\iota(T)$. If M is equipped with a polarisation over L, then we say that M is *polarised* over L, and if M admits a polarisation over L, then we say that M is *polarisable* over L.

Remark 3.11. A pairing $\langle \cdot, \cdot \rangle \colon M \otimes_R M \to L$ is a polarisation for $\mathbf{A}$ if and only if it is a polarisation for $\mathbf{T}$ that is W-invariant, i.e. such that for all $w \in W$ and all $m, m' \in M$ we have $\langle wm, wm' \rangle = \langle m, m' \rangle$.

Remark 3.12. If M is as in Definition 3.10, then $M^* = \mathrm{Hom}_R(M, R)$ is a graded $\mathbf{A}$-module, where for every $c \in C$, the homogeneous component $(M^*)_c$ consists of the homomorphisms $M \to R$ that factor through $M_{c^{-1}}$. A polarisation on M over a field L is the same as an isomorphism of (not graded) $\mathbf{A}_L$-modules $M_L \to (M^*)_L$ that sends M_c to $(M^*)_{c^{-1}}$.

Proposition 3.13. *Let M be a graded $\mathbf{A}$-module that is R-torsion-free, and let L be a field containing Q. Then M is polarisable over L if and only if for every $c \in C$, $(M_c)_L$ is a self-dual $(\mathbf{A}_1)_L$-module.*

Proof. The "only if" direction is clear, so we prove the other direction.

We may, without loss of generality, replace R by L, replace all the algebras and modules by their tensor product with L over R, and replace $\mathbf{T}$ by its image in $\mathbf{A}$. Then $\mathbf{T}$, being a reduced commutative algebra over a field by Assumption 3.9, is a product of fields. In particular, every quotient of $\mathbf{T}$ is also reduced. The algebra $\mathbf{T}_1$ is a product of factors B that are each either a field that is stabilised by ι, or a product $B = A_1 \times A_2$ of fields with ι inducing an isomorphism from A_1 to A_2 and from A_2 to A_1. Accordingly the tensor products $B \otimes_{\mathbf{T}_1} \bullet$ for all such factors B induce a direct product decomposition on the $\mathbf{T}_1$-algebras $\mathbf{A}_1$, $\mathbf{T}$, and $\mathbf{A}$, and a direct sum decomposition on the modules M and M_c for all $c \in C$. Clearly, M is polarisable if and only if for all B as above the module $B \otimes_{\mathbf{T}_1} M$ is, and similarly, for a given $c \in C$, the $\mathbf{A}_1$-module M_c is self-dual if and only if for all B, the $B \otimes_{\mathbf{T}_1} \mathbf{A}_1$-module $B \otimes_{\mathbf{T}_1} M_c$ is. Moreover, $\mathbf{T}$ being reduced implies that so is $B \otimes_{\mathbf{T}_1} \mathbf{T}$ for all B as above. Thus, we assume, without loss of generality, that $\mathbf{T}_1$ is either a field stabilised by ι or isomorphic to $A_1 \times A_2$ as just described.

Let
$$\tilde{C} = \{c \in C : \mathbf{A}_c M \neq 0\}.$$
For an arbitrary $T \in \mathbf{A}$, the non-degeneracy of the pairing on M implies that $TM \neq 0$ if and only $\iota(T)M \neq 0$. It follows that if $c \in C$, and $\mathbf{T}_1 = A_1 \times A_2$, then $(A_1 \otimes_{\mathbf{T}_1} \mathbf{A})_c$ does not annihilate M if and only if $(A_2 \otimes_{\mathbf{T}_1} \mathbf{A})_c$ does not. Therefore, Lemma 3.2 implies that we have
$$\tilde{C} = \{c \in C : \mathbf{A}_c \text{ contains an invertible element}\}.$$

In particular, $\tilde{C}$ is a subgroup of C.

Now, suppose that for every c, the $\mathbf{A}_1$-module $(M_c)_L$ is self-dual. Let D be a complete set of coset representatives for $C/\tilde{C}$, and for each $d \in D$, let $\langle \cdot, \cdot \rangle_d$ be a non-degenerate L-valued L-bilinear pairing on M_d such that for all $T \in \mathbf{A}_1$, the operator $\iota(T)$ is the adjoint of T. For every $c \in \tilde{C}$, fix an invertible element $T_c \in \mathbf{A}_c$. For $c \in \tilde{C}$ and $d \in D$, define a pairing $\langle \cdot, \cdot \rangle_{cd}$ on M_{cd} by setting, for every $m, m' \in M_{cd}$,

$$\langle m, m' \rangle_{cd} = \langle T_c^{-1} m, \iota(T_c) m' \rangle_d.$$

Let $\langle \cdot, \cdot \rangle$ be the unique pairing on M that restricts to $\langle \cdot, \cdot \rangle_c$ for all $c \in C$ and that makes the different homogeneous components pairwise orthogonal. It is clear that each $\langle \cdot, \cdot \rangle_c$ is non-degenerate, therefore so is $\langle \cdot, \cdot \rangle$. We claim that for all $T \in \mathbf{A}$, the adjoint of T with respect to this pairing is $\iota(T)$. It suffices to prove this for homogeneous elements T. Let $c, c' \in \tilde{C}$, $d \in D$, $T \in \mathbf{A}_c$, $m \in M_{c^{-1}c'd}$ and $m' \in M_{c'd}$. Then we have

$$
\begin{aligned}
\langle Tm, m' \rangle_{c'd} &= \langle T_{c'}^{-1} T m, \iota(T_{c'}) m' \rangle_d \\
&= \langle \underbrace{(T_{c'}^{-1} T T_{c'c^{-1}})}_{\in \mathbf{A}_1} T_{c'c^{-1}}^{-1} m, \iota(T_{c'}) m' \rangle_d \\
&= \langle T_{c'c^{-1}}^{-1} m, \iota(T_{c'}^{-1} T T_{c'c^{-1}}) \iota(T_{c'}) m' \rangle_d \\
&= \langle T_{c'c^{-1}}^{-1} m, \iota(T_{c'c^{-1}}) \iota(T) m' \rangle_d \\
&= \langle m, \iota(T) m' \rangle_{c^{-1}c'd},
\end{aligned}
$$

as required. This shows that $\langle \cdot, \cdot \rangle$ is a polarisation on M, and completes the proof. $\square$

Remark 3.14. For M as in Definition 3.10 to be polarisable over a field L, it is not sufficient that M_L be a self-dual $(\mathbf{A}_1)_L$-module. Indeed, let $W = 1$, $\mathbf{A}_1 = \mathbf{T}_1 = Q \times Q$, with ι swapping the two factors, $\mathbf{T} = \mathbf{A} = \mathbf{T}_1$ with trivial grading by a group $C = \{1, c\}$ of order 2, and let $M_1 = Q$ with $\mathbf{T}$ acting via projection onto the first factor Q, and $M_c = Q$ with $\mathbf{T}$ acting via projection onto the second factor. Then M_1 and M_c are dual to each other, so that $M = M_1 \oplus M_c$ is a self-dual $\mathbf{T}_1$-module. However, M_1 is not self-dual, so M is not polarisable over Q.

Corollary 3.15. *Assume that $W = 1$ and that either $\mathbf{T}_1$ is a field or ι is trivial. Let M be a graded $\mathbf{T}$-module that is R-torsion-free. Then M is polarisable over Q.*

Proof. Without loss of generality, assume that $R = Q$ and replace $\mathbf{T}$ by its image in $\mathbf{A}$. Under the assumptions, $\mathbf{T}_1$ is a product of fields, each preserved by ι, and M is polarisable over Q if and only if it is after taking the tensor product with each of these factors. Thus, it suffices to assume that $\mathbf{T}_1$ is a field. In that case, for every $c \in C$ the $\mathbf{T}_1$-vector spaces $(M_c)_Q$ and $(M_c^*)_Q$ have the same dimension, and are therefore isomorphic. The result follows from Proposition 3.13. $\square$

Proposition 3.16. *Let M be a graded $\mathbf{A}$-module that is R-torsion-free, and let L be a field containing Q. Then M is polarisable over Q if and only if M is polarisable over L.*

Proof. Assume that M is polarisable over L. Let $c \in C$ be arbitrary. Then the $(\mathbf{T}_1)_L$-module $(M_c)_L$ is self-dual, i.e. $(M_c)_L$ and $(M_c^*)_L$ are isomorphic. By the Deuring–Noether Theorem [37, Theorem 19.25] this implies that $(M_c)_Q$ and $(M_c^*)_Q$ are isomorphic, i.e. the $(\mathbf{T}_1)_Q$-module $(M_c)_Q$ is self-dual. By Proposition 3.13, the module M is polarisable over Q. $\square$

Definition 3.17. We will say that a graded **A**-module M is *polarisable* if it is polarisable over some field containing Q. In light of Proposition 3.16 this is equivalent to M being polarisable over Q.

3.3. **Regulator constants for graded modules.** In this subsection we continue to make Assumption 3.9.

Definition 3.18. Let M be a graded **A**-module that is free over R. Let $c, c' \in C$. Suppose that M is polarisable and that M_c is L-linked to $M_{c'}$ by $T \in (\mathbf{A}_{c'c^{-1}})_L$ for some field L containing Q. Then $\iota(T)$, being the adjoint of an isomorphism with respect to a polarisation, induces an isomorphism $(M_{c'})_L \to (M_c)_L$. Define the *regulator constant of M with respect to the pair* (c, c') by

$$\mathcal{C}_{c,c'}(M) = \frac{\det(T \colon (M_c)_L \to (M_{c'})_L)}{\det(\iota(T) \colon (M_{c'})_L \to (M_c)_L)} = \mathcal{D}_{T,\iota(T)} \in L^\times/(R^\times)^2,$$

using the notation of Section 2.1, where recall that the determinants are evaluated with respect to any bases B_1 of M_c and B_2 of $M_{c'}$.

Proposition 3.19. *For every polarisation $\langle \cdot, \cdot \rangle$ on M we have*

$$\mathcal{C}_{c,c'}(M) = \frac{\det(\langle \cdot, \cdot \rangle \mid M_c)}{\det(\langle \cdot, \cdot \rangle \mid M_{c'})}.$$

The value of $\mathcal{C}_{c,c'}(M)$ does not depend on T, nor on the polarisation.

Proof. Apply equation (2.2) and observations (O1) and (O2) of Section 2.1. $\quad\square$

Remark 3.20. Let M be a graded **A**-module that is free over R and polarisable over a field L. Then being L-linked is an equivalence relation on the homogeneous components of M, and for all $c_1, c_2, c_3 \in C$ such that M_{c_1}, M_{c_2}, and M_{c_3} are linked we have

$$\mathcal{C}_{c_1,c_2}(M) = \mathcal{C}_{c_2,c_1}(M)^{-1} \text{ and } \mathcal{C}_{c_1,c_3}(M) = \mathcal{C}_{c_1,c_2}(M)\mathcal{C}_{c_2,c_3}(M).$$

Corollary 3.21. *Let M be a polarisable graded **A**-module that is free over R, and let $c, c' \in C$ be such that M_c and $M_{c'}$ are linked. Then we have*

$$\mathcal{C}_{c,c'}(M) \in Q^\times/(R^\times)^2.$$

Proof. By Proposition 3.16 there exists a polarisation on M that takes values in Q, and by Proposition 3.19 the value of $\mathcal{C}_{c,c'}(M)$ does not depend on the polarisation. $\quad\square$

Lemma 3.22. *Let M be a graded **A**-module that is free over R. Let $c, c' \in C$, and set $c = c'c^{-1}$. Suppose that there exists $T \in \mathbf{A}_c$ such that $T \colon M \to M$ is surjective. Then one has $\mathcal{C}_{c,c'}(M) \in R^\times/(R^\times)^2$.*

Proof. By Nakayama's lemma [46, Theorem 2.4], the **T**-module endomorphism T of M, being surjective, is in fact an isomorphism. This also implies that $\iota(T)$ is an isomorphism. Thus $\det(T \colon M_c \to M_{c'})$ and $\det(\iota(T) \colon M_{c'} \to M_c)$, evaluated with respect to any R-bases on M_c and $M_{c'}$, are units in R, and the result follows. $\quad\square$

Lemma 3.23. *Let $R \hookrightarrow \tilde{R}$ be an embedding of commutative domains, let L be a field containing $\tilde{R}$, and let M be a graded **A**-module that is free over R and that is polarisable over L. Then for all $c, c' \in C$ one has*

$$\mathcal{C}_{c,c'}(M) = \mathcal{C}_{c,c'}(\tilde{R} \otimes_R M) \in L^\times/(\tilde{R}^\times)^2.$$

Proof. This is immediate from the definitions. $\quad\square$

The next result is not used in the sequel, but it illustrates a connection between Hecke eigenvalues and regulator ratios. Proposition J is a special case.

Proposition 3.24. *Let $R = Q$ be a field, let $\mathbf{T}_1$ be a field, take W to be trivial, let M be a graded $\mathbf{T}$-module. Let $c \in C$ have order 2, assume that M_1 and M_c are linked, by T, say, and let $T^2 = a \in \mathbf{T}_1$. Then M is polarisable, and we have*

$$\mathcal{C}_{1,c}(M) = \mathrm{Nm}_{\mathbf{T}_1/Q}(a)^n \bmod (Q^\times)^2,$$

where n is the common dimension of M_1 and M_c over $\mathbf{T}_1$, and where $\mathrm{Nm}_{\mathbf{T}_1/Q}$ denotes the Q-algebra norm on $\mathbf{T}_1$.

Proof. By Corollary 3.15, M is polarisable. Write the action of T on $M_1 \oplus M_c$ as a block-diagonal matrix $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ over the field $\mathbf{T}_1$ with A, B invertible. Then $T^2 = a$ acts as

$$\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \begin{pmatrix} a \cdot \mathrm{Id}_n & 0 \\ 0 & a \cdot \mathrm{Id}_n \end{pmatrix}.$$

Since ι is an automorphism of $\mathbf{T}$ that is trivial on Q, we have $\det_Q(\iota(B)) = \det_Q(B)$. We get $\mathcal{C}_{1,c}(M) = \det_Q(A)/\det_Q(\iota(B)) \equiv \det_Q(AB) \equiv \mathrm{Nm}_{\mathbf{T}_1/Q}(a^n) \bmod (Q^\times)^2$. This proves the result. $\qquad\square$

The next three results show that in many situations regulator constants of a module can be factored as products of regulator constants of "simpler" modules.

Lemma 3.25. *Let M be a polarisable graded $\mathbf{A}$-module that is free over R and that decomposes as a direct sum $M = \bigoplus_i M_i$ of polarisable graded $\mathbf{A}$-submodules. Let $c, c' \in C$, and assume that M_c and $M_{c'}$ are linked. Then for every i, $(M_i)_c$ and $(M_i)_{c'}$ are linked, and we have*

$$\mathcal{C}_{c,c'}(M) = \prod_i \mathcal{C}_{c,c'}(M_i).$$

Proof. The first assertion is clear. Also, by hypothesis, we may choose a polarisation on M that makes all summands M_i pairwise orthogonal, whence the second assertion follows. $\qquad\square$

Proposition 3.26. *Let $R = Q$ be a field, and let M be a polarisable graded $\mathbf{A}$-module that is free over R. Let $c, c' \in C$, and assume that M_c and $M_{c'}$ are linked. Let the image of $\mathbf{T}_1$ in $\mathbf{A}$ decompose as a direct product $\prod_i B_i$ of Q-algebras with involution, and for all i, let $M_i = B_i \otimes_{\mathbf{T}_1} M$. Then for all i, the graded $\mathbf{A}$-module M_i is polarisable, $(M_i)_c$ and $(M_i)_{c'}$ are linked, and we have*

$$\mathcal{C}_{c,c'}(M) = \prod_i \mathcal{C}_{c,c'}(M_i) \in Q^\times/(Q^\times)^2.$$

Proof. By Proposition 3.13, since M is polarisable, the M_i are polarisable. We have a decomposition $M = \bigoplus_i M_i$. The last two assertions follow from Lemma 3.25. $\quad\square$

Proposition 3.27. *Let $R = \mathbb{Z}_p$, and let M be a polarisable graded $\mathbf{A}$-module that is free over R. Let $\mathcal{N} = \{\mathfrak{n} \cap \iota(\mathfrak{n}) : \mathfrak{n} \in \mathrm{MaxSpec}(\mathbf{T}_1)\}$ let $M_{\mathfrak{n}'} = (\mathbf{T}_1)_{\mathfrak{n}'} \otimes_{\mathbf{T}_1} M$. Then for all $\mathfrak{n}' \in \mathcal{N}$, the graded $\mathbf{A}$-module $M_{\mathfrak{n}'}$ is polarisable.*

Let $c, c' \in C$, and assume that M_c and $M_{c'}$ are linked. Then for all $\mathfrak{n}' \in \mathcal{N}$, $(M_{\mathfrak{n}'})_c$ and $(M_{\mathfrak{n}'})_{c'}$ are linked, and we have

$$\mathcal{C}_{c,c'}(M) = \prod_{\mathfrak{n}' \in \mathcal{N}} \mathcal{C}_{c,c'}(M_{\mathfrak{n}'}) \in \mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2.$$

Proof. By [23, Corollary 7.6], there is a decomposition $\mathbf{T}_1 = \prod_{\mathfrak{n}} (\mathbf{T}_1)_{\mathfrak{n}}$, with the product running over the maximal ideals of $\mathbf{T}_1$. Accordingly, by grouping terms, there is a direct sum decomposition $M = \bigoplus_{\mathfrak{n}' \in \mathcal{N}} M_{\mathfrak{n}'}$. By Proposition 3.13, since M is polarisable, the $M_{\mathfrak{n}'}$ are polarisable. The last two statements follow from Lemma 3.25. $\qquad\square$

4. Vignéras pairs of manifolds

In the present section we will introduce the number theoretic situation we are interested in, a general version of the Vignéras construction, and apply the results of Section 3 to this situation, deriving inexplicit self-twist conditions for various kinds of isospectralities. We fix the following notation.

F	a number field.
$\mathbb{Z}_L$	the ring of integers of a number field or p-adic field L.
$\mathfrak{p}$	a maximal ideal of $\mathbb{Z}_F$.
$\mathbb{Z}_\mathfrak{p}$	ring of integers of the completion $F_\mathfrak{p}$.
$\mathfrak{N}$	a non-zero ideal of $\mathbb{Z}_F$.
∞	the set of all real places of F.
$\mathrm{Cl}_F(\mathfrak{N}\mathcal{V})$	the ray class group with modulus $\mathfrak{N}$ times a subset $\mathcal{V}$ of ∞
$U_F(\mathfrak{N}\mathcal{V})$	the group $\{u \in \mathbb{Z}_F^\times : u - 1 \in \mathfrak{N}$ and $v(u) > 0$ for all $v \in \mathcal{V}\}$.
$F_\mathbb{R}$	$\mathbb{R} \otimes_\mathbb{Q} F$.
$\mathbb{A}_{F,f}$	the ring $(\prod_p \mathbb{Z}_p) \otimes F$ of finite adèles of F.
$\mathbb{A}_F$	the ring $F_\mathbb{R} \times \mathbb{A}_{F,f}$ of adèles of F.
D	a division algebra over F of degree $d \geq 2$ that is not totally definite.
$\mathbb{H}$	the $\mathbb{R}$-algebra of Hamilton quaternions.
$\mathcal{V}_\mathbb{R}(D),$	the sets of infinite places of F that are real and split in D,
$\mathcal{V}_\mathbb{C}(D),$	respectively complex (and necessarily split in D),
$\mathcal{V}_\mathbb{H}(D)$	respectively ramified in D.
$\mathcal{O}$	a maximal order in D.
$\mathbb{G}$	the algebraic group corresponding to $D^\times$, i.e. representing the functor $(\bullet \otimes_F D)^\times$ on the category of F-algebras.
G_∞	$\mathbb{G}(F_\mathbb{R})$.
K_∞	a maximal compact subgroup of G_∞.
Z_∞	the centre of G_∞.
$\Lambda_\mathfrak{p}, d_\mathfrak{p}$	are defined up to isomorphism by $\mathbb{Z}_\mathfrak{p} \otimes_{\mathbb{Z}_F} \mathcal{O} \cong \mathrm{M}_{d_\mathfrak{p}}(\Lambda_\mathfrak{p})$, where $\Lambda_\mathfrak{p}$ is a local ring. From now on fix such isomorphisms.
$\Delta_\mathfrak{p}$	the division algebra $F_\mathfrak{p} \otimes_{\mathbb{Z}_\mathfrak{p}} \Lambda_\mathfrak{p}$.
δ_D	the finite product $\prod_\mathfrak{p} \mathfrak{p}^{d-d_\mathfrak{p}} \subset \mathbb{Z}_F$.
$K(\mathfrak{p}^i),$ $i \in \mathbb{Z}_{\geq 0}$	$\mathrm{GL}_{d_\mathfrak{p}}(\Lambda_\mathfrak{p})$ if $i = 0$, $1 + \mathrm{J}(\Lambda_\mathfrak{p})^i \mathrm{M}_{d_\mathfrak{p}}(\mathbb{Z}_\mathfrak{p})$ if $i > 0$ (i.e. the kernel of reduction modulo $\mathrm{J}(\Lambda_\mathfrak{p})^i$), where $\mathrm{J}(\Lambda_\mathfrak{p})$ is the maximal ideal of $\Lambda_\mathfrak{p}$.
$K(\mathfrak{N})$	$\prod_{\mathfrak{p}^i \| \mathfrak{N}} K(\mathfrak{p}^i)$, the product running over all maximal ideals of $\mathbb{Z}_F$.
K_f	an open subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$ containing $K(\mathfrak{N})$ and such that $K_f/\mathrm{Z}(K_f)$ is compact.
nrd	the reduced norm map from $\mathbb{G}$ to the multiplicative group $\mathbb{G}_m$.
$\mathbb{G}(\mathbb{A}_F)^+$	$\{g = (g_v)_v \in \mathbb{G}(\mathbb{A}_F) : \det(g_v) > 0$ for all $v \in \mathcal{V}_\mathbb{R}(D)\}$.
H^+	$H \cap \mathbb{G}(\mathbb{A}_F)^+$, where H is a subgroup of $\mathbb{G}(\mathbb{A}_F)$.
$\mathcal{Y}$	$\mathbb{G}(F)^+\backslash\mathbb{G}(\mathbb{A}_F)/Z_\infty K_\infty K_f$.
X	$G_\infty/(Z_\infty K_\infty)$.
C	$\mathrm{nrd}(\mathbb{G}(F)^+)\backslash\mathbb{A}_{F,f}^\times/\mathrm{nrd}(K_f)$.
$\Gamma_c, c \in C$	$\Gamma_g = \mathbb{G}(F)^+ \cap gK_f g^{-1}$ where the class of $\mathrm{nrd}(g)$ in C is c.
Y_c	$\Gamma_c\backslash X$.
$\mathrm{N}(K_f)$	the normaliser of K_f in $\mathbb{G}(\mathbb{A}_{F,f})$.
W	$F^\times\backslash(Z_\infty K_\infty \times \mathrm{N}(K_f)/K_f)$.
C_iso	see discussion after Proposition 4.2, in particular Equation (4.3).
$\bar{c}$	the class of c in C_iso, where $c \in C$.
$c^\perp$	the set of $\chi \in \widehat{C}_\mathrm{iso}$ such that $\chi(c) = 1$, where $c \in C$.
$\mathbb{T}$	the subalgebra of $\mathbb{Z}[K_f\backslash\mathbb{G}(\mathbb{A}_{F,f})/K_f]$ generated by the $K_f\mathbb{G}(F_\mathfrak{p})K_f$ for all $\mathfrak{p} \nmid \delta_D\mathfrak{N}$.

We will now recall some basic facts about these objects. We refer to [27, §2.6, §5.5, §15.2] for these facts and further material.

The double quotient $\mathcal{Y}$ is a compact orientable orbifold. Fix a set $\mathcal{R}$ of double coset representatives of $\mathbb{G}(F)^+\backslash\mathbb{G}(\mathbb{A}_{F,f})/K_f$. For each $g \in \mathcal{R}$, define $\Gamma_g = \mathbb{G}(F)^+ \cap gK_fg^{-1}$, and $Y_g = \Gamma_g\backslash X$. Then each Y_g is connected, and one has $\mathcal{Y} = \bigsqcup_{g\in\mathcal{R}} Y_g$. The reduced norm map induces a bijection between the component set $\mathcal{R}$ and the group C, and we will from now on implicitly use this bijection, and will index the connected components of $\mathcal{Y}$ by C. The group C is a quotient of

$$\mathrm{nrd}(\mathbb{G}(F)^+)\backslash\mathrm{nrd}(\mathbb{G}(\mathbb{A}_{F,f}))/\mathrm{nrd}(K(\mathfrak{N})) = \mathrm{Cl}_F(\mathfrak{N}\infty).$$

We have

$$G_\infty \cong \mathrm{GL}_d(\mathbb{R})^{\mathcal{V}_\mathbb{R}(D)} \times \mathrm{GL}_d(\mathbb{C})^{\mathcal{V}_\mathbb{C}(D)} \times \mathrm{GL}_{d/2}(\mathbb{H})^{\mathcal{V}_\mathbb{H}(D)}.$$

The maximal compact subgroup K_∞ is accordingly a product of maximal compact subgroups over the infinite places of F, is unique up to conjugation, and may be taken to be

$$K_\infty \cong \mathrm{O}_d(\mathbb{R})^{\mathcal{V}_\mathbb{R}(D)} \times \mathrm{U}_d(\mathbb{C})^{\mathcal{V}_\mathbb{C}(D)} \times \mathrm{U}_{d/2}(\mathbb{H})^{\mathcal{V}_\mathbb{H}(D)}.$$

Accordingly, the dimension of X (and therefore also of $\mathcal{Y}$) can be computed to be $\left(\frac{d^2+d-2}{2}\right)\cdot\#\mathcal{V}_\mathbb{R}(D) + \left(d^2-1\right)\cdot\#\mathcal{V}_\mathbb{C}(D) + \left(\frac{d^2-d-2}{2}\right)\cdot\#\mathcal{V}_\mathbb{H}(D)$. Note, in particular, that when $d = 2$, the condition on D to be not totally definite, equivalently the set $\mathcal{V}_\mathbb{R}(D) \cup \mathcal{V}_\mathbb{C}(D)$ to be non-empty, is precisely the condition to ensure that the dimension of $\mathcal{Y}$ is positive.

We now give an alternative description of $\mathcal{Y}$, which we will use in the next section.

Lemma 4.1. *We have $Z_\infty^+ K_\infty^+ = G_\infty^+ \cap (Z_\infty K_\infty)$.*

Proof. One inclusion is obvious. For the other, it suffices to argue place by place, and the statement is only non-empty at the places $v \in \mathcal{V}_\mathbb{R}(D)$. Locally at such a place, the claim is that we have $\mathbb{R}_{>0}^\times \mathrm{SO}_d(\mathbb{R}) = \mathrm{GL}_d(\mathbb{R})^+ \cap (\mathbb{R}^\times \mathrm{O}_d(\mathbb{R}))$, where $\mathrm{GL}_d(\mathbb{R})^+$ denotes the group of real $d \times d$ matrices with positive determinant, and where $\mathbb{R}^\times$ denotes the subgroup of $\mathrm{GL}_d(\mathbb{R})$ consisting of scalar matrices. To prove the claim, suppose that we have $z \in \mathbb{R}^\times$, $k \in \mathrm{O}_d(\mathbb{R})$ such that $\det(zk) > 0$. If we have $\det(z) > 0$, then we also have $\det(k) > 0$, hence $zk \in \mathbb{R}_{>0}^\times \mathrm{SO}_d(\mathbb{R})$. If, on the other hand, we have $\det(z) < 0$, so that in particular d is odd, then $-1 \in \mathbb{R}^\times$ has determinant -1, and is also an orthogonal matrix, so that we have $zk = (-1\cdot z)(-1\cdot k) \in \mathbb{R}_{>0}^\times \mathrm{SO}_d(\mathbb{R})$. This proves the claim, and hence the lemma. $\square$

Proposition 4.2. *We have canonical isomorphisms*

$$\mathcal{Y} = \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}_F)/Z_\infty^+ K_\infty^+ K_f$$

and

$$C = F_+^\times\backslash\mathbb{A}_{F,f}^\times/\mathrm{nrd}(K_f) = F^\times\backslash(\mathbb{A}_{F,f}^\times \times \{\pm 1\}^{\mathcal{V}_\mathbb{R}(D)\cup\mathcal{V}_\mathbb{H}(D)})/\mathrm{nrd}(K_f),$$

where $F_+^\times = \{\alpha \in F : v(\alpha) > 0 \text{ for all } v \in \mathcal{V}_\mathbb{R}(D) \cup \mathcal{V}_\mathbb{H}(D)\}$.

Proof. First we prove the first isomorphism. By Lemma 4.1, the inclusion $G_\infty^+ \hookrightarrow G_\infty$ induces an injection

$$\mathbb{G}(F)^+\backslash(G_\infty^+ \cdot \mathbb{G}(\mathbb{A}_{F,f}))/Z_\infty^+ K_\infty^+ K_f \hookrightarrow \mathbb{G}(F)^+\backslash(G_\infty \cdot \mathbb{G}(\mathbb{A}_{F,f}))/Z_\infty K_\infty K_f.$$

Since the map $\det\colon K_\infty \to \{\pm 1\}^{\mathcal{V}_\mathbb{R}(D)}$ is surjective, this injection is also a surjection.

Next, the same inclusion induces an injection

$$\mathbb{G}(F)^+\backslash(G_\infty^+ \cdot \mathbb{G}(\mathbb{A}_{F,f}))/Z_\infty^+ K_\infty^+ K_f \hookrightarrow \mathbb{G}(F)\backslash(G_\infty \cdot \mathbb{G}(\mathbb{A}_{F,f}))/Z_\infty^+ K_\infty^+ K_f.$$

By weak approximation, the map $\mathrm{sign}\,\mathrm{nrd}\colon \mathbb{G}(F) \to \{\pm 1\}^{\mathcal{V}_\mathbb{R}(D)}$ is also surjective, so this injection, too, is a surjection, and the first isomorphism is proven.

Next, we show that $C = F_+^\times \backslash \mathbb{A}_{F,f}^\times / \operatorname{nrd}(K_f)$. Indeed, by the Hasse–Schilling–Maass Theorem [57, Theorem 33.15], the image of $\mathbb{G}(F)^+$ under the reduced norm map is precisely $F_+^\times$.

The last isomorphism follows from the fact that the map $F^\times \to \{\pm 1\}^{\mathcal{V}_\mathbb{R}(D) \cup \mathcal{V}_\mathbb{H}(D)}$ is surjective. $\qquad\square$

Applying Proposition 4.2, we have a map

$$\nu \colon W \to C = F^\times \backslash (\{\pm 1\}^{\mathcal{V}_\mathbb{R}(D) \cup \mathcal{V}_\mathbb{H}(D)} \times \mathbb{A}_{F,f}^\times) / \operatorname{nrd}(K_f)$$

$$w = (w_\infty, w_f) \mapsto F^\times (\operatorname{sign} \operatorname{nrd}(w_\infty), \operatorname{nrd}(w_f))^{-1} \operatorname{nrd}(K_f),$$

and we define, as in Section 3, the quotient $C_{\mathrm{iso}} = C/\nu(W)$. We have a canonical isomorphism

$$(4.3) \qquad C_{\mathrm{iso}} \cong F^\times \backslash (\{\pm 1\}^{\mathcal{V}_\mathbb{H}(D)}) \times \mathbb{A}_{F,f}^\times / \operatorname{nrd}(\mathrm{N}(K_f)),$$

and since $\mathrm{N}(K_f)$ contains $\mathbb{A}_{F,f}^\times K_f$, the group C_{iso} is a quotient of

$$\operatorname{nrd}(\mathbb{G}(F)^+) \backslash \operatorname{nrd}(\mathbb{G}(\mathbb{A}_{F,f})) / \operatorname{nrd}(K_\infty K(\mathfrak{N}) \mathbb{A}_{F,f}^\times) = \mathrm{Cl}_F(\mathfrak{N}\mathcal{V}_\mathbb{H}(D)) / (\mathrm{Cl}_F(\mathfrak{N}\mathcal{V}_\mathbb{H}(D)))^d.$$

Elements of W act via isometries on $\mathcal{Y} = \mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}_F) / Z_\infty^+ K_\infty^+ K_f$ by right multiplication.

The algebra $\mathbb{T}$ is commutative. It is generated by double cosets

$$T_\mathcal{D} = K_f \operatorname{diag}(\mathcal{D}) K_f,$$

where $\mathcal{D} = (\mathfrak{a}_1, \dots, \mathfrak{a}_d) \in \left(\mathbb{A}_{F,f}^\times / \prod_\mathfrak{p} \mathbb{Z}_\mathfrak{p}^\times \right)^d$ is such that for every $\mathfrak{p} | \delta_D \mathfrak{N}$ and for every $\mathfrak{a}_i$ one has $\operatorname{ord}_\mathfrak{p}(\mathfrak{a}_i) = 0$, i.e. $\mathcal{D}$ is a d-tuple of non-zero fractional ideals of F that are coprime to $\delta_D \mathfrak{N}$. The algebra $\mathbb{T}$ is a subalgebra of $\mathbb{Z}[K_f \backslash \mathbb{G}(\mathbb{A}_{F,f}) / K_f]$.

There is an involution on $\mathbb{Z}[K_f \backslash \mathbb{G}(\mathbb{A}_{F,f}) / K_f]$ – see [65, Ch. 1, §8.6(d)] – given by inversion, which induces an involution ι on $\mathbb{T}$. Explicitly, it is given by

$$\iota \colon \mathbb{T} \to \mathbb{T},$$

$$T_\mathcal{D} \mapsto T_{\mathcal{D}^{-1}}.$$

This involution is extended to $\mathbb{T}[W]$ by defining it to be inversion on W.

The algebra $\mathbb{T}[W]$ is graded by C, the grading being defined on $\mathbb{T}$ by the condition that $T_\mathcal{D}$ for $\mathcal{D}$ as above belongs to the homogeneous component of the image of

$$[\mathcal{D}]^{-1} = \prod_{i=1}^d [\mathfrak{a}_i]^{-1} \in \mathrm{Cl}_F(\mathfrak{N}\infty)$$

in the quotient C, and being induced on W by the homomorphism ν.

Definition 4.4. Let L be a field. A *Hecke eigensystem over L* is an L-algebra homomorphism $a = (a_\mathcal{D})_\mathcal{D} \colon L \otimes \mathbb{T} \to L$, $1 \otimes T_\mathcal{D} \mapsto a_\mathcal{D}$. Let M an $(L \otimes \mathbb{T})$-module. The *multiplicity in M* of a Hecke eigensystem $(a_\mathcal{D})$ is the L-dimension of the space of all elements $f \in M$ satisfying

$$T_\mathcal{D} f = a_\mathcal{D} f$$

for all $\mathcal{D} \in \left(\mathbb{A}_{F,f}^\times / \prod_\mathfrak{p} \mathbb{Z}_\mathfrak{p}^\times \right)^d$. A *Hecke eigensystem in M* is a Hecke eigensystem that has multiplicity at least 1 in M.

Given a Hecke eigensystem $(a_\mathcal{D})_\mathcal{D}$ in M and $\chi \in \widehat{C}$, we say that $(a_\mathcal{D})_\mathcal{D}$ *admits a self-twist* by χ if for all $\mathcal{D} \in \left(\mathbb{A}_{F,f}^\times / \prod_\mathfrak{p} \mathbb{Z}_\mathfrak{p}^\times \right)^d$ satisfying $\chi([\mathcal{D}]) \neq 1$ one has $a_\mathcal{D} = 0$. In the case when R contains all n-th roots of unity, where n is the order of χ, we may view χ as taking values in $R^\times$, and then the condition of admitting a self-twist is equivalent to the condition that for all $\mathcal{D}$ one has $\chi([\mathcal{D}]) \cdot a_\mathcal{D} = a_\mathcal{D}$.

The following two lemmas will link the self-twist condition with Proposition 3.6.

Lemma 4.5. *Let R be a domain, let $\mathbf{T} = R \otimes \mathbb{T}$, let M be a $\mathbf{T}$-module that is finitely generated over R, let $\mathfrak{p} \in \mathrm{MaxSpec}(R)$, and let E be a simple $\mathbf{T}$-submodule of $R/\mathfrak{p} \otimes_R M$. Then there exists a finite field extension S of $R/\mathfrak{p}$ and a Hecke eigensystem $(a_{\mathcal{D}})_{\mathcal{D}}$ over S in $S \otimes_R M$ such that whenever $T_{\mathcal{D}} \in \mathbb{T}$ annihilates E, one has $a_{\mathcal{D}} = 0$.*

Proof. Let $\mathfrak{m}$ be the annihilator of E in $R/\mathfrak{p} \otimes \mathbf{T}$, and let $S = (R/\mathfrak{p} \otimes \mathbf{T})/\mathfrak{m}$ be the residue field of $\mathfrak{m}$. Since M is finitely generated over R, the field S is a finite extension of $R/\mathfrak{p}$. The module $S \otimes_{R/\mathfrak{p}} E$ is a 1-dimensional S-vector subspace of $S \otimes_R M$. For every $\mathcal{D}$, define $a_{\mathcal{D}}$ to be the image of $T_{\mathcal{D}}$ in S. Then $(a_{\mathcal{D}})_{\mathcal{D}}$ is a Hecke eigensystem with the claimed property. $\square$

Lemma 4.6. *Let M be a graded $\mathbb{T}[W]$-module, and let $c \in C$. Suppose that there exists a field L, a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$, and a Hecke eigensystem over L in $L \otimes M$ admitting a self-twist by χ. Then no element of $\mathbb{T}[W]_c$ acts invertibly on M.*

Proof. Let $f \in L \otimes M$ be an eigenvector whose eigensystem has a self-twist by χ. We will show that every $T \in \mathbb{T}[W]_c$ annihilates f. First, let $T = tw \in \mathbb{T}[W]_c$, where $w \in W$ and $t \in \mathbb{T}_{c\nu(w)^{-1}}$. Since $\chi(c\nu(w)^{-1}) = \chi(c) \neq 1$, we have $tf = 0$, and therefore $Tf = wtf = 0$. Since $\mathbb{T}[W]_c$ is spanned by elements T of this form, the proof of the claim is complete. Finally, if an element of $\mathbb{T}[W]_c$ acted invertibly on M, it would also act invertibly on $L \otimes M$; this proves the lemma. $\square$

Fix an integer $i \in \{0, \ldots, d\}$. As in the introduction, let Δ denote the Laplace operator. If Y is either $\mathcal{Y}$ or Y_c for some $c \in C$, and $\lambda \in \mathbb{R}$, we will consider the following R-modules $\mathcal{F}(Y)$ attached to Y for suitable rings R:

- $\mathcal{F}(Y) = \Omega^i_{\Delta=\lambda}(Y)$, the space of real differential i-forms on Y on which Δ acts by multiplication by λ, with $R = \mathbb{R}$,
- $\mathcal{F}(Y) = \mathcal{H}^i(Y) = \Omega^i_{\Delta=0}$, the space of real harmonic i-forms, with $R = \mathbb{R}$,
- the homology group $\mathcal{F}(Y) = H_i(Y, R)$, where R is any domain, e.g. $\mathbb{Z}$ or $\mathbb{Z}_p$.

For each of these collections $\mathcal{F}$ of R-modules one has a direct sum decomposition

$$\mathcal{F}(\mathcal{Y}) = \bigoplus_{c \in C} \mathcal{F}(Y_c),$$

which defines a C-grading on $\mathcal{F}(\mathcal{Y})$. We will apply the formalism of Section 3 with $\mathbf{T} = R \otimes \mathbb{T}$. The Hecke algebra $\mathbf{T}[W]$ naturally acts on $M = \mathcal{F}(\mathcal{Y})$, and the image $\mathbf{A}$ of $\mathbf{T}[W]$ in $\mathrm{End}(\mathcal{F}(\mathcal{Y}))$ is an R-algebra that is finitely generated as an R-module. The algebra $\mathbf{A}$ inherits the C-grading and the involution ι from $\mathbb{T}[W]$, and $\mathcal{F}(\mathcal{Y})$ is a graded $\mathbf{A}$-module in the sense of Section 3.3.

Theorem 4.7. *Let $c \in C$, and let $i \geq 0$ be an integer.*
(1) *Let $\lambda \in \mathbb{R}$. Then exactly one of the following two statements is true:*
   (i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem over $\mathbb{C}$ in $\mathbb{C} \otimes_{\mathbb{R}} \Omega^i_{\Delta=\lambda}(\mathcal{Y})$ admitting a self-twist by χ;*
   (ii) *there exists $T \in \mathbb{T}[W]_c$ inducing, for all $b \in C$, an isomorphism of $\mathbb{T}_1$-modules*
$$T \colon \Omega^i_{\Delta=\lambda}(Y_b) \to \Omega^i_{\Delta=\lambda}(Y_{cb}).$$
(2) *At least one of the following two statements is true:*
   (i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem over $\mathbb{C}$ in the module $\mathbb{C} \otimes_{\mathbb{R}} \bigoplus_{\lambda \in \mathbb{R}} \Omega^i_{\Delta=\lambda}(\mathcal{Y})$ admitting a self-twist by χ;*
   (ii) *for all $b \in C$, the manifolds Y_b and Y_{cb} are i-isospectral.*
(3) *Let p be a prime number. Then exactly one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$ and a Hecke eigensystem over $\overline{\mathbb{F}}_p$ in $\overline{\mathbb{F}}_p \otimes H_i(\mathcal{Y}, \mathbb{Z})$ admitting a self-twist by χ;*

(ii) *there exists $T \in \mathbb{T}[W]_c$ inducing, for all $b \in C$, an isomorphism of $\mathbb{T}_1$-modules*

$$T \colon H_i(Y_b, \mathbb{Z}_{(p)}) \to H_i(Y_{cb}, \mathbb{Z}_{(p)}).$$

(4) *Let $\mathfrak{n}$ be a maximal ideal of $\mathbb{T}_1$, let p be the characteristic of $\mathbb{T}_1/\mathfrak{n}$. Then exactly one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$ and a Hecke eigensystem over $\overline{\mathbb{F}}_p$ in $\overline{\mathbb{F}}_p \otimes H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}}$ admitting a self-twist by χ;*

(ii) *there exists $T \in \mathbb{T}[W]_c$ inducing, for all $b \in C$, an isomorphism of $\mathbb{T}_1$-modules*

$$T \colon H_i(Y_b, \mathbb{Z})_{\mathfrak{n}} \to H_i(Y_{cb}, \mathbb{Z})_{\mathfrak{n}}.$$

Proof.

(1) Take $R = \mathbb{R}$. Suppose that there are no χ and Hecke eigensystem as in the statement. By Lemma 4.5, the graded **A**-module $M = \Omega^i_{\Delta = \lambda}(\mathcal{Y})$ satisfies the hypotheses of Proposition 3.6, so there exists $T' \in \mathbf{A}_c$ that acts invertibly on M. The image of $\mathbb{T}[W]_c$ in **A** generates the real vector space $\mathbf{A}_c$, and being an isomorphism is a non-empty Zariski-open condition, so there also exists $T \in \mathbb{T}[W]_c$ realising an isomorphism as claimed. The converse follows from Lemma 4.6.

(2) This follows immediately from (1).

(3) Take $R = \mathbb{Z}_{(p)}$. Suppose, once again, that there are no χ and Hecke eigensystem as in the statement. By Lemma 4.5, the graded **A**-module $M = H_i(\mathcal{Y}, \mathbb{Z}_{(p)})$ satisfies the hypotheses of Proposition 3.6, so there exists $T' \in \mathbf{A}_c$ that acts invertibly on M. Clearing denominators and lifting to $\mathbb{T}[W]$, we obtain a $T \in \mathbb{T}[W]_c$ as required. The converse follows from Lemma 4.6.

(4) The proof is identical to that of the previous part, with $R = \mathbb{Z}_p$ and $M = H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{n}}$. $\square$

Remark 4.8. Notice that in part (2) there is no claim that the isospectrality is realised by a Hecke operator, and we do not get an equivalence but only one implication.

The group G_∞ acts by unitary operators on the complex Hilbert space

$$\mathrm{L}^2(\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f)$$

of square-integrable functions $\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f \to \mathbb{C}$, and also, for each $c \in C$, on the analogously defined space $\mathrm{L}^2(\Gamma_c \backslash G_\infty/Z_\infty)$.

Definition 4.9. Let $c, c' \in C$. We say that the groups Γ_c and $\Gamma_{c'}$ are *representation equivalent* if there is an isomorphism of unitary G_∞-representations

$$\mathrm{L}^2(\Gamma_c \backslash G_\infty/Z_\infty) \cong \mathrm{L}^2(\Gamma_{c'} \backslash G_\infty/Z_\infty).$$

The Hecke algebra $\mathbb{T}[W]$ naturally acts on $\mathrm{L}^2(\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f)$, making the L^2-space a C-graded $\mathbb{T}[W]$-module.

Theorem 4.10. *Let $c \in C$. Then at least one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$ and a Hecke eigensystem over $\mathbb{C}$ in the module $\mathrm{L}^2(\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f)$ admitting a self-twist by χ;*

(ii) *for all $b \in C$ the groups Γ_b and Γ_{cb} are representation equivalent.*

Proof. Suppose that there are no χ and Hecke eigensystem as in the statement. Fix $b \in C$. By [25] (see also [28]), there are decompositions into Hilbert orthogonal direct sums of isotypical unitary representations of G_∞ with finite multiplicities

$$\mathrm{L}^2(\mathbb{G}(F)^+\backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f) \cong \widehat{\bigoplus}_\Pi V(\Pi),$$

and, for all $c' \in C$,

$$\mathrm{L}^2(\Gamma_{c'}\backslash G_\infty/Z_\infty) \cong \widehat{\bigoplus}_\Pi V(\Pi)_{c'},$$

both sums indexed by pair-wise non-isomorphic irreducible unitary representations Π of $\mathbb{G}(\mathbb{A}_F)$. The action of $\mathrm{N}(K_f)/K_f \times Z_\infty$ commutes with that of G_∞, and K_∞ is contained in G_∞, so each $V(\Pi)$ is also a $\mathbb{T}[W]$-module, and is graded, with c'-component $V(\Pi)_{c'}$.

Let Π be an irreducible unitary representation of G_∞. We claim that Π occurs with the same multiplicities in $V(\Pi)_b$ and in $V(\Pi)_{cb}$. We have a Hilbert direct sum decomposition of isotypical unitary finite-dimensional representations of K_∞

$$V(\Pi)|_{K_\infty} \cong \widehat{\bigoplus}_\sigma V(\Pi)(\sigma),$$

with the sum running over pair-wise non-isomorphic irreducible unitary representations σ of K_∞, again compatible with the grading, and where each summand is preserved by $\mathbb{T}[W]$. Let σ be an irreducible unitary representation of K_∞ such that $V(\Pi)(\sigma) \neq 0$, so that it is enough to prove that $\dim V(\Pi)(\sigma)_b = \dim V(\Pi)(\sigma)_{cb}$. Let $R = \mathbb{C}$ and let $\mathbf{A}$ be the image of $\mathbb{C} \otimes \mathbb{T}[W]$ in the endomorphism algebra of the finite-dimensional $\mathbb{C}$-vector space $V(\Pi)(\sigma)$, so that $\mathbf{A}$ is a finite-dimensional $\mathbb{C}$-algebra. By Lemma 4.5 the graded $\mathbf{A}$-module $M = V(\Pi)(\sigma)$ satisfies the hypotheses of Proposition 3.6, so there exists $T \in \mathbf{A}_c$ that acts invertibly on M, therefore realising an isomorphism

$$T\colon V(\Pi)(\sigma)_b \to V(\Pi)(\sigma)_{cb}.$$

This proves the desired equality of dimensions, and therefore the desired equality of multiplicities for Π. Since Π was arbitrary, we obtain an isomorphism as claimed. $\square$

For the rest of the section, fix a degree $i \in \mathbb{Z}_{\geq 0}$ and a prime number p, and let $\mathbf{A}$ be the image of $\mathbb{T}[W]$ in $\mathrm{End}_\mathbb{Z} H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$, so that $H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$ is a graded $\mathbf{A}$-module. We will apply the results of Section 3 with $R = \mathbb{Z}$ and $\mathbf{T} = \mathbb{T}$. For every $c \in C$ there is a canonical positive definite pairing on $\mathcal{H}^i(Y_c)$ [4, Notation 3.2 and Lemma 3.3], [58]. By the Hodge and de Rham theorems, this harmonic forms pairing induces a non-degenerate $\mathbb{R}$-valued pairing on $H_i(Y_c, \mathbb{R})$. The *i-th regulator* $\mathrm{Reg}_i(Y_c)$ *of* Y_c is defined to be the covolume of the lattice $H_i(Y_c, \mathbb{Z})_{\mathrm{free}}$ in $H_i(Y_c, \mathbb{R})$.

Definition 4.11. Define $\mathcal{N} = \{\mathfrak{n} \cap \iota(\mathfrak{n}) : \mathfrak{n} \in \mathrm{MaxSpec}(\mathbb{Z}_p \otimes \mathbb{T}_1)\}$.

Lemma 4.12. *Define a pairing on $M = H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$ induced by the harmonic forms pairing on the summands $H_i(Y_c, \mathbb{Z})_{\mathrm{free}}$ for $c \in C$ and by making the distinct summands pairwise orthogonal. Then this defines a polarisation on M in the sense of Definition 3.10. Moreover, for every $\mathfrak{n}' \in \mathcal{N}$ the graded $\mathbb{Z}_p \otimes \mathbf{A}$-module $M_{\mathfrak{n}'}$ is polarisable.*

Proof. The adjoint of each $T \in \mathbb{T}$ with respect to the harmonic forms pairing is $\iota(T)$, and W acts by isometries (see also Remark 3.11), so the pairing defines a polarisation on the graded module $\mathcal{H}^i(\mathcal{Y})$. Moreover, the Hodge–de Rham isomorphism between $\mathcal{H}^i(\mathcal{Y})$ and $H_i(\mathcal{Y}, \mathbb{R})$ is $\mathbf{A}$-equivariant, so the harmonic forms pairing is an $\mathbb{R}$-valued polarisation on the $\mathbf{A}$-module M. The last assertion follows from Proposition 3.27. $\square$

Lemma 4.13. *Let c, $c' \in C$, let $M = H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$, and suppose that M_c and $M_{c'}$ are linked in the sense of Definition 3.1. Then:*

(1) *we have*
$$\mathcal{C}_{c,c'}(M) = \frac{\mathrm{Reg}_i(Y_c)^2}{\mathrm{Reg}_i(Y_{c'})^2},$$
where recall that the invariant $\mathcal{C}_{c,c'}(M)$ was defined in Definition 3.18;

(2) *we have*
$$\frac{\mathrm{Reg}_i(Y_c)^2}{\mathrm{Reg}_i(Y_{c'})^2} \in \mathbb{Q}^{\times};$$

(3) *for all $\mathfrak{n}' \in \mathcal{N}$, the homogeneous components $(M_{\mathfrak{n}'})_c$ and $(M_{\mathfrak{n}'})_{c'}$ are linked, and we have*
$$\mathcal{C}_{c,c'}(M) \equiv \prod_{\mathfrak{n}' \in \mathcal{N}} \mathcal{C}_{c,c'}(M_{\mathfrak{n}'}) \mod (\mathbb{Z}_p^{\times})^2,$$
where $\mathcal{C}_{c,c'}(M) \in \mathbb{Q}^{\times}$ is viewed as an element of $\mathbb{Q}_p^{\times}$.

Proof. (1) By definition, $\mathrm{Reg}_i(Y_c)^2$ is the determinant of the Gram matrix of the harmonic forms pairing with respect to any $\mathbb{Z}$-basis of $H_i(Y_c, \mathbb{Z})_{\mathrm{free}}$, and similarly for $Y_{c'}$. The claim therefore follows from Lemma 4.12 and Proposition 3.19.

(2) The assertion follows from combining part (1) and Corollary 3.21.

(3) The assertion follows from Lemma 3.23, Lemma 4.12, and Proposition 3.27. $\square$

Theorem 4.14. *Let $c \in C$. Then exactly one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem over $\mathbb{C}$ in $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{H}^i(\mathcal{Y})$ admitting a self-twist by χ;*

(ii) *for all $b \in C$ the homogeneous components $H_i(Y_b, \mathbb{Z})_{\mathrm{free}}$ and $H_i(Y_{cb}, \mathbb{Z})_{\mathrm{free}}$ are linked.*

Proof. Suppose that there are no χ and Hecke eigensystem as in the statement. The isomorphism between $\mathcal{H}^i(\mathcal{Y})$ and $H^i(\mathcal{Y}, \mathbb{R})$ is $\mathbf{A}$-equivariant. Lemma 4.5, applied to $M = H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$, and Corollary 3.7 imply that M_c and $M_{c'}$ are linked. The converse follows from Lemma 4.6. $\square$

Theorem 4.15. *Let $c \in C$, and let p be a prime number, let $M = H_i(\mathcal{Y}, \mathbb{Z})_{\mathrm{free}}$, and let $\mathcal{N}$ be as in Definition 4.11.*

(1) *Let $\mathfrak{n}' \in \mathcal{N}$. Then at least one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem over $\overline{\mathbb{F}}_p$ in $\overline{\mathbb{F}}_p \otimes (M/\mathfrak{n}'M)$ admitting a self-twist by χ;*

(ii) *for all $b \in C$, the homogeneous components $(M_{\mathfrak{n}'})_b$ and $(M_{\mathfrak{n}'})_{cb}$ are linked, and we have*
$$\mathcal{C}_{b,cb}(M_{\mathfrak{n}'}) \in \mathbb{Z}_p^{\times};$$

(2) *At least one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem over $\overline{\mathbb{F}}_p$ in $\overline{\mathbb{F}}_p \otimes M$ admitting a self-twist by χ;*

(ii) *for all $b \in C$, the homogeneous components M_b and M_{cb} are linked, and we have*
$$\frac{\mathrm{Reg}_i(Y_b)^2}{\mathrm{Reg}_i(Y_{cb})^2} \in \mathbb{Z}_{(p)}^{\times}.$$

Proof. (1) Suppose that there are no χ and Hecke eigensystem in $\overline{\mathbb{F}}_p \otimes (M/\mathfrak{n}'M)$ as in the statement. By Lemma 4.5 the assumptions of Proposition 3.6 are satisfied for $M_{\mathfrak{n}'}$, so there exists $T \in \mathbf{A}_c$ that acts invertibly on $M_{\mathfrak{n}'}$. The result follows from Lemma 3.22.

(2) Suppose that there are no χ and Hecke eigensystem in $\overline{\mathbb{F}}_p \otimes M$ as in the statement, and let $b \in C$. By the same argument as in part (1), the homogeneous components M_b and M_{cb} are linked. The claimed equality then follows by combining Lemma 4.13 and part (1) of the present theorem. $\square$

5. SELF-TWISTS IN CHARACTERISTIC 0 AND AUTOMORPHIC INDUCTION

Assumption 5.1. We keep the notation of Section 4, but we assume that $d = 2$, i.e. that D is a quaternion algebra. In particular we have $Z_\infty = Z_\infty^+$.

In this section we prove Theorem D for representation equivalence, $\Omega^\bullet$-, Ω^0-, and $\mathcal{H}^\bullet$-isospectrality, Theorem G parts (1) and (2), and Theorem C. To prove Theorem D we will use the results of Section 4 and make the self-twist conditions in those results explicit. To that end, we will relate the various $\mathbb{T}$-modules to automorphic representations, directly or using Matsushima's formula. By an automorphic induction theorem of Langlands, the self-twist condition yields the existence of Hecke characters of quadratic extensions of F with specific properties. To make these properties explicit, we inspect closely irreducible representations of $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$ and how they interact with automorphic induction, Jacquet–Langlands transfer, and Matsushima's formula. Theorem G parts (1) and (2) constitute a refinement of Theorem D for $\mathcal{H}^\bullet$-isospectrality. To deduce Theorem C, we count the Hecke characters that contribute to the various Casimir eigenspaces. This is achieved by proving local bounds, Lemmas 5.2 and 5.5, and then counting points in lattices.

The content of Sections 5.1–5.3 is well-known.

- In Section 5.1 we fix our notation and recall basic facts about Hecke characters.
- In Section 5.2.1 we recall the classification of irreducible representations of $GL_2(E)$ for an Archimedean local field E, including their K-type, Casimir eigenvalue, and central character, and how they arise from automorphic induction and Jacquet–Langlands transfer.
- In Section 5.2.2 we recall the classification of smooth irreducible representations of $GL_2(E)$ for a non-Archimedean local field E, and we state which ones arise as automorphic inductions, as well as some properties of conductors.
- In Section 5.3 we recall the results on automorphic representations of GL_2 that we need, namely strong multiplicity 1, automorphic induction, Jacquet–Langlands correspondence, and Matsushima's formula.

General references for these facts are [26, Appendix B] [27], [33].

In Section 5.4 we prove Theorem D for the isospectralities listed above and Theorem G parts (1) and (2). That section is divided into subsections, one for each type of isospectrality. Finally, in Section 5.5 we prove Theorem C.

Throughout Section 5 all group representations will be on complex Hilbert spaces.

5.1. **Automorphic representations of** GL_1**.** In this section, we fix some notation for Hecke characters. For a general reference, see e.g. [38, Ch. XIV].

Let G be a locally compact group. Recall that a *quasi-character* of G is a continuous homomorphism $G \to \mathbb{C}^\times$, and a *character* of G is a quasi-character that is unitary.

Every quasi-character of $\mathbb{R}^\times$ is of the form

$$\Psi_\mathbb{R}(k, s) \colon x \mapsto \operatorname{sign}(x)^k |x|^s$$

for a unique pair $(k, s) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}$.

Every quasi-character of $\mathbb{C}^\times$ is of the form

$$\Psi_{\mathbb{C}}(k,s)\colon z \mapsto \left(\frac{z}{|z|}\right)^k |z|^{2s}$$

for a unique pair $(k,s) \in \mathbb{Z} \times \mathbb{C}$.

In both cases, such a quasi-character is unitary if and only if one has $s \in i\mathbb{R}$.

If E is a non-Achimedean local field, $\mathfrak{p}$ is the maximal ideal of the ring of integers $\mathbb{Z}_E$ of E, and Ψ is a quasi-character of $E^\times$, then there exists $m \in \mathbb{Z}_{\geq 0}$ such that Ψ is trivial on $(1 + \mathfrak{p}^m) \cap \mathbb{Z}_E^\times$. The *conductor* of Ψ is $\mathfrak{p}^m$, where m is the smallest such integer.

A *Hecke quasi-character* of F is a quasi-character of $F^\times \backslash \mathbb{A}_F^\times$, and a *Hecke character* of F is a character of $F^\times \backslash \mathbb{A}_F^\times$.

Every Hecke quasi-character Ψ of F is of the form

$$\Psi = \prod_v \Psi_v$$

for quasi-characters Ψ_v of $F_v^\times$, where v runs over the places of F. For every infinite place v, we define k_v and s_v by writing $\Psi_v = \Psi_{F_v}(k_v, s_v)$. Note that this definition depends on identifications between F_v and $\mathbb{C}$ for every complex place v. Replacing a chosen identification with its complex conjugate does not change s_v but negates k_v. Everything that we write in this section is insensitive to this ambiguity, as long as we impose the following convention: whenever L/F is a quadratic extension of number fields, v is a complex place of F, and w, w' are places of L extending v, we always choose isomorphisms $L_w \cong \mathbb{C}$ and $L_{w'} \cong \mathbb{C}$ that extend *the same* arbitrarily chosen isomorphism $F_v \cong \mathbb{C}$.

A Hecke quasi-character Ψ is called *algebraic* if for every real place v we have $s_v \in \mathbb{Z}$ and for every complex place v we have $s_v + k_v/2 \in \mathbb{Z}$. If Ψ is an algebraic Hecke character, then for every embedding $\tau\colon F \to \mathbb{C}$ there exists a uniquely determined $q_\tau \in \mathbb{Z}$ such that for all $\alpha \in F^\times$ that are positive at all real places we have

$$\prod_{v \mid \infty} \Psi_v(\alpha) = \prod_{\tau\colon F \to \mathbb{C}} \tau(\alpha)^{q_\tau},$$

and the products run over all infinite places v, respectively embeddings $\tau\colon F \to \mathbb{C}$. We refer to the collection $(q_\tau)_{\tau \in \mathrm{Hom}(F,\mathbb{C})}$ as the *type* of an algebraic Hecke character.

5.2. Representations of GL_2 over local fields.

Throughout this subsection, let E be a local field. We will fix the required notation and briefly recall the relevant facts for irreducible representations of $\mathrm{GL}_2(E)$. General references for this subsection are [26, Appendix B] and [33, §5-6, 14-15].

Let L be a quadratic étale E-algebra, i.e. either a quadratic field extension of E or a direct product $E \times E$. There is an *automorphic induction* functor AI_E^L from the category of irreducible representations of $\mathrm{GL}_1(L)$ to that of irreducible representations of $\mathrm{GL}_2(E)$. If σ is the non-trivial E-linear automorphism of L, then for all characters Ψ of $L^\times$ we have $\mathrm{AI}_E^L(\Psi^\sigma) = \mathrm{AI}_E^L(\Psi)$.

Let A be a quaternion algebra over E. The local Jacquet–Langlands correspondence [33] attaches to every irreducible representation Π of $A^\times$ a representation $\mathrm{JL}_A(\Pi)$ of $\mathrm{GL}_2(E)$, well-defined up to isomorphism, sometimes referred to as the *Jacquet–Langlands transfer* of Π from $A^\times$ to GL_2. If A is split, then $\mathrm{JL}_A(\Pi)$ is isomorphic to Π. We will say more about the properties of $\mathrm{JL}_A(\Pi)$ in some special cases below.

5.2.1. *Archimedean fields.* In this subsection, assume that E is Archimedean.

Given an irreducible representation Π of $\mathrm{GL}_2(E)$ on a vector space V, its *central character* is the group homomorphism $\zeta_\Pi \colon E^\times \to \mathbb{C}^\times$ characterised by the property that for all $x \in E^\times \subset \mathrm{GL}_2(E)$ and all $v \in V$ one has $\Pi(x)(v) = \zeta_\Pi(x)v$.

Let K be a compact subgroup of $\mathrm{GL}_2(E)$. The restriction of every representation of $\mathrm{GL}_2(E)$ to K decomposes as a Hilbert direct sum of irreducible representations. Fix a full set of representatives u of isomorphism classes of irreducible representations of K. The K-*type* of a representation Π of $\mathrm{GL}_2(E)$ is the multiset of representatives u with multiplicities $\dim \mathrm{Hom}_K(u, \Pi)$.

Given an irreducible representation Π of $\mathrm{GL}_2(E)$ or of $\mathbb{H}^\times$, the *Casimir operator*, which we take to be the negation of [9, Chapters 1 and 2] (see also [10, Theorem 2.2.1] in the case of $\mathrm{GL}_2(\mathbb{R})$), acts as a scalar on a dense subspace of the underlying vector space. We call this scalar the *Casimir eigenvalue* of Π. The definition of the Casimir operator depends on a choice of scalar multiple of the Killing form, and we choose the normalisation for which, in the case of $\mathrm{GL}_2(E)$, the quotient by a maximal compact subgroup has constant curvature -1, and in the case of $\mathbb{H}^\times$, the form agrees with that for $\mathrm{GL}_2(\mathbb{R})$ after complexification.

Isomorphism classes of irreducible representations of $\mathrm{SO}_2(\mathbb{R})$ have dimension 1 and are parametrised by integers, with $k \in \mathbb{Z}$ corresponding to

$$r(k) \colon \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{ik\theta}.$$

Isomorphism classes of irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ are parametrised by non-negative integers, with $k \in \mathbb{Z}_{\geq 0}$ corresponding to the representation

$$s(k) = \mathrm{Sym}^k \mathbb{C}^2,$$

which has dimension $k+1$, where $\mathbb{C}^2$ is equipped with the standard action of $\mathrm{SU}_2(\mathbb{C})$.

Isomorphism classes of irreducible representations of $\mathbb{H}^\times$ are parametrised by elements of $\mathbb{Z}_{\geq 2} \times \mathbb{C}$, with $(k, \mu) \in \mathbb{Z}_{\geq 2} \times \mathbb{C}$ corresponding to the representation

$$t(k, \mu) = \mathrm{Sym}^{k-2} \mathbb{C}^2 \otimes_{\mathbb{C}} \mathrm{nrd}^{\mu/2},$$

which has dimension $k - 1$, where $\mathbb{C}^2$ is equipped with the action of $\mathbb{H}^\times$ induced by the map $\mathbb{H}^\times \to \mathrm{SL}_2(\mathbb{C})$ defined by $h \mapsto \mathrm{nrd}(h)^{-1/2} j(h)$ for some isomorphism $j \colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{M}_2(\mathbb{C})$. The eigenvalue of the Casimir operator on $t(k, \mu)$ is $\frac{k}{2}(1 - \frac{k}{2})$ and the central character of $t(k, \mu)$ is $\Psi_{\mathbb{R}}(k, \mu)$.

Every irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ is isomorphic to one from exactly one of the following families.

- Irreducible principal series $\mathrm{PS}_{\mathbb{R}}(\Psi_1, \Psi_2)$, where Ψ_1, Ψ_2 are quasi-characters $\Psi_i = \Psi_{\mathbb{R}}(k_i, s_i)$ such that $s = \frac{1}{2}(s_1 - s_2 + 1)$ and $k = k_1 - k_2$ satisfy $2s \notin k + 2\mathbb{Z}$.
- Discrete series and limits of discrete series $\mathrm{DS}(k, \mu)$, where $k \geq 1$ is an integer and $\mu \in \mathbb{C}$.
- Finite-dimensional representations $\mathrm{F}_{\mathbb{R}}(k, \Psi) = \mathrm{Sym}^{k-2} \mathbb{C}^2 \otimes_{\mathbb{C}} (\Psi \circ \det)$, where $k \geq 2$ is an integer and Ψ is a quasi-character of $\mathbb{R}^\times$.

Representation	$\mathrm{SO}_2(\mathbb{R})$-type	Casimir eigenvalue	central character
$\mathrm{PS}_{\mathbb{R}}(\Psi_1, \Psi_2)$	$\{r(m) : m \in k + 2\mathbb{Z}\}$	$s(1 - s)$	$\Psi_1 \Psi_2$
$\mathrm{DS}(k, \mu)$	$\{r(m) : m \in k + 2\mathbb{Z}, |m| \geq k\}$	$\frac{k}{2}(1 - \frac{k}{2})$	$\Psi_{\mathbb{R}}(k, \mu)$
$\mathrm{F}_{\mathbb{R}}(k, \Psi)$	$\{r(m) : m \in k + 2\mathbb{Z}, |m| < k\}$	$\frac{k}{2}(1 - \frac{k}{2})$	$\Psi_{\mathbb{R}}(k, k - 2)\Psi^2$

Every irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ is isomorphic to one from exactly one of the following families.

- Irreducible principal series $\mathrm{PS}_{\mathbb{C}}(\Psi_1, \Psi_2)$, where Ψ_1, Ψ_2 are quasi-characters $\Psi_i = \Psi_{\mathbb{C}}(k_i, s_i)$ such that $s = s_1 - s_2$ and $k = k_1 - k_2$ satisfy $2s \notin k + 2\mathbb{Z}$ or $|s| < 1 + \frac{|k|}{2}$.

- Finite-dimensional representations $F_{\mathbb{C}}(k, k', \Psi) = \mathrm{Sym}^{k-2}\,\mathbb{C}^2 \otimes \mathrm{Sym}^{k'-2}\,\bar{\mathbb{C}}^2 \otimes (\Psi \circ \det)$, where $k, k' \geq 2$ are integers, Ψ is a quasi-character of $\mathbb{C}^\times$, and $\mathbb{C}^2$ denotes the standard representation of $\mathrm{GL}_2(\mathbb{C})$, while $\bar{\mathbb{C}}^2$ denotes its composition with complex conjugation.

Repre-sentation	$\mathrm{SU}_2(\mathbb{C})$-type	Casimir eigenvalue	central character
$\mathrm{PS}_{\mathbb{C}}(\Psi_1, \Psi_2)$	$\{s(m) : m \in k + 2\mathbb{Z}, m \geq |k|\}$	$1 - \frac{k^2}{4} - s^2$	$\Psi_1 \Psi_2$
$\mathrm{F}_{\mathbb{C}}(k, k', \Psi)$	$\{s(m) : m \in k + k' + 2\mathbb{Z},$ $|k - k'| \leq m \leq k + k' - 4\}$	$k(1 - \frac{k}{2}) + k'(1 - \frac{k'}{2})$	$\Psi_{\mathbb{C}}(k - k',$ $\frac{k+k'-4}{2})\Psi^2$

Lemma 5.2. *Let V be a finite-dimensional representation of K_∞. Then there exists $\kappa_\infty > 0$ such that for every irreducible representation Π_∞ of G_∞ one has*

$$\dim \mathrm{Hom}_{K_\infty}(V, \Pi_\infty) \leq \kappa_\infty.$$

Proof. The assertion follows by inspection of the above lists of the irreducible representations of $\mathrm{GL}_2(E)$ for $E = \mathbb{R}$ and $E = \mathbb{C}$. $\qquad\square$

The automorphic inductions AI_E^L from $\mathrm{GL}_1(L)$ to $\mathrm{GL}_2(E)$ are as follows.

- Extension $\mathbb{R} \subset \mathbb{C}$: $\mathrm{AI}_{\mathbb{R}}^{\mathbb{C}}(\Psi_{\mathbb{C}}(k, s)) = \mathrm{DS}(1 + |k|, 2s)$.
- Extension $\mathbb{R} \subset \mathbb{R} \times \mathbb{R}$: let $\Psi_i = \Psi_{\mathbb{R}}(k_i, s_i)$ for $i = 1, 2$ be ordered such that $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2)$. Then:
  - $\mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_2, \Psi_1) = \mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_1, \Psi_2) = \mathrm{PS}_{\mathbb{R}}(\Psi_1, \Psi_2)$ if $s_1 - s_2 + 1 \notin k_1 + k_2 + 2\mathbb{Z}$;
  - $\mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_2, \Psi_1) = \mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_1, \Psi_2) = \mathrm{F}_{\mathbb{R}}(k, \Psi_{\mathbb{R}}(k_2, s_2 + \frac{1}{2}))$, where $k = s_1 - s_2 + 1$, if $k \in k_1 + k_2 + 2\mathbb{Z}$ and $k \neq 1$;
  - $\mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_2, \Psi_1) = \mathrm{AI}_{\mathbb{R}}^{\mathbb{R} \times \mathbb{R}}(\Psi_1, \Psi_2) = \mathrm{DS}(1, 2s_1) = \mathrm{DS}(1, 2s_2)$ if $s_1 = s_2$ and $k_1 \not\equiv k_2 \mod 2$.
- Extension $\mathbb{C} \subset \mathbb{C} \times \mathbb{C}$: let $\Psi_i = \Psi_{\mathbb{C}}(k_i, s_i)$ for $i = 1, 2$ be ordered such that $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2)$, $s = s_1 - s_2$, and $k = k_1 - k_2$. Then:
  - $\mathrm{AI}_{\mathbb{C}}^{\mathbb{C} \times \mathbb{C}}(\Psi_2, \Psi_1) = \mathrm{AI}_{\mathbb{C}}^{\mathbb{C} \times \mathbb{C}}(\Psi_1, \Psi_2) = \mathrm{PS}_{\mathbb{C}}(\Psi_1, \Psi_2)$ if $2s \notin k + 2\mathbb{Z}$ or $|s| < \frac{|k|}{2} + 1$;
  - $\mathrm{AI}_{\mathbb{C}}^{\mathbb{C} \times \mathbb{C}}(\Psi_2, \Psi_1) = \mathrm{AI}_{\mathbb{C}}^{\mathbb{C} \times \mathbb{C}}(\Psi_1, \Psi_2) = \mathrm{F}_{\mathbb{C}}(s + \frac{k}{2} + 1, s - \frac{k}{2} + 1, \Psi_{\mathbb{C}}(k_2, s_2 + \frac{1}{2}))$ otherwise.

The Jacquet–Langlands transfer $\mathrm{JL}_{\mathbb{H}}$ from $\mathbb{H}^\times$ to $\mathrm{GL}_2(\mathbb{R})$ is

$$\mathrm{JL}_{\mathbb{H}}(t(k, \mu)) = \mathrm{DS}(k, \mu) \text{ for } (k, \mu) \in \mathbb{Z}_{\geq 2} \times \mathbb{C}.$$

We now record the isomorphism classes of some representations of various compact groups.

Let $\mathfrak{so}_2(\mathbb{R})$ and $\mathfrak{gl}_2(\mathbb{R})$ be the Lie algebras of $\mathrm{SO}_2(\mathbb{R})$, respectively of $\mathrm{GL}_2(\mathbb{R})$, both equipped with the adjoint action of $\mathrm{SO}_2(\mathbb{R})$. We have the following isomorphisms of $\mathrm{SO}_2(\mathbb{R})$-representations.

i	$\mathbb{C} \otimes_{\mathbb{R}} \Lambda^i(\mathfrak{gl}_2(\mathbb{R})/\mathfrak{so}_2(\mathbb{R}))$
0	$r(0)$
1	$r(-2) \oplus r(2)$
2	$r(0)$

Now, let $\mathfrak{su}_2(\mathbb{C})$ and $\mathfrak{gl}_2(\mathbb{C})$ be the Lie algebras of $\mathrm{SU}_2(\mathbb{C})$, respectively of $\mathrm{GL}_2(\mathbb{C})$, both equipped with the adjoint action of $\mathrm{SU}_2(\mathbb{C})$. We have the following isomorphisms of $\mathrm{SU}_2(\mathbb{C})$-representations.

i	$\mathbb{C} \otimes_{\mathbb{R}} \Lambda^i(\mathfrak{gl}_2(\mathbb{C})/\mathfrak{su}_2(\mathbb{C}))$
0	$s(0)$
1	$s(2)$
2	$s(2)$
3	$s(0)$

5.2.2. *Non-Archimedean fields.* An additional general reference for this subsection is [11], particularly Chapters 8 and 13.

Let E be a p-adic field, let $\mathfrak{p}$ be its maximal ideal. Let A be the unique quaternion division algebra over E. A representation of $A^\times$ or of $\mathrm{GL}_2(E)$ is called *smooth* if every vector in the underlying vector space is fixed by some compact open subgroup. We will assume that all our representations are smooth without repeating it.

Every irreducible representation of $A^\times$ is finite-dimensional. Every irreducible representation of $\mathrm{GL}_2(E)$ is of exactly one of the following types:

- finite-dimensional representations: they are all of the form $g \mapsto \phi \circ \det$ for a quasi-character $\phi\colon E^\times \to \mathbb{C}^\times$,
- irreducible principal series $\mathrm{PS}_E(\Psi_1, \Psi_2)$, where $\Psi_1, \Psi_2\colon E^\times \to \mathbb{C}^\times$ are quasi-characters such that $\Psi_1\Psi_2^{-1} \notin \{|\cdot|^{\pm 1}\}$,
- special representations,
- supercuspidal representations.

The automorphic induction AI_E^L from $\mathrm{GL}_1(L)$ to $\mathrm{GL}_2(E)$ satisfies the following.

- An automorphic induction is never a special representation.
- Extension $E \subset E \times E$: let $\Psi_1, \Psi_2\colon E^\times \to \mathbb{C}^\times$ be quasi-characters.
  - $\mathrm{AI}_E^{E \times E}(\Psi_1, \Psi_2) = \mathrm{PS}_E(\Psi_1, \Psi_2)$ if $\Psi_1\Psi_2^{-1} \notin \{|\cdot|^{\pm 1}\}$.
  - $\mathrm{AI}_E^{E \times E}(\Psi_1, \Psi_2) = \phi \circ \det$ for some quasi-character $\phi\colon E^\times \to \mathbb{C}^\times$ otherwise.
- Quadratic field extension $E \subset L$: let $\sigma \in \mathrm{Gal}(L/E)$ be the non-trivial element, let $\chi\colon \mathrm{Gal}(L/E) \to \mathbb{C}^\times$ be the non-trivial quadratic character, and let $\Psi\colon L^\times \to \mathbb{C}^\times$ be a quasi-character.
  - $\mathrm{AI}_E^L(\Psi)$ is supercuspidal if $\Psi \neq \Psi^\sigma$.
  - $\mathrm{AI}_E^L(\Psi) = \mathrm{PS}_E(\phi, \phi\chi)$ for some quasi-character $\phi\colon E^\times \to \mathbb{C}^\times$ otherwise.

For $r \in \mathbb{Z}_{\geq 0}$, let
$$K_0(\mathfrak{p}^r) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c \equiv 0 \bmod \mathfrak{p}^r \right\} \subset \mathrm{GL}_2(\mathbb{Z}_E)$$
and
$$K_1(\mathfrak{p}^r) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : d \equiv 1 \bmod \mathfrak{p}^r, c \equiv 0 \bmod \mathfrak{p}^r \right\} \subset K_0(\mathfrak{p}^r).$$
If Π is an irreducible representation of $\mathrm{GL}_2(E)$, then its *conductor* is $\mathfrak{p}^n$ where $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is minimal subject to the condition $\Pi^{K_1(\mathfrak{p}^n)} \neq 0$.

Lemma 5.3. *Let Π be an irreducible representation, let $\mathfrak{p}^n$ be its conductor, and let $i \in \mathbb{Z}_{\geq n}$. Then the space $\Pi^{K_1(\mathfrak{p}^i)}$ of fixed points has dimension $i - n + 1$.*

Proof. See [17], see also [59, Theorem 1.2.1 (ii)]. $\square$

In $A^\times$, we define $K_0(\mathfrak{p}^0)$ to be the unit group of the unique maximal order in $A^\times$. When we use the notation $K_0(\mathfrak{p}^0)$, it will be clear from the context whether it refers to the subgroup of $A^\times$ or to $\mathrm{GL}_2(\mathbb{Z}_E)$.

Lemma 5.4. *Let L/E be an étale quadratic extension, let δ be its discriminant, let $\Psi\colon L^\times \to \mathbb{C}^\times$ be a quasi-character such that $\mathrm{AI}_E^L(\Psi)$ is infinite-dimensional, and let $\mathfrak{F}$ be the conductor of Ψ. Then the conductor of $\mathrm{AI}_E^L(\Psi)$ is $\delta \cdot \mathrm{Nm}_{L/E}(\mathfrak{F})$.*

Proof. See [59, Table at the end of §1]. $\square$

The Jacquet–Langlands transfer JL_A from $A^\times$ to $\mathrm{GL}_2(E)$ has the following property: an irreducible representation Π of $\mathrm{GL}_2(E)$ is in the image of JL_A if and only if Π is special or supercuspidal.

Lemma 5.5. *There exists $\kappa_f > 0$ such that for every irreducible representation Π_f of $\mathbb{G}(\mathbb{A}_{F,f})$ we have*

$$\dim \Pi_f^{K_f} \leq \kappa_f.$$

Proof. Recall that we have $K(\mathfrak{N}) \subset K_f$, so that for every irreducible representation Π_f of $\mathbb{G}(\mathbb{A}_{F,f})$ we have $\dim \Pi_f^{K_f} \leq \dim \Pi_f^{K(\mathfrak{N})}$. Write $K(\mathfrak{N}) = \prod_{\mathfrak{p}} K(\mathfrak{p}^{m_{\mathfrak{p}}})$ and $\Pi_f = \otimes'_{\mathfrak{p}} \Pi_{\mathfrak{p}}$, where $\otimes'$ denotes a restricted tensor product. Note that for every prime $\mathfrak{p}$ we have $\dim \Pi_{\mathfrak{p}}^{K(\mathfrak{p}^0)} \leq 1$, and we have $\dim \Pi_f^{K(\mathfrak{N})} = \prod_{\mathfrak{p}} \dim \Pi_{\mathfrak{p}}^{K(\mathfrak{p}^{m_{\mathfrak{p}}})}$. Let $\mathfrak{p}$ be such that $m_{\mathfrak{p}} > 0$.

- If $\mathfrak{p}$ is split in D, then by Lemma 5.3 we have $\dim \Pi_{\mathfrak{p}}^{K(\mathfrak{p}^{m_{\mathfrak{p}}})} \leq \dim \Pi_{\mathfrak{p}}^{K_1(\mathfrak{p}^{2m_{\mathfrak{p}}})} \leq 2m_{\mathfrak{p}} + 1$, which depends only on K_f.
- If $\mathfrak{p}$ is ramified in D, then consider the finite group $G = D_{\mathfrak{p}}^{\times}/(F_{\mathfrak{p}}^{\times} K(\mathfrak{p}^{m_{\mathfrak{p}}}))$, and let $d(G)$ be the maximal dimension of an irreducible representation of G. Since $\Pi_{\mathfrak{p}}$ is irreducible, $F_{\mathfrak{p}}^{\times}$ acts via a character ζ on $\Pi_{\mathfrak{p}}$. Therefore $\Pi_{\mathfrak{p}}^{K(\mathfrak{p}^{m_{\mathfrak{p}}})}\zeta^{-1}$ is an irreducible representation of G, hence of dimension at most $d(G)$, which also depends only on K_f.

This proves the existence of κ_f. $\qquad\square$

5.3. Automorphic representations of GL_2.

For the duration of the subsection, let $\mathbf{G}$ be one of the algebraic groups $\mathbb{G}$ or GL_2 over F.

For a general definition of an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ and of cuspidality see [27, Definitions 6.3.5 and 6.5.1]. For our purposes, simpler definitions in special cases will be sufficient.

A *discrete automorphic representation* of $\mathbf{G}(\mathbb{A}_F)$ is an irreducible subrepresentation of $\mathrm{L}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)/\mathbb{R}_{>0})$, where $\mathbb{R}_{>0} \subset \mathrm{Z}(\mathbf{G}(F_{\mathbb{R}})) = \prod_{v|\infty} \mathrm{Z}(\mathbf{G}(F_v))$ is embedded diagonally. For the relation with automorphic representations in the usual sense see [27, Theorem 6.6.4]. Every automorphic representation of $\mathbb{G}(\mathbb{A}_F)$ is discrete and cuspidal, and every cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ can be identified with a discrete automorphic representation, see [27, Theorem 6.5.3].

Let Π be a discrete automorphic representation of $\mathbf{G}(\mathbb{A}_F)$. Then Π can be written as a restricted tensor product $\Pi_{\infty} \otimes \Pi_f = \bigotimes'_v \Pi_v$ over the places v of F, where Π_{∞} is a representation of $\mathbf{G}(F_{\mathbb{R}})$, Π_f is a representation of $\mathbf{G}(\mathbb{A}_{F,f})$, and for each place v, the factor Π_v is a representation of $\mathbf{G}(F_v)$. The fixed point space

$$\Big(\bigotimes'_{\mathfrak{p} \nmid \delta_D \mathfrak{N}} \Pi_{\mathfrak{p}} \Big)^{K_f} = \bigotimes_{\mathfrak{p} \nmid \delta_D \mathfrak{N}} \Pi_{\mathfrak{p}}^{K_1(\mathfrak{p}^0)}$$

is 1-dimensional, and admits an action of $\mathbb{T}$, thus giving rise to a Hecke eigensystem.

If Π is a discrete automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ and χ is a Hecke character of F, we write $\Pi\otimes\chi$ as shorthand for the automorphic representation $\Pi\otimes_{\mathbb{C}}(\chi\circ\mathrm{nrd})$.

Theorem 5.6 (Strong multiplicity 1). *Every cuspidal automorphic representation occurs with multiplicity 1 in $\mathrm{L}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)/\mathbb{R}_{>0})$. Moreover, if Π and Π' are two cuspidal automorphic representations of $\mathbf{G}$ such that Π_v is isomorphic to Π'_v for all but finitely many places v of F, then Π and Π' are isomorphic.*

Proof. See [33], [3, Theorem 5.1 (b), (c)], see also [27, Theorems 11.4.3 and 11.7.2]. $\qquad\square$

Theorem 5.7 (Automorphic induction). *Let L/F be a quadratic extension of number fields and let σ be the generator of its Galois group, and let χ be the quadratic Hecke character corresponding to the extension L/F by class field theory. Let Ψ be a Hecke character of L. The automorphic induction $\mathrm{AI}_F^L(\Psi) = \otimes'_v \mathrm{AI}_{F_v}^{F_v \otimes_F L}(\Psi_v)$ is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. The representation $\mathrm{AI}_F^L(\Psi)$ is cuspidal*

if and only if $\Psi^\sigma \neq \Psi$. A cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$ is the automorphic induction of a Hecke character of L if and only if $\Pi \otimes \chi \cong \Pi$.

Proof. See [39], see also [27, Theorem 13.4.2]. $\qquad\square$

Theorem 5.8 (Jacquet–Langlands correspondence)**.** *There exists a unique injection JL_D from the class of isomorphism classes of automorphic representations of $\mathbb{G}(\mathbb{A}_F)$ to the class of isomorphism classes of discrete automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying $\mathrm{JL}_{D_v}(\Pi_v) = \mathrm{JL}_D(\Pi)_v$ for all places v. The representation $\mathrm{JL}_D(\Pi)$ is cuspidal if and only if Π is infinite-dimensional. When restricting to the class of infinite-dimensional representations, the image is the class of cuspidal automorphic representations Π such that for all places v, the local representation Π_v is in the image of JL_{D_v}.*

Proof. See [33], see also [27, Theorem 19.4.3]. $\qquad\square$

Corollary 5.9. *Keep the notation as in Theorem 5.7. Let Π be an automorphic representation of $\mathbb{G}$ such that $\Pi \otimes \chi \cong \Pi$. Then Π is infinite-dimensional, and there exists a Hecke character Ψ of L such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$.*

Proof. If Π is finite-dimensional, then it is 1-dimensional, and the hypothesis is never satisfied, therefore Π is infinite-dimensional. By Multiplicity 1, Theorem 5.6, the Jacquet–Langlands correspondence, being local-global compatible, is also compatible with twisting. Applying Theorems 5.7 and 5.8 therefore gives the statement. $\qquad\square$

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G_∞ and K_∞, respectively, both equipped with the adjoint action of K_∞^+.

Theorem 5.10 (Matsushima's formula)**.** *We have*

$$\Omega^i(\mathcal{Y})_{\mathbb{C}} \cong \bigoplus_\Pi \mathrm{Hom}_{K_\infty^+}(\mathbb{C} \otimes_{\mathbb{R}} \Lambda^i(\mathfrak{g}/\mathfrak{k}), \Pi_\infty) \otimes_{\mathbb{C}} \Pi_f^{K_f},$$

where the sum ranges over automorphic representations Π of $\mathbb{G}(\mathbb{A}_F)$ such that Π_∞ is trivial on Z_∞. Moreover, the isomorphism is equivariant with respect to:

- *the Laplace operator on the left hand side and the Casimir operator acting on Π_∞ on the right hand side, and*
- *the action of Hecke operators, acting on $\Pi_f^{K_f}$ on the right hand side.*

Proof. This is a well-known adélic reformulation of the classical Matsushima formula [47], see [8, §1.2]. The equivariance with respect to Hecke operators follows from the canonicity of the correspondence in ibid. Note that the statement simplifies in our situation due to Multiplicity 1, Theorem 5.6. $\qquad\square$

Definition 5.11. Given a collection $\underline{\boldsymbol{i}} = (i_v)_v$ of non-negative integers indexed by the infinite places v of F, we define

$$\Omega^{\underline{\boldsymbol{i}}}(\mathcal{Y})_{\mathbb{C}} = \bigoplus_\Pi \mathrm{Hom}_{K_\infty^+}\left(\bigotimes_v \mathbb{C} \otimes_{\mathbb{R}} \Lambda^{i_v}(\mathfrak{g}_v/\mathfrak{k}_v), \Pi_\infty\right) \otimes_{\mathbb{C}} \Pi_f^{K_f}.$$

Moreover, given a collection $\underline{\boldsymbol{\lambda}} = (\lambda_v)_v$ or real numbers indexed by the infinite places v of F, define $\Omega^{\underline{\boldsymbol{i}}}_{\Delta = \underline{\boldsymbol{\lambda}}}(\mathcal{Y})_{\mathbb{C}} \subset \Omega^{\underline{\boldsymbol{i}}}(\mathcal{Y})_{\mathbb{C}}$ to be the subspace consisting of Laplace eigenvectors on which the Laplace operator at each infinite place v has eigenvalue λ_v. Let $\mathcal{H}^{\underline{\boldsymbol{i}}}(\mathcal{Y})_{\mathbb{C}} = \Omega^{\underline{\boldsymbol{i}}}_{\Delta = \underline{\mathbf{0}}}(\mathcal{Y})_{\mathbb{C}}$ be the space of harmonic forms in $\Omega^{\underline{\boldsymbol{i}}}(\mathcal{Y})_{\mathbb{C}}$, where $\underline{\mathbf{0}}$ denotes the zero vector.

Corollary 5.12. *For every $i \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{R}$, we have*

$$\Omega^i_{\Delta = \lambda}(\mathcal{Y})_{\mathbb{C}} = \bigoplus_{\underline{\lambda}} \bigoplus_{\underline{i}} \Omega^{\underline{i}}_{\Delta = \underline{\lambda}}(\mathcal{Y})_{\mathbb{C}}, \quad \text{and} \quad \mathcal{H}^i(\mathcal{Y})_{\mathbb{C}} = \bigoplus_{\underline{i}} \mathcal{H}^{\underline{i}}(\mathcal{Y})_{\mathbb{C}},$$

where the sums run over all collections $\underline{\lambda} = (\lambda_v)_v$ satisfying $\sum_v \lambda_v = \lambda$, respectively all collections $\underline{i} = (i_v)_v$ satisfying $\sum_v i_v = i$.

5.4. Self-twist conditions.

In this subsection we prove Theorem D for representation equivalence, $\Omega^\bullet$-, Ω^0-, and $\mathcal{H}^\bullet$-isospectrality, and Theorem G parts (1) and (2).

5.4.1. *Representation equivalence.*

Proposition 5.13. *Let χ be an order 2 Hecke character, let L/F be the corresponding quadratic extension, and let σ denote the non-trivial automorphism of L/F. Let V be a K_∞^+-representation. Then every Hecke eigensystem over $\mathbb{C}$ in the module $M = \mathrm{Hom}_{K_\infty^+}(V, \mathrm{L}^2(\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f))$ that admits a self-twist by χ is attached to some automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ for a unitary Hecke character Ψ of L satisfying all of the following:*

(1) *for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi_w^\sigma \neq \Psi_w$;*

(2) *the parameters (k_w, s_w) of the character Ψ satisfy the following conditions:*
   - *for every real place v of F that extends to a complex place w of L, we have $s_w = 0$ and k_w is odd;*
   - *for every real place v of F that extends to two real places w, w' of L, we have $s_w + s_{w'} = 0$ and $k_w + k_{w'} = 0 \mod 2$;*
   - *for every complex place v of F that extends to two complex places w, w' of L, we have $s_w + s_{w'} = 0$ and $k_w + k_{w'} = 0$.*

Each such Hecke eigensystem has multiplicity $\dim \Pi_f^{K_f} \cdot \dim \mathrm{Hom}_{K_\infty^+}(V, \Pi_\infty)$ (which may be 0).

Proof. Hecke eigensystems over $\mathbb{C}$ in M are attached to automorphic representations $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that the central character of Π_∞ vanishes on Z_∞, and each such eigensystem has multiplicity $\dim \Pi_f^{K_f} \cdot \dim \mathrm{Hom}_{K_\infty^+}(V, \Pi_\infty)$. Hecke eigensystems over $\mathbb{C}$ in $\mathrm{L}^2(\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}_F)/Z_\infty K_f)$ that admit a self-twist by χ are exactly those attached to automorphic representations $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that $\Pi \otimes \chi \cong \Pi$ and the central character of Π_∞ vanishes on Z_∞. By Corollary 5.9, such Π are exactly the ones satisfying

   - Π is infinite-dimensional;
   - $\mathrm{JL}_D(\Pi) = \mathrm{AI}_{L/F}(\Psi)$ for some Hecke character Ψ of L;
   - the central character of Π_∞ vanishes on Z_∞.

By Theorem 5.8 and Theorem 5.7, the first two conditions are equivalent to the existence of a Hecke character Ψ of L such that we have $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ and $\Psi^\sigma \neq \Psi$. Let Ψ be a Hecke character of L such that $\Psi^\sigma \neq \Psi$. By Theorem 5.8 there exists Π such that $\mathrm{AI}_F^L(\Psi) = \mathrm{JL}_D(\Pi)$ if and only if $\mathrm{AI}_F^L(\Psi)$ is discrete series at the infinite places that are ramified in D, and special or supercuspidal at the finite places that are ramified in D. Let v be a place of F that ramifies in D. If v is an infinite place: by Subsection 5.2.1, the local component $\mathrm{AI}_F^L(\Psi)_v$ is a discrete series representation if and only if v extends to a complex place w in L and $k_w \neq 0$, equivalently there is a single place w of L above v and $\Psi_w^\sigma \neq \Psi_w$. If v is a finite place: by Subsection 5.2.2, the local component $\mathrm{AI}_F^L(\Psi)_v$ is special or supercuspidal if and only there is a single place w of L above v and $\Psi_w^\sigma \neq \Psi_w$. This proves that

condition (1) of the conclusion of the proposition holds. By Subsection 5.2.1, the central character of $\mathrm{AI}_F^L(\Psi)$ vanishing on Z_∞ is equivalent to condition (2) in the conclusion. Since Ψ is trivial on $L^\times$, all the s_w have the same real part. Therefore condition (2) implies that this real part is 0, so that Ψ is unitary. $\square$

Definition 5.14. If L/F is a quadratic extension, then an L^2-*shady character* of L is a unitary Hecke character Ψ of L such that L and Ψ have all of the following properties:

- the field L and the character Ψ satisfy the conditions (1) and (2) in Proposition 5.13, so that in particular, by Theorems 5.7 and 5.8, there exists a unique automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ satisfying $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$;
- we have $\Pi_f^{K_f} \neq 0$.

The following is Theorem D for representation equivalence.

Theorem 5.15. *Let $c \in C$. Then at least one of the following statements is true:*
- (i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$, with corresponding quadratic extension L/F, and an L^2-shady character of L;*
- (ii) *for all $b \in C$ the groups Γ_b and Γ_{cb} are representation equivalent.*

Proof. This is an immediate consequence of Theorem 4.10 and Proposition 5.13. $\square$

Remark 5.16. In Proposition 5.13, condition (1) at a real place v of F is equivalent to the following: if v ramifies in D, then v extends to a complex place w of L and we have $k_w \neq 0$.

It will sometimes be useful to reformulate the conditions of Proposition 5.13 as follows.

Lemma 5.17. *Let L/F be a quadratic extension, let σ denote the non-trivial automorphism of L/F, and let Ψ be a Hecke character of L. Then $\Psi^\sigma \Psi$ has finite order if and only if all of the following conditions hold:*
- *for every real place v of F that extends to a complex place w of L, we have $s_w = 0$;*
- *for every real place v of F that extends to two real places w, w' of L, we have $s_w + s_{w'} = 0$;*
- *for every complex place v of F that extends to two complex places w, w' of L, we have $s_w + s_{w'} = 0$ and $k_w + k_{w'} = 0$.*

Proof. At a place of F that extends to two places of L, the action of σ on Ψ swaps the parameters (k, s). At a real place v of F that extends to a complex place w of L, the action of σ on Ψ negates k_w and leaves s_w unchanged. Finally, a Hecke character of L has finite order if and only if all its s parameters are 0 and all its k parameters at complex places are 0. Putting these together gives the lemma. $\square$

Remark 5.18. When at least one real place of F ramifies in L, the Hecke characters appearing in Theorem 5.15 are *partially algebraic Hecke characters* in the sense of [48, Section 5.5]. In fact, they must come from the construction given in the proof of *ibid.*, Proposition 41.

By omitting conditions from Theorem 5.15, we recover previously known results.

Corollary 5.19. *Let $c \in C$. Then at least one of the following statements is true:*
- (i) *there exists a character $\chi \in \widehat{C}_{\mathrm{iso}}$, with corresponding quadratic extension L/F, such that*
  - (a) *$\chi(c) = -1$, i.e. $\chi \notin c^\perp$, and*
  - (b) *for every place v of F that ramifies in D, there is a single place of L above v;*

(ii) *for all $b \in C$ the groups Γ_b and Γ_{cb} are representation equivalent.*

Remark 5.20. Corollary 5.19 seems much weaker than Theorem 5.15, but is sufficient to recover previously known results. It would be interesting to know how much weaker it actually is, specifically whether, given L/F as in Corollary 5.19, there always exists a Ψ as in Theorem 5.15.

Corollary 5.21. *Let $\mathfrak{N}$ be a non-zero ideal of $\mathbb{Z}_F$ coprime to δ_D, and let $K_f = \prod_{\mathfrak{p}^i \| \mathfrak{N}} K_0(\mathfrak{p}^i)$. Let $c \in C$. Then at least one of the following statements is true:*
  (i) *we have that*
      (a) *D is unramified at all finite places of F, and*
      (b) *there exists a quadratic extension L/F that is ramified at exactly the same set of real places of F as D, such that all primes of F dividing $\mathfrak{N}$ with odd exponent are split in L, and such that all primes of F whose class in C_{iso} is $\bar{c}$ are inert in L;*
  (ii) *for all $b \in C$ the groups Γ_b and Γ_{cb} are representation equivalent.*

Proof. By Equation (4.3) and a local computation [67, eqns. (23.2.4) and (23.2.8), Proposition 23.4.14], we have $C_{\mathrm{iso}} = \mathrm{Cl}_F(\mathcal{V}_{\mathbb{H}}(D))/\langle \mathfrak{a}^2, \mathfrak{p} \text{ ramified in } D, \mathfrak{p}^e \| \mathfrak{N} \rangle$. Assume that for some $b \in C$ the groups Γ_b and Γ_{cb} are not representation equivalent. Let $\mathfrak{p}$ be a finite place of F that ramifies in D. Then the class of $\mathfrak{p}$ is trivial in C_{iso}, so $\mathfrak{p}$ splits in L; but this contradicts Corollary 5.19 (b). Therefore, no finite place ramifies in D. Let χ and L/F be as in Corollary 5.19. By the expression of C_{iso} as a class group, L/F is unramified at all finite places of F, split at all prime ideals dividing $\mathfrak{N}$ with odd exponent, and the set of real places of F that ramify in L is a subset of those that ramify in D. Corollary 5.19 (b) gives the other inclusion between those sets of ramified real places. Let $\mathfrak{p}$ be a finite place of F whose class in C_{iso} equals $\bar{c}$. Then $\chi(\mathfrak{p}) \neq 1$ by Corollary 5.19 (a), so that $\mathfrak{p}$ is inert in L. $\square$

Remark 5.22. The groups of the form Γ_c for $c \in C$ are exactly the ones of the form $\mathcal{O}^\times$, where $\mathcal{O}$ is an Eichler order of level $\mathfrak{N}$. Corollary 5.21 is essentially the criterion used in [44] to investigate isospectrality of a large number of pairs of hyperbolic orbifolds of dimension 2 and 3, using the selectivity method [44, Theorem 2.17 and Theorem 2.19].

Corollary 5.23. *Assume that there is a maximal ideal $\mathfrak{p}$ of $\mathbb{Z}_F$ that is ramified in D, and that there exists an element $g \in \mathrm{N}(K_f)$ such that the valuation $\mathrm{ord}_\mathfrak{q}(\mathrm{nrd}(g))$ is even for every finite place $\mathfrak{q} \neq \mathfrak{p}$ and such that $\mathrm{ord}_\mathfrak{p}(\mathrm{nrd}(g))$ is odd. Then for all $c, c' \in C$ the groups Γ_c and $\Gamma_{c'}$ are representation equivalent.*

Proof. Let $c, c' \in C$ and assume that Γ_c and $\Gamma_{c'}$ are not representation equivalent, and let L/F be as in Corollary 5.19. By definition of C_{iso}, the class of $\mathrm{nrd}(g)$ in C_{iso} is trivial. On the other hand, since C_{iso} has exponent dividing 2, the valuation assumptions imply that the class of $\mathrm{nrd}(g)$ equals the class of $\mathfrak{p}$. This implies that $\mathfrak{p}$ splits in L, contradicting Corollary 5.19 (b). Therefore, the groups Γ_c and $\Gamma_{c'}$ are representation equivalent. $\square$

Remark 5.24. Corollary 5.23 is essentially the same criterion as in [55, Section 3]: the condition (H3) in that paper (with $v_0 = \mathfrak{p}$) implies that we can take g to be a uniformiser of $D_\mathfrak{p}$ at v_0 and $g_v = 1$ for $v \neq v_0$. One superficial difference is that the representation equivalent groups in [55] are arithmetic subgroups of $\mathrm{SL}_1(D)$, whereas ours are subgroups of $\mathrm{GL}_1(D)$; by choosing appropriate levels K_f, our method can be adapted to the $\mathrm{SL}_1(D)$ setting, but we chose to stick to $\mathrm{GL}_1(D)$ for simplicity of the exposition. The similarity is not surprising: our Hecke eigenvalue systems with a self-twist exactly correspond to the endoscopic representations studied in [36] and playing an important role in [55]. However, our method proves

representation equivalence without an analysis of the Labesse–Langlands multiplicity formula.

5.4.2. *Differential forms.*

Proposition 5.25. *Let χ be an order 2 Hecke character, with corresponding quadratic extension L/F, let σ denote the non-trivial automorphism of L/F, and let $\underline{\boldsymbol{i}} = (i_v)_v$ and $\underline{\boldsymbol{\lambda}} = (\lambda_v)_v$ be as in Definition 5.11. Then the Hecke eigensystems over $\mathbb{C}$ in $\Omega^{\underline{\boldsymbol{i}}}_{\Delta = \underline{\boldsymbol{\lambda}}}(\mathcal{Y})_{\mathbb{C}}$ that admit a self-twist by χ are the Hecke eigenvalue systems attached to an automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}^L_F(\Psi)$ for some unitary Hecke character Ψ of L satisfying all of the following conditions:*

(1) *for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi^\sigma_w \neq \Psi_w$;*
(2) *for every real place v of F that extends to a complex place w of L we have*
   - $i_v = 0$ *if v is ramified in D;*
   - $i_v = 1$ *otherwise;*

   $\lambda_v = 0$, *and $\Psi_w = \Psi_{\mathbb{C}}(k, 0)$ with $k \in \{\pm 1\}$;*
(3) *for every real place of F that extends to two real places w, w' of L we have $i_v \in \{0, 1, 2\}$, $\Psi_w = \Psi_{\mathbb{R}}(k, s)$ and $\Psi_{w'} = \Psi_{\mathbb{R}}(k', s')$ where $k \equiv k' \mod 2$, and $s = it = -s'$ for $t \in \mathbb{R}$ satisfying $\frac{1}{4} + t^2 = \lambda_v$;*
(4) *for every complex place v of F, extending to places w and w' of L, we have $\Psi_w = \Psi_{\mathbb{C}}(k, s)$ and $\Psi_{w'} = \Psi_{\mathbb{C}}(k', s')$ where either*
   - $i_v \in \{0, 1, 2, 3\}$, $k = k' = 0$, *and $s = it = -s'$ with $t \in \mathbb{R}$ satisfying $1 + 4t^2 = \lambda_v$; or*
   - $i_v \in \{1, 2\}$, $k = -k' \in \{\pm 1\}$, *and $s = it = -s'$ with $t \in \mathbb{R}$ satisfying $4t^2 = \lambda_v$;*

each such Hecke eigenvalue system occurring with multiplicity $2^r \dim \Pi^{K_f}_f$, where r is the number of real places v of F for which one has $i_v = 1$.

Proof. We apply Proposition 5.13 with $V = \Omega^{\underline{\boldsymbol{i}}}_{\Delta = \underline{\boldsymbol{\lambda}}}(\mathcal{Y})_{\mathbb{C}}$. Let Ψ and Π be as in Proposition 5.13. For every infinite place w of L, write $s_w = it_w$ with $t_w \in \mathbb{R}$.

Write $K^+_\infty = \prod_v K_v$, with the product running over the infinite places of F, and for each such place v, let $H_v = \mathrm{Hom}_{K_v}(\Lambda^{i_v}(\mathfrak{g}_v/\mathfrak{k}_v), \Pi_v)$.

Let v be a real place of F that extends to a complex place w of L. Then we have $\mathrm{AI}^L_F(\Psi)_v = \mathrm{DS}(1 + |k_w|, 0)$. By the local computation in Section 5.2.1 the space H_v is non-trivial if and only if either

- the place v is unramified in D, $i_v = 1$, $k_w \in \{\pm 1\}$, and $\lambda_v = 0$, in which case $\dim H_v = 2$; or
- the place v is ramified in D, $i_v = 0$, $k_w \in \{\pm 1\}$, and $\lambda_v = 0$, in which case $\dim H_v = 1$.

Let v be a real place of F that extends to two real places w, w' of L. Then we necessarily have $\mathrm{AI}^L_F(\Psi)_v = \mathrm{PS}(\Psi_w, \Psi_{w'})$. Moreover, the space H_v is non-trivial if and only if $i_v \in \{0, 1, 2\}$ and $\lambda_v = \frac{1}{4} + t^2_w$. In this case, $\dim H_v = 2$ if $i_v = 1$, and $\dim H_v = 1$ otherwise.

Let v be a complex place of F, extending to two places w and w' of L. Then we necessarily have $\mathrm{AI}^L_F(\Psi)_v = \mathrm{PS}(\Psi_w, \Psi_{w'})$. Moreover, the space H_v is non-trivial if and only if one of the following holds:

- $k_w = 0$, $i_v \in \{0, 1, 2, 3\}$, $\lambda_v = 1 + 4t^2_w$;
- $k_w \in \{\pm 1\}$, $i_v \in \{1, 2\}$, $\lambda_v = 4t^2_w$.

When H_v is non-trivial, we have $\dim H_v = 1$. Taking the tensor product of the spaces H_v over all places $v \mid \infty$ gives the result. $\qquad\square$

Definition 5.26. If L/F is a quadratic extension, and $\underline{i}$ and $\underline{\lambda}$ are as in Proposition 5.25, then an $(\Omega^{\underline{i}}_{\Delta=\underline{\lambda}})$-*shady character* of L is a unitary Hecke character Ψ of L such that L and Ψ have all of the following properties:

- the field L and the character Ψ satisfy the conditions (1)–(4) in Proposition 5.25, so that in particular, by Theorems 5.7 and 5.8, there exists a unique automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ satisfying $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$;
- we have $\Pi_f^{K_f} \neq 0$.

We now prove a general version of Theorem D, which we will then specialise.

Theorem 5.27. *Let $c \in C$, let $i \in \mathbb{Z}_{\geq 0}$, and let $\lambda \in \mathbb{R}_{\geq 0}$. Then exactly one of the following statements is true:*

(i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$, with corresponding quadratic extension L/F, collections $\underline{\lambda} = (\lambda_v)_v$ and $\underline{i} = (i_v)_v$ of real numbers, respectively non-negative integers, indexed by the infinite places v of F, such that $\sum_v \lambda_v = \lambda$ and $\sum_v i_v = i$, and an $(\Omega^{\underline{i}}_{\Delta=\underline{\lambda}})$-shady character of L;*

(ii) *there exists $T \in \mathbb{T}[W]_c$ inducing, for all $b \in C$, an isomorphism of $\mathbb{T}_1$-modules*

$$T \colon \Omega^i_{\Delta=\lambda}(Y_b) \to \Omega^i_{\Delta=\lambda}(Y_{cb}).$$

Proof. This is an immediate consequence of Theorem 4.7 (1), Corollary 5.12, and Proposition 5.25. $\qquad\square$

Definition 5.28. If L/F is a quadratic extension, with non-trivial automorphism σ, then an $\Omega^\bullet$-*shady character* of L is a unitary Hecke character Ψ of L such that L and Ψ have all of the following properties:

(1) for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi_w^\sigma \neq \Psi_w$;

(2) for every real place v of F that extends to a complex place w of L we have $\Psi_w = \Psi_\mathbb{C}(k,0)$ with $k \in \{\pm 1\}$;

(3) for every real place of F that extends to two real places w, w' of L we have $\Psi_w = \Psi_\mathbb{R}(k,s)$ and $\Psi_{w'} = \Psi_\mathbb{R}(k',s')$ where $k \equiv k' \mod 2$ and $s = -s' \in i\mathbb{R}$;

(4) for every complex place v of F, extending to places w and w' of L, we have $\Psi_w = \Psi_\mathbb{C}(k,s)$ and $\Psi_{w'} = \Psi_\mathbb{C}(k',s')$ where $k = -k' \in \{-1,0,1\}$ and $s = -s' \in i\mathbb{R}$;

(5) the unique automorphic representation Π of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ satisfies $\Pi_f^{K_f} \neq 0$.

The following is Theorem D for $\Omega^\bullet$-isospectrality.

Corollary 5.29. *Let $c \in C$. Then at least one of the following statements is true:*

(i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$, with corresponding quadratic extension L/F with non-trivial automorphism σ, and an $\Omega^\bullet$-shady character of L;*

(ii) *for all $b \in C$ the manifolds Y_b and Y_{cb} are i-isospectral for all $i \geq 0$.*

Proof. Suppose that part (i) does not hold. Then nor does part (i) of Theorem 5.27 for any i and λ. Thus part (ii) of Theorem 5.27 holds for every i and λ, whence we deduce that part (ii) of the corollary holds. $\qquad\square$

Definition 5.30. If L/F is a quadratic extension, with non-trivial automorphism σ, then an Ω^0-*shady character* of L is a unitary Hecke character Ψ of L such that L and Ψ have all of the following properties:

(1) for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi_w^\sigma \neq \Psi_w$;

(2) for every real place v of F that extends to a complex place w of L, the algebra D is ramified at v and $\Psi_w = \Psi_\mathbb{C}(k,0)$ with $k \in \{\pm 1\}$;

(3) for every real place of F that extends to two real places w, w' of L we have $\Psi_w = \Psi_{\mathbb{R}}(k, s)$ and $\Psi_{w'} = \Psi_{\mathbb{R}}(k', s')$ where $k \equiv k' \mod 2$ and $s = -s' \in i\mathbb{R}$;

(4) for every complex place v of F, extending to places w and w' of L, we have $\Psi_w = \Psi_{\mathbb{C}}(0, s)$ and $\Psi_{w'} = \Psi_{\mathbb{C}}(0, s')$ where $s = -s' \in i\mathbb{R}$;

(5) the unique automorphic representation Π of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ satisfies $\Pi_f^{K_f} \neq 0$.

The following is Theorem D for Ω^0-isospectrality.

Corollary 5.31. *Let $c \in C$. Then at least one of the following statements is true:*

(i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$, with corresponding quadratic extension L/F, and an Ω^0-shady character Ψ of L;*

(ii) *for all $b \in C$ the manifolds Y_b and Y_{cb} are 0-isospectral.*

Proof. The proof is completely analogous to that of Corollary 5.29. $\square$

5.4.3. *Cohomology.*

Proposition 5.32. *Let χ be an order 2 Hecke character, with corresponding quadratic extension L/F, let σ denote the non-trivial automorphism of L/F, and let $\underline{i} = (i_v)_v$ be as in Definition 5.11. Then the Hecke eigensystems over $\mathbb{C}$ in $\mathcal{H}^{\underline{i}}(\mathcal{Y})_{\mathbb{C}}$ that admit a self-twist by χ are the Hecke eigenvalue systems attached to an automorphic representation $\Pi = \Pi_{\infty} \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ for some unitary Hecke character Ψ of L satisfying all of the following:*

(1) *for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi_w^{\sigma} \neq \Psi_w$;*

(2) *every real place v of F extends to a complex place w of L, and for all such places we have*
- $i_v = 0$ *if v is ramified in D;*
- $i_v = 1$ *otherwise;*

and $\Psi_w = \Psi_{\mathbb{C}}(k, 0)$ with $k \in \{\pm 1\}$;

(3) *for every complex place v of F, extending to places w and w' of L, we have $\Psi_w = \Psi_{\mathbb{C}}(k, 0)$ and $\Psi_{w'} = \Psi_{\mathbb{C}}(k', 0)$, where $i_v \in \{1, 2\}$, $k = -k' \in \{\pm 1\}$;*

each such Hecke eigenvalue system occurring with multiplicity $2^r \dim \Pi_f^{K_f}$, where r is the number of real places v of F for which one has $i_v = 1$.

Proof. The result is obtained by specialising Proposition 5.25 to $\underline{\lambda} = \underline{0}$. $\square$

Definition 5.33. If L/F is a quadratic extension, with non-trivial automorphism σ, then an $\mathcal{H}^{\bullet}$-*shady character* of L is a unitary Hecke character Ψ of L such that L and Ψ have all of the following properties:

(1) for every place v of F that ramifies in D, there is a single place w of L above v, and we have $\Psi_w^{\sigma} \neq \Psi_w$;

(2) every real place v of F extends to a complex place w of L, and for all such places we have $\Psi_w = \Psi_{\mathbb{C}}(k, 0)$ with $k \in \{\pm 1\}$;

(3) for every complex place v of F, extending to places w and w' of L, we have $\Psi_w = \Psi_{\mathbb{C}}(k, 0)$ and $\Psi_{w'} = \Psi_{\mathbb{C}}(k', 0)$, where $k = -k' \in \{\pm 1\}$;

(4) the unique automorphic representation Π of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_F^L(\Psi)$ satisfies $\Pi_f^{K_f} \neq 0$.

The following is Theorem D for $\mathcal{H}^{\bullet}$-isospectrality and, together with Lemma 4.13, implies Theorem G parts (1) and (2).

Theorem 5.34. *Let $c \in C$, and let $i \in \mathbb{Z}_{\geq 0}$. Then exactly one of the following statements is true:*

(i) *we have $\#\mathcal{V}_{\mathbb{R}} + \#\mathcal{V}_{\mathbb{C}} \leq i \leq \#\mathcal{V}_{\mathbb{R}} + 2\#\mathcal{V}_{\mathbb{C}}$ and there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$, with corresponding quadratic extension L/F, and an $\mathcal{H}^{\bullet}$-shady character of L;*

(ii) *for all $b \in C$ the homogeneous components $H_i(Y_b, \mathbb{Z})_{\mathrm{free}}$ and $H_i(Y_{cb}, \mathbb{Z})_{\mathrm{free}}$ are linked in the sense of Definition 3.1, and we have*

$$\frac{\mathrm{Reg}_i(Y_b)^2}{\mathrm{Reg}_i(Y_{cb})^2} \in \mathbb{Q}^{\times}.$$

Proof. This is an immediate consequence of Theorem 4.14, Corollary 5.12, Proposition 5.32, and Lemma 4.13. □

Remark 5.35. The range of i appearing in Theorem 5.34 is sometimes called the cuspidal range or the tempered range. Outside that range, the only automorphic representations contributing to harmonic forms are non-cuspidal, and moreover by Theorem 5.34 we get automatic rationality for the regulator quotient.

Definition 5.36. Let *Ю* and *Я* be fields and *Э/Ю* a quadratic étale algebra. A collection $(q_{\tau})_{\tau} \in \{0,1\}^{\mathrm{Hom}(\text{Э},\text{Я})}$ is *balanced* if for every $\tau \neq \tau' \in \mathrm{Hom}(\text{Э}, \text{Я})$ such that $\tau|_{\text{Ю}} = \tau'|_{\text{Ю}}$, we have $\{q_{\tau}, q_{\tau'}\} = \{0,1\}$.

Proposition 5.37. *Let L/F be a quadratic extension with L totally complex, let Ψ be a Hecke character of L, and let $\Psi^{\mathrm{alg}} = \Psi \| \cdot \|^{-1/2}$. Then Ψ satisfies (2) and (3) of Definition 5.33 if and only if Ψ^{alg} is an algebraic Hecke character whose type is balanced.*

Proof. This is well-known (see e.g. [48, Lemma 4]). The type $(q_{\tau})_{\tau}$ of Ψ^{alg} is related to the parameters $(k_w)_w$ of Ψ by the relations $q_{\tau} + q_{\bar{\tau}} = 1$ and $q_{\bar{\tau}} - q_{\tau} = k_{\tau}$. □

Remark 5.38. The conditions on Ψ^{alg} also imply that for every complex embedding τ of L we have $\{q_{\tau}, q_{\bar{\tau}}\} = \{0,1\}$.

The following corollary shows that $\mathcal{H}^{\bullet}$-isospectrality is in fact very frequent.

Corollary 5.39. *Let $c \in C$, and let $i \in \mathbb{Z}_{\geq 0}$. Then at least one of the following statements is true:*

(i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ corresponding to a quadratic extension L/F such that L contains a CM subfield not contained in F;*

(ii) *for all $b \in C$ we have*

$$\frac{\mathrm{Reg}_i(Y_c)^2}{\mathrm{Reg}_i(Y_{c'})^2} \in \mathbb{Q}^{\times}.$$

Proof. Assume that the second statement does not hold, and apply Theorem 5.34 and Proposition 5.37 to get corresponding characters Ψ and Ψ^{alg}. By the Artin–Weil theorem (see [68] for the statement and, for example, [51, Lemma 2.3.4] for several proofs), the existence of an algebraic Hecke character of non-parallel type forces L to contain a CM subfield, and therefore a maximal one, M, say. The type of Ψ^{alg} descends to M but not to F, so that F cannot contain M. □

Remark 5.40. As we consider weaker and weaker notions of isospectrality, there are fewer and fewer Hecke characters that can prevent us from proving the respective isospectrality. We find it instructive to list the conditions on the shady characters incrementally, as follows.

The shady characters Ψ appearing in Corollary 5.29 (i-isospectrality for all i) are exactly the characters appearing in Theorem 5.15 (representation equivalence) that satisfy the additional conditions:

- for every real place v of F that extends to a complex place w of L we have $k_w \in \{\pm 1\}$, and

- for every complex place v of F, extending to places w and w' of L, we have $k_w \in \{-1, 0, 1\}$.

Assuming that L and D are ramified at the same set of real places of F, the shady characters Ψ appearing in Corollary 5.31 (0-isospectrality) are exactly the characters appearing in Corollary 5.29 (i-isospectrality for all i) that satisfy the additional condition:

- for every complex place v of F, extending to places w and w' of L, we have $k_w = 0$.

Assuming that L is totally complex, the shady characters Ψ appearing in Theorem 5.34 (rationality of regulator quotients) are exactly the characters appearing in Corollary 5.29 (i-isospectrality for all i) that satisfy the additional condition:

- for every complex place v of F, extending to places w and w' of L, we have $k_w \in \{\pm 1\}$ and $s_w = 0$.

5.5. **Sparsity of non-matching Laplace eigenvalues.** In this subsection we show that Vignéras pairs of manifolds are always "almost isospectral". The main result is Theorem 5.45, which in particular implies Theorem C from the Introduction.

Notation 5.41. Let V be a finite-dimensional representation of K_∞ and Γ a discrete cocompact subgroup of G_∞. Then we set

$$\Omega(V, \Gamma) = \mathrm{Hom}_{K_\infty}(V, \mathrm{L}^2(\Gamma \backslash G_\infty / Z_\infty)).$$

By Δ we denote the operator on $\Omega(V, \Gamma)$ induced by the Casimir operator on the L^2-space. If A is a commutative ring, $a \colon A \to \mathbb{C}$ is a ring homomorphism, and Ω is an $A_\mathbb{C}$-module, we write $\Omega_{A=a}$ for the subspace of Ω consisting of all elements $\omega \in \Omega$ such that one has $t\omega = a(t)\omega$ for all $t \in A$.

Lemma 5.42. *Let V be a finite-dimensional representation of K_∞. Then there exists $\kappa > 0$ such that for every ring homomorphism $a \colon \mathbb{T}_1 \to \mathbb{C}$, we have*

$$\sum_{c \in C} \dim \Omega(V, \Gamma_c)_{\mathbb{T}_1 = a} \leq \kappa.$$

Proof. Let $M = \mathrm{Hom}_{K_\infty}(V, \mathrm{L}^2(\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F) / Z_\infty K_f))_{\mathbb{T}_1 = a}$, which is a $\mathbb{T}$-module that is finite-dimensional over $\mathbb{C}$, and satisfies

$$\dim M = \sum_{c \in C} \dim \Omega(V, \Gamma_c)_{\mathbb{T}_1 = a}.$$

The conclusion of the lemma is trivial if $M = 0$, so assume otherwise. Let $\mathbf{A}$ be the image of $\mathbb{T}_\mathbb{C}$ in $\mathrm{End}_\mathbb{C}(M)$, which inherits the grading from $\mathbb{T}$. Then $\mathbf{A}_1$ is a field, and $\mathbf{A}$ is a finite étale algebra over $\mathbf{A}_1$ of dimension at most $\#C$, so there are at most $\#C$ Hecke eigensystems in M. We have

$$\dim M = \sum_b \dim M_{\mathbb{T}=b},$$

where b ranges over Hecke eigensystems in M. Let $b = (b_\mathcal{D})_\mathcal{D}$ be a Hecke eigensystem in M. By Strong Multiplicity 1, Theorem 5.6, there exists a unique automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that Π_f corresponds to b, and Π has multiplicity 1 in $\mathrm{L}^2(\mathbb{G}(F)^+ \backslash \mathbb{G}(\mathbb{A}_F) / Z_\infty)$. We have

$$\dim M_{\mathbb{T}=b} = \dim \mathrm{Hom}_{K_\infty}(V, \Pi_\infty) \cdot \dim \Pi_f^{K_f} \leq \kappa_\infty \cdot \kappa_f,$$

where κ_∞ and κ_f are as in Lemmas 5.2 and 5.5. Putting everything together, we see that $\kappa = (\#C) \cdot \kappa_\infty \cdot \kappa_f$ satisfies the conclusion of the lemma. $\qquad\square$

We will need the following version of Theorem D.

Lemma 5.43. *Let V be a finite-dimensional representation of K_∞. Let $\lambda \geq 0$ and let $a\colon \mathbb{T}_1 \to \mathbb{C}$ be a ring homomorphism. Then at least one of the following statements holds:*

(i) *there exist a character $\chi \in \widehat{C}_{\mathrm{iso}}$, with corresponding quadratic extension L/F, and an L^2-shady character of L, such that the automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ with $\mathrm{JL}_D(\Pi) = \mathrm{AI}_{L/F}(\Psi)$ satisfies*
   - $\mathrm{Hom}_{K_\infty}(V, \Pi_\infty) \neq 0$,
   - *the Casimir eigenvalue of Π_∞ is λ, and*
   - $\mathbb{T}_1$ *acts on Π_f as a;*

(ii) *all spaces $\Omega(V, \Gamma_c)_{\Delta=\lambda, \mathbb{T}_1=a}$ for $c \in C$ have the same dimension.*

Proof. Apply Proposition 3.6, analogously to e.g. the proof of Theorem 4.7, and then apply Proposition 5.13. $\qquad\square$

We now count the relevant shady characters by reducing to lattice point counting.

Lemma 5.44. *Let $\chi \in \widehat{C}_{\mathrm{iso}}$, let L/F be the corresponding quadratic extension, and assume that there exists an L^2-shady character of L. Let Λ be the group of unitary Hecke characters Ψ of L such that*

- *for every real place v of F that extends to a complex place w of L, we have $s_w = 0$ and $k_w = 0$;*
- *for every real place v of F that extends to two real places w, w' of L, we have $s_w + s_{w'} = 0$;*
- *for every complex place v of F, extending to two complex places w, w' of L, we have $s_w + s_{w'} = 0$ and $k_w = k_{w'} = 0$;*

For a Hecke character Ψ of L, if there exists an automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ such that $\mathrm{JL}_D(\Pi) = \mathrm{AI}_{L/F}(\Psi)$, let $\lambda(\Psi)$ be the Casimir eigenvalue of Π_∞; otherwise let $\lambda(\Psi) = \infty$.

Let Ψ_0 be a unitary Hecke character of L. Then as $T \to \infty$, we have
$$\#\{\Psi \in \Lambda\colon \lambda(\Psi_0\Psi) \leq T\} = O(T^{r/2}),$$
where $r = \#\mathcal{V}_\mathbb{R}(D) + \#\mathcal{V}_\mathbb{C}(D)$.

Proof. Note that the definition of L^2-shady characters, specifically Proposition 5.13, property (1), and equation (4.3), imply that L and D are ramified at the same real places of F. By Lemma 5.17, a Hecke character Ψ of L is in Λ if and only if $\Psi^\sigma\Psi$ has finite order and one has $k_w = 0$ for all complex places w of L. Therefore the rank of Λ is $\mathrm{rk}\,\mathbb{Z}_L^\times - \mathrm{rk}\,\mathbb{Z}_F^\times = r$.

Let Ψ_0 be a unitary Hecke character of L. By the formulas in Section 5.2.1, there exists a quadratic form $Q\colon \Lambda \to \mathbb{R}$ that gives $\Lambda/\Lambda_{\mathrm{tors}}$ the structure of a Euclidean lattice and such that for all $\Psi \in \Lambda$, either $\lambda(\Psi_0\Psi) = \infty$ or
$$\lambda(\Psi_0\Psi) = Q(\Psi) + O(Q(\Psi)^{1/2}).$$
In particular, since Λ_{tors} is finite, we have
$$\#\{\Psi \in \Lambda\colon Q(\Psi) \leq T\} = O(T^{r/2})$$
in the limit as $T \to \infty$. This proves the lemma. $\qquad\square$

We can finally prove the following result, of which Theorem C is a special case.

Theorem 5.45. *Let V be a finite-dimensional representation of K_∞, and let $c, c' \in C$. Then as $T \to \infty$, we have*
$$\sum_{\lambda \leq T}\left|\dim \Omega(V, \Gamma_c)_{\Delta=\lambda} - \dim \Omega(V, \Gamma_{c'})_{\Delta=\lambda}\right| = O(T^{r/2}),$$
where $r = \#\mathcal{V}_\mathbb{R}(D) + \#\mathcal{V}_\mathbb{C}(D)$.

Proof. Let κ be as in Lemma 5.42. Then by Lemma 5.43 we have

$$\sum_{\lambda \leq T} \left| \dim \Omega(V, \Gamma_c)_{\Delta = \lambda} - \dim \Omega(V, \Gamma_{c'})_{\Delta = \lambda} \right|$$

$$= \sum_{\lambda \leq T} \sum_{a \colon \mathbb{T}_1 \to \mathbb{C}} \left| \dim \Omega(V, \Gamma_c)_{\Delta = \lambda, \mathbb{T}_1 = a} - \dim \Omega(V, \Gamma_{c'})_{\Delta = \lambda, \mathbb{T}_1 = a} \right|$$

$$(5.46) \qquad \leq \sum_{\chi \in \widehat{C}_{\mathrm{iso}}} \sum_{\Psi} \kappa,$$

where for each $\chi \in \widehat{C}_{\mathrm{iso}}$, with corresponding quadratic extension L/F, the last sum runs over L^2-shady characters Ψ of L such that the automorphic representation $\Pi = \Pi_\infty \otimes \Pi_f$ of $\mathbb{G}(\mathbb{A}_F)$ with $\mathrm{JL}_D(\Pi) = \mathrm{AI}_{L/F}(\Psi)$ satisfies

- $\mathrm{Hom}_{K_\infty}(V, \Pi_\infty) \neq 0$ and
- the Casimir eigenvalue of Π_∞ is at most T.

Let χ and L/F be as above and let σ be the non-trivial automorphism of L/F, and assume that there exists an L^2-shady character of L. By inspection of the tables in Section 5.2.1, we see that the condition $\mathrm{Hom}_{K_\infty}(V, \Pi_\infty) \neq 0$ implies that the k_w parameters at complex places w of L of all Ψ in the sum must belong to a finite set depending only on V. Let Λ be as in Lemma 5.44. The set of Hecke characters Ψ of L that appear in (5.46) is contained in a finite union of cosets of Λ, so Lemma 5.44 gives the desired bound. $\qquad\square$

Remark 5.47. Weyl's law implies that for all $c \in C$ there exists $a > 0$ such that as $T \to \infty$ we have

$$\sum_{\lambda \leq T} \dim \Omega(V, \Gamma_c)_{\Delta = \lambda} \sim a T^{d/2},$$

where $d = 2\#\mathcal{V}_\mathbb{R}(D) + 3\#\mathcal{V}_\mathbb{C}(D)$. In particular, it follows from Theorem 5.45 that the eigenvalues in the Laplace spectra of any two $\Omega(V, \Gamma_c)$ and $\Omega(V, \Gamma_{c'})$ that do not match up are sparse – have density 0 – among all eigenvalues.

Remark 5.48. It is instructive to compare Theorem 5.45 in the special case $r = 1$ to the results of [35]. In this case, [35, Theorem 1(2)] shows that if Γ_c and $\Gamma_{c'}$ are torsion-free, and for all finite-dimensional representations V of K_∞ one has

$$\sum_{\lambda \leq T} \left| \dim \Omega(V, \Gamma_c)_{\Delta = \lambda} - \dim \Omega(V, \Gamma_{c'})_{\Delta = \lambda} \right| = o(T^{r/2}),$$

then Γ_c and $\Gamma_{c'}$ are representation equivalent. On the other hand, there do exist torsion-free Γ_c and $\Gamma_{c'}$ arising from the Vignéras construction that are not representation equivalent – see [44] for examples and further references. This shows that the bound in [35, Theorem 1(2)] is sharp.

6. Self-twists in characteristic p and Galois representations

Assumption 6.1. We keep Assumption 5.1. In addition, we assume that we have $K_f = K_0(\mathfrak{N})$ for some non-zero ideal $\mathfrak{N}$ coprime to δ_D, where $K_0(\mathfrak{N}) = \prod_{\mathfrak{p}^i \| \mathfrak{N}} K_0(\mathfrak{p}^i)$.

In this section we prove Theorem D for $\mathbb{Z}_p$-isospectrality, Theorem F, Theorem G part (3), and Theorem H with a specified set S, all conditional on a conjecture on the existence of Galois representations with prescribed local behaviour attached to mod p Hecke eigenvalue systems, Conjecture 6.5. The basic idea for proving Theorem D is the same as in Section 5, but with the difference that we cannot use automorphic induction in the mod p setting. Instead, we go via Galois representations, where the self-twist condition implies that the representation is induced from a 1-dimensional Galois character. We then analyse the local properties of these

inductions. Theorem F and Theorem G part (3) are refinements of Theorem D for $\mathbb{Z}_p$-isospectrality. The proof of Theorem 6.22 is an effective version of an argument of Serre, [61, Proposition 20], showing that if one has mod p Hecke characters with bounded weights for lots of primes p, then one has a suitable characteristic 0 Hecke character.

In Section 6.1 we recall some basic definitions for Galois representations and recall a result on conductors of induced representations. In Section 6.2 we state Conjecture 6.5 and prove Theorem D for $\mathbb{Z}_p$-isospectrality, Theorem F, and Theorem G part (3). In Section 6.3 we collect local properties of Galois representations, which are then used in Section 6.4 to prove Theorem H.

6.1. **Basics on Galois representations.** All representations of Galois groups will be finite-dimensional and continuous with respect to the discrete topology on the target.

If L is a perfect field, then $\overline{L}$ denotes a fixed algebraic closure, G_L denotes the Galois group of $\overline{L}/L$, and, if p is a prime number, $\varepsilon_{L,p}$ denotes the mod p cyclotomic character $\varepsilon_{L,p} \colon G_L \to \mathbb{F}_p^\times$.

Definition 6.2. If E is a local field, we will write $\mathbb{E}$ for its residue field.

Let E be a local field. We denote by Frob_E an arbitrary choice of an element of G_E that acts as $x \mapsto x^{\#\mathbb{E}}$ on the residue field of $\overline{E}$, and call such an element a *Frobenius element*.

Let p be a prime number. We say that a representation (V, ρ) of G_E over $\mathbb{F}_p$ is *finite flat* if there exists a finite flat group scheme $\mathcal{G}$ over $\mathbb{Z}_E$ such that one has $V \cong \mathcal{G}(\overline{E})$ as representations of G_E.

Let q be a power of p. We say that a representation (V, ρ) of G_E over $\mathbb{F}_q$ is *finite flat* if the restriction of scalars $\mathrm{Res}_{\mathbb{F}_p}^{\mathbb{F}_q} V$ from $\mathbb{F}_q$ to $\mathbb{F}_p$, which is a representation of G_E over $\mathbb{F}_p$ of dimension $[\mathbb{F}_q : \mathbb{F}_p] \cdot \dim V$, is finite flat.

We say that a representation $\rho \colon G_E \to \mathrm{GL}_2(\mathbb{F}_q)$ is *ordinary* if it has a non-zero unramified quotient.

Definition 6.3. Let L be a number field, let p be a prime number, and $\mathfrak{p}$ a maximal ideal of $\mathbb{Z}_L$. We abbreviate $G_{L_\mathfrak{p}}$ to $G_\mathfrak{p}$. We denote by $\mathrm{Frob}_\mathfrak{p}$ a choice of Frobenius element in $G_\mathfrak{p}$, which, in turn, determines a well-defined conjugacy class in G_L.

Let q be a power of p. We say that a representation (V, ρ) of G_L over $\mathbb{F}_q$ is *finite flat at $\mathfrak{p}$* if the restriction $\rho|_{G_\mathfrak{p}}$ is finite flat.

We say that a representation $\rho \colon G_L \to \mathrm{GL}_2(\mathbb{F}_q)$ is *ordinary at $\mathfrak{p}$* if its restriction to $G_\mathfrak{p}$ is ordinary.

The *prime-to-p-conductor* of a representation ρ of G_L is the largest divisor of the Artin conductor that has no prime divisors lying above p.

Lemma 6.4. *Let p be a prime number. Let ψ be a character of G_L, and let $\mathfrak{M}$ be its prime-to-p-conductor. Then the representation $\rho = \mathrm{Ind}_{G_F/G_L} \psi$ has prime-to-p-conductor $\delta_{L/F} \mathrm{Nm}_{L/F}(\mathfrak{M})$, where $\delta_{L/F}$ denotes the relative discriminant of L/F.*

Proof. See [63]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2. **Self-twist conditions.** The following is a variant on conjectures formulated by Ash, Calegari–Geraghty, Calegari–Venkatesh, and others on Galois representations attached to cohomology classes [2], [13, Conjecture B], [14, Conjecture 2.2.5]. We claim no originality, except for possible mistakes.

Conjecture 6.5. *Under Assumption 6.1, let $i \geq 0$ be an integer, let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$, and let p be the residue characteristic of $\mathfrak{m}$. If $H_i(\mathcal{Y}, \mathbb{Z})_\mathfrak{m}$ is non-zero, then there exists a semisimple Galois representation*

$$\rho \colon G_F \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$$

with the following properties:

(1) ρ *is unramified outside* $p\delta_D\mathfrak{N}$, *and for all* $\mathfrak{q} \nmid p\delta_D\mathfrak{N}$ *we have*

$$\operatorname{Tr}\rho(\operatorname{Frob}_{\mathfrak{q}}) \equiv T_{\mathfrak{q}} \bmod \mathfrak{m};$$

(2) $\det\rho = \zeta \cdot \varepsilon_{F,p}$, *where* ζ *is an unramified character;*
(3) *either* ρ *is decomposable or for all* $\mathfrak{q} \mid \delta_D$ *with* $\mathfrak{q} \nmid p$ *we have*

$$\rho|_{G_{\mathfrak{q}}} \cong \begin{pmatrix} \theta\varepsilon_{F,p} & * \\ 0 & \theta \end{pmatrix}$$

with θ *unramified;*
(4) *for all* $\mathfrak{p} \nmid \delta_D\mathfrak{N}$ *with* $\mathfrak{p} \mid p$, *either*
 - ρ *is finite flat at* $\mathfrak{p}$, *or*
 - ρ *is ordinary at* $\mathfrak{p}$ *but not finite flat;*
(5) *the prime-to-p-conductor of* ρ *divides* $\delta_D\mathfrak{N}$.

Note that we do not make any conjectures about the properties of ρ at primes that divide both $\delta_D\mathfrak{N}$ and p, nor about the precise shape of ρ at $\mathfrak{p} \mid \delta_D$ when ρ is decomposable. The conjecture as stated will be sufficient for our purposes.

Versions of the conjecture are known in some special cases, see [60] for the first results of this nature and [15] for the most up-to-date results and overview of the state of knowledge.

Remark 6.6. In the literature, the most precise local properties of the conjectural Galois representations are typically formulated only for irreducible representations, and the most subtle properties are those for primes above p. Following a suggestion of Jack Thorne, we have performed the following sanity check: according to [32] the boundary contribution to the cohomology of arithmetic subgroups of $\operatorname{GL}_2(F)$ yields reducible Galois representations whose reduction modulo $\mathfrak{p}$ satisfies condition (4) of the conjecture.

Lemma 6.7. *Let* $p > 2$ *be a prime number. Let* G *be a group and* H *an index* 2 *subgroup. Let* $\chi\colon G \to \{\pm 1\}$ *be the corresponding quadratic character. Let* $\rho\colon G \to \operatorname{GL}_2(\overline{\mathbb{F}}_p)$ *be a semisimple representation such that*

$$\rho \otimes \chi \cong \rho.$$

Then there exists $\psi\colon H \to \overline{\mathbb{F}}_p^{\times}$ *such that*

$$\rho \cong \operatorname{Ind}_{G/H}\psi.$$

Proof. First suppose that ρ is decomposable, so that $\rho \cong \theta \oplus \theta\chi$ for some character θ of G. Since $p \neq 2$, we have $\overline{\mathbb{F}}_p[G/H] \cong \mathbf{1} \oplus \chi$, and therefore $\operatorname{Ind}_{G/H}(\theta|_H) \cong \theta \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p[G/H] \cong \rho$, as claimed.

Now suppose that ρ is irreducible. Then we have

$$\operatorname{Hom}_H(\rho|_H, \rho|_H) \cong \operatorname{Hom}_G(\rho, \operatorname{Ind}_{G/H}(\rho|_H)) \cong \operatorname{Hom}_G(\rho, \rho \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p[G/H]).$$

Since ρ is irreducible, we have $\dim \operatorname{Hom}_G(\rho, \rho) = 1$, and therefore

$$\dim \operatorname{Hom}_H(\rho|_H, \rho|_H) = 2.$$

The restriction $\rho|_H$ is therefore reducible, so there exists a 1-dimensional subrepresentation ψ of $\rho|_H$. We then have

$$\operatorname{Hom}_G(\operatorname{Ind}_{G/H}\psi, \rho) \cong \operatorname{Hom}_H(\psi, \rho|_H) \neq 0,$$

so there exists a non-zero homomorphism $f\colon \operatorname{Ind}_{G/H}\psi \to \rho$, which must be an isomorphism by irreducibility of ρ and dimension comparison. $\square$

Lemma 6.8. *Let L be a number field, let p be a prime number, let $\mathfrak{F}$ be an ideal of $\mathbb{Z}_L$ supported at the primes above p, let $\mathfrak{M}$ be an ideal of $\mathbb{Z}_L$ coprime to p, and let $\mathcal{V}$ be a set of real places of L. Then every Hecke character $\Psi\colon \mathrm{Cl}_L(\mathfrak{F}\mathfrak{M}\mathcal{V}) \to \overline{\mathbb{F}}_p^{\times}$ has modulus dividing $p\mathfrak{M}\mathcal{V}$.*

Proof. The character Ψ is at most tamely ramified at all primes of $\mathbb{Z}_L$ above p, hence the result follows. $\qquad\square$

Definition 6.9. Let L/F be a quadratic extension. A $\mathbb{Z}_p$-*shady character* of L is a Hecke character
$$\Psi\colon \mathrm{Cl}_L(p\delta_D\mathfrak{N}\mathbb{Z}_L\infty) \to \overline{\mathbb{F}}_p^{\times}$$
such that $\rho = \mathrm{Ind}_{G_F/G_L}\psi$ satisfies conditions (2)–(5) from Conjecture 6.5, where $\psi\colon G_L \to \overline{\mathbb{F}}_p^{\times}$ is the corresponding Galois character.

We now prove Theorem D for $\mathbb{Z}_p$-isospectrality.

Theorem 6.10. *Let $p > 2$ be a prime number. Assume that Conjecture 6.5 holds for all $i \geq 0$ and all maximal ideals $\mathfrak{m}$ of $\mathbb{T}$ with residue characteristic p. Let $c \in C$. Then at least one of the following two statements is true:*

(i) *there exist $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a $\mathbb{Z}_p$-shady character of the quadratic extension determined by χ;*

(ii) *there exists $T \in \mathbb{T}[W]_c$ such that for every integer $i \geq 0$ and every $b \in C$, all of the following hold:*

  (a) *T induces an isomorphism of $\mathbb{T}_1$-modules*
$$T\colon H_i(Y_b, \mathbb{Z}_{(p)}) \to H_i(Y_{cb}, \mathbb{Z}_{(p)});$$

  (b) *one has*
$$\frac{\mathrm{Reg}_i(Y_{cb})^2}{\mathrm{Reg}_i(Y_b)^2} \in \mathbb{Z}_{(p)}^{\times}.$$

Proof. Suppose that (ii) does not hold, so that at least one of (a), (b) does not hold. Then by combining Theorem 4.7(3) and Theorem 4.15(2) we deduce that there exists $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^{\perp}$ and a Hecke eigensystem $a = (T_{\mathcal{D}} \mapsto a_{\mathcal{D}})_{\mathcal{D}}$ over $\overline{\mathbb{F}}_p$ in $\overline{\mathbb{F}}_p \otimes H_i(\mathcal{Y}, \mathbb{Z})$ admitting a self-twist by χ.

Let $\mathfrak{m} \subset \mathbb{T}$ be the kernel of $a\colon \mathbb{T} \to \overline{\mathbb{F}}_p$. Since the image of $\mathbb{T}$ under a is a subring of $\overline{\mathbb{F}}_p$, and hence a field, the ideal $\mathfrak{m}$ is maximal, and by assumption we have $H_i(\mathcal{Y}, \mathbb{Z})_{\mathfrak{m}} \neq 0$. Let ρ be a Galois representation as in the conclusion of Conjecture 6.5. By property (1) and Chebotarev's density theorem we have $\rho \otimes \chi \cong \rho$. By Lemma 6.7 we have $\rho \cong \mathrm{Ind}_{G_F/G_L}\psi$, where L/F is the quadratic extension cut out by χ. By combining properties (1) and (5) of ρ with Lemmas 6.4 and 6.8 we deduce that ψ has conductor dividing $p\delta_D\mathfrak{N}\mathbb{Z}_L\infty$, so that by class field theory it corresponds to a $\mathbb{Z}_p$-shady character of L. $\qquad\square$

Lemma 6.11. *For all $T \in \mathbb{T}_1$, the endomorphisms T and $\iota(T)$ of $H_i(\mathcal{Y}, \mathbb{Z})$ are the same.*

Proof. Let $T = \prod_{j=1}^k T_{\mathcal{D}_j} \in \mathbb{T}_1$ where the $\mathcal{D}_j = (\mathfrak{a}_j, \mathfrak{b}_j)$ are such that the class of $\mathfrak{d} = \prod_{j=1}^k \mathfrak{a}_j\mathfrak{b}_j$ in C is trivial. We have
$$\iota(T) = \prod_{j=1}^k T_{\mathcal{D}_j^{-1}} = T_{(\mathfrak{d},\mathfrak{d})}^{-1} \cdot T.$$

Since the class of $\mathfrak{d}$ in C is trivial, by Proposition 4.2 there exists $\alpha \in F_+^{\times}$ such that we have $\mathfrak{d} = \alpha\mathbb{Z}_F$. But the corresponding Hecke operator $T_{(\alpha,\alpha)}$ acts trivially on $H_i(\mathcal{Y}, \mathbb{Z})$. This proves the conclusion of the lemma for T. Since Hecke operators of this form generate $\mathbb{T}_1$, this proves the lemma. $\qquad\square$

If p is a prime number, $\mathfrak{n}$ is a maximal ideal of $\mathbb{Z}_p \otimes \mathbb{T}_1$ and L/F is a quadratic extension, then we say that $\mathfrak{n}$ *corresponds to* a $\mathbb{Z}_p$-shady character Ψ of L if for every prime ideal $\mathfrak{q}$ of $\mathbb{Z}_F$ that splits in L, we have $T_\mathfrak{q} \bmod \mathfrak{n} = \Psi(\mathfrak{Q}) + \Psi(\mathfrak{Q}')$, where $\mathfrak{q}\mathbb{Z}_L = \mathfrak{Q}\mathfrak{Q}'$. We have the following local version of Theorem 6.10, proving Theorem F and Theorem G part (3).

Theorem 6.12. *Let $p > 2$ be a prime number, and let $\mathfrak{n}$ be a maximal ideal of $\mathbb{Z}_p \otimes \mathbb{T}_1$. Assume that Conjecture 6.5 holds for all $i \geq 0$ and all maximal ideals $\mathfrak{m}$ of $\mathbb{T}$ above $\mathfrak{n}$. Let $c \in C$. Then at least one of the following two statements is true:*

  (i) *there exists $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$, with corresponding quadratic extension L/F, such that $\mathfrak{n}$ corresponds to a $\mathbb{Z}_p$-shady character Ψ of L;*

  (ii) *there exists $T \in \mathbb{T}[W]_c$ such that for every integer $i \geq 0$ and every $b \in C$, all of the following hold:*

    (a) *T induces an isomorphism of $\mathbb{T}_1$-modules*

$$T \colon H_i(Y_b, \mathbb{Z})_\mathfrak{n} \to H_i(Y_{cb}, \mathbb{Z})_\mathfrak{n};$$

    (b) *we have $\mathcal{C}_{b,cb}(H_i(\mathcal{Y}, \mathbb{Z})_\mathfrak{n}) \in \mathbb{Z}_p^\times$.*

Proof. The proof follows the same pattern as that of Theorem 6.10.

Suppose that (ii) does not hold. Then in particular $H_i(\mathcal{Y}, \mathbb{Z})_\mathfrak{n}$ is non-zero, so that $\mathfrak{n}$ is lifted from a maximal ideal of the image of $\mathbb{T}_1$ in $\operatorname{End} H_i(\mathcal{Y}, \mathbb{Z}_p)$. By Lemma 6.11 we have $\iota(\mathfrak{n}) = \mathfrak{n}$. By Theorem 4.7 (4) and Theorem 4.15 (1) there is a character $\chi \in \widehat{C}_{\mathrm{iso}} \setminus c^\perp$, and a Hecke eigensystem a with a self-twist by χ. Let $\mathfrak{m} \subset \mathbb{T}$ be the kernel of $a \colon \mathbb{T} \to \overline{\mathbb{F}}_p$, let L/F be the field cut out by χ, and $\rho = \operatorname{Ind}_{G_F/G_L} \psi$ be the Galois representation attached by Conjecture 6.5. Then an easy calculation (see also the proof of Theorem 6.22, specifically equation (6.25)) shows that $\mathfrak{n}$ corresponds to the Hecke character Ψ attached to ψ by class field theory. $\qquad\square$

6.3. **Local representations.** For the duration of this subsection, let $p > 2$ be a prime, let E be a p-adic field, we write $\mathbb{Z}_E$ for its ring of integers, π for its uniformiser, e for its ramification index over $\mathbb{Q}_p$, so that $p \in \pi^e \mathbb{Z}_E^\times$, f for the residue field degree $[\mathbb{E} : \mathbb{F}_p]$, and we take $\overline{\mathbb{E}}$ to be the residue field of the maximal unramified extension E_{nr} of E. For each $n \geq 1$, let $\mathbb{E}_n$ be the subfield of $\overline{\mathbb{E}}$ of degree n over $\mathbb{F}_p$ (so that $\mathbb{E} = \mathbb{E}_f$).

Let $I \supset P$ be the inertia group, respectively the wild inertia inside G_E, and let $I_\mathrm{t} = I/P$.

In this subsection we abbreviate $\varepsilon_{E,p}$ to ε and Frob_E to Frob.

Let $n \in \mathbb{Z}_{\geq 1}$, let $q = p^n$ and $d \mid q - 1$. Define a character $\theta_d \colon I_\mathrm{t} \to \mathbb{E}_n^\times$ by

$$\frac{s(\pi^{1/d})}{\pi^{1/d}} = \theta_d(s) \in \mu_d(E_{\mathrm{nr}}) \cong \mu_d(\mathbb{E}_n) \text{ for all } s \in I_\mathrm{t}.$$

A *fundamental character of level n* is a character $I_\mathrm{t} \to \overline{\mathbb{F}}_p^\times$ of the form

$$\omega_\tau = \tau \circ \theta_{q-1}$$

for some $\tau \in \operatorname{Hom}(\mathbb{E}_n, \overline{\mathbb{F}}_p)$. Let φ be the absolute Frobenius automorphism $x \mapsto x^p$ of $\overline{\mathbb{F}}_p$. If one chooses a $\tau_0 \in \operatorname{Hom}(\mathbb{E}_n, \mathbb{F}_q)$, then every character $\chi \colon I_\mathrm{t} \to \mathbb{F}_q^\times$ is of the form

$$\chi = \omega_{\tau_0}^a$$

for some $a \in \mathbb{Z}$.

Remark 6.13. If we have an equality

$$\omega_{\tau_0}^a = \prod_{\tau \in \operatorname{Hom}(\mathbb{E}_n, \mathbb{F}_q)} \omega_\tau^{h_\tau}$$

for $h_\tau \in \mathbb{Z}$, then the exponents satisfy $a \equiv \sum_{i=0}^{n-1} h_{\varphi^i \circ \tau_0} p^i \mod (q-1)$. Since ω_{τ_0} has order $q-1$, we may always choose $a \in \{0, \ldots, q-2\}$ and the $h_\tau \in \{0, \ldots, p-1\}$, in which case the h_τ are uniquely determined by χ, except that $(0, \ldots, 0)$ and $(p-1, \ldots, p-1)$ both represent the trivial character.

If $n, m \in \mathbb{Z}_{\geq 1}$ are such that $n \mid m$, and $\tau \in \mathrm{Hom}(\mathbb{E}_n, \overline{\mathbb{F}}_p)$, then

$$\omega_\tau = \prod_{\tau' \in \mathrm{Hom}(\mathbb{E}_m, \overline{\mathbb{F}}_p), \tau'|_{\mathbb{E}_n} = \tau} \omega_{\tau'}.$$

By abuse of notation, whenever χ is a character of $I_{\mathrm{t}} = I/P$, we will also denote by χ its inflation to I that is trivial on P.

Let $n \in \mathbb{Z}_{\geq 1}$, set $q = p^n$, and let $\psi \colon G_E \to \mathbb{F}_q^\times$ be a character. Then ψ is tamely ramified since the order of $\mathbb{F}_q^\times$ is coprime to p, and for every choice $\mathrm{Frob} \in G_E$ of Frobenius element and for every $s \in I$ we have

$$\psi(s) = \psi(\mathrm{Frob} \cdot s \cdot \mathrm{Frob}^{-1}) = \psi(s)^{\#\mathbb{E}}.$$

Thus, we may always assume that $n \mid f$. In particular, we can write

$$\psi|_I = \prod_{\tau \in \mathrm{Hom}(\mathbb{E}, \mathbb{F}_{p^f})} \omega_\tau^{h_\tau}.$$

In other words, we can decompose $\psi|_I$ using only fundamental characters of level f. This also implies that

$$(\omega_\tau)^\sigma = \omega_{\tau \circ \sigma}$$

for every $\tau \in \mathrm{Hom}(\mathbb{E}, \overline{\mathbb{F}}_p)$ and all $\sigma \in G_E$.

Proposition 6.14. *The mod p cyclotomic character satisfies $\varepsilon|_I = \theta_{p-1}^e$.*

Proof. See [61, Proposition 8]. $\qquad\square$

We will typically use this proposition in the following form: for every $n \in \mathbb{Z}_{\geq 1}$ we have $\varepsilon|_I = \prod_\tau \omega_\tau^e$ where the product is over $\tau \in \mathrm{Hom}(\mathbb{E}_n, \mathbb{F}_{p^n})$.

Theorem 6.15 (Raynaud). *Let $\rho \colon G_E \to \mathrm{GL}_n(\mathbb{F}_p)$ be a representation. If ρ is finite flat, then so is its semisimplification. If ρ is semisimple, then it is finite flat if and only if each of its irreducible factors is finite flat. If ρ is irreducible, then it is finite flat if and only if there exists a character $\psi \colon G_E \to \mathbb{F}_q^\times$, where $q = p^n$, satisfying*

$$\psi|_I = \prod_{\tau \in \mathrm{Hom}(\mathbb{E}_n, \mathbb{F}_q)} \omega_\tau^{h_\tau}$$

with $0 \leq h_\tau \leq e$, such that

$$\rho \cong \mathrm{Res}_{\mathbb{F}_p}^{\mathbb{F}_q} \psi.$$

Proof. [Raynaud, Corollaire 3.3.7 and Théorème 3.4.3] $\qquad\square$

Lemma 6.16. *Let $\rho \colon G_E \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an ordinary representation such that $\det \rho = \zeta \cdot \varepsilon$, where ζ is an unramified character, and such that ρ is either induced from a 1-dimensional character or decomposable. Then ρ is finite flat.*

Proof. If ρ is induced from a 1-dimensional character, then it is semisimple. Since it is also ordinary, it has a stable line, and therefore is decomposable into a direct sum of 1-dimensional characters. The proof thus reduces to the case that ρ is decomposable.

Assume that ρ is a direct sum of two 1-dimensional characters. Since ρ is ordinary, one of these characters is unramified. Due to the determinant assumption, the other one is of the form $\theta\varepsilon$, where θ is an unramified character. By Theorem 6.15 and Proposition 6.14, ρ is finite flat. $\qquad\square$

Lemma 6.17. *Suppose there exists a finite flat character $\theta\colon G_E \to \overline{\mathbb{F}}_p^\times$ and an unramified character $\phi\colon G_E \to \overline{\mathbb{F}}_p^\times$ such that $\theta^2\phi = \varepsilon$. Then we have $e > 1$.*

Proof. Suppose that $e = 1$. Let $n \in \mathbb{Z}_{\geq 1}$ be such that θ takes values in $\mathbb{F}_{p^n}^\times$. By Theorem 6.15 we may write

$$\theta|_I = \prod_{\tau \in \mathrm{Hom}(\mathbb{E}_n, \overline{\mathbb{F}}_p)} \omega_\tau^{h_\tau}$$

with $h_\tau \in \{0, 1\}$. By assumption and by Proposition 6.14 we have

$$\prod_{\tau \in \mathrm{Hom}(\mathbb{E}_n, \overline{\mathbb{F}}_p)} \omega_\tau^{2h_\tau} = (\theta^2\phi)|_I = \varepsilon|_I = \prod_{\tau \in \mathrm{Hom}(\mathbb{E}_n, \overline{\mathbb{F}}_p)} \omega_\tau.$$

All exponents of ω_τ appearing in the equality are in $\{0, 1, \ldots, p-1\}$, and $1 \notin \{0, p-1\}$, so Remark 6.13 implies that these exponents are uniquely determined. Thus, we have $2h_\tau = 1$ for all τ, which is impossible. $\square$

6.4. **Global representations.** If L is a number field and $\mathfrak{P}$ is a maximal ideal of $\mathbb{Z}_L$, then we denote by $\mathbb{F}_{\mathfrak{P}}$ the residue field $\mathbb{Z}_L/\mathfrak{P}$, and by $I_{\mathfrak{P}} \subset G_{\mathfrak{P}}$ the inertia subgroup at $\mathfrak{P}$.

In this subsection we prove the main result of the section, Theorem 6.22, which is a precise version of Theorem H from the introduction.

Lemma 6.18. *Let L/F be a quadratic extension of number fields, let $p > 2$ be a prime number that is unramified in L, let $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a Galois representation of the form $\rho = \mathrm{Ind}_{G_F/G_L} \psi$, where $\psi\colon G_L \to \overline{\mathbb{F}}_p^\times$ is a character, and let $\mathfrak{p} \mid p$. Assume that ρ is finite flat at $\mathfrak{p}$ and that $\det \rho = \zeta \cdot \varepsilon_{F,p}$ for an unramified character ζ. Then for every prime $\mathfrak{P} \mid \mathfrak{p}$ of L we have*

$$\psi|_{I_{\mathfrak{P}}} = \prod_{\tau \in \mathrm{Hom}(\mathbb{F}_{\mathfrak{P}}, \overline{\mathbb{F}}_p)} \omega_\tau^{h_\tau}$$

for some collection $(h_\tau)_\tau \in \{0, 1\}^{\mathrm{Hom}(\mathbb{Z}_L/\mathfrak{p}\mathbb{Z}_L, \overline{\mathbb{F}}_p)} = \prod_{\mathfrak{P}|\mathfrak{p}} \{0, 1\}^{\mathrm{Hom}(\mathbb{F}_{\mathfrak{P}}, \overline{\mathbb{F}}_p)}$ that is balanced, as in Definition 5.36.

Proof. Let σ be the non-trivial element of the Galois group. Fixing an extension of the discrete $\mathfrak{p}$-adic valuation on F to $\overline{F}$ identifies $G_{\mathfrak{p}}$ with a subgroup of G_F, and determines a prime $\mathfrak{P}$ of L above $\mathfrak{p}$. We have the inclusions

$$
\begin{array}{ccccc}
I_{\mathfrak{p}} & \subset & G_{\mathfrak{p}} & \subset & G_F \\
\cup & & \cup & & \cup \\
I_{\mathfrak{P}} & \subset & G_{\mathfrak{P}} & \subset & G_L.
\end{array}
$$

Of the vertical inclusions, the right hand one has index 2, so that G_L is a normal subgroup of G_F, while the left one is an equality, since L/F is unramified at $\mathfrak{p}$. These inclusions also induce inclusions $\mathbb{F}_{\mathfrak{p}} \subset \mathbb{F}_{\mathfrak{P}} \cong \mathbb{F}_{\mathfrak{P}^\sigma} \subset \overline{\mathbb{F}}_p$, of which the first one is either an equality or an inclusion of index 2, and where the middle isomorphism is induced by σ.

Since $\mathfrak{p}$ is unramified in L/F, a full set of double coset representatives for $I_{\mathfrak{p}} \backslash G_F / G_L$ is given by $\{1, \sigma\}$, and we have $I_{\mathfrak{p}} = I_{\mathfrak{P}} = I_{\mathfrak{p}} \cap G_L = I_{\mathfrak{p}} \cap G_L^\sigma$. Mackey's formula therefore implies that we have

$$\rho|_{I_{\mathfrak{p}}} \cong \psi|_{I_{\mathfrak{P}}} \oplus \psi^\sigma|_{I_{\mathfrak{P}}}.$$

Theorem 6.15 implies that $\psi|_{I_{\mathfrak{P}}}$ is of the form

$$\psi|_{I_{\mathfrak{P}}} = \prod_{\tau \in \mathrm{Hom}(\mathbb{F}_{\mathfrak{P}}, \overline{\mathbb{F}}_p)} \omega_\tau^{h_\tau}$$

with $h_\tau \in \{0, 1\}$ for all τ.

Since we have $\det \rho = \zeta \cdot \varepsilon_{F,p}$ for an unramified character ζ, we obtain, using Proposition 6.14,

$$\varepsilon_{F,p}|_{I_\mathfrak{p}} = \prod_\tau \omega_\tau = \prod_\tau \omega_\tau^{h_\tau} \prod_\tau (\omega_\tau^\sigma)^{h_\tau} = \prod_\tau \omega_\tau^{h_\tau} \omega_{\tau \circ \sigma}^{h_\tau} = \prod_\tau \omega_\tau^{h_\tau + h_{\tau \circ \sigma}},$$

where the products run over $\tau \in \mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)$. Since we have $p-1 \geq 2$, Remark 6.13 implies that $h_\tau + h_{\tau \circ \sigma} = 1$ for all τ, i.e. the collection $(h_\tau)_\tau$ is balanced. $\square$

Definition 6.19. Given a finite extension L/F of number fields, a non-zero ideal $\mathfrak{M}$ of $\mathbb{Z}_L$, and a set $\mathcal{V}$ of real places of L, we denote by $U_L^{(F)}(\mathfrak{M}\mathcal{V})$ the group of units of $\mathbb{Z}_L$ that are congruent to an element of $\mathbb{Z}_F$ modulo $\mathfrak{M}$ and that are positive at all places in $\mathcal{V}$.

Lemma 6.20. *Let L/F be a quadratic extension of number fields that is unramified at all finite places, p a prime number unramified in L, and $\mathfrak{M}$ a non-zero ideal of $\mathbb{Z}_L$. Let $(h_\tau)_\tau$ be a collection of integers indexed by $\tau \in \mathrm{Hom}(\mathbb{Z}_L/p, \overline{\mathbb{F}}_p) = \bigsqcup_{\mathfrak{P}|p} \mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)$. Let $\psi \colon G_L \to \overline{\mathbb{F}}_p^\times$ be a character with prime-to-p-conductor dividing $\mathfrak{M}$ such that the character $\det \mathrm{Ind}_{G_F/G_L} \psi$ of G_F is unramified at all primes $\mathfrak{q}$ of $\mathbb{Z}_F$ satisfying $\mathfrak{q} \mid \mathfrak{M} \cap \mathbb{Z}_F$ and $\mathfrak{q} \nmid p$, and such that for all prime ideals $\mathfrak{P} \mid p$ of $\mathbb{Z}_L$ we have*

$$\psi|_{I_\mathfrak{P}} = \prod_{\tau \in \mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)} \omega_\tau^{h_\tau}.$$

Let $\tilde{L}$ be the Galois closure of L over $\mathbb{Q}$ and let $\tilde{\mathfrak{P}} \mid p$ be a prime of $\tilde{L}$, inducing a bijection $\beta \colon \mathrm{Hom}(L, \tilde{L}) \cong \mathrm{Hom}(\mathbb{Z}_L/p, \overline{\mathbb{F}}_p)$. Then for all $u \in U_L^{(F)}(\mathfrak{M}\infty)$, we have

$$\prod_{\tau \in \mathrm{Hom}(L, \tilde{L})} \tau(u)^{h_{\beta(\tau)}} \equiv 1 \bmod \tilde{\mathfrak{P}}.$$

Proof. Without loss of generality, we may replace $\mathfrak{M}$ by its coprime-to-p part. By class field theory, ψ corresponds to a character

$$\mathrm{Cl}_L(\mathfrak{F}\mathfrak{M}\infty) \to \overline{\mathbb{F}}_p^\times,$$

where $\mathfrak{F}$ is an ideal supported at the primes above p, and in fact by Lemma 6.8 it corresponds to a character

$$\Psi \colon \mathrm{Cl}_L(p\mathfrak{M}\infty) \to \overline{\mathbb{F}}_p^\times.$$

Moreover, by [24] the assumption on $\det \mathrm{Ind}_{G_F/G_L} \psi$ is equivalent to Ψ factoring through $\mathrm{Cl}_L(p\mathfrak{M}\infty)/(\mathbb{Z}_F/(\mathfrak{M} \cap \mathbb{Z}_F))^\times$. We have an exact sequence

$$(6.21) \qquad 1 \to \frac{\prod_{\mathfrak{P}|p} \mathbb{F}_\mathfrak{P}^\times}{U_L^{(F)}(\mathfrak{M}\infty)} \to \frac{\mathrm{Cl}_L(p\mathfrak{M}\infty)}{(\mathbb{Z}_F/(\mathfrak{M} \cap \mathbb{Z}_F))^\times} \to \frac{\mathrm{Cl}_L(\mathfrak{M}\infty)}{(\mathbb{Z}_F/(\mathfrak{M} \cap \mathbb{Z}_F))^\times} \to 1.$$

By [61, Section 1.5, Proposition 3], the restriction of Ψ to $\prod_{\mathfrak{P}|p} \mathbb{F}_\mathfrak{P}^\times$ satisfies

$$\Psi|_{\prod_{\mathfrak{P}|p} \mathbb{F}_\mathfrak{P}^\times} = \prod_{\mathfrak{P}|p} \prod_{\tau \in \mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)} \tau^{-h_\tau}.$$

This forces $\prod_{\tau \in \mathrm{Hom}(\mathbb{Z}_F/p, \overline{\mathbb{F}}_p)} \tau^{-h_\tau}$ to be trivial on the image of $U_L^{(F)}(\mathfrak{M}\infty)$, which is equivalent to the claimed statement. $\square$

We now prove Theorem H from the introduction.

Theorem 6.22. *Assume Conjecture 6.5. Let x be the exponent of Cl_F, let $\chi \in \widehat{C}_{\mathrm{iso}}$, let L/F be the corresponding quadratic extension, let $\tilde{L}$ denote the Galois closure of L over $\mathbb{Q}$. Let S be the union of the following sets of prime numbers: the primes that are ramified in L; the prime numbers that divide $2\,\mathrm{Nm}_{F/\mathbb{Q}}(\delta_D \mathfrak{N})$; and the prime numbers p for which there exists an ideal $\mathfrak{M}$ of $\mathbb{Z}_L$ such that $\mathrm{Nm}_{L/F}\mathfrak{M}$ divides $\delta_D\mathfrak{N}$, and a balanced collection $(h_\tau)_\tau \in \{0,1\}^{\mathrm{Hom}(L,\tilde{L})}$ such that all of the following hold:*

(a) *for all units $u \in U_L^{(F)}(\mathfrak{M}\infty)$, the prime p divides*

$$(6.23) \qquad \mathrm{Nm}_{\tilde{L}/\mathbb{Q}}\left(\prod_{\tau \in \mathrm{Hom}(L,\tilde{L})} \tau(u)^{h_\tau} - 1\right);$$

(b) *for every prime ideal $\mathfrak{q}$ of $\mathbb{Z}_F$ dividing δ_D, let $\mathfrak{Q}$ be a prime of $\mathbb{Z}_L$ above $\mathfrak{q}$, let k be the order of $\mathfrak{Q}$ in $\mathrm{Cl}_L(\mathfrak{N}\mathbb{Z}_L\infty)/(\mathbb{Z}_F/\mathfrak{N})^\times$, and let $\alpha \in \mathbb{Z}_F + \mathfrak{N}\mathbb{Z}_L$ be a totally positive generator of $\mathfrak{Q}^k$; then p divides*

$$(6.24) \qquad \mathrm{Nm}_{\tilde{L}/\mathbb{Q}}\left(\left(\prod_\tau \tau(\alpha)^{2h_\tau x} - \mathrm{Nm}_{L/\mathbb{Q}}(\mathfrak{Q})^{2kx}\right)\left(\prod_\tau \tau(\alpha)^{2h_\tau x} - 1\right)\right).$$

Then:

(1) *every prime number p such that there exists a $\mathbb{Z}_p$-shady character of L is contained in S;*
(2) *the set S is infinite if and only if there exists an $\mathcal{H}^\bullet$-shady character of L.*

Proof. Let σ be the non-trivial automorphism of L/F. We first prove (1). Let p be a prime number such that there exists a $\mathbb{Z}_p$-shady character Ψ. If $p|2\,\mathrm{Nm}_{F/\mathbb{Q}}(\delta_D\mathfrak{N})$, then it is in S. Assume otherwise.

Let $\psi \colon G_L \to \overline{\mathbb{F}}_p^\times$ be the corresponding Galois character, and let $\rho = \mathrm{Ind}_{G_F/G_L}\psi$, which, in particular, satisfies the properties of Conjecture 6.5. By property (2) we have $\det \rho = \zeta \cdot \varepsilon_{F,p}$, where ζ is an unramified character. Let $\mathfrak{p}$ be any prime ideal of $\mathbb{Z}_F$ above p. By Mackey's formula the restriction $\rho|_{G_\mathfrak{p}}$ is either also induced from a 1-dimensional character or decomposable. By property (4) and Lemma 6.16 the representation ρ is finite flat at $\mathfrak{p}$. Now consider two cases.

First suppose that ρ is decomposable. Since $p \neq 2$, we have $\rho \cong \theta \oplus \theta\chi$ for some character $\theta \colon G_F \to \overline{\mathbb{F}}_p^\times$ that is finite flat at $\mathfrak{p}$. By Lemma 6.17, applied with $\phi = \chi\zeta^{-1}$, the prime $\mathfrak{p}$ is ramified in F, so that we have $p \in S$.

Next suppose that ρ is irreducible. If p ramifies in L, then we have $p \in S$. Assume otherwise. By Lemma 6.18 we may write, for every prime ideal $\mathfrak{P}$ of $\mathbb{Z}_L$ above $\mathfrak{p}$,

$$\psi|_{I_\mathfrak{P}} = \prod_{\tau \in \mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)} \omega_\tau^{h'_\tau},$$

where the collection $(h'_\tau)_\tau \in \{0,1\}^{\mathrm{Hom}(\mathbb{Z}_L/\mathfrak{p}\mathbb{Z}_L, \overline{\mathbb{F}}_p)} = \prod_{\mathfrak{P}|\mathfrak{p}}\{0,1\}^{\mathrm{Hom}(\mathbb{F}_\mathfrak{P}, \overline{\mathbb{F}}_p)}$ is balanced. Let $\mathfrak{M}$ be the prime-to-p-conductor of Ψ. Since L/F is unramified at all finite primes, Lemma 6.4 and property (5) of ρ imply that we have $(\mathrm{Nm}_{L/F}\mathfrak{M})|\delta_D\mathfrak{N}$. Since $\mathfrak{p}|p$ was arbitrary, we may apply Lemma 6.20 to obtain $\tilde{\mathfrak{P}}$ and β as in Lemma 6.20 such that for every $u \in U_L^{(F)}(\mathfrak{M}\infty)$ we have

$$\prod_{\tau \in \mathrm{Hom}(L,\tilde{L})} \tau(u)^{h'_{\beta(\tau)}} \equiv 1 \bmod \tilde{\mathfrak{P}}.$$

Reduction mod $\tilde{\mathfrak{P}}$ also induces a bijection $\beta_F \colon \mathrm{Hom}(F, \tilde{L}) \cong \mathrm{Hom}(\mathbb{Z}_F/p, \overline{\mathbb{F}}_p)$ such that the following square is commutative:

$$
\begin{array}{ccc}
\mathrm{Hom}(L, \tilde{L}) & \xrightarrow{\ \text{restriction}\ } & \mathrm{Hom}(F, \tilde{L}) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_F} \\
\mathrm{Hom}(\mathbb{Z}_L/p, \overline{\mathbb{F}}_p) & \xrightarrow{\ \text{res.}\ } & \mathrm{Hom}(\mathbb{Z}_F/p, \overline{\mathbb{F}}_p).
\end{array}
$$

Therefore the collection $(h_\tau = h'_{\beta(\tau)})_\tau$ indexed by $\tau \in \mathrm{Hom}(L, \tilde{L})$ is balanced, and for all $u \in U_L^{(F)}(\mathfrak{M}\infty)$ the prime p divides

$$
\mathrm{Nm}_{\tilde{L}/\mathbb{Q}}\left(\prod_{\tau \in \mathrm{Hom}(L, \tilde{L})} \tau(u)^{h_\tau} - 1 \right).
$$

This proves divisibility (a), and it remains to prove divisibility (b).

Let $\mathfrak{q}$ be a prime ideal of $\mathbb{Z}_F$ dividing δ_D. Since we have

$$
C_{\mathrm{iso}} = \mathrm{Cl}_F(\mathcal{V}_{\mathbb{H}}(D))/\langle \mathfrak{a}^2, \mathfrak{p} \text{ ramified in } D, \mathfrak{p}^e \| \mathfrak{N} \rangle,
$$

the prime $\mathfrak{q}$ splits in L. Let $\mathfrak{Q}$ be a prime ideal of $\mathbb{Z}_L$ above $\mathfrak{q}$. By Mackey's formula, we have

$$
(6.25) \qquad \rho|_{G_\mathfrak{q}} \cong \psi|_{G_\mathfrak{Q}} \oplus \psi^\sigma|_{G_\mathfrak{Q}}.
$$

Since $\mathfrak{q}$ does not divide p and ρ is irreducible, property (3) of ρ says

$$
\rho|_{G_\mathfrak{q}} \cong \begin{pmatrix} \theta\varepsilon_{F,p} & * \\ 0 & \theta \end{pmatrix}
$$

with θ unramified. But $\rho|_{G_\mathfrak{q}}$ is semisimple, so we have

$$
(6.26) \qquad \rho|_{G_\mathfrak{q}} \cong \theta\varepsilon_{F,p} \oplus \theta.
$$

In particular ρ is unramified at $\mathfrak{q}$, and therefore ψ is unramified at $\mathfrak{Q}$. Since $\mathfrak{q}|\delta_D$ and $\mathfrak{Q}|\mathfrak{q}$ were arbitrary, the conductor of Ψ divides $p\mathfrak{N}\mathbb{Z}_L\infty$. By property (2) of ρ, we have $\theta^2 = \zeta|_{G_\mathfrak{q}}$, so that, in particular, we have $\theta^{2x} = 1$. By equations (6.25) and (6.26) we get

$$
\{\psi(\mathrm{Frob}_\mathfrak{Q})^{2x}, \psi^\sigma(\mathrm{Frob}_\mathfrak{Q})^{2x}\} = \{\varepsilon_{L,p}(\mathrm{Frob}_\mathfrak{Q})^{2x}, 1\},
$$

in other words,

$$
(6.27) \qquad \{\Psi(\mathfrak{Q})^{2x}, \Psi^\sigma(\mathfrak{Q})^{2x}\} = \{\mathrm{Nm}_{L/\mathbb{Q}}(\mathfrak{Q})^{2x}, 1\}.
$$

Let k and α be as in (b). By the exact sequence (6.21), applied with $\mathfrak{M} = \mathfrak{N}\mathbb{Z}_L$, we have

$$
\prod_{\tau \in \mathrm{Hom}(\mathbb{Z}_L/p, \overline{\mathbb{F}}_p)} \tau(\alpha)^{h'_\tau} = \Psi(\mathfrak{Q}^k).
$$

Combining this with equation (6.27) we deduce that one of the following congruences holds:

$$
\prod_{\tau \in \mathrm{Hom}(L, \tilde{L})} \tau(\alpha)^{2h_\tau x} \equiv \mathrm{Nm}_{L/\mathbb{Q}}(\mathfrak{Q})^{2kx} \mod \tilde{\mathfrak{P}}, \text{ or}
$$

$$
\prod_{\tau \in \mathrm{Hom}(L, \tilde{L})} \tau(\alpha)^{2h_\tau x} \equiv 1 \mod \tilde{\mathfrak{P}},
$$

which proves (b).

Now we prove (2). For a number field M and a non-zero ideal $\mathfrak{A}$ of $\mathbb{Z}_M$, define

$$
\hat{U}_M(\mathfrak{A}) = \{(u_v)_v \in \prod_{v \nmid \infty} \mathbb{Z}_{M,v}^\times : u_v \equiv 1 \mod \mathfrak{A}\mathbb{Z}_{M,v}\} \subset \mathbb{A}_M^\times.
$$

If Ψ is a Hecke character of L, then the central character of $\mathrm{AI}_F^L(\Psi)$ is $\chi \cdot \Psi|_{\mathbb{A}_F^\times}$ (see e.g. [59] or [26, Appendix B]).

First suppose that there exists an $\mathcal{H}^\bullet$-shady character Ψ of L. By Corollary 5.21 this implies that we have $\delta_D = \mathbb{Z}_F$, so that condition (b) on p is empty. By Proposition 5.37 the character $\Psi^{\mathrm{alg}} = \Psi\| \cdot \|^{-1/2}$ is an algebraic Hecke character whose type $(h_\tau)_{\tau \in \mathrm{Hom}(L,\tilde{L})}$ is balanced. This implies that we have $(\Psi^{\mathrm{alg}})^\sigma \neq \Psi^{\mathrm{alg}}$, so by Theorem 5.7 the automorphic induction $\Pi = \mathrm{AI}_F^L(\Psi^{\mathrm{alg}})$ is infinite-dimensional. By Lemma 5.4 the conductor $\mathfrak{M}$ of Ψ^{alg} satisfies $\mathrm{Nm}_{L/F}(\mathfrak{M})|\mathfrak{N}$. Moreover, the central character $\chi \cdot \Psi^{\mathrm{alg}}|_{\mathbb{A}_F^\times}$ of Π vanishes on $K_0(\mathfrak{N}) \cap \mathbb{A}_F^\times = \hat{U}_F(1)$, so that Ψ^{alg} vanishes on $\hat{U}_F(1)\hat{U}_L(\mathfrak{M})$. This implies that for all $u \in U_L^{(F)}(\mathfrak{M}\infty)$ the infinity-component of Ψ^{alg} satisfies

$$\prod_{v|\infty} \Psi_v^{\mathrm{alg}}(u) = 1,$$

so that all prime numbers p satisfy divisibility (a). This shows that the set S is infinite.

Conversely, suppose that S if infinite. There are only finitely many distinct balanced collections $(h_\tau)_\tau \in \{0,1\}^{\mathrm{Hom}(L,\tilde{L})}$, and only finitely many ideals $\mathfrak{M}$ of $\mathbb{Z}_L$ such that $\mathrm{Nm}_{L/F} \mathfrak{M}$ divides $\delta_D\mathfrak{N}$. Pick a collection $(h_\tau)_\tau$ and an ideal $\mathfrak{M}$ for which infinitely many primes satisfy the divisibilities (a) and (b). Suppose that there exists a prime ideal $\mathfrak{q}|\delta_D$, and let $\mathfrak{Q}$, k and α be as in (b). Since the collection $(h_\tau)_\tau$ is balanced, we have

$$\left(\prod_\tau (\tau(\alpha)\tau(\alpha)^\sigma)^{h_\tau} \right)^{2x} = \left(\mathrm{Nm}_{L/\mathbb{Q}}\,\alpha \right)^{2x} = \mathrm{Nm}_{L/\mathbb{Q}}(\mathfrak{Q}^{2kx}).$$

Since α generates a non-trivial ideal, we have $\prod_\tau(\tau(\alpha)^{2h_\tau x}) \neq 1$, and similarly for α^σ in place of α, so that the expression in (6.24) is non-zero. This contradicts that it is divisible by infinitely many primes. Hence we have $\delta_D = \mathbb{Z}_F$.

Next, for every $u \in U_L^{(F)}(\mathfrak{M}\infty)$, infinitely many primes dividing the expression (6.23) forces it to be 0. This implies that there exists an algebraic Hecke character Ψ^{alg} of L with conductor dividing $\mathfrak{M}$, with type $(h_\tau)_\tau$, and such that Ψ^{alg} vanishes on $\hat{U}_F(1)$. Note that we have $K_0(\mathfrak{N}) = K_1(\mathfrak{N}) \cdot (K_0(\mathfrak{N}) \cap \mathbb{A}_F^\times)$. By Proposition 5.37 and Lemma 5.4 the Hecke character $\Psi = \Psi^{\mathrm{alg}}\| \cdot \|^{1/2}$ is an $\mathcal{H}^\bullet$-shady character of L, as required. $\square$

7. Curiosity cabinet

We continue to make Assumption 6.1. Given a number field L and a non-zero ideal $\mathfrak{F}$ of $\mathbb{Z}_L$, we write $\mathbf{HC}_{L,\mathfrak{F}}$ for the group of Hecke characters of L of conductor dividing $\mathfrak{F}$. If Γ is a subgroup of $D^\times$, let $P\Gamma$ denote the image of Γ in $D^\times/F^\times$.

In Subsection 7.1, we first explain that the existence of the various kinds of shady characters is computable, and then give a criterion to prove non-isospectrality. In Subsection 7.2, we give a series of examples illustrating our methods, and we prove Theorems A, B and E.

7.1. **Toolkit.** By [48], there exist algorithms performing the following tasks:
- given $(L, \mathfrak{F})$, compute the structure of $\mathbf{HC}_{L,\mathfrak{F}}$;
- given $\Psi \in \mathbf{HC}_{L,\mathfrak{F}}$, compute the multiplicative order of Ψ;
- given $\Psi \in \mathbf{HC}_{L,\mathfrak{F}}$, a non-zero ideal $\mathfrak{a}$, and a target precision, compute $\Psi(\mathfrak{a}) \in \mathbb{C}^\times$ to the required precision;
- given $\Psi \in \mathbf{HC}_{L,\mathfrak{F}}$, an infinite place v of L, and a target precision, compute the parameters $(k_v, s_v) \in \mathbb{Z} \times \mathbb{C}$ to the required precision.

This is not quite enough to decide existence of shady characters, because their definitions involve vanishing conditions on the parameters s_v, so having access to approximations is not sufficient. We obtain decision algorithms as follows:

- For L^2-, $\Omega^\bullet$ and Ω^0-shady characters, by Lemma 5.17 it is sufficient to be able to compute the action of a given automorphism of L on $\mathbf{HC}_{L,\mathfrak{F}}$. This is possible using the description of $\mathbf{HC}_{L,\mathfrak{F}}$ in [48] as the dual of a group on which the action can clearly be computed.
- For $\mathcal{H}^\bullet$-shady characters, by Proposition 5.37 it is sufficient to be able to compute groups of algebraic characters, for which an algorithm is described in [48].
- For $\mathbb{Z}_p$-shady characters, it is sufficient to be able to compute ray class groups, for which there are standard algorithms [19, Chapter 4].

Definition 7.1. Let L/F be a quadratic extension, let $i \in \mathbb{Z}_{\geq 0}$, and $\lambda \in \mathbb{R}$. Then an $\Omega^i_{\Delta=\lambda}$-*shady character* of L is a $(\Omega^{\boldsymbol{i}}_{\Delta=\boldsymbol{\lambda}})$-shady character of L for some $\boldsymbol{i} = (i_v)_v$, $\boldsymbol{\lambda} = (\lambda_v)_v$ satisfying $\sum_v i_v = i$ and $\sum_v \lambda_v = \lambda$, with both sums running over the infinite places v of F.

The following result will be used to prove non-i-isospectrality.

Lemma 7.2. *Assume that $\mathcal{V}_\mathbb{R}(D) = \emptyset$, that $\delta_D = 1$, and that C has order 2, let $c \in C$ be the non-trivial element, and L/F the corresponding quadratic extension. Let $i \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{R}$. Suppose that there exists a unique pair $\{\Psi, \Psi^{-1}\}$ of $\Omega^i_{\Delta=\lambda}$-shady characters of L. Suppose in addition that the conductor $\mathfrak{F}$ of Ψ satisfies that $\mathfrak{N}/\operatorname{Nm}_{L/F}(\mathfrak{F})$ is the square of an ideal of $\mathbb{Z}_F$. Then the orbifolds Y_1 and Y_c are not i-isospectral.*

Proof. It suffices to show that the multiplicity of λ in the Δ-spectrum on $\Omega^i(\mathcal{Y})$ is odd. Let $\chi \in \widehat{C}$ be the unique non-trivial character.

First let $\Pi = \Pi_\infty \otimes \Pi_f$ be an automorphic representation of $\mathbb{G}(\mathbb{A}_F)$ such that $\Pi \otimes \chi \not\cong \Pi$. Then Π and $\Pi \otimes \chi$ have the same Casimir eigenvalue. Moreover, since the Hecke character χ is trivial on $\operatorname{nrd}(K_f)$, the dimensions of $\Pi_f^{K_f}$ and of $(\Pi_f \otimes \chi)^{K_f}$ are equal. Thus, the contribution from Π and from its twist by χ to the λ-eigenspace under the isomorphism of Corollary 5.12 is even-dimensional.

By assumption, there exists a unique automorphic representation Π of $\mathbb{G}(\mathbb{A}_F)$ such that $\Pi_f^{K_f}$ is non-trivial, $\Pi \otimes \chi \cong \Pi$, and Π contributes to $\Omega^i_{\Delta=\lambda}$. We now apply Proposition 5.25 in conjunction with Corollary 5.12. The hypothesis on $\mathcal{V}_\mathbb{R}(D)$ ensures that, in the notation of Proposition 5.25, we have $r = 0$. Moreover, by Lemmas 5.3 and 5.4, the assumption on the conductor of Ψ implies that the contribution from Π to multiplicity of the λ-eigenspace is odd-dimensional. $\square$

7.2. **Collection.** The following example reproduces [44, Example 6.1] with our method.

Example 7.3. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^6 - 2\alpha^5 - \alpha^4 + 4\alpha^3 - 4\alpha^2 + 1 = 0$. The field F is the unique number field of discriminant -974528 and signature $(4, 1)$ (LMFDB `6.4.974528.1`). Let D be the unique quaternion division algebra ramified at every real place and no finite place of F. Let $\mathfrak{N} = (1)$. We have $C \cong C_{\text{iso}} \cong C_2$, which therefore has a single non-trivial character χ, corresponding to the quadratic extension $L = F(\sqrt{2\alpha - 1})$.

Let $C = \{1, c\}$. We have

$$\operatorname{vol}(Y_1) = \operatorname{vol}(Y_c) = \frac{974528^{3/2}\zeta_F(2)}{2^{12}\pi^{10}} = 2.834032\ldots.$$

The extension L/F is ramified at exactly 2 of the 4 real places of F. By Corollary 5.21, the groups Γ_1 and Γ_c are representation equivalent.

The following example proves Theorem A. It is the smallest example we could find of a pair of hyperbolic 3-orbifolds that are i-isospectral for all i but not representation equivalent.

Example 7.4. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^4 - \alpha^3 + \alpha^2 + 4\alpha - 4 = 0$, which is also the field $\mathbb{Q}(\sqrt{-10 - 14\sqrt{5}})$. The field F is the unique number field of discriminant -1375 and signature $(2,1)$ (LMFDB `4.2.1375.1`). Let D be the unique quaternion division algebra ramified at every real place and no finite place of F. Let $\mathfrak{N} = (1)$. We have $C \cong C_{\mathrm{iso}} \cong C_2$, which therefore has a single non-trivial character χ, corresponding to the quadratic extension $L = F(\zeta_{10})$. Let σ be the non-trivial automorphism of L/F. Let $C = \{1, c\}$. We have

$$\mathrm{vol}(Y_1) = \mathrm{vol}(Y_c) = \frac{1375^{3/2}\zeta_F(2)}{2^8\pi^6} = 0.2510654\ldots.$$

By [44, Theorem C], the groups Γ_1 and Γ_c are not representation equivalent. Take representatives $(\tau_k)_k$ of $\mathrm{Hom}(L, \mathbb{C})$ modulo complex conjugation such that

$$(\tau_k(\zeta_{10}))_k = (e^{-\frac{2\pi}{10}i}, e^{\frac{2\pi}{10}i}, e^{3\frac{2\pi}{10}i}, e^{-3\frac{2\pi}{10}i}),$$
$$(\tau_k(\alpha))_k \approx (0.809 - 1.607i, 0.809 - 1.607i, 0.845, -1.463).$$

We compute the group $\mathbf{HC}_{L,(1)}$ of unitary Hecke characters. It is isomorphic to $\mathbb{Z}^7 \times \mathbb{R}$. Let $\Psi_1, \ldots, \Psi_7$ denote the computed basis of the canonical complement of $\|\cdot\|^{i\mathbb{R}}$ (see e.g. [48, Section 3.5]). For each character Ψ and complex embedding τ, we display an approximation of the pair $(k, t) \in \mathbb{Z} \times \mathbb{R}$ such that $\Psi_\tau = \Psi_{\mathbb{C}}(k, it)$.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7
τ_1	$(2,0)$	$(-2,0)$	$(0,0)$	$(-3,-0.898)$	$(1,-0.367)$	$(1,-0.367)$	$(-2,1.149)$
τ_2	$(-2,0)$	$(2,0)$	$(2,0)$	$(3,0.898)$	$(-2,-1.265)$	$(0,-1.265)$	$(0,-1.149)$
τ_3	$(4,0)$	$(-9,0)$	$(-2,-0.611)$	$(-10,0)$	$(5,-0.015)$	$(3,1.647)$	$(-1,-0.525)$
τ_4	$(-4,0)$	$(9,0)$	$(2,0.611)$	$(12,0)$	$(-6,1.647)$	$(-4,-0.015)$	$(3,0.525)$

We then compute the subgroup $\mathbf{HC}_{L,(1)}^-$ of unitary Hecke characters Ψ such that $\Psi^\sigma\Psi$ has finite order. It admits a basis $\Psi_1', \ldots, \Psi_4'$, where

$$\Psi_1' = \Psi_1, \quad \Psi_2' = \Psi_2, \quad \Psi_3' = \Psi_4, \quad \Psi_4' = \Psi_3^{-1}\Psi_5\Psi_6^{-1}\Psi_7^{-2}.$$

We display the Ψ_j' as above.

	Ψ_1'	Ψ_2'	Ψ_3'	Ψ_4'
τ_1	$(2,0)$	$(-2,0)$	$(-3,-0.898)$	$(4,-2.298)$
τ_2	$(-2,0)$	$(2,0)$	$(3,0.898)$	$(-4,2.298)$
τ_3	$(4,0)$	$(-9,0)$	$(-10,0)$	$(6,0)$
τ_4	$(-4,0)$	$(9,0)$	$(12,0)$	$(-10,0)$

Projecting on the values $(k_{\tau_1}, k_{\tau_3}, k_{\tau_4})$, we obtain a lattice generated by the columns of a matrix, say B, and with LLL-reduced basis with matrix B', where

$$B = \begin{pmatrix} 2 & -2 & -3 & 4 \\ 4 & -9 & -10 & 6 \\ -4 & 9 & 12 & -10 \end{pmatrix}, \qquad B' = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 1 & 2 & 3 \end{pmatrix}.$$

From this it is clear that there is no $\Omega^\bullet$-shady character of L, as such a character would have to satisfy $k_{\tau_1} \in \{-1, 0, 1\}$ and $k_{\tau_3}, k_{\tau_4} \in \{\pm 1\}$. By Corollary 5.29, the orbifolds Y_1 and Y_c are therefore i-isospectral for all $i \geq 0$.

The maximal cyclic subgroups of $P\Gamma_1$ have order 2, 3 or 5, the maximal cyclic subgroups of $P\Gamma_c$ have order 2, 3 or 10, and we have

$$H_1(P\Gamma_1, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \quad \text{and} \quad H_1(P\Gamma_c, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$
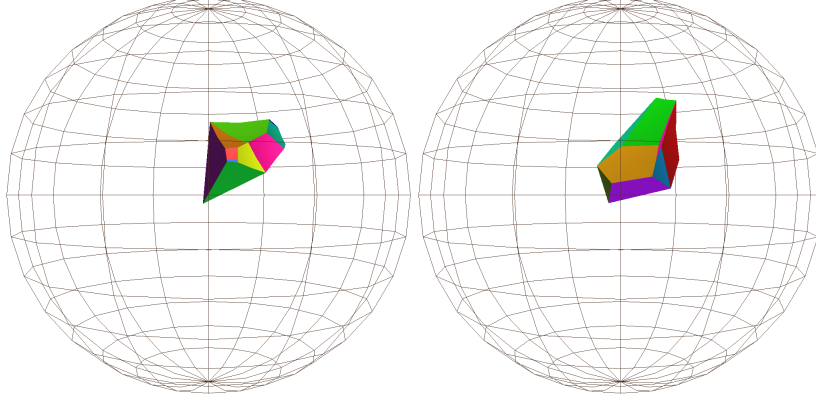
FIGURE 1. Isospectral and 1-isospectral but not representation equivalent 3-orbifolds (volume ≈ 0.251)

Remark 7.5. The above example is, in particular, an example of a pair of orbifolds that are i-isospectral for all i, but not strongly isospectral. Wolf had asked in [69] whether such examples exist, and their existence was first shown in [42] in all dimensions greater than 4.

The following example proves Theorem B. It is the smallest example we could find of a pair of hyperbolic 3-orbifolds that are 0-isospectral but not not 1-isospectral.

Example 7.6. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^4 - 3\alpha^2 - 2\alpha + 1 = 0$. The field F is the unique number field of discriminant -1328 and signature $(2,1)$ (LMFDB `4.2.1328.1`). Let D be the unique quaternion division algebra ramified at every real place and no finite place of F. Let $\mathfrak{N} = (1)$. We have $C \cong C_{\mathrm{iso}} \cong C_2$, which therefore has a single non-trivial character χ, corresponding to the quadratic extension $L = F(\zeta_4)$. Let σ be the non-trivial automorphism of L/F. Let $C = \{1, c\}$. We have

$$\mathrm{vol}(Y_1) = \mathrm{vol}(Y_c) = \frac{1328^{3/2}\zeta_F(2)}{2^8\pi^6} = 0.2461808\ldots.$$

Take representatives $(\tau_k)_k$ of $\mathrm{Hom}(L, \mathbb{C})$ modulo complex conjugation such that

$$(\tau_k(\zeta_4))_k = (i, -i, i, i),$$
$$(\tau_k(\alpha))_k \approx (-1.138 + 0.485i, -1.138 + 0.485i, 1.940, 0.337).$$

We compute the group $\mathbf{HC}_{L,(1)}$ of unitary Hecke characters. It is isomorphic to $\mathbb{Z}^7 \times \mathbb{R}$. Let $\Psi_1, \ldots, \Psi_7$ denote the computed basis of the canonical complement of $\|\cdot\|^{i\mathbb{R}}$. We display the Ψ_j as in the previous example.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7
τ_1	$(4,0)$	$(-23,-0.651)$	$(-31,0.550)$	$(2,-0.453)$	$(-31,0.531)$	$(-6,-1.266)$	$(19,1.114)$
τ_2	$(-4,0)$	$(21,-0.651)$	$(30,1.266)$	$(-2,-0.453)$	$(30,-1.634)$	$(7,-0.550)$	$(-20,-2.218)$
τ_3	$(4,0)$	$(-22,0.737)$	$(-31,-0.568)$	$(2,-1.120)$	$(-32,-0.191)$	$(-7,0.568)$	$(21,-0.191)$
τ_4	$(4,0)$	$(-22,0.564)$	$(-32,-1.248)$	$(2,2.026)$	$(-31,1.295)$	$(-8,1.248)$	$(20,1.295)$

We then compute the subgroup $\mathbf{HC}^-_{L,(1)}$ of unitary Hecke characters Ψ such that $\Psi^\sigma\Psi$ has finite order. It admits a basis $\Psi'_1, \ldots, \Psi'_4$, where

$$\Psi'_1 = \Psi_1, \quad \Psi'_2 = \Psi_3\Psi_6, \quad \Psi'_3 = \Psi_5^{-1}\Psi_7, \quad \Psi'_4 = \Psi_2^{-1}\Psi_4^{-1}\Psi_5\Psi_7.$$

We display the Ψ'_j as above.
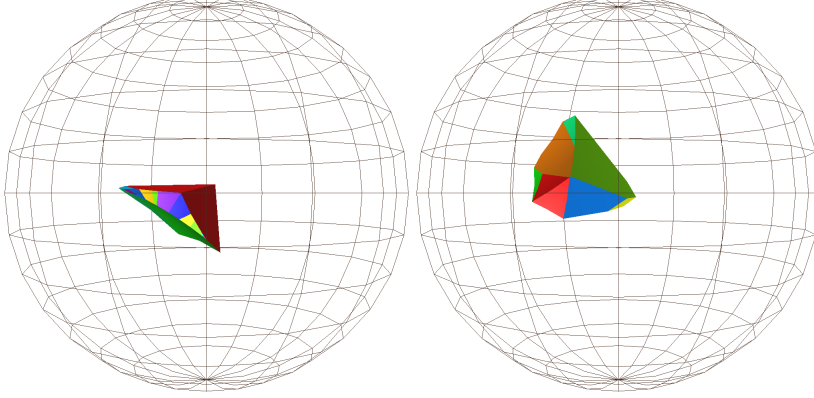
FIGURE 2. Isospectral but not 1-isospectral 3-orbifolds (volume ≈ 0.246)

	Ψ'_1	Ψ'_2	Ψ'_3	Ψ'_4
τ_1	$(4,0)$	$(-37,-0.716)$	$(50,0.584)$	$(9,2.748)$
τ_2	$(-4,0)$	$(37,0.716)$	$(-50,-0.584)$	$(-9,-2.748)$
τ_3	$(4,0)$	$(-38,0)$	$(53,0)$	$(9,0)$
τ_4	$(4,0)$	$(-40,0)$	$(51,0)$	$(9,0)$

Projecting on the values $(k_{\tau_1}, k_{\tau_3}, k_{\tau_4})$, we obtain a lattice generated by the columns of a matrix, say B, and with LLL-reduced basis with matrix B', where

$$B = \begin{pmatrix} 4 & -37 & 50 & 9 \\ 4 & -38 & 53 & 9 \\ 4 & -40 & 51 & 9 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

From this it is clear that there is no Ω^0-shady character of L, as such a character would have to satisfy $k_{\tau_1} = 0$ and $k_{\tau_3}, k_{\tau_4} \in \{\pm 1\}$. By Corollary 5.31, the orbifolds Y_1 and Y_c are therefore 0-isospectral. However, there exists an $\Omega^\bullet$-shady character, namely $\Psi_{\text{shady}} = (\Psi'_1)^{-2}\Psi'_4$, and the set of $\Omega^\bullet$-shady characters is $\Psi_{\text{shady}}^{\pm 1}\Psi_0^{\mathbb{Z}}$, where $\Psi_0 = (\Psi'_1)^{-9}(\Psi'_4)^4$:

	Ψ_{shady}	Ψ_0
τ_1	$(1,2.748)$	$(0,10.994)$
τ_2	$(-1,-2.748)$	$(0,-10.994)$
τ_3	$(1,0)$	$(0,0)$
τ_4	$(1,0)$	$(0,0)$

For all $n \in \mathbb{Z}$, the character $\Psi_{\text{shady}}\Psi_0^n$ contributes an eigenvalue $\lambda(n) \approx 4(2.748 + 10.994n)^2$ to the Laplace spectrum of $\Omega^1(\mathcal{Y})$ by Proposition 5.25. The first few corresponding eigenvalues are

$$30.2167\ldots, \quad 271.9505\ldots, \quad 755.4182\ldots, \quad 1480.6196\ldots, \quad 2447.5549\ldots.$$

The quadratic growth of these eigenvalues is an example of the phenomenon described in Theorem C. Only $\Psi_{\text{shady}}^{\pm 1}$ contribute to the eigenvalue $\lambda(0)$, so by Lemma 7.2, the orbifolds Y_1 and Y_c are not 1-isospectral.

The maximal cyclic subgroups of both $P\Gamma_1$ and $P\Gamma_c$ have order 2, 3 or 4, and we have

$$H_1(P\Gamma_1, \mathbb{Z}) \cong H_1(P\Gamma_c, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

The following is an example of 0-isospectral 3-orbifolds with distinct Betti numbers (in particular, they are not 1-isospectral). Note that Tenie [64] has constructed

a pair of isospectral 3-manifolds that have non-isomorphic rational cohomology rings.

Example 7.7. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^6 - \alpha^5 - 3\alpha^4 + 2\alpha^2 + 4\alpha + 1 = 0$. The field F is the unique number field of discriminant -958527 and signature $(4,1)$ (LMFDB `6.4.958527.1`). Let D be the unique quaternion division algebra ramified at every real place and no finite place of F. Let $\mathfrak{N} = (1)$. We have $C \cong C_{\mathrm{iso}} \cong C_2$, which therefore has a single non-trivial character χ, corresponding to the quadratic extension $L = F(\zeta_6)$. Let σ be the non-trivial automorphism of L/F.

Let $C = \{1, c\}$. We have

$$\mathrm{vol}(Y_1) = \mathrm{vol}(Y_c) = \frac{958527^{3/2}\zeta_F(2)}{2^{12}\pi^{10}} = 3.397413\ldots.$$

Take representatives $(\tau_k)_k$ of $\mathrm{Hom}(L, \mathbb{C})$ modulo complex conjugation such that

$$(\tau_k(\zeta_6))_k = (e^{-\frac{2\pi}{6}i}, e^{\frac{2\pi}{6}i}, e^{\frac{2\pi}{6}i}, e^{-\frac{2\pi}{6}i}, e^{-\frac{2\pi}{6}i}, e^{\frac{2\pi}{6}i}),$$
$$(\tau_k(\alpha))_k \approx (1.959, -0.411 + 0.835i, -0.411 - 0.835i, -0.287, 1.511, -1.361).$$

We compute the group $\mathbf{HC}_{L,(1)}$ of unitary Hecke characters. It is isomorphic to $\mathbb{Z}^{11} \times \mathbb{R}$. Let $\Psi_1, \ldots, \Psi_{11}$ denote the computed basis of the canonical complement of $\|\cdot\|^{i\mathbb{R}}$. We display the Ψ_j as above.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6
τ_1	$(1,0)$	$(-1,-0.610)$	$(-1,0.610)$	$(1,0.242)$	$(-1,0.072)$	$(2,-0.036)$
τ_2	$(-1,0)$	$(0,1.032)$	$(0,0.078)$	$(0,0.299)$	$(0,0.007)$	$(-1,-0.421)$
τ_3	$(-1,0)$	$(0,-0.078)$	$(0,-1.032)$	$(-2,0.299)$	$(2,0.007)$	$(-2,0.414)$
τ_4	$(1,0)$	$(0,-0.395)$	$(0,0.395)$	$(1,-0.847)$	$(-1,-1.271)$	$(0,0.635)$
τ_5	$(1,0)$	$(1,-0.292)$	$(1,0.292)$	$(1,-0.643)$	$(-1,0.259)$	$(1,-0.130)$
τ_6	$(-1,0)$	$(0,0.344)$	$(0,-0.344)$	$(-1,0.650)$	$(1,0.926)$	$(0,-0.463)$

	Ψ_7	Ψ_8	Ψ_9	Ψ_{10}	Ψ_{11}
τ_1	$(-1,0.157)$	$(-1,0.261)$	$(1,-0.356)$	$(-2,-0.428)$	$(1,0.261)$
τ_2	$(0,-0.762)$	$(1,0.282)$	$(-1,-0.154)$	$(2,0.374)$	$(-1,1.0138)$
τ_3	$(0,1.068)$	$(1,1.014)$	$(0,0.381)$	$(1,-0.160)$	$(-1,0.282)$
τ_4	$(1,-1.059)$	$(-2,-0.304)$	$(2,-0.722)$	$(-3,0.549)$	$(2,-0.304)$
τ_5	$(0,-0.192)$	$(-1,-0.721)$	$(2,1.013)$	$(-3,0.754)$	$(1,-0.721)$
τ_6	$(0,0.788)$	$(0,-0.532)$	$(0,-0.162)$	$(1,-1.088)$	$(0,-0.532)$

We then compute the subgroup $\mathbf{HC}^-_{L,(1)}$ of unitary Hecke characters Ψ such that $\Psi^\sigma\Psi$ has finite order. It admits a basis $\Psi'_1, \ldots, \Psi'_6$, where

$$\Psi'_1 = \Psi_1, \Psi'_2 = \Psi_2\Psi_3, \Psi'_3 = \Psi_8^{-1}\Psi_{11}, \Psi'_4 = \Psi_5^{-1}\Psi_9\Psi_{10}^{-1}, \Psi'_5 = \Psi_5\Psi_6^2, \Psi'_6 = \Psi_4\Psi_5\Psi_7^{-2}.$$

We display the Ψ'_j as above.

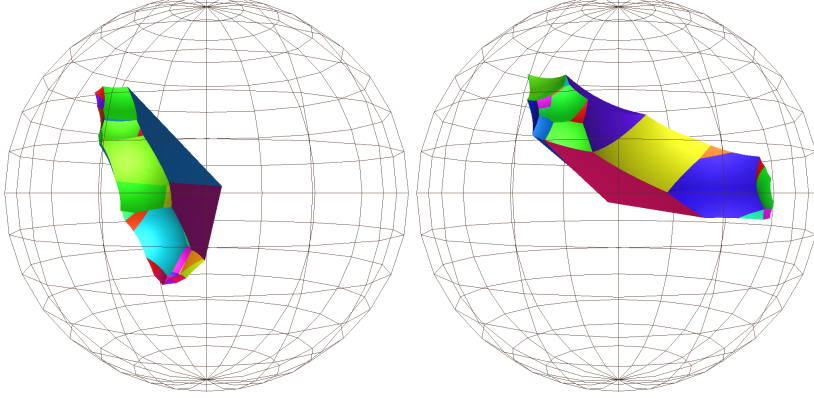|	Ψ'_1	Ψ'_2	Ψ'_3	Ψ'_4	Ψ'_5	Ψ'_6
τ_1	$(1,0)$	$(-2,0)$	$(2,0)$	$(4,0)$	$(3,0)$	$(2,0)$
τ_2	$(-1,0)$	$(0,1.110)$	$(-2,0.732)$	$(-3,-0.534)$	$(-2,-0.835)$	$(0,1.830)$
τ_3	$(-1,0)$	$(0,-1.110)$	$(-2,-0.732)$	$(-3,0.534)$	$(-2,0.835)$	$(0,-1.830)$
τ_4	$(1,0)$	$(0,0)$	$(4,0)$	$(6,0)$	$(-1,0)$	$(-2,0)$
τ_5	$(1,0)$	$(2,0)$	$(2,0)$	$(6,0)$	$(1,0)$	$(0,0)$
τ_6	$(-1,0)$	$(0,0)$	$(0,0)$	$(-2,0)$	$(1,0)$	$(0,0)$

FIGURE 3. Isospectral 3-orbifolds with distinct Betti numbers (volume ≈ 3.397)

Projecting on the values $(k_{\tau_1}, k_{\tau_2}, k_{\tau_4}, k_{\tau_5}, k_{\tau_6})$, we obtain a lattice generated by the columns of a matrix, say B, and with LLL-reduced basis with matrix B', where

$$B = \begin{pmatrix} 1 & -2 & 2 & 4 & 3 & 2 \\ -1 & 0 & -2 & -3 & -2 & 0 \\ 1 & 0 & 4 & 6 & -1 & -2 \\ 1 & 2 & 2 & 6 & 1 & 0 \\ -1 & 0 & 0 & -2 & 1 & 0 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

From this it is clear that there is no Ω^0-shady character of L, as such a character would have to satisfy $k_{\tau_2} = 0$ and $k_{\tau_1}, k_{\tau_4}, k_{\tau_5}, k_{\tau_6} \in \{\pm 1\}$. However, there exists an $\mathcal{H}^\bullet$-shady character, namely $\Psi_{\mathrm{shady}} = \Psi_1$. Moreover, $\Psi_{\mathrm{shady}}^{\pm 1}$ are the only $\mathcal{H}^\bullet$-shady characters of L, so that the 1st Betti numbers of Y_1 and Y_c differ by 1. In fact, we can independently compute the homology and see that we have

$$H_1(P\Gamma_1, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4 \quad \text{and} \quad H_1(P\Gamma_c, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}.$$

The maximal cyclic subgroups of $P\Gamma_1$ have order 2 or 6, and the maximal cyclic subgroups of $P\Gamma_c$ have order 2 or 3.

The eigenvalues of the Hecke operators acting on $H_1(\mathcal{Y}, \mathbb{Q}) = \mathbb{Q}$ are rational, so one expects them to correspond to an elliptic curve E over F. In this case, since this cohomology class comes from a Hecke character, CM theory implies that this elliptic curve exists, and one can find it explicitly:

$$E : y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where

$$a_2 = \alpha^5 - \alpha^4 - 3\alpha^3 + \alpha^2 + 3\alpha + 2,$$
$$a_3 = \alpha^5 - \alpha^4 - 3\alpha^3 + \alpha^2 + \alpha + 2,$$
$$a_4 = \alpha^3 - 2\alpha,$$
$$a_6 = 12159\alpha^5 - 15642\alpha^4 - 31998\alpha^3 + 9172\alpha^2 + 21688\alpha + 42423.$$

In other words, the fact that the elliptic curve $y^2 + y = x^3$ with j-invariant 0, after base-change to F and a sextic twist, acquires everywhere good reduction, is an algebraic explanation for the fact that Y_1 and Y_c are not 1-isospectral!

The following example proves Theorem E. It is an example of non-isospectral 3-orbifolds forming a Vignéras pair with no $\mathcal{H}^*$-shady character and non-zero 1st Betti numbers; in particular their regulator ratio is rational for a non-trivial reason.

Example 7.8. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^4 - 2\alpha^3 + 7\alpha^2 - 6\alpha - 3 = 0$. The field F is a number field of discriminant -10224 and signature $(2,1)$ (LMFDB `4.2.10224.2`). Let D be the unique quaternion division algebra ramified at every real place and no finite place of F. Let $\mathfrak{N} = (1)$. We have $C \cong C_{\text{iso}} \cong C_2$, which therefore has a single non-trivial character χ, corresponding to the quadratic extension $L = F(\zeta_{12})$. Let σ be the non-trivial automorphism of L/F.

Let $C = \{1, c\}$. We have

$$\text{vol}(Y_1) = \text{vol}(Y_c) = \frac{10224^{3/2}\zeta_F(2)}{2^8\pi^6} = 5.902455\ldots.$$

Take representatives $(\tau_k)_k$ of $\text{Hom}(L, \mathbb{C})$ modulo complex conjugation such that

$$(\tau_k(\zeta_{12}))_k = (e^{-5\frac{2\pi}{12}i}, e^{-5\frac{2\pi}{12}i}, e^{\frac{2\pi}{12}i}, e^{-\frac{2\pi}{12}i}),$$
$$(\tau_k(\alpha))_k \approx (1.345, -0.345, 0.500 - 2.493i, 0.500 - 2.493i).$$

We compute the group $\mathbf{HC}_{L,(1)}$ of unitary Hecke characters. It is isomorphic to $\mathbb{Z}^7 \times \mathbb{R}$. Let $\Psi_1, \ldots, \Psi_7$ denote the computed basis of the canonical complement of $\|\cdot\|^{i\mathbb{R}}$. We display the Ψ_j as above.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7
τ_1	$(1,0)$	$(3,0)$	$(-2,0)$	$(-2, 0.714)$	$(-3, -0.140)$	$(0, -0.140)$	$(0, 0.953)$
τ_2	$(1,0)$	$(3,0)$	$(-2,0)$	$(-2, -0.714)$	$(-4, 0.140)$	$(1, 0.140)$	$(0, 0.239)$
τ_3	$(-19,0)$	$(-33,0)$	$(26, -0.480)$	$(26, 0)$	$(43, -0.215)$	$(-9, 0.215)$	$(-6, -0.356)$
τ_4	$(19,0)$	$(33,0)$	$(-26, 0.480)$	$(-26, 0)$	$(-42, 0.215)$	$(10, -0.215)$	$(6, -0.836)$

We then compute the subgroup $\mathbf{HC}^{-}_{L,(1)}$ of unitary Hecke characters Ψ such that $\Psi^\sigma\Psi$ has finite order. It admits a basis $\Psi'_1, \ldots, \Psi'_4$, where

$$\Psi'_1 = \Psi_1, \quad \Psi'_2 = \Psi_2, \quad \Psi'_3 = \Psi_3, \quad \Psi'_4 = \Psi_5^{-1}\Psi_6.$$

We display the Ψ'_j as above.

	Ψ'_1	Ψ'_2	Ψ'_3	Ψ'_4
τ_1	$(1,0)$	$(3,0)$	$(-2,0)$	$(3,0)$
τ_2	$(1,0)$	$(3,0)$	$(-2,0)$	$(5,0)$
τ_3	$(-19,0)$	$(-33,0)$	$(26, -0.480)$	$(-52, 0.430)$
τ_4	$(19,0)$	$(33,0)$	$(-26, 0.480)$	$(52, -0.430)$

Projecting on the values $(k_{\tau_1}, k_{\tau_2}, k_{\tau_3})$, we obtain a lattice generated by the columns of a matrix, say B, and with LLL-reduced basis with matrix B', where

$$B = \begin{pmatrix} 1 & 3 & -2 & 3 \\ 1 & 3 & -2 & 5 \\ -19 & -33 & 26 & -52 \end{pmatrix}, \qquad B' = \begin{pmatrix} -1 & -2 & 1 \\ 1 & -2 & 1 \\ 0 & 2 & 5 \end{pmatrix}.$$

From this it is clear that there is no $\mathcal{H}^\bullet$-shady character of L, as such a character would have to satisfy $k_\tau \in \{\pm1\}$ for all $\tau \in \text{Hom}(L, \mathbb{C})$. However, there exists an $\Omega^\bullet$-shady character of L, namely $\Psi_{\text{shady}} = (\Psi'_1)^{-1}(\Psi'_2)^{-1}\Psi'_4$, and the set of $\Omega^\bullet$-shady characters is $\Psi_{\text{shady}}^{\pm1}\Psi_0^{\mathbb{Z}}$, where $\Psi_0 = (\Psi'_1)^{-1}(\Psi'_2)^{-1}(\Psi'_3)^{-2}$:

	Ψ_{shady}	Ψ_0
τ_1	$(-1,0)$	$(0,0)$
τ_2	$(1,0)$	$(0,0)$
τ_3	$(0, 0.430)$	$(0, 0.960)$
τ_4	$(0, -0.430)$	$(0, -0.960)$

For all $n \in \mathbb{Z}$, the character $\Psi_{\text{shady}}\Psi_0^n$ contributes an eigenvalue $\lambda(n) \approx 1 + 4(0.430 + 0.960n)^2$ to the Laplace spectrum of $\Omega^1(\mathcal{Y})$ by Proposition 5.25. The first few corresponding eigenvalues are

$$1.741\ldots, \quad 2.123\ldots, \quad 8.735\ldots, \quad 9.883\ldots, \quad 23.107\ldots, \quad 25.020\ldots.$$
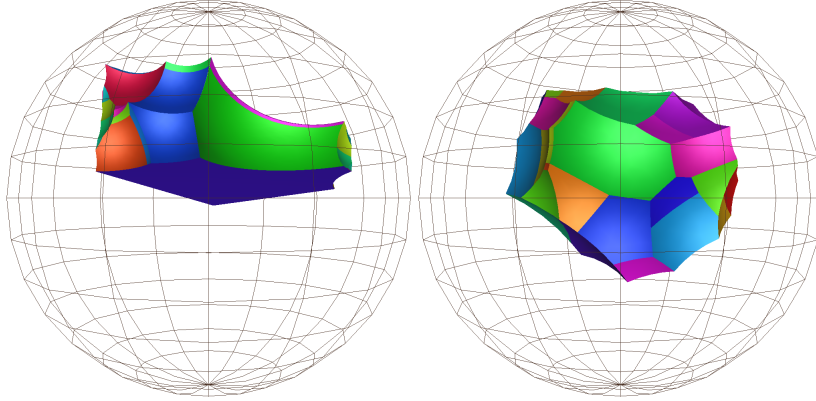
FIGURE 4. Non-isospectral 3-orbifolds with Betti number 1 and rational regulator square quotient (volume ≈ 5.902)

Only $\Psi_{\mathrm{shady}}^{\pm 1}$ contribute to the eigenvalue $\lambda(0)$, so by Lemma 7.2, the orbifolds Y_1 and Y_c are not 1-isospectral. Since these characters are also Ω^0-shady, by the same argument these orbifolds are also not 0-isospectral.

Since there is no $\mathcal{H}^\bullet$-shady character of L, by Theorem 5.34, we have

$$\frac{\mathrm{Reg}_1(Y_1)^2}{\mathrm{Reg}_1(Y_c)^2} \in \mathbb{Q}^\times,$$

and the Betti numbers of Y_1 and Y_c are equal. In fact, we have

$$H_1(P\Gamma_1, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z} \quad \text{and} \quad H_1(P\Gamma_c, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z},$$

and in particular both 1st Betti numbers are 1, so the rationality of the ratio of regulators is a non-trivial statement. Moreover, since the orbifolds are not isospectral, the Cheeger–Müller theorem does not say anything about this rationality.

The maximal cyclic subgroups of $P\Gamma_1$ have order 2, 3, 4 or 12, and the maximal cyclic subgroups of $P\Gamma_c$ have order 2 or 3.

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