Algebraic number theory

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Introduction: why algebraic number theory?

Consider the following statement, today known as Fermat's last theorem:

Theorem 0.0.1. *Let $n \geqslant 3$ be an integer. Then the Diophantine equation*

$$x^n + y^n = z^n$$

has no nontrivial solution, i.e.

$$x^n + y^n = z^n,\ x, y, z \in \mathbb{Z} \Longrightarrow xyz = 0.$$

In 1847, while this theorem still had not been proved, the French mathematician Gabriel Lamé had an idea for the case where n is an odd prime. Suppose we have $x^n + y^n = z^n$ with x, y and z all nonzero integers, which we may assume are relatively prime. Let $\zeta = e^{2\pi i/n}$, so that $\zeta^n = 1$, and consider

$$\mathbb{Z}[\zeta] = \{P(\zeta),\ P \in \mathbb{Z}[x]\},$$

the smallest subring of $\mathbb{C}$ containing ζ. Then, in this ring, we have

$$x^n = z^n - y^n = \prod_{k=1}^{n}(z - \zeta^k y).$$

Lamé claimed that to conclude that each factor $z - \zeta^k y$ is an n^{th} power, it suffices to show that these factors are pairwise coprime. If this were true, then we would be able to find integers x', y' and z', smaller than x,y and z but all nonzero, such that $x'^n + y'^n = z'^n$; this would lead to an "infinite descent" and thus prove the theorem.

Unfortunately, Lamé's claim relied on the supposition that factorisation into irreducibles is unique, and while this is true in $\mathbb{Z}$, we now know that it need not be true in more general rings such as $\mathbb{Z}[\zeta]$. For instance, in the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5},\ a, b \in \mathbb{Z}\}$$

where $\sqrt{-5}$ means $i\sqrt{5}$, we have the two different factorisations

$$6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5}),$$

where each factor is irreducible. Lamé was thus unable to justify his claim, and the theorem remained unproved for almost 150 years.

The numbers ζ and $\sqrt{-5}$ are examples of *algebraic numbers*. The goal of this course is to study the property of such numbers, and of rings such that $\mathbb{Z}[\zeta]$, so as to know what we are allowed to do with them, and what we are not. As an application, we will see how to solve certain Diophantine equations.

References

This module is based on the book *Algebraic Number Theory and Fermat's Last Theorem*, by I.N. Stewart and D.O. Tall, published by A.K. Peters (2001). The contents of the module forms a proper subset of the material in that book. (The earlier edition, published under the title Algebraic Number Theory, is also suitable.)

For alternative viewpoints, students may also like to consult the books *A Brief Guide to Algebraic Number Theory*, by H.P.F. Swinnerton-Dyer (LMS Student Texts # 50, CUP), or *Algebraic Number Theory*, by A. Fröhlich and M.J. Taylor (CUP). Finally, students interested in the algorithmic side of things should consult *A course in computational algebraic number theory* by H. Cohen (Graduate Texts in Mathematics # 138, Springer).

Chapter 1

Number fields

1.1 Resultants

Before we actually get started with number theory, let us introduce a tool which will turn out to be very valuable.

Definition 1.1.1. Let K be a field, and let $A = \sum_{j=0}^{m} a_j x^j$ and $B = \sum_{k=0}^{n} b_k x^k$ be two polynomials with coefficients in K. The *resultant* of A and B is the $(m+n) \times (m+n)$ determinant

$$\mathrm{Res}(A,B) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_0 \end{vmatrix},$$

where the first n rows contain the coefficients of A and the m last ones contain those of B.

The main properties of the resultant are the following:

Theorem 1.1.2.

- $\mathrm{Res}(A,B) \in K$, *and in fact, if the coefficients of both* A *and* B *lie in a subring* $\mathcal{R}$ *of* K, *then* $\mathrm{Res}(A,B) \in \mathcal{R}$.
- *If we can factor (over* K *or over a larger field)* A *and* B *as*

$$A = a\prod_{j=1}^{\deg A}(x-\alpha_j) \text{ and } B = b\prod_{k=1}^{\deg B}(x-\beta_k),$$

then

$$\begin{aligned}\mathrm{Res}(A,B) &= a^{\deg B}\prod_{j=1}^{\deg A}B(\alpha_j) = a^{\deg B}b^{\deg A}\prod_{j=1}^{\deg A}\prod_{k=1}^{\deg B}(\alpha_j-\beta_k)\\ &= (-1)^{\deg A\deg B}b^{\deg A}\prod_{k=1}^{\deg B}A(\beta_k) = (-1)^{\deg A\deg B}\mathrm{Res}(B,A).\end{aligned}$$

- $\mathrm{Res}(A,B) = 0$ *if and only if* A *and* B *have a common factor in* $K[x]$.

Example 1.1.3. Take $K=\mathbb{Q}$, $A = x^2-2 \in \mathbb{Q}[x]$ and $B = x^2+1 \in \mathbb{Q}[x]$. Since actually A and B lie in $\mathbb{Z}[x]$, we have $\mathrm{Res}(A,B) \in \mathbb{Z}$; this is simply because by definition,

$$\mathrm{Res}(A,B) = \begin{vmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 0 & -2\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{vmatrix}.$$

Besides, since we have

$$A = (x-\sqrt{2})(x+\sqrt{2}) \text{ and } B = (x-i)(x+i)$$

over $\mathbb{C}$, we find that

$$\mathrm{Res}(A,B) = B(\sqrt{2})B(-\sqrt{2}) = A(i)A(-i) = (\sqrt{2}-i)(\sqrt{2}+i)(-\sqrt{2}-i)(-\sqrt{2}+i) = 9.$$

Example 1.1.4. Suppose we have $A = BQ + R$ in $K[x]$, and let b be the leading coefficient of B. Then

$$\mathrm{Res}(A,B) = (-1)^{\deg A\deg B}b^{\deg A-\deg R}\mathrm{Res}(B,R).$$

This gives a way to compute $\mathrm{Res}(A,B)$ by performing successive Euclidian divisions, which is more efficient (at least for a computer) than computing a large determinant when the degrees of A and B are large.

1.2 Field extensions

1.2.1 Notation

Let K and L be fields such that $K \subseteq L$. One says that K is a *subfield* of L, and that L is an *extension* of K.

In what follows, whenever $\alpha \in L$ (resp. $\alpha_1, \alpha_2, \cdots \in L$), we will write $K(\alpha)$ (resp. $K(\alpha_1, \alpha_2, \cdots)$) to denote the smallest subfield of L containing K as well as α (resp. $\alpha_1, \alpha_2, \cdots$). For example, we have $\mathbb{C} = \mathbb{R}(i)$, and $K(\alpha) = K$ if anf only if $\alpha \in K$.

Also, when $\mathcal{R}$ is a subring of K, we will write

$$\mathcal{R}[\alpha] = \{P(\alpha),\ P \in \mathcal{R}[x]\}$$

to denote the smallest subring of L containing $\mathcal{R}$ as well as α, and similarly

$$\mathcal{R}[\alpha_1, \cdots, \alpha_n] = \{P(\alpha_1, \cdots, \alpha_n),\ P \in \mathcal{R}[x_1, \cdots, x_n]\}.$$

Example 1.2.1. The ring $K[\alpha]$ is a subring of the field $K(\alpha)$.

1.2.2 Algebraic elements, algebraic extensions

Definition 1.2.2. Let $\alpha \in L$. Then set of polynomials $P \in K[x]$ such that $P(\alpha) = 0$ is an ideal V_α of $K[x]$, and one says that α is *algebraic* over K if this ideal is nonzero, that is to say if there exists a nonzero $P \in K[x]$ which vanishes at α. Else one says that α is transcendental over K, or just transcendental (for short) when $K = \mathbb{Q}$.

In the case when α is algebraic over K, the ideal V_α can be generated by one polynomial since the ring $K[x]$ is a PID. This polynomial is unique up to scaling, so there is a unique *monic* polynomial $m_\alpha(x)$ that generates V_α. This polynomial $m_\alpha(x)$ is called the *minimal polynomial* of α over K. One then says that α is algebraic over K of *degree* n, where $n = \deg m_\alpha \in \mathbb{N}$, and one writes $\deg_K \alpha = n$. When $K = \mathbb{Q}$, one says for short that α is algebraic of degree n.

If every element of L is algebraic over K, one says that L is an *algebraic extension* of K.

Theorem 1.2.3. *Let L/K be a field extension, and let $\alpha \in L$ be algebraic over K of degree n. Then $K[\alpha]$ is a field, so it agrees with $K(\alpha)$. It is also a vector space of dimension n over K, with basis*

$$1, \alpha, \alpha^2, \cdots, \alpha^{n-1},$$

which we write as

$$K(\alpha) = K[\alpha] = \bigoplus_{j=0}^{n-1} K\alpha^j.$$

Remark 1.2.4. On the other hand, if $\alpha \in L$ is transcendental over K, then it is not difficult to see that

$$K(\alpha) = \{r(\alpha),\ r \in K(x)\}$$

is isomorphic to the field $K(x)$ of rational fractions over K, whence the notation $K(\alpha)$; in particular, it is infinite-dimensional as a K-vector space, and $K[\alpha]$ is a strict subring of $K(\alpha)$.

Proof of theorem 1.2.3. Let us begin with the second equality. Let $m(x) = m_\alpha(x) \in K[x]$ be the minimal polynomial of α over K, an irreducible polynomial of degree n. For all $P(x) \in K[x]$, euclidian division in $K[x]$ tells us that we may write

$$P(x) = m(x)Q(x) + R(x)$$

where $Q(x), R(x) \in K[x]$ and $\deg R(x) < n$. Evaluating at $x = \alpha$, we find that $P(\alpha) = R(\alpha)$, so that

$$K[\alpha] = \left\{ \sum_{j=0}^{n-1} \lambda_j \alpha^j,\ \lambda_j \in K \right\}.$$

Besides, if we had a relation of the form

$$\sum_{j=0}^{n-1} \lambda_j \alpha^j = 0$$

with the λ_j in K and not all zero, this would mean that the nonzero polynomial

$$\sum_{j=0}^{n-1} \lambda_j x^j \in K[x]$$

vanishes at $x = \alpha$, and since its degree is $< n$, this would contradict the definition of the minimal polynomial.

Therefore, the $(\alpha^j)_{0 \leqslant j < n}$ span $K[\alpha]$ as a K-vector space and are linearly independent over K, so they form a K-basis of $K[\alpha]$.

For the first equality, we must prove that the ring $K[\alpha]$ is actually a field. Let us thus prove that any nonzero $\beta \in K[\alpha]$ is invertible in $K[\alpha]$. We know from the above that $\beta = P(\alpha)$ for some nonzero $P(x) \in K[x]$ of degree $< n$. Since $m(x)$ is irreducible over K and $\deg P(x) < \deg m(x) = n$, it follows that $P(x)$ and $m(x)$ are coprime, so that there exist $U(x)$ and $V(x)$ in $K[x]$ such that

$$U(x)P(x) + V(x)m(x) = 1.$$

Evaluating at $x = \alpha$, we find that $U(\alpha)P(\alpha) + 0 = 1$, which proves that $U(\alpha) \in K[\alpha]$ is the inverse of $\beta = P(\alpha)$. □

Example 1.2.5. Let $\alpha = \sqrt{2}$. Then α is a root of $x^2 - 2 \in \mathbb{Q}[x]$. SInce this polynomial is of degree only 2, if it were reducible, it would split into fsctors of degree 1; since $\alpha \notin \mathbb{Q}$, we conclude that $x^2 - 2$ is irreducible, so it is the minimal polynomial of α, which is thus is algebraic of degree 2. In particular, we have

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}] = \mathbb{Q} \oplus \mathbb{Q}\sqrt{2},$$

which means that every element of $\mathbb{Q}(\sqrt{2})$ can be written in a unique way as $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$.

Similarly, since $i^2 = -1$, i is algebraic of degree 2, and its minimal polynomial is x^2+1. It is also algebraic of degree 2 over $\mathbb{R}$, with the same minimal polynomial $x^2 + 1$, but which is this time seen as lying in $\mathbb{R}[x]$. We deduce that

$$\mathbb{Q}(i) = \mathbb{Q}[i] = \mathbb{Q} \oplus \mathbb{Q}i$$

and that

$$\mathbb{C} = \mathbb{R}(i) = \mathbb{R}[i] = \mathbb{R} \oplus \mathbb{R}i.$$

We thus recover the well-known fact that every complex number can be written uniquely as $a + bi$ with $a, b \in \mathbb{R}$.

On the contrary, one can prove that π is transcendental (but it is not easy). In particular, $\mathbb{R}$ is not an algebraic extension of $\mathbb{Q}$.

Finally, one can prove that $\sqrt{3}$ is algebraic of degree 2 over $\mathbb{Q}(\sqrt{2})$. This amounts to say that $x^2 - 3$, which is irreducible over $\mathbb{Q}$, remains irreducible

over $\mathbb{Q}(\sqrt{2})$. It follows that

$$\mathbb{Q}(\sqrt{2})(\sqrt{3}) = \mathbb{Q}(\sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})\sqrt{3}$$

as a vector space over $\mathbb{Q}(\sqrt{2})$, so that every element of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ can be written in a unique way as $a + b\sqrt{3}$ with $a, b \in \mathbb{Q}(\sqrt{2})$.

Theorem 1.2.6. *Let L/K be a field extension. The sum, difference, product, and quotient*[1] *of two elements of L which are algebraic over K are algebraic over K.*

Proof. Let α (resp. β) be algebraic over K, so that there exists a nonzero polynomial $A(x) \in K[x]$ (resp. $B(x) \in K[x]$) such that $A(\alpha) = 0$ (resp. $B(\beta) = 0$). Factor $A(x)$ and $B(x)$ in some algebraic closure of K,

$$A(x) = \prod_{j=1}^{m}(x - \alpha_j), \quad B(x) = \prod_{k=1}^{n}(x - \beta_k),$$

with $\alpha = \alpha_1$ and $\beta = \beta_1$, and consider the polynomials $A(y)$ and $B(x - y)$ as polynomials in y over the field $K(x)$. Their resultant

$$C(x) = \operatorname{Res}\big(A(y), B(x - y)\big)$$

lies in $K(x)$, and actually even in $K[x]$ according to theorem 1.1.2, since the coefficients of $A(y)$ and $B(x - y)$ (still seen as polynomials in y) lie in $K[x]$. Besides, still according to theorem 1.1.2, we have

$$C(x) = \prod_{j=1}^{m} B(x - y)|_{y=\alpha_j} = \prod_{j=1}^{m} B(x - \alpha_j) = \prod_{j=1}^{m}\prod_{k=1}^{n}(x - \alpha_j - \beta_k),$$

so that $\alpha + \beta$ is a root of $C(x)$ and is thus algebraic over K.

The cases of $\alpha - \beta$, $\alpha\beta$ and α/β can be dealt with similarly. □

A consequence of this theorem is that the set of complex numbers which are algebraic over $\mathbb{Q}$ is actually a subfield of $\mathbb{C}$.

Example 1.2.7. According to this theorem, $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic. However, the computations needed to exhibit a nonzero polynomial vanishing at α require a bit of effort. We will actually determine the degree and the minimal polynomial of α by another method in example 1.2.24 below.

[1] Not by 0, of course.

1.2.3 The degree of an extension

Let L be an extension of a field K. If we forget temporarily about the multiplication on L, so that only addition is left, then L can be seen as a vector space over K.

Definition 1.2.8. The *degree* of L over K is the dimension (finite or infinite) of L seen as a K-vector space. It is denoted by $[L : K]$.

If this degree is finite, one says that L is a *finite extension* of K.

Example 1.2.9. Let $\alpha \in L$. If α is algebraic over K with minimal polynomial $m_\alpha(x) \in K[x]$ of degree n, then theorem 1.2.3 tells us that

$$K(\alpha) = \left\{ \sum_{k=0}^{n-1} \lambda_k \alpha^k,\ \lambda_k \in K \right\} = K \oplus K\alpha \oplus \cdots \oplus K\alpha^{n-1},$$

so $[K(\alpha) : K] = n = \deg_K \alpha$. On the other hand, if α is transcendental over K, then $K(\alpha)$ is isomorphic to the rational fraction field $K(x)$, so $[K(\alpha) : K] = \infty$.

Remark 1.2.10. Clearly, the only extension L of a field K such that $[L : K] = 1$ is $L = K$ itself.

Proposition 1.2.11 (Multiplicativity of the degree)**.** *Let $K \subseteq L \subseteq M$ be finite extensions, let $(l_i)_{1 \leqslant i \leqslant [L:K]}$ be a K-basis of L, and let $(m_j)_{1 \leqslant j \leqslant [M:L]}$ be an L-basis of M. Then $(l_i m_j)_{\substack{1 \leqslant i \leqslant [L:K] \\ 1 \leqslant j \leqslant [M:L]}}$ is a K-basis of M. In particular, $[M : K] = [M : L][L : K]$.*

Proof. Let $m \in M$. Since $(m_j)_{1 \leqslant j \leqslant [M:L]}$ is an L-basis of M, we have

$$m = \sum_{j=1}^{[M:L]} \lambda_j m_j$$

for some $\lambda_j \in L$, and since $(l_i)_{1 \leqslant i \leqslant [L:K]}$ is a K-basis of L, each λ_j can be written

$$\lambda_j = \sum_{i=1}^{[L:K]} \mu_{i,j} l_i.$$

Thus we have

$$m = \sum_{j=1}^{[M:L]} \sum_{i=1}^{[L:K]} \mu_{i,j} l_i m_j,$$

which proves that the $l_i m_j$ span M over K.

Besides, if we had a linear dependency relation

$$\sum_{j=1}^{[M:L]} \sum_{i=1}^{[L:K]} \mu_{i,j} l_i m_j = 0$$

with $\mu_{i,j} \in K$, then we would have

$$\sum_{j=1}^{[M:L]} \lambda_j m_j = 0$$

where

$$\lambda_j = \sum_{i=1}^{[L:K]} \mu_{i,j} l_i \in L.$$

Since $(m_j)_{1 \leqslant j \leqslant [M:L]}$ is an L-basis of M, this would imply that

$$0 = \lambda_j = \sum_{i=1}^{[L:K]} \mu_{i,j} l_i \in L$$

for all j; and since $(l_i)_{1 \leqslant i \leqslant [L:K]}$ is a K-basis of L, this means that the $\mu_{i,j}$ are all zero. Thus the $l_i m_j$ are linearly independent over K. □

Example 1.2.12. According to example 1.2.5 above,

$$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2,$$

and

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2.$$

It then follows from proposition 1.2.11 that

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 2 \times 2 = 4.$$

1.2.4 The trace, norm, and characteristic polynomial

Definition 1.2.13. Let L be a finite extension of K, and let $\alpha \in L$. Then multiplication by α induces a K-endomorphism of L, denoted by

$$\begin{array}{rccc} \mu_\alpha \colon & L & \longrightarrow & L \\ & \xi & \longmapsto & \alpha\xi. \end{array}$$

The *trace*, *norm*, and *characteristic polynomial* of α (with respect to the extension L/K) are, respectively, the trace, determinant, and characteristic polynomial of this endomorphism. They are denoted respectively by $\mathrm{Tr}_K^L(\alpha) \in K$, $N_K^L(\alpha) \in K$, and $\chi_K^L(\alpha) \in K[x]$. When the extension L/K is clear from the context, we will just write $\mathrm{Tr}(\alpha)$, $N(\alpha)$ and $\chi(\alpha)$.

Remark 1.2.14. Note that, as K-endomorphisms of L, $\mu_{\alpha+\beta} = \mu_\alpha + \mu_\beta$ and $\mu_{\alpha\beta} = \mu_\alpha \circ \mu_\beta$ for all $\alpha, \beta \in L$; this translates respectively into the identities $\mathrm{Tr}(\alpha+\beta) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\beta)$ and $N(\alpha\beta) = N(\alpha)N(\beta)$. Thus the trace is an additive group homomorphism from L to K, whereas the norm is a multiplicative group homomorphism from $L^\times$ to $K^\times$.

Also note that $N(\alpha) = 0$ if and only if $\alpha = 0$.

Example 1.2.15. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, seen for now as an extension of $\mathbb{Q}$, and let $\alpha = \sqrt{2} + \sqrt{3} \in L$. With respect to the $\mathbb{Q}$-basis $(1, \sqrt{2}, \sqrt{3}, \sqrt{2}\sqrt{3})$ of L, the matrix of μ_α is

$$\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix};$$

therefore $\mathrm{Tr}_{\mathbb{Q}}^L(\alpha) = 0$, $N_{\mathbb{Q}}^L(\alpha) = 1$, and $\chi_{\mathbb{Q}}^L(\alpha) = x^4 - 10x^2 + 1$.

On the other hand, if we have the extension L/K in mind, where $K = \mathbb{Q}(\sqrt{2})$, then we find that the matrix of μ_α with respect to the K-basis $(1, \sqrt{3})$ of L is

$$\begin{pmatrix} \sqrt{2} & 3 \\ 1 & \sqrt{2} \end{pmatrix},$$

so that $\mathrm{Tr}_K^L(\alpha) = 2\sqrt{2}$, $N_K^L(\alpha) = -1$, and $\chi_{\mathbb{Q}}^L(\alpha) = x^2 - 2\sqrt{2}x - 1$.

Proposition 1.2.16. *Let L/K be a finite extension, let α be an element of L, and let $\chi = \chi_K^L(\alpha) \in K[x]$ be its characteristic polynomial. Then χ vanishes at α.*

Proof. Since χ is the characteristic polynomial of the endomorphism μ_α, we have $\chi(\mu_\alpha) = 0$ by Cayley-Hamilton. But $P(\mu_\alpha) = \mu_{P(\alpha)}$ for every polynomial $P \in K[x]$, so in particular $0 = \chi(\mu_\alpha) = \mu_{\chi(\alpha)}$, which means that $\chi(\alpha) = 0$. □

Corollary 1.2.17. *If an extension is finite, then it is algebraic.*

The converse does not hold; for instance $\overline{\mathbb{Q}}$ is an algebraic extension of $\mathbb{Q}$, but it is not a finite extension.

Definition 1.2.18. A *number field* is a finite extension of $\mathbb{Q}$.

Thus every element of a number field is algebraic, and conversely every algebraic number α spans a number field $\mathbb{Q}(\alpha)$. In a nutshell, it can be said that this course is about the arithmetic properties of number fields.

Example 1.2.19. The fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are number fields, of respective degrees 1, 2, 2, and 4. On the contrary, neither $\mathbb{R}$ nor $\mathbb{C}$ are number fields, for instance because $\pi \in \mathbb{R}$ so that $\mathbb{Q}(\pi) \subset \mathbb{R} \subset \mathbb{C}$ so $[\mathbb{R} : \mathbb{Q}]$ and $[\mathbb{C} : \mathbb{Q}]$ are infinite because $[\mathbb{Q}(\pi) : \mathbb{Q}] = \infty$ since π is transcendental. Similarly, the finite fields are not number fields, since they are not extensions of $\mathbb{Q}$.

Proposition 1.2.16 above tells us in particular that the characteristic polynomial is a multiple of the minimal polynomial. In fact, more is true:

Proposition 1.2.20 (Characteristic poly. vs. minimal poly.)**.** *Let L/K be a finite extension of degree n. Let $\alpha \in L$, let $m(x)$ be its minimal polynomial over K, and let $\chi(x)$ be its characteristic polynomial (with respect to the extension L/K). Let $d = \deg m(x)$. Then d divides n, and $\chi(x) = m(x)^{n/d}$.*

Proof. We have a double extension $K \subset K(\alpha) \subset L$. By hypothesis, $[L : K] = n$; besides, we have $[K(\alpha) : K] = \deg m(x) = d$. Therefore, $[L : K(\alpha)] = n/d$ is an integer.

Next, multiplication by α defines a K-endomorphism of $K(\alpha)$. The characteristic polynomial of this endomorphism is monic, has degree $d = \deg m(x)$, and vanishes at α by proposition 1.2.16; therefore, this polynomial agrees with $m(x)$.

Let now $M_\alpha \in \mathrm{Mat}_{d\times d}(K)$ be the matrix of this endomorphism on a K-basis $(e_j)_{1\leqslant j\leqslant d}$ of $K(\alpha)$, and let $(f_k)_{1\leqslant k\leqslant n/d}$ be a $K(\alpha)$-basis of L. Then,

by proposition 1.2.11, $(e_j f_k)_{\substack{1\leqslant j\leqslant d \\ 1\leqslant k\leqslant n/d}}$ is a K-basis of L, and in this basis, the matrix of the multiplication-by-α endomorphism of L is

$$\begin{pmatrix} M_\alpha & & 0 \\ & \ddots & \\ 0 & & M_\alpha \end{pmatrix},$$

a diagonal of n/d copies of M_α. Since $m(x)$ is the characteristic polynomial of M_α and $\chi(x)$ is the characteristic polynomial of this big block-diagonal matrix, the result follows. □

1.2.5 Primitive elements

Definition 1.2.21. Let L/K be a field extension, and let $\alpha \in L$, so that $K(\alpha) \subseteq L$. One says that α is a *primitive element* for L over K (or just a primitive element, when $K = \mathbb{Q}$) if $K(\alpha) = L$.

Remark 1.2.22. A primitive element for L/K, when it exists, has no reason to be unique (and in fact it is never unique).

By looking at the degrees and in view of proposition 1.2.20, one immediately gets the following

Proposition 1.2.23. *Let L/K be a finite extension, and let $\alpha \in L$. Then the following are equivalent:*

- *α is a primitive element for L/K,*
- *$\deg_K \alpha = [L : K]$,*
- *the characteristic polynomial $\chi_K^L(\alpha)$ is irreducible over K,*
- *the characteristic polynomial $\chi_K^L(\alpha)$ is squarefree over K,*
- *the characteristic polynomial $\chi_K^L(\alpha)$ agrees with the minimal polynomial of α over K.*

Example 1.2.24. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, and let $\alpha = \sqrt{2} + \sqrt{3} \in L$. We know from example 1.2.15 above that α is a root of

$$P(x) = x^4 - 10x^2 + 1 \in \mathbb{Q}[x].$$

Since $\gcd\big(P(x), P'(x)\big) = 1$, $P(x)$ is squarefree. It follows that it is irreducible over $\mathbb{Q}$, and that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$, so that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Thus α is a primitive element for the number field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Besides, it follows that the degree of α is 4, and that its minimal polynomial is $P(x)$. Furthermore, since (for instance) $\sqrt{2} \in L$, this also means that there exists a polynomial $F(x) \in \mathbb{Q}[x]$ such that $F(\sqrt{2} + \sqrt{3}) = \sqrt{2}$, a fact which was not obvious.

On the other hand, neither $\sqrt{2}$ nor $\sqrt{3}$ are primitive elements for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, since they span strictly smaller fields over $\mathbb{Q}$.

However, $\sqrt{3}$ is a primitive element for L over $\mathbb{Q}(\sqrt{2})$.

Example 1.2.25. i is a primitive element for $\mathbb{C}$ over $\mathbb{R}$, but certainly not over $\mathbb{Q}$ since $\mathbb{Q}(i)$ is much smaller than $\mathbb{C}$.

The following theorem guarantees the existence of primitive elements in many cases.

Theorem 1.2.26 (Primitive element theorem)**.** *Let K be a field of characteristic 0. If L is a finite extension of K, then there exists a primitive element for L/K.*

In fact, one can even prove that with these hypotheses, "most" elements of L are primitive elements.

Remark 1.2.27 (Technical, feel free to skip this)**.** Without the characteristic 0 hypothesis, this theorem is false. A classical counterexample is the extension L/K, where K is the 2-variable rational fraction field $K = \mathbb{F}_p(x, y)$ over the finite field $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$ with p elements, where $p \in \mathbb{N}$ is prime, and $L = \mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$.

In particular, every number field K has primitive elements, so it can be written in the form $K = \mathbb{Q}(\alpha)$, where $\alpha \in K$ is an algebraic number of the same degree as K. This means that K can be seen as the quotient $\mathbb{Q}(\alpha) \simeq \mathbb{Q}[x]/m(x)$ where $m(x) = m_\alpha(x)$ is the minimal polynomial of α over $\mathbb{Q}$. Thus K is "$\mathbb{Q}$ adjoined some abstract element α, which is entirely characterised by the relation $m(\alpha) = 0$". In particular, we can (and should) think of K abstractly, as opposed to as a subfield of $\mathbb{C}$.

Conversely, a convenient way of specifying a number field up to isomorphism is to give it in the form $\mathbb{Q}(\alpha)$, where α is a root of some irreducible polynomial $m(x)$, which is thus the minimal polynomial of α up to scaling. It is important that $m(x)$ be irreducible over $\mathbb{Q}$, since otherwise the number field is not well-defined.

1.3 Complex embeddings

1.3.1 Extension of complex embeddings

Let K be a number field, and let L be a finite extension of K. Let $\alpha \in L$ be a primitive element of the extension L/K, and let $m(x) \in K[x]$ be its minimal polynomial over K. Then $m(x)$ is irreducible over K; in particular, it is coprime with $m'(x)$. Suppose that we have field embeddings $\sigma : K \hookrightarrow \mathbb{C}$ and $\tau\colon L \hookrightarrow \mathbb{C}$. If $\tau|_K = \sigma$, we say that τ *extends* σ.

Since $m(x) \in K[x]$, we may apply σ to its coefficients, which yields a polynomial $m^\sigma(x) \in \mathbb{C}[x]$. If τ extends σ, then $\tau(\alpha) \in \mathbb{C}$ must be a root of $m^\sigma(x)$; conversely, for each complex root $z \in \mathbb{C}$ of $m^\sigma(x)$, we can define an embedding of L into $\mathbb{C}$ which extends σ, by the formula

$$\begin{array}{rcl} L & \hookrightarrow & \mathbb{C} \\ \sum_k \lambda_k \alpha^k & \longmapsto & \sum_k \sigma(\lambda_k) z^k, \end{array}$$

where the λ_k lie in K. The polynomial $m^\sigma(x)$ is of degree $[L : K]$; besides, it is coprime with $m'^\sigma(x)$ (apply σ to a relation $U(x)m(x) + V(x)m'(x) = 1$), so it has no multiple roots in $\mathbb{C}$. We thus get the following result.

Theorem 1.3.1 (Extension of complex embeddings)**.** *Let K be a number field, $\sigma\colon K \hookrightarrow \mathbb{C}$ an embedding of K into $\mathbb{C}$, and L a finite extension of K. Then there are exactly $[L : K]$ embeddings of L into $\mathbb{C}$ that extend σ.*

Corollary 1.3.2. *Let K be a number field. Then K can be embedded into $\mathbb{C}$ in exactly $[K : \mathbb{Q}]$ different ways.*

Proof. Apply the previous theorem to the extension $K/\mathbb{Q}$, and note that $\mathbb{Q}$ can be embedded into $\mathbb{C}$ only in one way. □

In particular, every number field K can be embedded into $\mathbb{C}$ in at least one way, so it may be seen as a subfield of $\mathbb{C}$. However, if $K \neq \mathbb{Q}$, then there are several embeddings into $\mathbb{C}$, so there are several inequivalent ways of seeing K as a subfield of $\mathbb{C}$, none of which is better than the other ones. As a consequence, it is better to think of number fields as abstract extensions of $\mathbb{Q}$ rather than as subfields of $\mathbb{C}$ whenever possible.

Example 1.3.3. Let $K = \mathbb{Q}(\alpha)$ where $\alpha^2 - 3 = 0$. This is a number field of degree 2, so there are 2 distinct ways of seeing K as a subfield of $\mathbb{C}$, namely by interpreting α as $\sqrt{3}$ or as $-\sqrt{3}$.

Let $L = K(\beta)$, where $\beta^2 - 4 - \alpha = 0$. One can prove that $[L : K] = 2$, so for each embedding of K into $\mathbb{C}$ there are 2 embeddings of L into $\mathbb{C}$ that extend it.

Namely, the $[K : \mathbb{Q}] = 2$ embeddings of K into $\mathbb{C}$ are

$$\sigma_1 \colon a + b\alpha \mapsto a + b\sqrt{3} \quad \text{and} \quad \sigma_2 \colon a + b\alpha \mapsto a - b\sqrt{3}$$

where $(a, b \in \mathbb{Q})$; each of them can be extended to L in $[L : K] = 2$ ways, respectively by

$$c + d\beta \mapsto \sigma_1(c) \pm \sigma_1(d)\sqrt{4 + \sqrt{3}} \quad \text{and by} \quad c + d\beta \mapsto \sigma_2(c) \pm \sigma_2(d)\sqrt{4 - \sqrt{3}}$$

where $c, d \in K$. We thus recover all $[L : \mathbb{Q}] = 4$ embeddings of L into $\mathbb{C}$.

1.3.2 The signature of a number field

Definition 1.3.4. An embedding of a number field into $\mathbb{C}$ is *real* if its image is contained in $\mathbb{R}$. Nonreal embeddings come in conjugate pairs, so we can define the *signature* of a number field K to be the pair (r_1, r_2), where r_1 is the number of real embeddings of K, and r_2 is the number of conjugate pairs of nonreal embeddings of K.

A number field is said to be *totally real* if $r_2 = 0$, and *totally complex* if $r_1 = 0$.

Obviously, one has the relation $[K : \mathbb{Q}] = r_1 + 2r_2$.

Example 1.3.5. The number field $\mathbb{Q}(\sqrt{2})$ has signature (2,0) and is thus totally real, whereas $\mathbb{Q}(i)$ has signature (0,1) and is thus totally complex.

On the other hand, $\mathbb{Q}(\sqrt[3]{2})$ has signature (1,1), and is thus neither totally real nor totally complex.

More generally, the signature of $\mathbb{Q}(\alpha)$ is (r_1, r_2), where r_1 (reps. r_2) is the number of real roots (resp. the number of conjugate pairs of complex nonreal roots) of the minimal polynomial of α.

1.3.3 Traces and norms vs. complex embeddings

The trace, norm, and characteristic polynomial are related to the complex embeddings by the following formulae:

Theorem 1.3.6. *Let K be a number field, $\sigma\colon K \hookrightarrow \mathbb{C}$ an embedding of K into $\mathbb{C}$, and let L a finite extension of K. If Σ denotes the set of the $[L:K]$ embeddings of L into $\mathbb{C}$ that extend σ, then we have*

$$\sigma\big(\operatorname{Tr}_K^L(\alpha)\big) = \sum_{\tau\in\Sigma} \tau(\alpha),$$

$$\sigma\big(N_K^L(\alpha)\big) = \prod_{\tau\in\Sigma} \tau(\alpha),$$

and

$$\chi_K^L(\alpha)^\sigma(x) = \prod_{\tau\in\Sigma_\tau} \big(x - \tau(\alpha)\big).$$

Corollary 1.3.7 (Transitivity of traces and norms)**.** *Suppose we have a double extension $K \subset L \subset M$, and let $\alpha \in M$. Then we have*

$$\operatorname{Tr}_K^M(\alpha) = \operatorname{Tr}_K^L\big(\operatorname{Tr}_L^M(\alpha)\big)$$

and

$$N_K^M(\alpha) = N_K^L\big(N_L^M(\alpha)\big).$$

Proof. Let $\sigma\colon K \hookrightarrow \mathbb{C}$ be an embedding. Then we have

$$\begin{aligned}
\sigma\big(\operatorname{Tr}_K^M(\alpha)\big) &= \sum_{\substack{\rho\colon M\hookrightarrow\mathbb{C}\\ \rho_{|K}=\sigma}} \rho(\alpha) \\
&= \sum_{\substack{\tau\colon L\hookrightarrow\mathbb{C}\\ \tau_{|K}=\sigma}} \sum_{\substack{\rho\colon M\hookrightarrow\mathbb{C}\\ \rho_{|L}=\tau}} \rho(\alpha) \\
&= \sum_{\substack{\tau\colon L\hookrightarrow\mathbb{C}\\ \tau_{|K}=\sigma}} \tau\big(\operatorname{Tr}_L^M(\alpha)\big) \\
&= \sigma\big(\operatorname{Tr}_K^L(\operatorname{Tr}_L^M(\alpha))\big),
\end{aligned}$$

whence the result for traces since σ, being an embedding, is one-to-one. The proof for the norms is the same, with products instead of the sums. □

Corollary 1.3.8. *Let K be a number field, and let Σ be the set of all its embeddings into $\mathbb{C}$. Then for all $\alpha \in K$, we have*

$$\operatorname{Tr}_{\mathbb{Q}}^K(\alpha) = \sum_{\sigma\in\Sigma} \sigma(\alpha),$$

$$N_{\mathbb{Q}}^K(\alpha) = \prod_{\sigma \in \Sigma} \sigma(\alpha),$$

and

$$\chi_{\mathbb{Q}}^K(\alpha)(x) = \prod_{\sigma \in \Sigma} \big(x - \sigma(\alpha)\big).$$

Example 1.3.9. Let $K = \mathbb{Q}(\sqrt{-7}) = \{a + b\sqrt{-7},\ a, b \in \mathbb{Q}\}$. Then $[K : \mathbb{Q}] = 2$, so K has 2 embeddings into $\mathbb{C}$, namely

$$a + b\sqrt{-7} \longmapsto a + bi\sqrt{7}$$

and

$$a + b\sqrt{-7} \longmapsto a - bi\sqrt{7}.$$

Thus we have

$$\mathrm{Tr}_{\mathbb{Q}}^K(a + b\sqrt{-7}) = (a + bi\sqrt{7}) + (a - bi\sqrt{7}) = 2a,$$

$$N_{\mathbb{Q}}^K(a + b\sqrt{-7}) = (a + bi\sqrt{7})(a - bi\sqrt{7}) = a^2 + 7b^2,$$

and

$$\chi_{\mathbb{Q}}^K(a + b\sqrt{-7}) = \big(x - (a + bi\sqrt{7})\big)\big(x - (a - bi\sqrt{7})\big) = x^2 - 2ax + a^2 + 7b^2.$$

Chapter 2

Algebraic integers

2.1 The ring of integers

In the previous chapter, we have defined number fields, which can be seen as generalisations of $\mathbb{Q}$. In order to perform arithmetic there, we would now like to study subrings of number fields which are the analogue of $\mathbb{Z} \subset \mathbb{Q}$. The question we are asking is thus: what is the generalisation of the notion of integer to number fields?

2.1.1 Monic polynomials

Definition 2.1.1. Let α be an algebraic number. One says that α is an *algebraic integer* if its *monic* minimal polynomial, which a priori lies in $\mathbb{Q}[x]$, actually lies in $\mathbb{Z}[x]$.

Recall that $\mathbb{Z}[x]$ is a UFD. Clearly, the factorisation in $\mathbb{Z}[x]$ of a *monic* polynomial can only involve *monic* factors (up to sign); besides, this factorisation is the same as in $\mathbb{Q}[x]$. This and proposition 1.2.20 imply the following characterisation of integers:

Theorem 2.1.2. *Let K be a number field, and let $\alpha \in K$. The following are equivalent:*

- *α is an algebraic integer,*
- *There exists a nonzero **monic** polynomial $P \in \mathbb{Z}[x]$ such that $P(\alpha) = 0$,*
- *The characteristic polynomial of α lies in $\mathbb{Z}[x]$.*

Example 2.1.3. In $\mathbb{Q}(\sqrt{-7})$, $\alpha = \sqrt{-7}$ is an algebraic integer, because it is a root of the monic polynomial $x^2 + 7$ (actually, this polynomial happens to be the minimal polynomial of α, but we do not need that here!). On the contrary, $\frac{1}{2}\sqrt{-7}$ is not an algebraic integer, because its characteristic polynomial $x^2 + \frac{7}{4}$ does not lie in $\mathbb{Z}[x]$; we could prove this by noticing that its *monic* minimal polynomial (which in this case is also $x^2 + \frac{7}{4}$) does not lie in $\mathbb{Z}[x]$. Finally, $\frac{1+\sqrt{-7}}{2}$ is an algebraic integer, since its characteristic polynomial $x^2 - x + 2$ lies in $\mathbb{Z}[x]$.

2.1.2 The ring of integers

The main point of this definition is that algebraic integers form a ring, just like classical integers.

Theorem 2.1.4. *The sum, difference, and product of two algebraic integers is an algebraic integer. As a consequence, the set of elements of a number field K which are algebraic integers forms a subring of K.*

This can be proved just like theorem 1.2.6, by considering resultants of polynomials in $\mathbb{Z}[x][y]$.

Remark 2.1.5 (Sanity check)**.** The reason why the same proof fails (as it should !) to show that the quotient two algebraic integers α and β is an algebraic integer is that although we can use resultants to produce a polynomial in $\mathbb{Z}[x]$ which vanishes at α/β, this polynomial is *not monic* in general.

Definition 2.1.6. The subring of a number field K formed by the elements which are algebraic integers is called the *ring of integers* of K. It is denoted by $\mathbb{Z}_K$ (some people use the notation $\mathcal{O}_K$).

Proposition 2.1.7. *We have $\mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$.*

Proof. Let $\alpha \in K$. If α happens to lie in $\mathbb{Q}$, then its characteristic polynomial is $(x - \alpha)^{[K:\mathbb{Q}]}$, which lies in $\mathbb{Z}[x]$ if and only if $\alpha \in \mathbb{Z}$. □

It is clear that the image of an algebraic integer by a complex embedding is an algebraic integer: just look at its minimal polynomial. In view of theorem 1.3.6, we deduce the following:

Proposition 2.1.8 (Integrality vs. trace and norm, relative)**.** *Let L/K be a finite extension of number fields, and let $\alpha \in L$. If $\alpha \in \mathbb{Z}_L$, then $\mathrm{Tr}_K^L(\alpha) \in \mathbb{Z}_K$, $N_K^L(\alpha) \in \mathbb{Z}_K$, and $\chi_K^L(\alpha) \in \mathbb{Z}_K[x]$.*

Corollary 2.1.9 (Integrality vs. trace and norm, absolute)**.** *Let K be a number field. For all $\alpha \in \mathbb{Z}_K$, we have $\mathrm{Tr}_{\mathbb{Q}}^K(\alpha) \in \mathbb{Z}$, $N_{\mathbb{Q}}^K(\alpha) \in \mathbb{Z}$, and $\chi_{\mathbb{Q}}^K(\alpha) \in \mathbb{Z}[x]$.*

Note that we already knew that for the characteristic polynomial.

2.2 Orders and discriminants

2.2.1 Linear algebra over $\mathbb{Z}$

Definition 2.2.1. Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, so that $\mathbb{Z} \subset \mathbb{K}$, and let V be a $\mathbb{K}$-vector space of finite dimension n. A *lattice* in V is a sub-additive-group of V of the form

$$I = \Big\{ \sum_{j=1}^{n} \lambda_j v_j,\ \lambda_j \in \mathbb{Z} \Big\},$$

where $(v_j)_{1 \leqslant j \leqslant n}$ is a $\mathbb{K}$-basis of V. We thus have

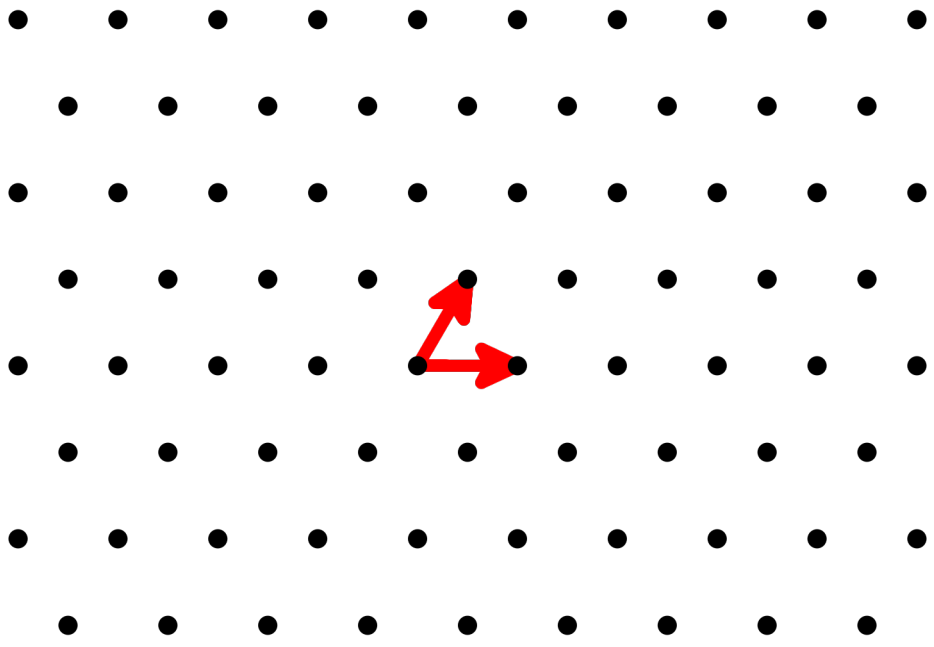
$$I = \Big\{ \sum_{j=1}^{n} \lambda_j v_j,\ \lambda_j \in \mathbb{Z} \Big\} \subsetneq V = \Big\{ \sum_{j=1}^{n} \lambda_j v_j,\ \lambda_j \in \mathbb{K} \Big\}.$$

We will write

$$I = \bigoplus_{j=1}^{n} \mathbb{Z} v_j = \mathbb{Z} v_1 \oplus \cdots \oplus \mathbb{Z} v_n,$$

and call $(v_j)_{1 \leqslant j \leqslant n}$ a $\mathbb{Z}$-basis of I.

Example 2.2.2. Here is an example of a lattice in dimension $n = 2$:



The red vectors form a $\mathbb{K}$-basis of $\mathbb{K}^2$, so they span a lattice, and form a $\mathbb{Z}$-basis of this lattice. You should keep this picture in mind, and try to imagine what a lattice looks like in dimension 3 and higher.

Just as a vector space, a lattice has many different bases. Any two $\mathbb{Z}$-bases of the lattice differ by an element of

$$\mathrm{GL}_n(\mathbb{Z}) = \{A \in \mathrm{GL}_n(\mathbb{Q}) \mid A, A^{-1} \in M_n(\mathbb{Z})\},$$

and it is not difficult to prove that

$$\mathrm{GL}_n(\mathbb{Z}) = \{A \in \mathrm{GL}_n(\mathbb{Q}) \mid \det A = \pm 1\}.$$

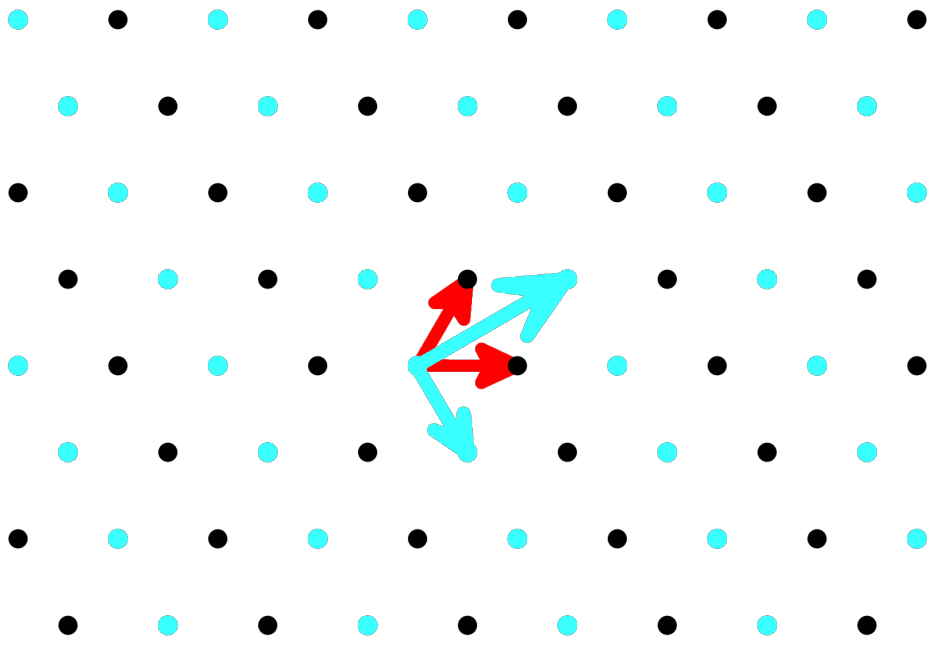
Suppose now that in $\mathbb{K}^n$, we have a lattice J which is contained in another lattice I; we then say that J is a *sublattice* of I. In particular, J is an additive subgroup of I, so it makes sense to consider the quotient group I/J; the cardinal of this quotient is then called the *index* of J in I, and we write it $[I : J]$.

Remark 2.2.3. This is the same notation as for the degree of a field extension, although these are rather different notions ! Unfortunately, these notations are well-established, so we cannot change them.

Theorem 2.2.4. *[Computation of the index] Suppose we have a lattice I in $\mathbb{K}^n$, and a sublattice $J \subseteq I$ be a sublattice. Fix a $\mathbb{Z}$-basis $v_1, \cdots, v_n$ of I, and a $\mathbb{Z}$-basis $w_1, \cdots w_n$ of J, and form the $n \times n$ matrix A expressing the w_j in terms of the v_i. Since $J \subset I$, the entries of A all lie in $\mathbb{Z}$; besides, $\det A \neq 0$ since the v_i and the w_j are two $\mathbb{K}$-bases of the vector space $\mathbb{K}^n$. The index of J in I is then given by $[I : J] = |\det A|$; in particular, it is always finite.*

We thus recover the fact that $I = J$ if and only if $\det A = \pm 1$.

Example 2.2.5. Let I be the lattice in the previous picture, and let us pick two linearly independent vectors (in blue) in I:



Since this vectors are linearly independent, they form another $\mathbb{K}$-basis $\mathbb{K}^2$, so they span another lattice J. Besides, since these vectors lie in I, the lattice J is contained in I, so J is a sublattice of J.

In order to know whether J agrees I or is a strict sublattice, and more generally to compute the index of J in I, we form the matrix A expressing the $\mathbb{Z}$-basis of J in terms of the $\mathbb{Z}$-basis of I, in other words the blue vectors in terms of the red vectors:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and we compute

$$|\det A| = |-2| = 2.$$

This result means that J vectors has index 2 in I; in particular, the containment $J \subsetneq I$ is proper. The fact that the index is 2 expresses that precisely half of the points I actually lie J, which becomes clear if we colourise the points of J in blue as on the picture above.

2.2.2 Orders

Definition 2.2.6. Let K be a number field of degree n. An *order* in K is a subring $\mathcal{O}$ of K which is also a lattice in the $\mathbb{Q}$-vector space K.

In particular, if $\mathcal{O} \subset K$ is an order, then for all $\alpha \in K$ there exists an $n \in \mathbb{N}$ such that $n\alpha \in \mathcal{O}$. Thus the fraction field of $\mathcal{O}$ is K itself.

Example 2.2.7. For example, $\mathbb{Z}[\sqrt{5}]$ is an order in $\mathbb{Q}(\sqrt{5})$, whereas $\mathbb{Z}[\frac{1}{2}\sqrt{5}]$ is not because it is not a lattice, and neither is $\mathbb{Z} \oplus \mathbb{Z}\frac{1}{2}\sqrt{5}$ because it is not a subring. Finally, $\mathbb{Z}[\sqrt{2}]$ is not an order in $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, because it does not span all of K over $\mathbb{Q}$.

Lattices provide yet another characterisation of algebraic integers.

Proposition 2.2.8. *Let α be an algebraic number. The following are equivalent:*

1. *α is an algebraic integer,*
2. *$\mathbb{Z}[\alpha]$ is an order in $\mathbb{Q}(\alpha)$,*

3. *There exist a number field K containing α and a lattice I in K which is stable under multiplication by α, i.e. such that $\alpha I \subseteq I$.*

Proof.

- $1 \Longrightarrow 2$: Let $m(x) = x^n + \sum_{j=0}^{n-1} \lambda_j x^j \in \mathbb{Q}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. If α is an algebraic integer, then the λ_j lie in fact in $\mathbb{Z}$, and the relation $m(\alpha) = 0$ shows that

$$\mathbb{Z}[\alpha] = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha^2 \oplus \cdots \oplus \mathbb{Z}\alpha^{n-1}.$$

  Since $\mathbb{Z}[\alpha]$ is also a ring, it is thus an order in

$$\mathbb{Q}(\alpha) = \mathbb{Q} \oplus \mathbb{Q}\alpha \oplus \mathbb{Q}\alpha^2 \oplus \cdots \oplus \mathbb{Q}\alpha^{n-1}.$$

- $2 \Longrightarrow 3$: Simply take $K = \mathbb{Q}(\alpha)$ and $I = \mathbb{Z}[\alpha]$.

- $3 \Longrightarrow 1$: If $\alpha I \subset I$, then the matrix of the multiplication-by-α map μ_α of L with respect to a $\mathbb{Z}$-basis of I is a matrix with coefficients in $\mathbb{Z}$, so by definition $\chi_{\mathbb{Q}}^K(\alpha)$ lies in $\mathbb{Z}[x]$, which implies that α is an algebraic integer.

□

Corollary 2.2.9. *Let $\mathcal{O}$ be an order in a number field K. Every element of $\mathcal{O}$ is an algebraic integer, so $\mathcal{O} \subseteq \mathbb{Z}_K$.*

Corollary 2.2.10. *Let α be an element of a number field K. The subring $\mathbb{Z}[\alpha]$ of K is an order in K if and only if α is both a primitive element for $K/\mathbb{Q}$ and an algebraic integer.*

2.2.3 Discriminants, part I

Definition 2.2.11. Let K be a number field of degree n. The *trace pairing* is the bilinear form

$$\begin{array}{rccc} \mathrm{Tr}\colon & K \times K & \longrightarrow & \mathbb{Q} \\ & (\alpha, \beta) & \longmapsto & \mathrm{Tr}_{\mathbb{Q}}^K(\alpha\beta) \end{array}.$$

Let $\alpha_1, \cdots, \alpha_n$ be n elements of K. Their *discriminant* is the determinant

$$\mathrm{disc}(\alpha_1, \cdots, \alpha_n) = \det\left(\mathrm{Tr}_{\mathbb{Q}}^K(\alpha_i\alpha_j)\right)_{1 \leqslant i,j \leqslant n} \in \mathbb{Q}.$$

Proposition 2.2.12. $\operatorname{disc}(\alpha_1, \cdots, \alpha_n) \neq 0$ *if and only if the* α_j *are* $\mathbb{Q}$*-linearly independent, that is to say if and only if they form a* $\mathbb{Q}$*-basis of* K*.*

Proof. Let $T \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ be the matrix such that $T_{i,j} = \mathrm{Tr}_{\mathbb{Q}}^K(\alpha_i\alpha_j)$, so that $\operatorname{disc}(\alpha_1, \cdots, \alpha_n) = \det T$.

If the α_j are $\mathbb{Q}$-linearly dependent, then we get a relation of linear dependency on the columns (and also on the rows) of T, so $\det T = 0$.

Conversely, suppose that the α_j form a $\mathbb{Q}$-basis of K. If we had $\det T = 0$, we would deduce a linear dependency relation between the columns of T; in other words, we would have $\mathrm{Tr}_{\mathbb{Q}}^K(\alpha_i\alpha) = 0$ for all i for some $\alpha = \sum_{j=1}^n \lambda_j\alpha_j$ with $\lambda_j \in \mathbb{Q}$ not all zero. Since the α_j form a $\mathbb{Q}$-basis of K, this means that $\alpha \neq 0$, and that $\mathrm{Tr}_{\mathbb{Q}}^K(\beta\alpha) = 0$ for all $\beta \in K$. But this is absurd (take $\beta = \frac{1}{\alpha}$, and note that $\mathrm{Tr}_{\mathbb{Q}}^K(1) = n \neq 0$). □

Definition 2.2.13. Let $\mathcal{O}$ be an order in a number field K, and let $(\omega_j)_{1\leqslant j\leqslant n}$ be a $\mathbb{Z}$-basis of $\mathcal{O}$. The *discriminant* of $\mathcal{O}$ is

$$\operatorname{disc}\mathcal{O} = \operatorname{disc}(\omega_1, \cdots, \omega_n) \in \mathbb{Z},$$

a non-zero integer. Since using another $\mathbb{Z}$-basis of $\mathcal{O}$ amounts to conjugating by a matrix in $\mathrm{GL}_n(\mathbb{Z})$, whose determinant will be ± 1, this does not depend on the choice of the basis of $\mathcal{O}$.

We now arrive to the central result of this chapter:

Theorem 2.2.14. *The ring of integers of a number field is an order in this number field.*

Example 2.2.15. Let us continue with example 1.3.9. An element $\alpha = a+b\sqrt{-7}$, $a, b \in \mathbb{Q}$ of $K = \mathbb{Q}(\sqrt{-7})$ lies in $\mathbb{Z}_K$ if and only if its characteristic polynomial $x^2 - 2ax + a^2 + 7b^2$ lies in $\mathbb{Z}[x]$, that is to say if and only if $2a \in \mathbb{Z}$ and $a^2 + 7b^2 \in \mathbb{Z}$. One checks easily that this condition is equivalent to $2a \in \mathbb{Z}$, $2b \in \mathbb{Z}$, and $a + b \in \mathbb{Z}$; thus

$$\mathbb{Z}_K = \mathbb{Z} \oplus \mathbb{Z}\frac{1+\sqrt{-7}}{2} = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right].$$

The middle term makes it apparent that $\mathbb{Z}_K$ is a lattice, with $\mathbb{Z}$-basis $\{1, \frac{1+\sqrt{-7}}{2}\}$; the right term makes it clear that $\mathbb{Z}_K$ is a ring.

In order to prove theorem 2.2.14, we need the following lemma, which is important in its own right:

Lemma 2.2.16. *Let K be a number field, and let α in K. There exists an integer $d \in \mathbb{N}$ such that $d\alpha$ is an algebraic integer.*

Proof. Let $\chi(x) \in \mathbb{Q}[x]$ be the characteristic polynomial of α, and let $d \in \mathbb{N}$ be a common denominator for the coefficients of $\chi(x)$, so that we may write

$$0 = \chi(\alpha) = \alpha^n + \sum_{j=0}^{n-1} \frac{\lambda_j}{d} \alpha^j$$

where $n = [K : \mathbb{Q}]$ and the λ_j are integers. Multiplying by d^n, we get

$$0 = d^n \chi(\alpha) = (d\alpha)^n + \sum_{j=0}^{n-1} d^{n-j} \lambda_j (d\alpha)^j,$$

which proves that $d\alpha$ is an algebraic integer. □

Note that this implies in particular that the fraction field of $\mathbb{Z}_K$ is K itself.

Proof of theorem 2.2.14. We already know that $\mathbb{Z}_K$ is a subring, and we must show that it is also a lattice. According to the lemma, we can find a $\mathbb{Q}$-basis $(\omega_j)_{1 \leqslant j \leqslant n}$ of K formed of algebraic integers; let

$$\Omega = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$$

be the lattice it spans. Let now $\alpha \in \mathbb{Z}_K$. We may write

$$\alpha = \sum_{j=1}^{n} \lambda_j \omega_j$$

where the λ_j are rational numbers. Multiplying by ω_k, $1 \leqslant k \leqslant n$, and taking the trace yields the $n \times n$ system of linear equations

$$\sum_{j=1}^{n} \lambda_j \operatorname{Tr}(\omega_j \omega_k) = \operatorname{Tr}(\alpha \omega_k) \quad (1 \leqslant k \leqslant n)$$

over $\mathbb{Z}$, of which the λ_j form a solution. The determinant of this system is $\Delta = \operatorname{disc}(\omega_1, \cdots, \omega_n)$, which is a nonzero integer since the ω_j are algebraic integers and form a $\mathbb{Q}$-basis of K. By inverting the matrix of the system, we thus see that the λ_j lie in $\frac{1}{\Delta}\mathbb{Z}$. We thus have

$$\Omega \subset \mathbb{Z}_K \subset \frac{1}{\Delta}\Omega,$$

which shows that $\mathbb{Z}_K$, being cornered between two lattices, is itself a lattice. □

Definition 2.2.17. In view of corollary 2.2.10, the ring of integers $\mathbb{Z}_K$ of K is thus an order which contains all the other orders of K as sublattices; it is therefore sometimes called the *maximal order* of K.

Definition 2.2.18. By theorem 2.2.4, every order is contained in $\mathbb{Z}_K$ with a finite index, and this index is called the *index* of the order.

Definition 2.2.19. The *discriminant* of K is defined as the discriminant of its ring of integers seen as an order, i.e.

$$\operatorname{disc} K = \operatorname{disc} \mathbb{Z}_K \in \mathbb{Z},$$

a nonzero integer.

Example 2.2.20. Let us continue with example 2.2.15. We now know that the ring of integers of $K = \mathbb{Q}(\sqrt{-7})$ is

$$\mathbb{Z}_K = \mathbb{Z}[\alpha] = \mathbb{Z} \oplus \mathbb{Z}\alpha,$$

where $\alpha = \frac{1+\sqrt{-7}}{2}$. As a consequence, we find that

$$\operatorname{disc} K = \operatorname{disc} \mathbb{Z}_K = \begin{vmatrix} \operatorname{Tr}_{\mathbb{Q}}^K(1) & \operatorname{Tr}_{\mathbb{Q}}^K(1 \cdot \alpha) \\ \operatorname{Tr}_{\mathbb{Q}}^K(\alpha \cdot 1) & \operatorname{Tr}_{\mathbb{Q}}^K(\alpha^2) \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -7.$$

We will soon see a much more efficient way to get to the same result.

To conclude this section, let us mention a famous result of Hermite's:

Theorem 2.2.21 (Hermite-Minkowski)**.** *Up to isomorphism, there are finitely many number fields of given discriminant.*

2.3 Computing the maximal order

Given a number field K, it is (usually) not difficult to find orders in K. For instance, if it is given in the form $K = \mathbb{Q}(\alpha)$, we may assume without loss of generality that the primitive element α is integral thanks to lemma 2.2.16, and then $\mathcal{O} = \mathbb{Z}[\alpha]$ is clearly an order in K. It may not, however, be the full ring of integers of K.

2.3.1 Denominators vs. the index

Here is a consequence of the definition of the index of an order.

Proposition 2.3.1. *Let $\mathcal{O}$ be an order in a number field K of degree n, let $f \in \mathbb{N}$ be its index, and let $(\omega_j)_{1\leqslant j\leqslant n}$ be any $\mathbb{Z}$-basis of $\mathcal{O}$. In particular, $(\omega_j)_{1\leqslant j\leqslant n}$ is also a $\mathbb{Q}$-basis of K, so every element of K can be uniquely written in the form $\sum_{j=1}^n \lambda_j\omega_j$ with $\lambda_j \in \mathbb{Q}$. Let $\operatorname{denom}_{\mathcal{O}}\left(\sum_{j=1}^n \lambda_j\omega_j\right)$ denote the lcm of the denominators of the λ_j. Then $\operatorname{denom}_{\mathcal{O}}\left(\sum_{j=1}^n \lambda_j\omega_j\right)$ depends on $\mathcal{O}$, but not on the choice of the $\mathbb{Z}$-basis $(\omega_j)_{1\leqslant j\leqslant n}$ of $\mathcal{O}$. Furthermore, for all $p \in \mathbb{N}$ prime, the following are equivalent:*

1. *$p \mid f$,*
2. *$\exists\, \beta \in \mathbb{Z}_K : p \mid \operatorname{denom}_{\mathcal{O}}(\beta)$,*
3. *$\exists\, \beta \in \mathbb{Z}_K :\ \operatorname{denom}_{\mathcal{O}}(\beta) = p$.*

In other words, when the elements of $\mathbb{Z}_K$ are put in the form $\sum_j \lambda_j\omega_j$, the primes $p \in \mathbb{N}$ that divide the denominator of one (or more) of the λ_j are exactly the ones that divide f. For example, if $\mathcal{O}$ is of the form $\mathbb{Z}[\alpha]$, we may consider the $\mathbb{Z}$-basis $1, \alpha, \cdots \alpha^{n-1}$ of $\mathcal{O}$, and $\operatorname{denom}_{\mathcal{O}}(\beta)$ is then the common denominator of the coefficients of β expressed as a polynomial of degree $< n$ in α.

Proof. (Non examinable) Think of $\mathcal{O}$, $\mathbb{Z}_K$ and K as additive groups. Then for every $\beta \in K$, $\operatorname{denom}_{\mathcal{O}}(\alpha)$ is the order of this element in the quotient group $K/\mathcal{O}$, since

$$\forall n \in \mathbb{N}, \quad n\beta = 0 \text{ in } K/\mathcal{O} \iff n\beta \in \mathcal{O} \iff \operatorname{denom}_{\mathcal{O}}(\beta).$$

Therefore, $\operatorname{denom}_{\mathcal{O}}(\beta)$ depends on $\mathcal{O}$ but not on the choice of a $\mathbb{Z}$-basis of $\mathcal{O}$.

Besides, $\mathbb{Z}_K/\mathcal{O}$ is an Abelian group whose order is by definition f, so the order of every element of $\mathbb{Z}_K$ divides f by Lagrange's theorem; conversely, for all $p \mid f$ prime, Cauchy's theorem tells us that there exists an element of $\mathbb{Z}_K/\mathcal{O}$ of order exactly p. □

Example 2.3.2. In $K = \mathbb{Q}(\sqrt{-7})$, the order $\mathcal{O} = \mathbb{Z}[\sqrt{-7}]$ is not the full ring of integers of K; in fact, we know from example 2.2.20 that $\mathbb{Z}_K = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$, so $\mathcal{O}$ is a non-maximal order of index 2. By the theorem, if we express the elements of K in the form $a + b\sqrt{-7}$ with $a, b \in \mathbb{Q}$, then the denominator of every element of $\mathbb{Z}_K$ is a power of 2; for instance, $\mathbb{Z}_K \ni \frac{1+\sqrt{-7}}{2} = \frac{1}{2} + \frac{1}{2}\sqrt{-7}$.

2.3.2 Discriminants, part II

The general approach to compute $\mathbb{Z}_K$ consists in starting with an order of the form $\mathbb{Z}[\alpha]$, and enlarging it until it becomes maximal. We need to know when we can stop, that is to say when the order is maximal. For this, the following relation between the discriminant of and order and its index is primordial.

Theorem 2.3.3. *Let K be a number field, and let $\mathcal{O}$ be an order in K of index $f \in \mathbb{N}$. Then*

$$\operatorname{disc} \mathcal{O} = f^2 \operatorname{disc} K.$$

Proof. Let $(\omega_j)_{1 \leqslant j \leqslant n}$ be a $\mathbb{Z}$-basis of $\mathbb{Z}_K$, and let T be the matrix of the $\operatorname{Tr}_{\mathbb{Q}}^K(\omega_i \omega_j)$, so that $\det T = \operatorname{disc} K$. If $P \in \operatorname{Mat}_{n \times n}(\mathbb{Z})$ is the change of basis matrix expressing a $\mathbb{Z}$-basis of $\mathcal{O}$ on the ω_i, then we have $\det P = \pm f$ by theorem 2.2.4, so $\operatorname{disc} \mathcal{O} = \det({}^t PTP) = f^2 \operatorname{disc} K$. □

Definition 2.3.4. For prime $p \in \mathbb{N}$, let us say that $\mathcal{O}$ is *p-maximal* if p does not divide the index of $\mathcal{O}$.

Corollary 2.3.5. *Let $\mathcal{O}$ be an order in a number field K. If $\mathcal{O}$ is not p-maximal, then $p^2 | \operatorname{disc} \mathcal{O}$. In particular, if $\operatorname{disc} \mathcal{O}$ is squarefree, then $\mathcal{O} = \mathbb{Z}_K$ and $\operatorname{disc} K = \operatorname{disc} \mathcal{O}$.*

This allows us to compute the ring of integers when the discriminant of the field is squarefree; unfortunately, it is usually not the case. We will see other criteria for the p-maximality of orders in the next chapter.

To use theorem 2.3.3, we need to be able to compute discriminants of orders of the form $\mathbb{Z}[\alpha]$. In this view, we introduce the discriminant of a polynomial.

Definition 2.3.6. Let $\mathcal{R}$ be an integral domain, and let $A(x) \in \mathcal{R}[x]$ be a monic polynomial of degree $n \in \mathbb{N}$ and leading coefficient $a \in \mathcal{R}$. The *discriminant* of $A(x)$ is

$$\operatorname{disc} A = \frac{(-1)^{n(n-1)/2}}{a} \operatorname{Res}(A, A').$$

Remark 2.3.7. It can be easily seen on the definition of the resultant as a determinant that $\operatorname{Res}(A, A')$ must be divisible by a, so $\operatorname{disc} A$ lies in $\mathcal{R}$.

Example 2.3.8. Let $A(x) = ax^2 + bx + c$, $a \neq 0$. Then $A'(x) = 2ax + b$, so that

$$\operatorname{Res}(A, A') = \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix} = 4a^2c - ab^2,$$

so we recover the well-known formula

$$\operatorname{disc} A = \frac{-1}{a} \operatorname{Res}(A, A') = b^2 - 4ac.$$

Theorem 2.3.9. *Let K be a field, $A(x) \in K[x]$ a polynomial of degree $n \in \mathbb{N}$ and leading coefficient $a \in K$, and let $\alpha_1, \ldots, \alpha_n$ be the roots of $A(x)$ (repeated with multiplicity) in some algebraically closed field Ω containing K. Then*

$$\begin{aligned} \operatorname{disc} A &= (-1)^{n(n-1)/2} a^{n-2} \prod_{j=1}^{n} P'(\alpha_j) \\ &= (-1)^{n(n-1)/2} a^{2n-2} \prod_{j \neq k} (\alpha_j - \alpha_k) \\ &= a^{2n-2} \prod_{j<k} (\alpha_j - \alpha_k)^2. \end{aligned}$$

In particular, $\operatorname{disc} A \neq 0$ *if and only if $A(x)$ has no multiple roots in Ω.*

Proof. The first equality is just an application of theorem 1.1.2. Then, since

$$A(x) = a \prod_{j=1}^{n} (x - \alpha_j),$$

we have

$$P'(x) = a \sum_{j=1}^{n} \prod_{k \neq j} (x - \alpha_k)$$

so

$$P'(\alpha_j) = a \prod_{k \neq j} (\alpha_j - \alpha_k),$$

whence the result. □

As discriminants can be tedious to compute explicitly, we establish once and for all the following formula.

Proposition 2.3.10. *(Non examinable) For all $n \in \mathbb{N}$ and $b, c \in \mathbb{Q}$, we have*

$$\operatorname{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big).$$

Proof. Let us introduce $\zeta = e^{2\pi i/(n-1)}$, and $\beta \in \mathbb{C}$ such that $\beta^{n-1} = -b/n$.

According to theorem 1.1.2, the resultant of P and P' can be computed in two ways: as the product of the values of P at the roots of P' (essentially), and vice versa. Here, the first way is easier, because the roots of P' are easy to express and manipulate. Explicitly, we have $P'(x) = nx^{n-1} + b$, whose complex roots are the $\zeta^k \beta$, $0 \leqslant k < n-1$, and

$$P(\zeta^k \beta) = \zeta^{kn}\beta^n + b\zeta^k\beta + c = \zeta^k\left(-\frac{\beta}{n}\right) + b\zeta^k\beta + c = \left(1 - \frac{1}{n}\right)\beta\zeta^k b + c.$$

Therefore,

$$\begin{aligned}
\operatorname{Res}(P, P') &= n^n \prod_{k=0}^{n-2} P(\zeta^k\beta) \qquad \text{because the leading coefficient of } P' \text{ is } n \\
&= n^n \prod_{k=0}^{n-2} \left(\left(1 - \frac{1}{n}\right)\beta\zeta^k b + c\right) \\
&= n^n (-1)^{n-1} \prod_{k=0}^{n-2} \left(-c - \zeta^k\left(1 - \frac{1}{n}\right)\beta b\right) \\
&= n^n (-1)^{n-1} \left(\big(-c\big)^{n-1} - \big((1-1/n)\beta b\big)^{n-1}\right) \text{ as } \prod_{k=0}^{n-2}(x - \zeta^k y) = x^{n-1} - y^{n-1} \\
&= n^n c^{n-1} - n^n \beta^{n-1} b^{n-1} (1/n - 1)^{n-1} \\
&= n^n c^{n-1} - n\left(-\frac{b}{n}\right)(1-n)^{n-1} b^{n-1} \\
&= n^n c^{n-1} + (1-n)^{n-1} b^n.
\end{aligned}$$

The result then follows since $\operatorname{disc} P = (-1)^{n(n-1)/2} \operatorname{Res}(P, P')$. □

Corollary 2.3.11. *(Examinable) In particular, we obtain the important formula*

$$\operatorname{disc}(x^3 + bx + c) = -4b^3 - 27c^2,$$

which you should learn by heart.

So far, we have introduced two notions of discriminants, one for orders, and one for polynomials. We now show that these notions coincide.

Theorem 2.3.12. *Let $K = \mathbb{Q}(\alpha)$ be a number field, where α is integral, and let $m(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. Then*

$$\operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} m.$$

Proof. Let $\alpha_1, \cdots, \alpha_n$ be the complex roots of $m(x)$ where $n = [K : \mathbb{Q}]$, and consider the matrix

$$A = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

Vandermonde tells us that $\det A = \prod_{j<k}(\alpha_k - \alpha_j)$. Besides, if $B = {}^{\mathrm{t}}AA$, then we have

$$B_{i,j} = \sum_{k=1}^{n} A_{k,i}A_{k,j} = \sum_{k=1}^{n} \alpha_k^{i-1}\alpha_k^{j-1} = \sum_{k=1}^{n} \alpha_k^{i+j-2} = \operatorname{Tr}_{\mathbb{Q}}^{K}(\alpha^{i+j-2}) = \operatorname{Tr}_{\mathbb{Q}}^{K}(\alpha^{i-1}\alpha^{j-1})$$

according to corollary 1.3.8. Since $1, \alpha, \alpha^2, \cdots, \alpha^{n-1}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\alpha]$, we conclude that

$$\operatorname{disc} \mathbb{Z}[\alpha] = \det B = (\det A)^2 = \prod_{j<k}(\alpha_j - \alpha_k)^2 = \operatorname{disc} m.$$

□

We immediately deduce the following consequence:

Proposition 2.3.13. *Let K be a number field of signature (r_1, r_2). Then the sign of* $\operatorname{disc} K$ *is $(-1)^{r_2}$.*

Proof. Again, write $K = \mathbb{Q}(\alpha)$ where α is integral, let $m(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α, and let $\alpha_1, \cdots, \alpha_n$ be its complex roots, ordered so that $\alpha_1, \cdots, \alpha_{r_1}$ are real and $\overline{\alpha_{r_1+j}} = \alpha_{r_1+r_2+j}$ for $1 \leqslant j \leqslant r_2$. We have

$$\operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} m = \prod_{j<k} (\alpha_j - \alpha_k)^2.$$

When j and k are both less than r_1, $(\alpha_j - \alpha_k)^2$ is the square of a real number and is thus positive. The other terms can be grouped in conjugate pairs and produce factors $\big(\pm |\alpha_j - \alpha_k|^2 \big)^2$, which are also positive, except when $j, k > r_1$ and $k = j + r_2$, in which case we get $(\alpha_j - \alpha_k)^2 = (\alpha_j - \overline{\alpha_j})^2 < 0$. As a result, the sign of $\operatorname{disc} m$ is $(-1)^{r_2}$. Since $\operatorname{disc} K$ differs of $\operatorname{disc} \mathbb{Z}[\alpha]$ by a square (thus positive) factor, the result follows. □

Example 2.3.14. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $f(x) = x^3 + x^2 - 2x + 8 = 0$. Since $f(x)$ is irreducible, this number field is well-defined and has degree 3, so its signature is either $(3, 0)$ or $(1, 1)$. One may compute that

$$\operatorname{disc}(x^3 + x^2 - 2x + 8) = -2012 = -2^2 \cdot 503.$$

Since this is negative, we can conclude that the signature of K is $(1, 1)$, which means that the polynomial $f(x)$ has one real root and one pair of complex conjugate nonreal roots.

Besides, as 503 is prime, theorem 2.3.3 implies that the order $\mathcal{O} = \mathbb{Z}[\alpha]$ is p-maximal for all p except maybe $p = 2$, and that the index of $\mathcal{O}$ divides 2. As a result, either $\mathcal{O} = \mathbb{Z}_K$ and $\operatorname{disc} K = -2012$, or $\mathcal{O}$ has index 2 and $\operatorname{disc} K = -503$.

In fact, since it can be checked that $\beta = \frac{\alpha^2+\alpha}{2}$ is an algebraic integer, $\mathcal{O}' = \mathbb{Z}[\alpha, \beta]$ is an order in which $\mathcal{O}$ has index 2 (if this is not obvious to you, write down the matrix expressing a $\mathbb{Z}$-basis of $\mathcal{O}$ on a $\mathbb{Z}$-basis of $\mathcal{O}'$, and check that its determinant is ± 2), so $\mathbb{Z}_K = \mathcal{O}'$ and $\operatorname{disc} K = -503$.

This example has the particularity that no order of the form $\mathbb{Z}[\gamma]$ is 2-maximal, whatever the algebraic integer $\gamma \in \mathbb{Z}_K$ is; we will see why in the next chapter (example 3.7.5). In particular, $\mathbb{Z}_K$ cannot be written in the form $\mathbb{Z}[\gamma]$ in the case of this number field.

2.4 The case of quadratic fields

Definition 2.4.1. A *quadratic field* is a number field of degree 2.

The classical formulae used to solve equations of degree 2 show that every quadratic field is of the form $\mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is squarefree and different from 0 and 1.

Quadratic fields are small enough that their ring of integers can be determined explicitly.

Theorem 2.4.2. *Let $d \in \mathbb{Z}$, $d \neq 1$ be a squarefree integer, and let $K = \mathbb{Q}(\sqrt{d})$. If $d \equiv 1 \bmod 4$, then $\mathbb{Z}_K = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ and $\operatorname{disc} K = d$, whereas if $d \not\equiv 1 \bmod 4$, then $\mathbb{Z}_K = \mathbb{Z}[\sqrt{d}]$ and $\operatorname{disc} K = 4d$.*

Proof. First, note that the discriminant of the order $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ is

$$\operatorname{disc} \mathcal{O} = \operatorname{disc}(x^2 - d) = 4d,$$

so $\mathcal{O}$ is maximal except maybe at $p = 2$ since d is squarefree.

In fact, an element $a + b\sqrt{d}$ of K (where $a, b \in \mathbb{Q}$) is an integer if and only if its characteristic polynomial

$$x^2 - 2ax + a^2 - b^2 d$$

has coefficient in $\mathbb{Z}$, thus if and only if $2a \in \mathbb{Z}$ and $a^2 - b^2 d \in \mathbb{Z}$. In particular, a and b must be half-integers. Since the squares of $\mathbb{Z}/4\mathbb{Z}$ are 0 and 1, we see that a and b must be integers except when $d \equiv 1 \mod 4$, in which case they must be half-integers such that $a + b \in \mathbb{Z}$. The claim on $\mathbb{Z}_K$ follows, and theorem 2.3.3 allows us to compute $\operatorname{disc} K$. □

Example 2.4.3. The ring of integers of $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$, and its discriminant is -4. Similarly, the ring of integers of $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Z}[\sqrt{2}]$, and its discriminant is 8, but the ring of integers of $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, and its discriminant is 5.

2.5 The case of cyclotomic fields

Cyclotomic fields are another important class of number fields whose ring of integers is easily described.

Definition 2.5.1. An algebraic number ζ satisfying $\zeta^n = 1$ for some $n \in \mathbb{N}$ is called an *n^{th} root of unity*. If $\zeta^m \neq 1$ for $m < n$, then it is a *primitive* n^{th} root of unity. Thus in $\mathbb{C}$ the n^{th} roots of unity are the $e^{2k\pi i/n}$, $0 \leqslant k < n$, and the primitive ones are the ones for which k and n are coprime.

The n^{th} *cyclotomic polynomial* is

$$\Phi_n(x) = \prod_{\substack{\zeta \text{ primitive } n^{\text{th}} \\ \text{root of unity} \in \mathbb{C}}} (x - \zeta) = \prod_{k \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - e^{2k\pi i/n}).$$

It has degree $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ (Euler's phi function). Moreover, it lies in $\mathbb{Z}[x]$, and it is irreducible over $\mathbb{Z}$ (and hence over $\mathbb{Q}$).

The n^{th} *cyclotomic field* is $\mathbb{Q}(\zeta)$, where ζ is any primitive n^{th} root of unity. It is thus a number field of degree $\varphi(n)$.

Theorem 2.5.2. *Let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive n^{th} root of unity, $n \in \mathbb{N}$. Then*

1. *$\mathbb{Z}_K = \mathbb{Z}[\zeta]$, and*
2. *(Non examinable)* $\operatorname{disc} K = \dfrac{(-1)^{\varphi(n)/2} n^{\varphi(n)}}{\prod_{p|n} p^{\varphi(n)/(p-1)}}$, *where the product ranges over the prime divisors p of n. In particular, when $n = p^v$ is a prime power, then* $\operatorname{disc} K = \pm p^{p^{v-1}(pv-v-1)}$.

Chapter 3

Ideals and factorisation

In the previous chapter, we have defined the ring of integers $\mathbb{Z}_K$ of a number field K. We are now going to investigate the properties of this ring.

In general, it is not a UFD, as demonstrated by the following example.

Example. Take $K = \mathbb{Q}(\sqrt{-5})$, so that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$ by theorem 2.4.2. In $\mathbb{Z}_K$, a number $\alpha = a + b\sqrt{-5}$ $(a, b \in \mathbb{Z})$ is invertible if and only if its norm $N_{\mathbb{Q}}^K(\alpha) = a^2 + 5b^2$ is 1, as can be easily seen from the formulae $N_{\mathbb{Q}}^K(\alpha\beta) = N_{\mathbb{Q}}^K(\alpha)N_{\mathbb{Q}}^K(\beta)$ and $\alpha^{-1} = \frac{\bar{\alpha}}{N_{\mathbb{Q}}^K(\alpha)}$. In particular two associate elements must have the same norm.

Consider the two factorisations

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

No factor of the first one is associate to a factor of the second one, because the norms of these factors are respectively 4, 9, 6 and 6. Besides, these factors are irreducible: if they were not, by taking norms, we would get integer solutions to $a^2 + 5b^2 = 2$ or 3, which is clearly impossible. We thus have two complete and yet distinct factorisations of 6 in $\mathbb{Z}_K$.

A great insight came from Kummer, who imagined that there should exist what he called "ideal numbers" $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ and $\mathfrak{p}_4$ such that $2 = \mathfrak{p}_1\mathfrak{p}_2$, $3 = \mathfrak{p}_3\mathfrak{p}_4$, $1 + \sqrt{-5} = \mathfrak{p}_1\mathfrak{p}_3$, and $1 - \sqrt{-5} = \mathfrak{p}_2\mathfrak{p}_4$. Indeed, this would allow us to recover a unique factorisation

$$6 = (\mathfrak{p}_1\mathfrak{p}_2)(\mathfrak{p}_3\mathfrak{p}_4) = (\mathfrak{p}_1\mathfrak{p}_3)(\mathfrak{p}_2\mathfrak{p}_4).$$

We will see that the ring of integers is what is called a Dedekind domain, which means that it enjoys very nice properties (in some sense, a nice factorisation theory of those "ideal numbers") which make it almost as good as

a UFD. The failure of this ring to be a UFD is measured by the so-called *class group*, which we will study in chapter 4.

3.1 Reminder on finite fields

Recall that any two finite fields of the same cardinal are isomorphic, and that a finite field of size $n \in \mathbb{N}$ exists if and only if n is a prime power. This justifies the notation $\mathbb{F}_q$ for "the" finite field of size q when $q \in \mathbb{N}$ is a prime power. For instance, we have $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$ for all prime $p \in \mathbb{N}$.

Besides, when q and r are prime powers, $\mathbb{F}_q$ is isomorphic to a subfield of $\mathbb{F}_r$ if and only if r is a power of q. In particular, if $q = p^f$ with $p \in \mathbb{N}$ prime and $f \in \mathbb{N}$, then $\mathbb{F}_q$ contains a copy of $\mathbb{F}_p$, and is thus of characteristic p.

Finally, recall that the multiplicative group of every finite field is cyclic, i.e. $\mathbb{F}_q^\times \simeq \mathbb{Z}/(q-1)\mathbb{Z}$.

3.2 Reminder on ideals

Throughout this section, we let $\mathcal{R}$ be a commutative ring. When $\alpha_1, \cdots, \alpha_m$ are elements of a $\mathcal{R}$, we will denote by $(\alpha_1, \cdots, \alpha_m)$ the ideal of $\mathcal{R}$ generated by these elements. We will say that an ideal of $\mathcal{R}$ is nonzero if it is not reduced to $\{0\}$, and that it is nontrivial if it is not the whole of $\mathcal{R}$.

Definition 3.2.1. Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$ be ideals of $\mathcal{R}$. Their sum and product are defined to be

$$\mathfrak{a}_1 + \cdots + \mathfrak{a}_m = \{a_1 + \cdots + a_m \mid a_1 \in \mathfrak{a}_1,\ \cdots,\ a_m \in \mathfrak{a}_m\}$$

and

$$\mathfrak{a}_1 \cdots \mathfrak{a}_m = \left\{ \sum_{j=1}^{n} a_{1,j} \cdots a_{m,j} \mid n \in \mathbb{N}, \text{ and } \forall j,\ a_{1,j} \in \mathfrak{a}_1,\ \cdots,\ a_{m,j} \in \mathfrak{a}_m \right\}.$$

Both are ideals of $\mathcal{R}$, and $\mathfrak{a}_1 \cdots \mathfrak{a}_m \subset \mathfrak{a}_i \subset \mathfrak{a}_1 + \cdots + \mathfrak{a}_m$ for all $1 \leqslant i \leqslant m$.

Example 3.2.2. We have

$$(\alpha_1, \cdots, \alpha_m) = \alpha_1\mathcal{R} + \cdots + \alpha_m\mathcal{R} = (\alpha_1) + \cdots + (\alpha_m),$$

and

$$(\alpha_1 \cdots \alpha_m) = (\alpha_1) \cdots (\alpha_m).$$

Theorem 3.2.3 (Chinese remainders)**.** *Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$ be ideals of $\mathcal{R}$ which are pairwise coprime, i.e. such that $\mathfrak{a}_i + \mathfrak{a}_j = \mathcal{R}$ for all $i \neq j$, and let $\mathfrak{b} = \mathfrak{a}_1 \cdots \mathfrak{a}_n$. The canonical projections induce a ring isomorphism*

$$\mathcal{R}/\mathfrak{b} \simeq \prod_{i=1}^{m} \mathcal{R}/\mathfrak{a}_i.$$

Definition 3.2.4. Let $\mathfrak{a} \subset \mathcal{R}$ be a nontrivial ideal of $\mathcal{R}$.

1. One says that $\mathfrak{a}$ is a *prime* ideal (or just a prime, for short) of $\mathcal{R}$ if for all $r, s \in \mathcal{R}$, $rs \in \mathcal{R}$ implies $r \in \mathcal{R}$ or $s \in \mathcal{R}$.

2. One says that $\mathfrak{a}$ is a *maximal* ideal of $\mathcal{R}$ if whenever $\mathfrak{b}$ is an ideal such that $\mathfrak{a} \subset \mathfrak{b} \subset \mathcal{R}$, then $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b} = \mathcal{R}$.

Theorem 3.2.5. *Let $\mathfrak{a} \subset \mathcal{R}$ be a nontrivial ideal of $\mathcal{R}$.*

1. *$\mathfrak{a}$ is prime if and only if the quotient ring $\mathcal{R}/\mathfrak{a}$ is an integral domain.*

2. *$\mathfrak{a}$ is maximal if and only if the quotient ring $\mathcal{R}/\mathfrak{a}$ is a field.*

Proof. Whenever $x \in \mathcal{R}$, let $\overline{x}$ denote the class of x in $\mathcal{R}/\mathfrak{a}$.

1. $\mathfrak{a}$ is prime if and only if $rs \in \mathfrak{a} \Rightarrow r \in \mathfrak{a}$ or $s \in \mathfrak{a}$, if and only if $\overline{rs} = \overline{0} \Rightarrow \overline{r} = \overline{0}$ or $\overline{s} = \overline{0}$, if and only if $R/\mathfrak{a}$ is an integral domain.

2. Suppose $\mathfrak{a}$ is maximal, and let $\overline{r} \in \mathcal{R}/\mathfrak{a}$ be nonzero. Then $r \notin \mathfrak{a}$, so the ideal spanned by r and $\mathfrak{a}$ must be the whole of $\mathcal{R}$. In particular this ideal contains 1, so that there exist $s \in \mathcal{R}$ and $a \in \mathfrak{a}$ such that $1 = rs + a$. But then we have $\overline{rs} = \overline{1}$, which proves that $\mathcal{R}/\mathfrak{a}$ is a field.

   Conversely, suppose that $\mathcal{R}/\mathfrak{a}$ is a field, and let $\mathfrak{b}$ be an ideal such that $\mathfrak{a} \subsetneq \mathfrak{b}$, and let us prove that $\mathfrak{b} = \mathcal{R}$. There exists $r \in \mathfrak{b}$, $r \notin \mathfrak{a}$. We then have $\overline{r} \neq \overline{0}$, so there exists $s \in \mathcal{R}$ such that $\overline{rs} = 1$ since $\mathcal{R}/\mathfrak{a}$ is a field. This means that $rs = 1 + a$ for some $a \in \mathfrak{a}$, so $1 = rs - a \in \mathfrak{b}$, which proves that $\mathfrak{b} = \mathcal{R}$.

□

Corollary 3.2.6. *Every maximal ideal is prime.*

Example 3.2.7. Let $\mathcal{R} = \mathbb{Z}[x]$, and let $p \in \mathbb{Z}$ be a prime number. Then the ideals (p) and (x) are prime but are not maximal, because the respective quotients, $\mathbb{F}_p[x]$ and $\mathbb{Z}$, are integral domains but are not fields. On the contrary, the ideal (p, x) is maximal, because the corresponding quotient is $\mathbb{F}_p$, which is a field.

3.3 Integral closure

Definition 3.3.1. Let $\mathcal{R} \subset \mathcal{S}$ be integral domains, and let $s \in \mathcal{S}$. One says that s is *integral* over $\mathcal{R}$ if there exists a nonzero *monic* polynomial $P \in \mathcal{R}[x]$ such that $P(s) = 0$.

If every $s \in \mathcal{S}$ is integral over $\mathcal{R}$, one says that $\mathcal{S}$ is an *integral extension* of $\mathcal{R}$.

Conversely, if every element of $\mathcal{S}$ which is integral over $\mathcal{R}$ lies in fact in $\mathcal{R}$, one says that $\mathcal{R}$ is *integrally closed* in $\mathcal{S}$.

In particular, one says for short that $\mathcal{R}$ is *integrally closed* if it is integrally closed in its fraction field $\operatorname{Frac} \mathcal{R}$.

Thus, for example, the set of the elements of a number field K which are integral over $\mathbb{Z}$ is precisely $\mathbb{Z}_K$. Also, $\mathbb{Z}$ is integrally closed.

Example 3.3.2. The ring $\mathcal{R} = \mathbb{Z}[\sqrt{5}]$ is not integrally closed. Indeed, $\alpha = \frac{1+\sqrt{5}}{2}$ lies in $\operatorname{Frac} \mathcal{R}$, and satisfies $\alpha^2 - \alpha - 1 = 0$ so α is integral over $\mathcal{R}$ (and even over $\mathbb{Z}$), yet $\alpha \notin \mathcal{R}$.

Proposition 3.3.3. *Every UFD is integrally closed.*

Proof. Let $\mathcal{R}$ be a UFD, and let $F = \operatorname{Frac} \mathcal{R}$ be its field of fractions. We must show that if $\alpha \in F$ satisfies $P(\alpha) = 0$ for some nonzero monic polynomial

$$P(x) = x^n + \sum_{j=0}^{n-1} r_j x^j \in \mathcal{R}[x],$$

then α lies in fact in $\mathcal{R}$.

Since $\mathcal{R}$ is a UFD, we may write $\alpha = a/d$, where $a, d \in \mathcal{R}$ are coprime. Clearing denominators, we get

$$a^n + d \sum_{j=0}^{n-1} r_j a^j d^{n-1-j} = 0,$$

which implies that d divides a^n. Therefore, d must be invertible in $\mathcal{R}$. □

Unfortunately, the converse does not hold.

Theorem 3.3.4. *Let K be a number field, and let $\mathbb{Z}_K$ be its ring of integers. Then $\mathbb{Z}_K$ is integrally closed.*

Proof. Let $\alpha \in K$ be such that there exists a nonzero monic polynomial $P(x) \in \mathbb{Z}_K[x]$ such that $P(\alpha) = 0$. For each embedding σ of K into $\mathbb{C}$, let $P^\sigma(x) \in \mathbb{C}[x]$ be the polynomial obtained by applying σ to the coefficients of $P(x)$. Then the coefficients of

$$Q(x) = \prod_{\sigma\colon K \hookrightarrow \mathbb{C}} P^\sigma(x)$$

lie in $\mathbb{Q}$ and are algebraic integers, so that $Q(x) \in \mathbb{Z}[x]$. Besides, $Q(x)$ is clearly monic, and furthermore, if we embed K into $\mathbb{C}$, we see that $Q(\alpha) = 0$. Therefore, α is an algebraic integer. □

Corollary 3.3.5. *Let $\mathcal{O}$ be an order in a number field K. Then $\mathcal{O}$ is integrally closed if and only if $\mathcal{O} = \mathbb{Z}_K$.*

Remark 3.3.6. This proof, which I admit is not great, is not the standard one. The usual way of proving this theorem consists in proving that when we have integral domains $\mathcal{R} \subset \mathcal{S} \subset \mathcal{T}$ such that $\mathcal{S}$ is integral over $\mathcal{R}$ and $\mathcal{T}$ is integral over $\mathcal{S}$, then $\mathcal{T}$ is integral over $\mathcal{R}$; unfortunately, the proof of this fact requires using the notion of module over a ring, which is beyond the scope of this course. For those of you who do know this theory, here is the (of course non examinable) proof:

First, if $\mathcal{R} \subset \mathcal{S}$ are two commutative rings, given finitely many elements $s_1, \cdots, s_n \in \mathcal{S}$ we have the equivalence

$$s_1, \cdots, s_n \text{ integral over } \mathcal{R} \iff \mathcal{R}[s_1, \cdots, s_n] \text{ is a finitely generated } \mathcal{R}\text{-module.}$$

Indeed, $\Rightarrow$ is immediate from the definition of integrality, and $\Leftarrow$ follows from Cayley-Hamilton (cf. the proof of proposition 2.2.8). As a result, if we suppose that $\mathcal{R} \subset \mathcal{S} \subset \mathcal{T}$ are such that $\mathcal{S}$ is integral over $\mathcal{R}$ and $\mathcal{T}$ is integral over $\mathcal{S}$, then for all $t \in \mathcal{T}$, we have a relation $t^n + \sum_{i=0}^{n-1} s_i t^i = 0$ for some $s_i \in \mathcal{S}$; if we let $M = \mathcal{R}[s_0, \cdots, s_{n-1}]$, then M is a finitely generated $\mathcal{R}$-module, and $M[t]$ is a finitely generated M-module, so that $M[t] = \mathcal{R}[s_0, \cdots, s_{n-1}, t]$ is a finitely generated $\mathcal{R}$-module, which proves that t is integral over $\mathcal{R}$.

3.4 Dedekind domains

Proposition 3.4.1. *Let $\mathcal{O}$ be an order in a number field K, and let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}$. Then $\mathfrak{a}$ is a lattice in K; in particular, the index $[\mathcal{O} : \mathfrak{a}]$ is finite.*

Proof. Let $\alpha \in \mathfrak{a}$, $\alpha \neq 0$, so that $\alpha\mathbb{Z}_K \subset \mathfrak{a} \subset \mathbb{Z}_K$. We know that $\mathbb{Z}_K$ is a lattice in K; let $\omega_1, \cdots, \omega_n$ be a $\mathbb{Z}$-basis of it, where $n = [K : \mathbb{Q}]$. Then $\alpha\omega_1, \cdots, \alpha\omega_n$ is a $\mathbb{Z}$-basis of $\alpha\mathbb{Z}_K$, which is thus also a lattice in K. Since $\mathfrak{a}$ is an additive subgroup of K which is cornered between the two lattices $\alpha\mathbb{Z}_K$ and $\mathbb{Z}_K$, it is itself a lattice. □

This prompts the following definition.

Definition 3.4.2. Let $\mathcal{O}$ be an order in a number field K, and let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}$. The *norm* of $\mathfrak{a}$ is the index $[\mathcal{O} : \mathfrak{a}]$. It is denoted by $N(\mathfrak{a}) \in \mathbb{N}$. By convention the norm of the zero ideal is 0.

Proposition 3.4.3. *Let $\mathcal{O}$ be an order in a number field K, and let $\mathfrak{a} \subset \mathcal{O}$ be an ideal. Then $N(\mathfrak{a}) \in \mathfrak{a}$.*

Proof. By definition, $N(\mathfrak{a})$ is the order of the finite additive group $\mathcal{O}/\mathfrak{a}$, so the image of the integer $N(\mathfrak{a})$ in $\mathcal{O}/\mathfrak{a}$ is 0 by Lagrange's theorem. □

Theorem 3.4.4. *Let $\mathcal{O}$ be an order in a number field K, let $\alpha \in \mathcal{O}$, and let $\mathfrak{a}$ be the ideal $\alpha\mathcal{O}$ of $\mathcal{O}$. Then*

$$N(\mathfrak{a}) = |N_{\mathbb{Q}}^K(\alpha)|.$$

Proof. Let $(\omega_j)_{1 \leqslant j \leqslant [K:\mathbb{Q}]}$ be a $\mathbb{Z}$-basis of $\mathcal{O}$. Then $(\alpha\omega_j)_{1 \leqslant j \leqslant [K:\mathbb{Q}]}$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$, and the change-of-basis matrix between these two bases is the matrix of the multiplication-by-α map with respect to the basis $(\omega_j)_{1 \leqslant j \leqslant [K:\mathbb{Q}]}$. The index $[\mathcal{O} : \mathfrak{a}]$ is the absolute value of the determinant of this matrix, but by definition this determinant is $N_{\mathbb{Q}}^K(\alpha)$. □

Lemma 3.4.5. *Let $\mathcal{O}$ be an order in a number field K and let $\mathfrak{a} \subset \mathfrak{b} \subset \mathcal{O}$ be ideals. Then $N(\mathfrak{b})$ divides $N(\mathfrak{a})$, with equality if and only if $\mathfrak{a} = \mathfrak{b}$.*

Proof. Since $\mathfrak{a} \subset \mathfrak{b} \subset \mathcal{O}$ we have $[\mathcal{O} : \mathfrak{a}] = [\mathcal{O} : \mathfrak{b}][\mathfrak{b} : \mathfrak{a}]$, that is $N(\mathfrak{a}) = N(\mathfrak{b})[\mathfrak{b} : \mathfrak{a}]$. This proves the divisibility, and there is equality if and only if $[\mathfrak{b} : \mathfrak{a}] = 1$, if and only if $\mathfrak{a} = \mathfrak{b}$. □

Proposition 3.4.6. *Let $\mathcal{O}$ be an order in a number field K, $\mathfrak{a} \subset \mathcal{O}$ an ideal, and let $\alpha \in \mathfrak{a}$, $\alpha \neq 0$. Then $N(\mathfrak{a})$ divides $|N_{\mathbb{Q}}^{K}(\alpha)|$, with equality if and only if $\mathfrak{a} = \alpha\mathcal{O}$.*

Proof. This follows immediately from Theorem 3.4.4 and Lemma 3.4.5. □

Proposition 3.4.7. *Let $\mathcal{O}$ be an order in a number field K. Then $\mathcal{O}$ is a Noetherian ring.*

Proof. Consider a chain of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$. The norms of these ideals form a nondecreasing sequence of positive integers, hence stabilises, so by Lemma 3.4.5 the chain of ideals must stabilise. □

Lemma 3.4.8. *Every finite integral domain is a field.*

Proof. Let $\mathcal{R}$ be a finite integral domain, let $r \in \mathcal{R}$ be nonzero, and consider the multiplication-by-r map

$$\begin{array}{rccc} \mu_r \colon & \mathcal{R} & \longrightarrow & \mathcal{R} \\ & x & \longmapsto & rx \end{array}.$$

Since $\mathcal{R}$ is a domain, μ_r is injective. In addition $\mathcal{R}$ is finite, so μ_r is a bijection. In particular, the element 1 has a preimage, which proves that r is invertible. □

Proposition 3.4.9. *Let $\mathcal{O}$ be an order in a number field K. Then every nonzero prime ideal of $\mathcal{O}$ is maximal.*

Proof. Let $\mathfrak{p} \subset \mathcal{O}$ be a nonzero prime ideal. Then $\mathcal{R} = \mathcal{O}/\mathfrak{p}$ is an integral domain, and by Proposition 3.4.1 the ring $\mathcal{R}$ is finite. Lemma 3.4.8 proves that $\mathcal{R}$ is a field and therefore $\mathfrak{p}$ is a maximal ideal. □

Definition 3.4.10. An integral domain $\mathcal{R}$ is called a *Dedekind domain* if it is Noetherian, integrally closed, and if every nonzero prime ideal of $\mathcal{R}$ is maximal.

Theorem 3.4.11. *The ring of integers of a number field is a Dedekind domain.*

Proof. This is the conjunction of Proposition 3.4.7, Theorem 3.3.4 and Proposition 3.4.9. □

3.5 Factorisation theory in Dedekind domains

The reason why we introduced the notion of Dedekind domain is the following major result.

Theorem 3.5.1. *Let $\mathcal{R}$ be a Dedekind domain. Every ideal $\mathfrak{a}$ of $\mathcal{R}$ is a product of prime ideals,*

$$\mathfrak{a} = \prod_{j=1}^{m} \mathfrak{p}_j.$$

Furthermore, this factorisation is unique[1]*.*

Thanks to this theorem, we can perform arithmetic in a Dedekind domain, but on ideals, not on numbers. The usual notions of divisibility, gcd, lcm... can be translated in terms of operations on ideals:

Theorem 3.5.2. *Let $\mathcal{R}$ be a Dedekind domain.*

1. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of $\mathcal{R}$. Then $\mathfrak{a}$ divides $\mathfrak{b}$ (meaning there exists an ideal $\mathfrak{c}$ such that $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$) if and only if $\mathfrak{a} \supset \mathfrak{b}$.*

2. *If $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$ are ideals of $\mathcal{R}$, then*

$$\gcd(\mathfrak{a}_1, \cdots, \mathfrak{a}_m) = \mathfrak{a}_1 + \cdots + \mathfrak{a}_m$$

and

$$\operatorname{lcm}(\mathfrak{a}_1, \cdots, \mathfrak{a}_m) = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_m.$$

Example 3.5.3. For $\mathcal{R} = \mathbb{Z}$, this translates into the following more familiar statements:

$$a \mid b \iff a\mathbb{Z} \supset b\mathbb{Z},$$

$$\gcd(a_1, \cdots, a_m)\mathbb{Z} = a_1\mathbb{Z} + \cdots + a_m\mathbb{Z},$$

and

$$\operatorname{lcm}(a_1, \cdots, a_m)\mathbb{Z} = a_1\mathbb{Z} \cap \cdots \cap a_m\mathbb{Z}.$$

Example 3.5.4. Let $\mathfrak{p}$, $\mathfrak{p}'$ be two prime ideals. If $\mathfrak{p}$ and $\mathfrak{p}'$ are distinct, then they are coprime, which means that $\mathfrak{p} + \mathfrak{p}' = \mathbb{Z}_K$ and that $\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{p}\mathfrak{p}'$. On the other hand, if $\mathfrak{p} = \mathfrak{p}'$, then we have $\mathfrak{p} + \mathfrak{p} = \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$. Note that $\mathfrak{p}^2$ is an indeal contained in $\mathfrak{p}$; in fact, this containment is strict by the uniqueness of the factorisation of ideals.

[1] Up to the order of the terms, of course.

In fact, it can be shown that the factorisation property presented in theorem 3.5.1 characterises Dedekind domains. Briefly,

- because containment means divisibility, the fact that ideals factor into primes implies that the domain is Noetherian,
- if a domain is not integrally closed, then using elements of the field of fractions which are integral but not in the domain, one can exhibit ideals which do not factor properly,
- in view of the equivalence between containment and divisibility, the fact that nonzero prime ideals play the role of irreducibles means that they are maximal.

As a first application of this, we can prove that although Dedekind domains are not in general principal, every ideal can be generated by at most two elements. In fact, even more is true:

Proposition 3.5.5. *Let $\mathfrak{a}$ be an ideal in a Dedekind domain $\mathcal{R}$, and let $\alpha \in \mathfrak{a}$, $\alpha \neq 0$. Then there exists $\beta \in \mathfrak{a}$ such that $\mathfrak{a} = (\alpha, \beta)$.*

Proof. Since $\alpha \in \mathfrak{a}$, we have $\alpha\mathcal{R} \subset \mathfrak{a}$, so in the factorisations

$$\alpha\mathcal{R} = \prod_{j=1}^{m} \mathfrak{p}_j^{a_j}, \quad \mathfrak{a} = \prod_{j=1}^{m} \mathfrak{p}_j^{e_j},$$

we have $e_j \leqslant a_j$ for all j. By uniqueness of the factorisation, for each j there exists $\beta_j \in \mathfrak{p}_j^{e_j} \setminus \mathfrak{p}_j^{e_j+1}$, and by Chinese remainders we may find $\beta \in \mathcal{R}$ such that for all j, $\beta \equiv \beta_j \bmod \mathfrak{p}_j^{e_j+1}$. In particular, for all j we have $\beta \in \mathfrak{p}_j^{e_j} \setminus \mathfrak{p}_j^{e_j+1}$, so we have the factorisation

$$\beta\mathcal{R} = \mathfrak{b} \prod_{j=1}^{m} \mathfrak{p}_j^{e_j},$$

where $\mathfrak{b}$ is coprime to the $\mathfrak{p}_j$. As a result, we have

$$(\alpha, \beta) = \alpha\mathcal{R} + \beta\mathcal{R} = \gcd(\alpha\mathcal{R}, \beta\mathcal{R}) = \prod_{j=1}^{m} \mathfrak{p}_j^{e_j} = \mathfrak{a},$$

as wanted. □

We now take $\mathcal{R} = \mathbb{Z}_K$ to be the ring of integers of some number field K.

Theorem 3.5.6 (Multiplicativity of the norm)**.** *Let $\mathfrak{a}, \mathfrak{b} \subset \mathbb{Z}_K$ be ideals of the ring of integers of a number field K. Then $N(\mathfrak{ab}) = N(\mathfrak{a})N(\mathfrak{b})$.*

Proof. It is enough to prove that

$$N\left(\prod_{j=1}^{m} \mathfrak{p}_j^{e_j}\right) = \prod_{j=1}^{m} N(\mathfrak{p}_j)^{e_j}$$

whenever the $\mathfrak{p}_j$ are pairwise distinct nonzero primes and the e_j are positive integers.

By uniqueness of factorisation we have $\mathfrak{p}_i^{e_i} + \mathfrak{p}_j^{e_j} = \mathbb{Z}_K$, i.e. the $\mathfrak{p}_j$ are pairwise coprime. The Chinese remainder theorem 3.2.3 then tells us that

$$N\left(\prod_{j=1}^{m} \mathfrak{p}_j^{e_j}\right) = \prod_{j=1}^{m} N(\mathfrak{p}_j^{e_j}),$$

so to conclude we must show that $N(\mathfrak{p}_j^{e_j}) = N(\mathfrak{p}_j)^{e_j}$.

Let $\mathfrak{p} \subset \mathbb{Z}_K$ be a nonzero prime. By uniqueness of the factorisation of ideals, there exists $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. The factorisation of the ideal $\pi\mathbb{Z}_K$ must then be $\pi\mathbb{Z}_K = \mathfrak{p}\mathfrak{a}$, where the ideal $\mathfrak{a}$ is coprime to $\mathfrak{p}$ (else π would lie in $\mathfrak{p}^2$), i.e. $\mathfrak{a} + \mathfrak{p} = \mathbb{Z}_K$. Fix $n \in \mathbb{N}$, and consider the homomorphism of additive groups

$$\begin{array}{ccc} \mathbb{Z}_K/\mathfrak{p} & \xrightarrow{\sim} & \mathfrak{p}^n/\mathfrak{p}^{n+1} \\ x & \longmapsto & \pi^n x. \end{array}$$

The kernel of this map is

$$\left\{x \in \mathbb{Z}_K \colon \pi^n x \in \mathfrak{p}^{n+1}\right\}/\mathfrak{p} = \left\{x \in \mathbb{Z}_K \colon \mathfrak{p}^{n+1} \mid (\pi^n x)\right\}/\mathfrak{p} = \mathfrak{p}/\mathfrak{p} = \{0\}$$

so this homomorphism is injective, and its image is

$$(\pi^n\mathbb{Z}_K + \mathfrak{p}^{n+1})/\mathfrak{p}^{n+1} = \big(\mathfrak{p}^n(\mathfrak{a} + \mathfrak{p})\big)/\mathfrak{p}^{n+1} = \mathfrak{p}^n/\mathfrak{p}^{n+1}$$

so it is also surjective. It is thus an isomorphism, so that

$$[\mathfrak{p}^n : \mathfrak{p}^{n+1}] = \#\mathfrak{p}^n/\mathfrak{p}^{n+1} = \#\mathbb{Z}_K/\mathfrak{p} = N(\mathfrak{p})$$

for all $n \in \mathbb{N}$. As a result, for all $e \in \mathbb{N}$, in view of the chain $\mathbb{Z}_K \supsetneq \mathfrak{p} \supsetneq \cdots \supsetneq \mathfrak{p}^e$ we have

$$N(\mathfrak{p}^e) = [\mathbb{Z}_K : \mathfrak{p}^e] = [\mathbb{Z}_K : \mathfrak{p}][\mathfrak{p} : \mathfrak{p}^2] \cdots [\mathfrak{p}^{e-1} : \mathfrak{p}^e] = N(\mathfrak{p})^e,$$

so the proof is complete. □

Example 3.5.7 (Failure of multiplicativity for a non-maximal order)**.** Let $\mathcal{O} = \mathbb{Z}[\alpha]$ where $\alpha = \sqrt{-3}$ and let $\mathfrak{a} = (2, \alpha + 1) = 2\mathbb{Z} + (\alpha + 1)\mathbb{Z} \subset \mathcal{O}$. Then $N(\mathfrak{a}) = \left|\det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\right| = 2$ but $\mathfrak{a}^2 = (2, \alpha + 1)^2 = (4, 2\alpha + 2, 2\alpha - 2) = 4\mathbb{Z} + (2\alpha + 2)\mathbb{Z}$ so $N(\mathfrak{a}^2) = \left|\det \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}\right| = 8$.

3.6 Decomposition of primes

Since the primes (meaning the prime ideals) of $\mathbb{Z}_K$ are the building blocks of all the nonzero ideals, they deserve particular attention.

Lemma 3.6.1. *Let $\mathfrak{a} \subset \mathbb{Z}_K$ be a nonzero ideal. Then $\mathfrak{a} \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$ of the form $a\mathbb{Z}$ for some nonzero $a \in \mathbb{N}$. Besides, if $\mathfrak{a}$ is a prime ideal, then a is a prime number.*

Proof. Let $f \colon \mathbb{Z} \to \mathbb{Z}_K/\mathfrak{a}$ be the canonical ring homomorphism. Then $\mathfrak{a} \cap \mathbb{Z} = \ker f$, so it is an ideal of $\mathbb{Z}$. Since $\mathbb{Z}$ is a PID, this ideal is of the form $a\mathbb{Z}$ with $a \in \mathbb{Z}$, and we can assume $a \geq 0$.

Now f induces an injective ring homomorphism $\bar{f} \colon \mathbb{Z}/a\mathbb{Z} \hookrightarrow \mathbb{Z}_K/\mathfrak{a}$. Since the ring $\mathbb{Z}_K/\mathfrak{a}$ is finite by Proposition 3.4.1, the ring $\mathbb{Z}/a\mathbb{Z}$ is also finite so $a > 0$.

If $\mathfrak{a}$ is prime, then $\mathbb{Z}_K/\mathfrak{a}$ is an integral domain, so the subring $\mathbb{Z}/a\mathbb{Z}$ is also an integral domain and therefore $a\mathbb{Z}$ is prime. □

Remark 3.6.2. Another way to see this is that a is the characteristic of the finite ring $\mathbb{Z}_K/\mathfrak{a}$, and is therefore nonzero; moreover it is a prime number when $\mathbb{Z}_K/\mathfrak{a}$ is an integral domain.

Definition 3.6.3. Let $\mathfrak{p}$ be a prime of $\mathbb{Z}_K$, and let $p \in \mathbb{N}$ be the prime number such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. One says that p is the prime number *below* $\mathfrak{p}$, and that $\mathfrak{p}$ is a prime ideal *above* p.

By Lemma 3.4.8, for all nonzero prime $\mathfrak{p}$, the quotient $\mathbb{Z}_K/\mathfrak{p}$ is a finite field, called the *residue field* of $\mathfrak{p}$. The characteristic of this field is by definition the prime number p below $\mathfrak{p}$, so this field is isomorphic to $\mathbb{F}_q$, where $q = p^f$ for some $f \in \mathbb{N}$ called the *residue degree* (some people say *inertial degree*) of $\mathfrak{p}$.

Remark 3.6.4. The prime number below a prime ideal $\mathfrak{p}$ is the unique p such that $N(\mathfrak{p})$ is a power of $\mathfrak{p}$.

By definition, an integer is a prime if it does not factor in $\mathbb{Z}$. It may very well, however, factor in the larger ring $\mathbb{Z}_K$. The following results tells us what kind of decompositions occur.

Theorem 3.6.5. *Let $p \in \mathbb{N}$ be prime, and let K be a number field. Then we have the factorisation*

$$p\mathbb{Z}_K = \prod_{j=1}^{g} \mathfrak{p}_j^{e_j},$$

where the $\mathfrak{p}_j$ are exactly the primes of $\mathbb{Z}_K$ above p and $e_j \geqslant 1$ for all j. Besides, if we let f_j be the inertial degree of $\mathfrak{p}_j$, we have the identity

$$\sum_{j=1}^{g} e_j f_j = [K : \mathbb{Q}].$$

Proof. Let $\mathfrak{p}$ be an ideal above p. Then by definition $\mathfrak{p} \supset p\mathbb{Z}_K$, so $\mathfrak{p}$ divides $p\mathbb{Z}_K$. This proves that the integers e_j are all nonzero.

Next, by definition, we have $N(\mathfrak{p}_j) = p^{f_j}$, so

$$N\left(\prod_{j=1}^{g} \mathfrak{p}_j^{e_j}\right) = \prod_{j=1}^{g} N(\mathfrak{p}_j)^{e_j} = \prod_{j=1}^{g} p^{e_j f_j}$$

by theorem 3.5.6. On the other hand, we also have

$$N(p\mathbb{Z}_K) = |N_{\mathbb{Q}}^K(p)| = |p^{[K:\mathbb{Q}]}|$$

by theorem 3.4.4, so the identity follows. □

Definition 3.6.6. The *ramification index* of a prime $\mathfrak{p}$ is the exponent $e \geqslant 1$ of $\mathfrak{p}$ in the decomposition of $p\mathbb{Z}_K$, where p is the prime below $\mathfrak{p}$. If $e \geqslant 2$, we say that $\mathfrak{p}$ is *ramified*; otherwise we say that $\mathfrak{p}$ is *unramified*. Let $n = [K : \mathbb{Q}]$.

- If there exists at least one ramified prime $\mathfrak{p}$ above p, we say that p *ramifies* in K (or that K ramifies at p).
- If $p\mathbb{Z}_K = \mathfrak{p}^n$ (so that the ramification index of $\mathfrak{p}$ is $e = n$ and its residue degree is $f = 1$), then we say that p is *totally ramified* in K (or that K is totally ramified at p).

- When $p\mathbb{Z}_K$ is a prime ideal, so that the above decomposition is simply $p\mathbb{Z}_K = \mathfrak{p}$, then we say that p is *inert* in K (or that K is inert at p).
- If p is neither inert nor ramified, we say that p *splits* in K (or that K splits at p).
- If we have a decomposition $p\mathbb{Z}_K = \prod_{j=1}^{n} \mathfrak{p}_j$ with the $\mathfrak{p}_j$ all distinct (so that their residue degrees are all 1), we say that p *splits completely* in K (or that K splits completely at p).

3.7 Practical factorisation

We are now going to see how to factor ideals explicitly. Let us start with the ideals $p\mathbb{Z}_K$, where $p \in \mathbb{N}$ is prime.

Theorem 3.7.1. *Let K be a number field, $p \in \mathbb{N}$ be prime, $\mathcal{O}$ be an order in K of the form $\mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{Z}_K$, and let $m(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. If $\mathcal{O}$ is p-maximal, and if*

$$\overline{m}(x) = \prod_{j=1}^{g} \overline{m_j}(x)^{e_j}$$

is the full factorisation of $\overline{m}(x) = m(x) \bmod p$ (that is to say, in $\mathbb{F}_p[x]$), then the full factorisation of $p\mathbb{Z}_K$ is

$$p\mathbb{Z}_K = \prod_{j=1}^{g} \mathfrak{p}_j^{e_j},$$

where $\mathfrak{p}_j = \big(p, m_j(\alpha)\big)$ and $m_j(x)$ denotes any lift of $\overline{m_j}(x)$ to $\mathbb{Z}[x]$. Besides, the residue degree of $\mathfrak{p}_j$ is $\deg \overline{m_j}(x)$.

Proof. (non-examinable) First, the prime divisors of $p\mathbb{Z}_K$ are in one-to-one correspondence with quotients of $\mathbb{Z}_K/p\mathbb{Z}_K$ that are integral domains. The correspondence is given by

$$\mathfrak{p} \mapsto (\mathbb{Z}_K/p\mathbb{Z}_K \to \mathbb{Z}_K/\mathfrak{p})$$

and

$$(\mathbb{Z}_K/p\mathbb{Z}_K \to A) \mapsto \ker(\mathbb{Z}_K \to \mathbb{Z}_K/p\mathbb{Z}_K \to A).$$

Since $\mathcal{O}$ is maximal at p, if we expressed a $\mathbb{Z}$-basis of $\mathbb{Z}_K$ in terms of a $\mathbb{Z}$-basis of $\mathcal{O}$, we would get coefficients which are rational numbers whose denominators are all coprime with p. As a result, the map $\mathcal{O}/p\mathcal{O} \longrightarrow \mathbb{Z}_K/p\mathbb{Z}_K$ induced by the inclusion $\mathcal{O} \subset \mathbb{Z}_K$ is an isomorphism.

We have

$$\mathcal{O} = \mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/\big(m(x)\big),$$

and therefore

$$\begin{aligned}\mathbb{Z}_K/p\mathbb{Z}_K \cong \mathcal{O}/p\mathcal{O} &\cong \mathbb{Z}[x]/\big(p, m(x)\big)\\ &\cong \mathbb{F}_p[x]/\big(\overline{m}(x)\big)\\ &= \mathbb{F}_p[x]/\prod_{j=1}^{g}\big(\overline{m_j}(x)^{e_j}\big)\\ &\cong \prod_{j=1}^{g}\mathbb{F}_p[x]/\big(\overline{m_j}(x)^{e_j}\big).\end{aligned}$$

In every quotient of $\mathbb{Z}_K/p\mathbb{Z}_K$ we have the relation $\prod_{j=1}^{g}\overline{m_j}(x)^{e_j} = 0$, so in every such quotient that is an integral domain one of the $\overline{m_j}(x)$ is zero, that is, every such quotient is of the form $F_j = \mathbb{F}_p[x]/\big(\overline{m_j}(x)\big)$ for some j. Such an F_j is indeed a field since $\overline{m_j}(x)$ is irreducible, and the corresponding prime ideal is the kernel $\mathfrak{p}_j = (p, m_j(\alpha))$ of the morphism $\mathbb{Z}_K \to F_j$. The residue field F_j has degree $\deg \overline{m_j}(x)$ over $\mathbb{F}_p$, which proves the claim about the residue degree.

Finally we must prove that the ramification indices are the e_j. First write

$$p\mathbb{Z}_K = \prod_{j=1}^{g}\mathfrak{p}_j^{e'_j}.$$

Then we have an isomorphism

$$\prod_{j=1}^{g}\mathbb{Z}_K/\mathfrak{p}_j^{e'_j} \cong \mathbb{Z}_K/p\mathbb{Z}_K \cong \prod_{j=1}^{g}\mathbb{F}_p[x]/\big(\overline{m_j}(x)^{e_j}\big),$$

hence for all j we get an isomorphism of the minimal quotient in which the image of $m_j(\alpha)$ is nilpotent:

$$\mathbb{Z}_K/\mathfrak{p}_j^{e'_j} \cong \mathbb{F}_p[x]/\big(\overline{m_j}(x)^{e_j}\big),$$

and we get $e'_j = e_j$ by comparing cardinalities. □

In general, to factor an integral ideal $\mathfrak{a}$, we can use the fact that the norm is multiplicative, and that every prime ideal having norm a power of a prime p appears in the factorisation of $p\mathbb{Z}_K$. The method consists in first factoring $N(\mathfrak{a})$ in $\mathbb{Z}$; then, for each prime p that appears in the factorisation, decomposing $p\mathbb{Z}_K$; and finally collecting the prime ideals that divide $\mathfrak{a}$.

Example 3.7.2. Let us come back to the problem of factoring 6 in $\mathbb{Z}[\sqrt{-5}] = \mathbb{Z}_K$, where $K = \mathbb{Q}(\sqrt{-5})$. Since

$$x^2 + 5 \equiv (x+1)^2 \bmod 2$$

we have

$$2\mathbb{Z}_K = (2, 1+\sqrt{-5})^2,$$

and 2 is totally ramified in K.

Since

$$x^2 + 5 \equiv (x-1)(x+1) \bmod 3,$$

we have

$$3\mathbb{Z}_K = (3, 1+\sqrt{-5})(3, 1-\sqrt{-5}),$$

and 3 is totally split in K.

We obtain

$$6\mathbb{Z}_K = (2\mathbb{Z}_K)(3\mathbb{Z}_K) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}).$$

Furthermore, since the ideals $(1 \pm \sqrt{-5})\mathbb{Z}_K$ are of norm 6, they must each factor into the product of a prime of norm 2 times a prime of norm 3, namely

$$(1 \pm \sqrt{-5})\mathbb{Z}_K = (2, 1+\sqrt{-5})(3, 1 \pm \sqrt{-5}).$$

Thus the factorisations

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

lead to the same decomposition into primes, as they should. Kummer's insight about "ideal numbers" was right!

Example 3.7.3. Consider the number field $K = \mathbb{Q}(\alpha)$ where α is a root of $f(x) = x^3 - x + 1$. For K to be well-defined, we need to prove that $x^3 - x + 1$ is irreducible; in fact it is already irreducible modulo 2 as we will see below. The discriminant of $f(x)$ is $-4(-1)^3 - 27 \cdot 1^2 = -23$, and this is also the discriminant of $\mathbb{Z}[\alpha]$. Since the index $f = [\mathbb{Z}_K : \mathbb{Z}[\alpha]]$ satisfies $-23 = \operatorname{disc}\mathbb{Z}[\alpha] = f^2 \operatorname{disc} K$, we must have $f = 1$: we have $\mathbb{Z}_K = \mathbb{Z}[\alpha]$. Let us look at some decompositions of primes that can occur.

- at 2: the polynomial $x^3 - x + 1$ has no root in $\mathbb{F}_2$. If it were reducible over $\mathbb{F}_2$, it would have a linear factor. So $f(x) \bmod 2$ is irreducible and $2\mathbb{Z}_K$ is prime: it has residue degree 3 and ramification index 1: the prime 2 is inert in K.

- at 5: the polynomial $x^3 - x + 1$ has 3 as a root modulo 5, yielding a factorisation $f(x) \equiv (x-3)(x^2+3x+3) \bmod 5$. Since x^2+3x+3 has no root modulo 5 and has degree 2, it is irreducible. We obtain $5\mathbb{Z}_K = \mathfrak{p}\mathfrak{p}'$ where $\mathfrak{p} = (5, \alpha - 3)$ has residue degree 1 and ramification index 1 and $\mathfrak{p}' = (5, \alpha^2 + 3\alpha + 3)$ has residue degree 2 and ramification index 1: the prime 5 is split but not totally split in K.

- at 23: we have $f(x) = (x+10)^2(x+3) \bmod 23$. We obtain $23\mathbb{Z}_K = \mathfrak{q}^2\mathfrak{q}'$ where $\mathfrak{q} = (23, \alpha + 10)$ has residue degree 1 and ramification index 2, and $\mathfrak{q}' = (23, \alpha + 3)$ has residue degree 1 and ramification index 1: the prime 23 is ramified but not totally ramified in K.

- at 59: we have $f(x) = (x + 4)(x + 13)(x + 42) \bmod 59$. We obtain $59\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ where $\mathfrak{p}_1 = (59, \alpha + 4)$ has residue degree 1 and ramification index 1, $\mathfrak{p}_2 = (59, \alpha+13)$ has residue degree 1 and ramification index 1, and $\mathfrak{p}_3 = (59, \alpha+42)$ has residue degree 1 and ramification index 1: the prime 59 is totally split in K.

Example 3.7.4. Let $K = \mathbb{Q}(\sqrt{-7})$ and let $\alpha = \frac{\sqrt{-7}+13}{2}$. Let us compute the factorisation of the ideal $\mathfrak{a} = (\alpha)$ in $\mathbb{Z}_K$. Since $-7 \equiv 1 \bmod 4$ we have $\mathbb{Z}_K = \mathbb{Z}[\omega]$ where $\omega = \frac{1+\sqrt{-7}}{2}$. Since $\alpha = \omega + 6 \in \mathbb{Z}_K$, $\mathfrak{a}$ is an integral ideal of $\mathbb{Z}_K$. We compute the norm of this ideal using Theorem 3.4.4:

$$N(\mathfrak{a}) = \left|N_{\mathbb{Q}}^K(\alpha)\right| = \frac{1}{4}(7 + 13^2) = 44 = 2^2 \cdot 11.$$

From this we get that $\mathfrak{a}$ is a product of some primes above 2 (one prime of norm 4 or two primes of norm 2, possibly equal) and one prime above 11. We therefore decompose the primes 2 and 11 in K. The minimal polynomial of ω is $m(x) = x^2 - x + 2$ and $\mathbb{Z}_K = \mathbb{Z}[\omega]$, so we can apply Theorem 3.7.1.

- $p = 2$: we have $m(x) \equiv x(x - 1) \bmod 2$, so 2 splits completely in K and $2\mathbb{Z}_K = \mathfrak{p}\mathfrak{p}'$ where $\mathfrak{p} = (2, \omega)$ and $\mathfrak{p}' = (2, \omega - 1)$.

- $p = 11$: we have $m(x) \equiv (x+4)(x+6) \bmod 11$, so 11 splits completely in K and $11\mathbb{Z}_K = \mathfrak{q}\mathfrak{q}'$ where $\mathfrak{q} = (11, \omega + 4)$ and $\mathfrak{q}' = (11, \omega + 6)$.

Now we know that $\mathfrak{a}$ is a product of two primes above 2 (since they both have norm 2) and one prime above 11, and we must determine which ones.

We clearly have $\alpha = \omega + 2 \cdot 3 \in \mathfrak{p}$. On the other hand we have $\alpha = \omega + 6 \equiv 1 \bmod \mathfrak{p}'$ since $\omega \equiv 1 \bmod \mathfrak{p}'$, so $\alpha \notin \mathfrak{p}'$, in other words $\mathfrak{p}'$ does not divide $\mathfrak{a}$. So $\mathfrak{p}$ divides $\mathfrak{a}$ with exponent 2. In addition we have $\alpha = \omega + 6 \in \mathfrak{q}'$. We conclude that $\mathfrak{a} = \mathfrak{p}^2\mathfrak{q}'$.

Example 3.7.5. Let$f(x) = x^3 + x^2 - 2x + 8$, and let $K = \mathbb{Q}(\alpha)$ where $f(\alpha) = 0$. We saw in example 2.3.14 that the order $\mathbb{Z}[\alpha]$ is p-maximal for all $p \neq 2$, so for instance we may compute that since $f(x)$ remains irreducible mod 3, the ideal $3\mathbb{Z}_K$ is prime, so that 3 is inert in K. Similarly, the full factorisation of $f(x)$ mod 5 is

$$f(x) \equiv (x + 1)(x^2 + 3) \bmod 5,$$

so

$$5\mathbb{Z}_K = (5, \alpha + 1)(5, \alpha^2 + 3)$$

splits as the product of a prime of degree 1 times another prime of degree 2.

With the help of a computer, we can try increasing the value of p until we find one which is totally split. We find that the smallest totally split $p \geqslant 3$ is $p = 59$, because

$$f(x) \equiv (x + 11)(x + 20)(x + 29) \bmod 59$$

splits completely mod 59 but does not for any smaller p. In particular, we have

$$59\mathbb{Z}_K = (59, \alpha + 11)(59, \alpha + 20)(59, \alpha + 29),$$

a product of three distinct primes of degree 1.

Finally, we know from example 2.3.14 that $\operatorname{disc} K = -503$, so 503 is the only prime $p \in \mathbb{N}$ which ramifies in K. More precisely, we can check that

$$f(x) = (x + 354)^2(x + 299) \bmod 503,$$

so that

$$503\mathbb{Z}_K = (503, \alpha + 354)^2(503, \alpha + 299).$$

However, we cannot see how $2\mathbb{Z}_K$ factors in $\mathbb{Z}_K$ by factoring $f(x)$ mod 2, because $\mathbb{Z}[\alpha]$ is not maximal at 2. In principle, we could look for another primitive element $\beta \in \mathbb{Z}_K$ such that the order $\mathbb{Z}[\beta]$ is maximal at 2, and then

determine the decomposition of $2\mathbb{Z}_K$ by factoring the minimal polynomial of β mod 2. This approach usually works, but unfortunately it does not in this particular case; indeed it can be proved that 2 splits completely in K, so if $\mathbb{Z}[\beta]$ were maximal at 2, then the minimal polynomial of β would be completely split and squarefree mod 2, which is impossible since the only possible linear factors mod 2 are x and $x+1$. Therefore, an order in this field of the form $\mathbb{Z}[\beta]$ is never maximal at 2.

Remark 3.7.6. In the case when for some $p \in \mathbb{N}$, no order of the form $\mathbb{Z}[\beta]$ and maximal at p exists (or is known), it is still possible to determine explicitly the decomposition of $p\mathbb{Z}_K$, but the method is much more complicated.

3.8 Ramification

Ramification is an important type of behaviour that can occur when decomposing rational primes in a number field. In this section we give a few more properties of ramification.

Proposition 3.8.1. *Let $K \subset L$ be two number fields, and let p be a prime number. If p ramifies in K then p ramifies in L.*

Proof. Let

$$p\mathbb{Z}_K = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i}$$

be the decomposition of $p\mathbb{Z}_K$ in prime ideals of $\mathbb{Z}_K$, and let, for each i,

$$\mathfrak{p}_i\mathbb{Z}_L = \prod_{j=1}^{g_i} \mathfrak{P}_{i,j}^{e_{i,j}}$$

be the decomposition of the ideal $\mathfrak{p}_i\mathbb{Z}_L$ of $\mathbb{Z}_L$ in prime ideals of $\mathbb{Z}_L$. We have the decomposition

$$p\mathbb{Z}_L = \prod_{i=1}^{g} \left(\prod_{j=1}^{g_i} \mathfrak{P}_{i,j}^{e_{i,j}} \right)^{e_i} = \prod_{i=1}^{g} \prod_{j=1}^{g_i} \mathfrak{P}_{i,j}^{e_i e_{i,j}}.$$

In particular, if p ramifies in K, then one of the e_i is at least 2, and hence $\mathfrak{P}_{i,1}$ appears with an exponent at least 2, so p ramifies in L. □

The case when p is totally ramified is a very special one, which allows us in particular to take a nice shortcut when computing the ring of integers of K.

Definition 3.8.2. Let $P(x) = x^n + \sum_{j=0}^{n-1} \lambda_j x^j \in \mathbb{Z}[x]$ be a monic polynomial, and let $p \in \mathbb{N}$ be a prime. We say that $P(x)$ is *Eisenstein* at p if p divides all the λ_j, but $p^2 \nmid \lambda_0$.

Example 3.8.3. $P(x) = x^2 - 84x + 90$ is Eisenstein at 2, but not at 5 because $5 \nmid 84$, nor at 7 because $7 \nmid 90$, nor at 3 because $3^2 \mid 90$.

Theorem 3.8.4 (Eisenstein's criterion)**.** *Let p be a prime number, and let $P(x) \in \mathbb{Z}[x]$ be a monic polynomial. If $P(x)$ is Eisenstein at p, then it is irreducible over $\mathbb{Q}$ (and thus also over $\mathbb{Z}$). Moreover, let $K = \mathbb{Q}(\alpha)$ where α is an algebraic number such that $P(\alpha) = 0$; then K is totally ramified at p and the order $\mathbb{Z}[\alpha]$ is maximal at p.*

Remark 3.8.5 (non-examinable)**.** Conversely, if K is a number field which is totally ramified at p, then there exists a primitive element $\alpha \in \mathbb{Z}_K$ whose minimal polynomial over $\mathbb{Q}$ is Eisenstein at p.

The proof of this theorem being a bit more technical than the rest of this section, we only give it here for reference; it is not examinable.

Proof. Suppose that $P(x) \in \mathbb{Z}[x]$ is Eisenstein at p, and let n be its degree. If $P(x) = Q(x)R(x)$ were reducible over $\mathbb{Z}$, then we would have $Q(x)R(x) = P(x) \equiv x^n \bmod p$, so that $Q(x) \equiv x^q$ and $R(x) \equiv x^r \bmod p$ for some nonzero integers q, r. But this would mean that p divides the constant terms of $Q(x)$ and $R(x)$, so that p^2 would divide the constant term of $P(x)$, which contradicts the fact that P is Eisenstein at p. Therfore, $P(x)$ is irreducible over $\mathbb{Z}$ (and thus also over $\mathbb{Q}$).

In particular, $K = \mathbb{Q}(\alpha)$ is a well-defined number field of degree n. If the order $\mathbb{Z}[\alpha]$ were not maximal at p, then by Proposition 2.3.1 there would exist integers λ_j not all divisible by p such that $\sum_{j=0}^{n-1} \frac{\lambda_j}{p} \alpha^j \in \mathbb{Z}_K$ is an integer. Then, if j_0 is the smallest integer such that $p \nmid \lambda_j$, then after subtract an element of $\mathbb{Z}[\alpha] \subset \mathbb{Z}_K$ and multiplying by α^{n-1-j_0}, we would get

$$\frac{\lambda_{j_0}}{p} \alpha^{n-1} + \frac{\alpha^n}{p} \sum_{k=0}^{n-j_0-2} \lambda_{j_0+1+k} \alpha^k \in \mathbb{Z}_K.$$

However, the relation $P(\alpha) = 0$ and the fact that p divides the coefficients of $P(x)$ imply that $\frac{\alpha^n}{p} \in \mathbb{Z}[\alpha] \subset \mathbb{Z}_K$, so we would have $\frac{\lambda_{j_0}}{p}\alpha^{n-1} \in \mathbb{Z}_K$. Taking the norm, this would mean that $\frac{\lambda_{j_0}^n N_{\mathbb{Q}}^K(\alpha)^{n-1}}{p^n} \in \mathbb{Z}$. Now, $N_{\mathbb{Q}}^K(\alpha)$ is, up to sign, the constant coefficient of $P(x)$, so it is of the form pq for some integer $q \in \mathbb{Z}$ which is coprime to p. We would then have $\frac{\lambda_{j_0}^n q^{n-1}}{p} \in \mathbb{Z}$, a contradiction.

It follows that the order $\mathbb{Z}[\alpha]$ is maximal at p, so we may apply theorem 3.7.1 below and deduce that since $P(x) \equiv x^n \bmod p$, the decomposition of p in K is $p\mathbb{Z}_K = \mathfrak{p}^n$, $\mathfrak{p} = (p, \alpha)$. In particular, p is totally ramified in K.

Conversely, suppose now that K is a number field in which $p \in \mathbb{N}$ is totally ramified, say $p\mathbb{Z}_K = \mathfrak{p}^n$ where $n = [K : \mathbb{Q}]$. For nonzero $x \in \mathbb{Z}_K$, let $v_{\mathfrak{p}}(x)$ be the largest nonnegative integer such that $\mathfrak{p}^{v_{\mathfrak{p}}(x)} \mid x\mathbb{Z}_K$ (i.e $v_{\mathfrak{p}}(x)$ is the exponent of $\mathfrak{p}$ in the factorisation of the ideal $x\mathbb{Z}_K$), and set $v_{\mathfrak{p}}(0) = +\infty$. Clearly, for every finite family of algebraic integers $x_i \in \mathbb{Z}_K$, we have

$$v_{\mathfrak{p}}\left(\prod_i x_i\right) = \sum_i v_{\mathfrak{p}}(x_i) \text{ and } v_{\mathfrak{p}}\left(\sum_i x_i\right) \geqslant \min_i v_{\mathfrak{p}}(x_i).$$

Besides, if $\sum_i x_i = 0$, then the minimum $\min_i v_{\mathfrak{p}}(x_i)$ must be attained for at least two values of i, for if not, say $v_{\mathfrak{p}}(x_1) < v_{\mathfrak{p}}(x_i)$ for all $i \geqslant 2$ for instance, then we would have

$$v_{\mathfrak{p}}(x_1) = v_{\mathfrak{p}}\left(-\sum_{i \geqslant 2} x_i\right) \geqslant \min_{i \geqslant 2} v_{\mathfrak{p}}(x_i) > v_{\mathfrak{p}}(x_1),$$

a contradiction. Note that for all $m \in \mathbb{Z}$, we have $v_{\mathfrak{p}}(m) = n \cdot v_p(m)$, where $v_p(m)$ denotes the exponent of p in the factorisation of m in $\mathbb{Z}$; in particular, $v_{\mathfrak{p}}(p) = n$.

Since $p \in \mathfrak{p}$, according to proposition 3.5.5 there exists $\alpha \in \mathbb{Z}_K$ such that $\mathfrak{p} = (p, \alpha)$. This means that $\gcd(p\mathbb{Z}_K, \alpha\mathbb{Z}_K) = \mathfrak{p}$, so we must have $v_{\mathfrak{p}}(\alpha) = 1$. Let $P(x) = x^m + \sum_{i<m} \lambda_i x^i \in \mathbb{Z}[x]$ be the minimal polynomial of α, where $m \leqslant n$ is the degree of α. Then, in the relation

$$\alpha^m + \sum_{i<m} \lambda_i \alpha^i = 0,$$

the minimum of $v_{\mathfrak{p}}$ must be attained at least twice; but since $v_{\mathfrak{p}}(\alpha^i) = i \cdot v_{\mathfrak{p}}(\alpha) = i$ and $p \mid v_{\mathfrak{p}}(\lambda_i)$ for all i, this forces $m = n$ and $p \mid \lambda_i$

for all i. In particular, this shows that α is a primitive element for $K/\mathbb{Q}$. Finally, the constant coefficient λ_0 is, up to sign, the norm of α, which is, up to sign, the norm of the ideal $\alpha\mathbb{Z}_K$ by theorem 3.4.4. Since $v_{\mathfrak{p}}(\alpha) = 1$ and $\mathfrak{p}$ is the only prime above p, this proves that $p^2 \nmid \lambda_0$, so that $P(x)$ is Eisenstein at p. □

The following theorem is important but we will not prove it.

Theorem 3.8.6. *The primes $p \in \mathbb{Z}$ which ramify in K are exactly the ones which divide the discriminant of K. In particular, there are only finitely many of them.*

Remark 3.8.7. Theorem 3.7.1 yields a special case of this theorem. Assume $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{Z}_K$ with minimal polynomial $m(x) \in \mathbb{Z}[x]$. In this case, p is ramified in $\mathbb{Z}_K$ if and only if $m(x)$ has repeated factors modulo p, if and only if $\operatorname{disc} m = 0 \bmod p$. On the other hand we have $\operatorname{disc} m = \operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} K$.

To conclude, we should mention the following result of Minkowski's.

Theorem 3.8.8. *If $K \neq \mathbb{Q}$ is a number field, then $|\operatorname{disc} K| > 1$, so there is at least one ramified prime $p \in \mathbb{N}$.*

This can be rephrased by saying that there is no nontrival unramified number field. We postpone the proof of this result to the next chapter.

3.9 The case of quadratic fields

In this section, we let $K = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is a squarefree integer different from 0 and 1, and we let $p \in \mathbb{N}$ be an odd prime.

Recall that the *Legendre symbol* is defined by

$$\left(\frac{d}{p}\right) = \begin{cases} 0 & \text{if } p \mid d, \\ +1 & \text{if } d \text{ is a nonzero square mod } p, \\ -1 & \text{if } d \text{ is a nonzero nonsquare mod } p. \end{cases}$$

The Legendre symbols tells us exactly how odd primes split in quadratic field.

Theorem 3.9.1. *Let $p \in \mathbb{N}$, $p \neq 2$ be a prime.*

- *If* $\left(\frac{d}{p}\right) = 0$*, then* $p\mathbb{Z}_K = \mathfrak{p}^2$ *is totally ramified in* K*.*
- *If* $\left(\frac{d}{p}\right) = +1$*, then* $p\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2$ *splits completely in* K*.*
- *If* $\left(\frac{d}{p}\right) = -1$*, then* $p\mathbb{Z}_K = \mathfrak{p}$ *is inert in* K*.*

Proof. Since $p \neq 2$, the order $\mathbb{Z}[\sqrt{d}]$ is p-maximal, so theorem 3.7.1 applies, and tells us that the decomposition of $p\mathbb{Z}_K$ is governed by the splitting behaviour of $x^2 - d \bmod p$. When $\left(\frac{d}{p}\right) = 0$, $+1$, or -1, $x^2 - d \bmod p$ is respectively a square, a product of two distinct linear factors, or irreducible, hence the result. □

Remark 3.9.2. For the case $\left(\frac{d}{p}\right) = 0$, we could also have used Eisentein's criterion (theorem 3.8.4).

The case of $p = 2$ is special and must be stated separately.

Theorem 3.9.3.

- *If* $d \equiv 2$ *or* $3 \bmod 4$*, then* $2\mathbb{Z}_K = \mathfrak{p}^2$ *is totally ramified in* K*.*
- *If* $d \equiv 1 \bmod 8$*, then* $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2$ *splits completely in* K*.*
- *If* $d \equiv 5 \bmod 8$*, then* $2\mathbb{Z}_K = \mathfrak{p}$ *is inert in* K*.*

Proof. If $d \equiv 2$ or $3 \bmod 4$, then theorem 2.4.2 tells us that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{d}]$, so we may apply theorem 3.7.1. Furthermore, we have $x^2 - d \equiv (x-d)^2 \bmod 2$, so 2 is totally ramified in K.

Let us now suppose that $d \equiv 1 \bmod 4$. Then $\mathbb{Z}_K = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$, so theorem 3.7.1 does not apply to $\mathbb{Z}[\sqrt{d}]$; on the other hand, it does apply to $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$. The minimal polynomial of $\frac{1+\sqrt{d}}{2}$ is $x^2 - x - \frac{d-1}{4}$, which reduces mod 2 to $x(x-1)$ when $d \equiv 1 \bmod 8$, and to $x^2 + x + 1$ which is irreducible over $\mathbb{F}_2$ when $d \equiv 5 \bmod 8$. □

The point of this is that, thanks to quadratic reciprocity, the decomposition type of p can be read off the class of p modulo $4d$. If you have not seen quadratic reciprocity, do not worry: we will not ask you to use it in assignments or exams.

Example 3.9.4. In $K = \mathbb{Q}(\sqrt{-5})$, for all prime $p \in \mathbb{N}$ we have

- p is (totally) split in K if and only if $p \equiv 1, 3, 7$ or $9 \bmod 20$,
- p is inert in K if and only if $p \equiv 11, 13, 17$ or $19 \bmod 20$,
- p is (totally) ramified in K if and only if $p \equiv 2$ or $5 \bmod 20$.

3.10 The case of cyclotomic fields

In this section, we fix an integer $n \geqslant 3$, and we let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive n^{th} root of unity. The law governing the splitting of primes in K is the following.

Theorem 3.10.1. *Let $p \in \mathbb{N}$ be a prime number, let p^v be the largest power of p which divides n (so in particular $v = 0$ if $p \nmid n$), let $m = n/p^v$, and let $f \in \mathbb{N}$ be the smallest nonzero integer such that $p^f \equiv 1 \bmod m$, i.e. the order of p in the group $(\mathbb{Z}/m\mathbb{Z})^\times$. Then the decomposition of $p\mathbb{Z}_K$ is*

$$p\mathbb{Z}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^{\varphi(p^v)},$$

where the $\mathfrak{p}_j$ are distinct primes which are all of inertial degree f. In particular, $g = \varphi(m)/f$.

Proof. (Non examinable) We know from theorem 2.5.2 that $\mathbb{Z}_K = \mathbb{Z}[\zeta]$, so by theorem 3.7.1 the decomposition of $p\mathbb{Z}_K$ corresponds to the factorisation of the cyclotomic polynomial $\Phi_n(x)$ mod p. When ξ_i (resp. η_j) ranges over the set of primitive m^{th} (resp. $(p^v)^{\text{th}}$) roots of unity, then the products $\xi_i\eta_j$ range over the set of primitive n^{th} roots of unity[2], so

$$\Phi_n(x) = \prod_{i,j}(x - \xi_i\eta_j).$$

Note that this factorisation takes place in $\mathbb{Z}_K[x]$, so it makes sense to reduce it modulo ideals of $\mathbb{Z}_K$. Let $\mathfrak{p}$ be a prime of $\mathbb{Z}_K$ above p. Since $\Phi_{p^v}(x) \mid (x^{p^v} - 1) \equiv (x-1)^{p^v} \bmod p$ and thus also mod $\mathfrak{p}$, we have $\Phi_{p^v}(x) = \prod_j (x -$

[2]This is just the isomorphism $(\mathbb{Z}/n\mathbb{Z})^\times \simeq (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/p^v\mathbb{Z})^\times$ from Chinese remainders in disguise.

$\eta_j) \equiv (x-1)^{\varphi(p^v)} \bmod \mathfrak{p}$, and so $\eta_j \equiv 1 \bmod \mathfrak{p}$ for all j since $\mathbb{Z}_K/\mathfrak{p}$ is a field. As a result,

$$\Phi_n(x) \equiv \prod_{i,j}(x - \xi_i \cdot 1) = \Phi_m(x)^{\varphi(p^v)} \bmod \mathfrak{p},$$

so the coefficients of the difference $\Phi_n(x) - \Phi_m(x)^{\varphi(p^v)}$ lie in $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$, i.e.

$$\Phi_n(x) \equiv \Phi_m(x)^{\varphi(p^v)} \bmod p.$$

We are thus led to studying how $\Phi_m(x)$ factors mod p. Now, $\Phi_m(x)$ divides $x^m - 1$, which is coprime mod p with its derivative mx^{m-1} since $p \nmid m$; therefore $x^m - 1$ is squarefree mod p, and so is $\Phi_m(x)$. In other words, reduction mod p is injective on m^{th} roots of unity, and in particular primitive m^{th} roots remain primitive mod p.

Since the multiplicative group of a finite field is cyclic, the field $\mathbb{F}_{p^a}$ contains a primitive m^{th} root of unity if and only if $m \mid p^a - 1$, if and only if $p^a \equiv 1 \bmod m$, if and only if $f \mid a$.

On the other hand, if $\Phi_m(x) \equiv f_1(x) \dots f_k(x) \bmod p$ is the factorisation of Φ_m into irreducibles over $\mathbb{F}_p[x]$, then Φ_m has a root in $\mathbb{F}_{p^a}$ if and only if one of the f_i has a root in $\mathbb{F}_{p^a}$, if and only if there is an i such that $\deg f_i \mid a$.

Putting these together, we obtain that for all i, $f \mid \deg f_i$. Moreover, the field $\mathbb{F}_{p^f}$ contains a primitive m-th root of unity, and therefore contains all of them, so Φ_m splits completely over $\mathbb{F}_{p^f}$, and therefore we have $\deg f_i = f$ for all i. As a result, the primes above $\mathfrak{p}$ all have inertial degree f. The fact that $g = \varphi(m)/f$ follows from Theorem 3.6.5 since

$$[K : \mathbb{Q}] = \varphi(p^v m) = \varphi(p^v)\varphi(m)$$

as p^v and m are coprime. □

Example 3.10.2. Let $n = 15$ and $K = \mathbb{Q}(\zeta_n)$. Let us compute the decomposition of some small primes.

- $p = 2$: we have $m = 15$, and by computing the powers of 2 mod 15 we see that it has order $f = 4$. We therefore have $g = \varphi(m)/f = 8/4 = 2$. The decomposition of 2 is

  $$2\mathbb{Z}_K = \mathfrak{p}_2\mathfrak{p}_2',$$

  and both these primes have residue degree 2 and ramification index 1. The prime 2 splits in K but is not totally split.

- $p = 3$: we have $m = 5$, and $3 \mod 5$ has order $f = 4$. We therefore have $g = \varphi(m)/f = 4/4 = 1$. The decomposition of 3 is

$$3\mathbb{Z}_K = \mathfrak{p}_3^2 \text{ since } \varphi(3) = 2,$$

  and $\mathfrak{p}_3$ has residue degree 4 and ramification index 2. The prime 3 is ramified but not totally ramified in K.

- $p = 5$: we have $m = 3$, and $5 \equiv -1 \bmod 3$ has order $f = 2$. We therefore have $g = \varphi(m)/f = 2/2 = 1$. The decomposition of 5 is

$$5\mathbb{Z}_K = \mathfrak{p}_5^4 \text{ since } \varphi(5) = 4,$$

  and $\mathfrak{p}_5$ has residue degree 2 and ramification index 4. The prime 5 is ramified but not totally ramified in K.

Corollary 3.10.3. *A prime $p \in \mathbb{N}$ splits completely in K if and only if $p \equiv 1 \bmod n$.*

Corollary 3.10.4. *A prime $p \in \mathbb{N}$ ramifies in K if and only if it divides n, except for $p = 2$, which ramifies in K if and only if $4 \mid n$.*

This last point was already more or less obvious from the formula for $\operatorname{disc} K$ from theorem 2.5.2. In the case when $n = p^v$ is itself a prime power, we can say a little more:

Theorem 3.10.5. *If $n = p^v$, then $p\mathbb{Z}_K = (\zeta - 1)^{\varphi(n)}$. In particular, p is totally ramified in K. Moreover, the minimal polynomial $\Phi_n(x + 1)$ of $\zeta - 1$ is Eisenstein at p.*

Proof. We already know that p is totally ramified by Theorem 3.10.1, and that $p\mathbb{Z}_K = \mathfrak{p}^{\varphi(n)}$ where $\mathfrak{p} = (p, \zeta - 1)$, since $\Phi_n(x) \equiv (x - 1)^{\varphi(n)} \bmod p$. We have

$$\Phi_n(x + 1) \equiv x^{\varphi(n)} \bmod p.$$

Besides,

$$\Phi_n = \frac{x^{p^v} - 1}{x^{p^{v-1}} - 1} = \sum_{j=0}^{p-1} x^{p^{v-1}j},$$

so the constant term of $\Phi_n(x + 1)$ is $\Phi_n(1) = p$, so that $N_{\mathbb{Q}}^K(\zeta - 1) = \pm p$. Therefore, $\Phi_n(x+1)$ is indeed Eisenstein at p. In addition, the inertial degree of the prime $\mathfrak{p}$ is 1 so its norm is p, and since $\zeta - 1 \in \mathfrak{p}$ has norm $\pm p$, we can conclude that $\mathfrak{p} = (\zeta - 1)$. □

Chapter 4

The class group

4.1 UFDs. vs. PID. vs. Dedekind domains

Dedekind domains are in general not principal ideal domains. In fact,

Proposition 4.1.1. *A Dedekind domain is a PID if and only if it is a UFD.*

Proof. Every PID is a UFD. Conversely, let $\mathcal{R}$ be a Dedekind domain which is a UFD, let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{R}$, and let $\alpha \in \mathfrak{a}$, $\alpha \neq 0$. Factor α into irreducibles

$$\alpha = \prod_{j=1}^{m} \pi_j^{a_j}.$$

Since the π_j are irreducible and $\mathcal{R}$ is a UFD, the ideals (π_j) are prime, so

$$\prod_{j=1}^{m} (\pi_j)^{a_j}$$

is the decomposition of the ideal (α) into primes. Since $\mathfrak{a}$ divides (α), we have

$$\mathfrak{a} = \prod_{j=1}^{m} (\pi_j)^{e_j}$$

for some $e_j \leqslant a_j$. It follows that

$$\mathfrak{a} = \left(\prod_{j=1}^{m} \pi_j^{e_j} \right)$$

is principal. Therefore $\mathcal{R}$ is a PID as claimed. □

As we saw in Example 3.7.2, there are number fields whose ring of integers is not a PID. We would like to be able to decide whether such a ring of integers is a PID. In fact, we would like to have a way of deciding whether, in certain situations where we are not working with a PID, we can use factorisation techniques to prove results about algebraic integers. In this chapter we will introduce a tool to do this: the *class group*.

4.2 Ideal inversion

Definition 4.2.1. Let K be a number field. A *fractional ideal* of K (or of $\mathbb{Z}_K$) is a lattice in K of the form $\frac{1}{d}\mathfrak{a} = \left\{\frac{\alpha}{d}, \alpha \in \mathfrak{a}\right\}$ for some ideal $\mathfrak{a} \subset \mathbb{Z}_K$.

To avoid confusion, ideals of $\mathbb{Z}_K$ in the usual sense are also called *integral ideals*.

Example 4.2.2. The fractional ideals of $\mathbb{Q}$ are the $x\mathbb{Z}$, $x \in \mathbb{Q}^\times$.

The sum and product of a finite family of fractional ideals are defined the same way as for integral ideals, and are fractional ideals. More generally, the notations $(\alpha) = \alpha\mathbb{Z}_K$ and $(\alpha_1, \cdots, \alpha_m) = (\alpha_1) + \cdots + (\alpha_m)$ initially used for integral ideals (so for $\alpha, \alpha_1, \cdots, \alpha_m \in \mathbb{Z}_K$) can be extended to fractional ideals, i.e. to $\alpha, \alpha_1, \ldots, \alpha_m \in K^\times$. For instance, the fractional ideal generated by an element $\alpha \in K^\times$ is

$$(\alpha) = \alpha\mathbb{Z}_K \subset K.$$

Indeed, it is clear by lemma 2.2.16 that $(\alpha_1, \ldots, \alpha_m)$ is a fractional ideal for all $\alpha_1, \ldots, \alpha_m \in K^\times$. It is also clear that this ideal is an integral ideal if and only if $\alpha_1, \ldots, \alpha_m$ lie all in $\mathbb{Z}_K$.

Theorem 4.2.3. *Let K be a number field. Every fractional ideal $\mathfrak{a}$ of K is* invertible*, meaning there exists a fractional ideal $\mathfrak{b}$ such that $\mathfrak{ab} = \mathbb{Z}_K$. This ideal $\mathfrak{b}$ is unique, and is denoted by $\mathfrak{a}^{-1}$.*

Proof. It is enough to show that every nonzero *integral* ideal is invertible as a fractional ideal. Indeed, every fractional ideal $\mathfrak{b}$ is by definition of the form $\frac{1}{d}\mathfrak{a}$ for some nonzero integer $d \in \mathbb{N}$ and nonzero integral ideal $\mathfrak{a} \subseteq \mathbb{Z}_K$, but if $\mathfrak{a}^{-1}$ is an inverse of $\mathfrak{a}$, then clearly $d\mathfrak{a}^{-1}$ is an inverse of $\mathfrak{b}$. So let $\mathfrak{a}$ be a nonzero integral ideal. Then $N(\mathfrak{a}) \in \mathfrak{a}$ by proposition 3.4.3, so $\mathfrak{a} \mid N(\mathfrak{a})\mathbb{Z}_K$, which means that there exists an ideal $\mathfrak{b}$ such that $\mathfrak{ab} = N(\mathfrak{a})\mathbb{Z}_K$. As a result, $\frac{1}{N(\mathfrak{a})}\mathfrak{b}$ is an inverse of $\mathfrak{a}$. The fact that inverses are unique follows from the associativity of ideal multiplication. □

Remark 4.2.4. One can prove that

$$\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subset \mathbb{Z}_K\}.$$

Remark 4.2.5. Clearly, the inverse of an integral ideal contains $\mathbb{Z}_K$, and vice versa.

Definition 4.2.6. Let K be a number field. The *group of fractional ideals* $\mathcal{I}_K$ is the set of fractional ideals of K, equipped with ideal multiplication.

Proposition 4.2.7. *Let K be a number field. Every fractional ideal $\mathfrak{a}$ of K admits a unique factorisation*

$$\mathfrak{a} = \prod_i \mathfrak{p}_i^{e_i}$$

where $e_i \in \mathbb{Z}$ and $\mathfrak{p}_i$ are prime ideals of $\mathbb{Z}_K$.

Proof. Let $\mathfrak{a} = \frac{1}{d}\mathfrak{b}$ where $d \in \mathbb{Z}_{>0}$ and $\mathfrak{b}$ is an integral ideal. By factoring $\mathfrak{b}$ and (d) we obtain a factorisation of $\mathfrak{a}$. The uniqueness statement follows from uniqueness of factorisations of integral ideals. □

Remarks 4.2.8.

- Proposition 4.2.7 says that we have an isomorphism of abelian groups

$$\mathcal{I}_K \cong \bigoplus_{\mathfrak{p} \text{ prime}} \mathbb{Z}.$$

- To compute the factorisation of a fractional ideal, just put it in the form $\frac{1}{d}\mathfrak{a}$ with $d \in \mathbb{N}$ and $\mathfrak{a}$ integral, factor $\mathfrak{a}$ and d separately, and divide their factorisations.

4.3 The class group

Recall that in a ring $\mathcal{R}$ we say that an ideal $\mathfrak{a} \subset \mathcal{R}$ is principal if it is generated by one element, i.e. if there is $x \in \mathcal{R}$ such that $\mathfrak{a} = x\mathbb{Z}_K$.

Definition 4.3.1. Let K be a number field, and let $\mathfrak{a} \subset K$ be a fractional ideal. We say that $\mathfrak{a}$ is *principal* if it is generated by one element, i.e. if there is $x \in K^\times$ such that $\mathfrak{a} = x\mathbb{Z}_K$.

Note that for an integral ideal, it is not clear that this is the same notion as before since we allow generators that are not integers. However, we have the following.

Lemma 4.3.2. *Let K be a number field, and let $\mathfrak{a} \subset \mathbb{Z}_K$ be an integral ideal. Then $\mathfrak{a}$ is principal as a fractional ideal if and only if it is principal in the usual sense.*

Proof. Assume that $\mathfrak{a}$ is principal, and let $x \in K^\times$ be a generator. Then $x\mathbb{Z}_K = \mathfrak{a} \subset \mathbb{Z}_K$, so that x is an algebraic integer. The converse is clear. □

Example 4.3.3. Let $K = \mathbb{Q}(\sqrt{-5})$, so that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$ and $\operatorname{disc} K = -20$. Let $\mathfrak{p}_2$ (resp. $\mathfrak{p}_5$) be the unique prime above 2 (resp. above 5) coming from the decomposition of those primes (Theorems 3.9.1 and 3.9.3). Since $N(\mathfrak{p}_2) = 2$, if $\mathfrak{p}_2$ admitted a generator $x + \sqrt{-5}y \in \mathbb{Z}_K$ then the integers x, y have to satisfy $N_{\mathbb{Q}}^K(x + \sqrt{-5}y) = x^2 + 5y^2 = 2$, which is clearly impossible. So $\mathfrak{p}_2$ is not principal. On the other hand, $\mathfrak{p}_5 = (\sqrt{-5})$ is principal. However, $\mathfrak{p}_2^2 = (2)$ is principal.

Example 4.3.4. Let $K = \mathbb{Q}(\sqrt{-23})$, so that $\mathbb{Z}_K = \mathbb{Z}[\omega]$ where $\omega = \frac{1+\sqrt{-23}}{2}$, and $\operatorname{disc} K = -23$. Since $-23 \equiv 1 \bmod 8$, the prime $2 = \mathfrak{p}_2\mathfrak{p}_2'$ splits completely in K by Theorem 3.9.3. To see whether $\mathfrak{p}_2$ is principal, let us compute the norm of a generic element $z = x + \omega y \in \mathbb{Z}_K$. We have

$$N_{\mathbb{Q}}^K(z) = (x + \tfrac{1}{2}y)^2 + \frac{23}{4}y^2 = x^2 + xy + 6y^2.$$

If $N_{\mathbb{Q}}^K(z) = 2$, then $\frac{23}{4}y^2 \leq 2$ so $|y| \leq 2\sqrt{\frac{2}{23}} \approx 0.59$. So we must have $y = 0$, but then $z \in \mathbb{Z}$ cannot have norm 2. Therefore $\mathfrak{p}_2$ is not principal.

In the same way, if $\mathfrak{p}_2^2$ were principal, then its generator z would have norm 4, but again $|y| \leq \frac{2}{\sqrt{23}} \approx 0.83$ and z has to be an integer. Therefore the only element of norm 4 are ± 2, but $(2) = \mathfrak{p}_2\mathfrak{p}_2' \neq \mathfrak{p}_2^2$ by uniqueness of factorisation.

However, for $(x, y) = (1, 1)$ we get an element $z \notin \mathbb{Z}$ of norm 8. The ideals of norm 8 are $\mathfrak{p}_2^3$, $\mathfrak{p}_2^2\mathfrak{p}_2' = 2\mathfrak{p}_2$, $\mathfrak{p}_2{\mathfrak{p}_2'}^2 = 2\mathfrak{p}_2'$ and ${\mathfrak{p}_2'}^3$. Since z is not in $2\mathbb{Z}_K$ (because its coefficients on the $\mathbb{Z}$-basis of $\mathbb{Z}_K$ are not divisible by 2), the ideal (z) cannot be $2\mathfrak{p}_2$ or $2\mathfrak{p}_2'$. To determine whether (z) is $\mathfrak{p}_2^3$ or ${\mathfrak{p}_2'}^3$, we must distinguish them by giving explicit generators.

The minimal polynomial of ω is $x^2 - x + 6$, which factors modulo 2 as $x(x-1) \bmod 2$. We have $\mathfrak{p}_2 = (2, \omega)$ and $\mathfrak{p}_2' = (2, \omega - 1)$. Now $z = \omega + 1 = \omega - 1 + 2 \in \mathfrak{p}_2'$, so that $(z) = {\mathfrak{p}_2'}^3$.

Finally, $(8/z) = (\mathfrak{p}_2\mathfrak{p}_2')^3/{\mathfrak{p}_2'}^3 = \mathfrak{p}_2^3$ is principal.

In the last two examples, for each ideal we considered, a power of that ideal was principal. This is a general phenomenon, and suggests that we should look at the multiplicative structure of ideals and principal ideals. This motivates the following definition.

Definition 4.3.5. Let K be a number field. Let $\mathcal{I}_K$ be the group of fractional ideals of K and $\mathcal{P}_K$ be the subgroup of principal fractional ideals of K. The *class group* of K is

$$\mathrm{Cl}(K) = \mathcal{I}_K/\mathcal{P}_K.$$

An *ideal class* is a class in this quotient. We say that two ideals are *equivalent* if they are in the same class. We write $[\mathfrak{a}]$ for the ideal class of the fractional ideal $\mathfrak{a}$.

With this definition, $\mathbb{Z}_K$ is a PID if and only if $\mathrm{Cl}(K)$ is trivial: we say that the class group measures the obstruction for $\mathbb{Z}_K$ to be a PID. By definition, a fractional ideal $\mathfrak{a}$ is principal if and only if the class $[\mathfrak{a}]$ is trivial.

Note that every ideal class is represented by an integral ideal: a fractional ideal $\frac{1}{d}\mathfrak{a}$ with $\mathfrak{a} \subset \mathbb{Z}_K$ is in the same class as $\mathfrak{a}$.

4.4 Finiteness of the class group: the Minkowski bound

The most important result about the class group is that it is always finite.

Theorem 4.4.1 (Minkowski)**.** *Let K be a number field of signature (r_1, r_2) and degree $n = r_1 + 2r_2$. Let*

$$M_K = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}\sqrt{|\operatorname{disc} K|}.$$

Then every ideal class is represented by an integral ideal of norm at most M_K.

The number M_K is called the Minkowski bound (or the Minkowski constant). We will prove this theorem after introducing "geometry of numbers" techniques in chapter 6.

Corollary 4.4.2. *The class group* $\mathrm{Cl}(K)$ *is finite.*

Proof. It suffices to prove that there are finitely many integral ideals of a given norm, but that follows from the factorisation theorem. □

Because of this finiteness result, the following definition makes sense.

Definition 4.4.3. The *class number* h_K of K is the order of $\mathrm{Cl}(K)$.

Corollary 4.4.4. *The class group of* K *is generated by the classes of the prime ideals of norm at most* M_K*.*

Proof. By the factorisation theorem, every integral ideal of norm at most M_K is a product of the primes of norm at most M_K. Passing to the quotient gives the result. □

As claimed, we obtain that every ideal has a power that is principal.

Corollary 4.4.5. *For every fractional ideal* $\mathfrak{a}$ *of* K*, the fractional ideal* $\mathfrak{a}^{h_K}$ *is principal.*

Proof. Since the group $\mathrm{Cl}(K)$ is finite, every element has finite order, and that order is a divisor of the order h_K of the group. □

Corollary 4.4.6. *For every fractional ideal* $\mathfrak{a}$ *of* K *and every integer* m *coprime to* h_K*, if* $\mathfrak{a}^m$ *is principal then* $\mathfrak{a}$ *is principal.*

Proof. Since m is coprime to h_K, there exists integers $u, v \in \mathbb{Z}$ such that $um + vh_K = 1$. We get

$$[\mathfrak{a}] = [\mathfrak{a}]^{um+vh_K} = ([\mathfrak{a}]^m)^u([\mathfrak{a}]^{h_K})^v = 1,$$

where $[\mathfrak{a}]^m = 1$ by hypothesis and $[\mathfrak{a}]^{h_K} = 1$ by Corollary 4.4.5. This says exactly that $\mathfrak{a}$ is principal. □

Theorem 3.8.8 is now a consequence of

Corollary 4.4.7. *If* K *is a number field of degree* $n \geq 2$ *and discriminant* $\operatorname{disc} K$*, then*

$$|\operatorname{disc} K| \geq \frac{4}{e^3}\left(\frac{\pi e^{3/2}}{4}\right)^n > 1.$$

Proof. Since every integral ideal has norm at least one, we have $M_K \geq 1$. We can rewrite this as

$$|\operatorname{disc} K| \geq \frac{n^{2n}}{(n!)^2}\left(\frac{\pi}{4}\right)^{2r_2} \geq \frac{n^{2n}}{(n!)^2}\left(\frac{\pi}{4}\right)^{n}.$$

Let $u_n = \frac{n^{2n}}{(n!)^2}\left(\frac{\pi}{4}\right)^n$. For all $n \geq 2$ we have

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{\pi}{4}\left(1+\frac{1}{n}\right)^{2n} = \frac{\pi}{4}\exp\left(2n\log\Big(1+\frac{1}{n}\Big)\right) \geq \frac{\pi}{4}\exp\left(2n\Big(\frac{1}{n}-\frac{1}{2n^2}\Big)\right)\\ &= \frac{\pi}{4}\exp\Big(2-\frac{1}{n}\Big) \geq \frac{\pi}{4}\exp\Big(\frac{3}{2}\Big).\end{aligned}$$

Since $u_2 = \pi^2/4$, we obtain the result. □

For examples of computations of class groups, see Section 7.3.

4.5 Applications: Diophantine equations

4.5.1 Sums of two squares

The problem in this section is to determine the integers that are sums of two squares. In other words, for each integer $n \in \mathbb{Z}$ we want to determine whether the equation

$$x^2 + y^2 = n, \quad x, y \in \mathbb{Z} \tag{4.1}$$

has a solution. An obvious necessary condition is that $n \geq 0$. Moreover, since Equation 4.1 clearly has a solution for $n = 0$, we can assume that $n > 0$. To study this equation, we remark that we can factor it as

$$n = x^2 + y^2 = (x+yi)(x-yi) = N_{\mathbb{Q}}^{\mathbb{Q}(i)}(x+yi), \quad x, y \in \mathbb{Z}.$$

Since the ring of integers of $K = \mathbb{Q}(i)$ is $\mathbb{Z}_K = \mathbb{Z}[i]$, we see that Equation 4.1 is actually a special case of a *norm equation*:

$$N_{\mathbb{Q}}^{K}(z) = n, \quad z \in \mathbb{Z}_K. \tag{4.2}$$

What we are going to see on this particular example is a general method to solve norm equations, although we may need to adapt it to the particular situation.

Note that a nice consequence of Equation 4.2 is that the set of solutions is multiplicative: a product of solutions is again a solution. This was not obvious from Equation 4.1.

The first step is to find which positive integers n are the norm of an integral ideal in $\mathbb{Z}_K$.

Lemma 4.5.1. *Let $n \in \mathbb{Z}_{>0}$, and let $n = \prod_i p_i^{a_i}$ be its factorisation into distinct primes. Then n is the norm of an integral ideal in $\mathbb{Z}[i]$ if and only if for every p_i that is inert in K, the exponent a_i is even.*

Proof. Let $\mathfrak{a} = \prod_j \mathfrak{q}_j^{b_j}$ be an integral ideal of $\mathbb{Z}_K$, and for all j let q_j be the prime below $\mathfrak{q}_j$ and f_j be the inertial degree of $\mathfrak{q}_j$, i.e. $f_j = 2$ if q_j is inert and $f_j = 1$ otherwise. Then we have

$$N(\mathfrak{a}) = \prod_j q_j^{f_j b_j},$$

so the condition of the lemma is necessary.

Let us prove that the condition is sufficient. By multiplicativity of the norm, it is enough to prove it for n a prime power, say $n = p^a$. If p is not inert then there is a prime $\mathfrak{p}$ above p of inertial degree 1, and hence of norm p, so that $N(\mathfrak{p}^a) = p^a$. If p is inert, the condition says that $a = 2b$ is even, and $N\big((p)^b\big) = p^a$. □

Theorem 4.5.2. *An integer $n > 0$ is a sum of two squares if and only if for every prime p dividing n and congruent to 3 modulo 4, the exponent of p in the factorisation of n is even.*

Proof. First, note that a prime number p is inert in K if and only if $\left(\frac{-1}{p}\right) = -1$, if and only if $p \equiv 3 \bmod 4$.

By Lemma 4.5.1, the condition is necessary. Conversely, if n satisfies the condition, then by Lemma 4.5.1 there exists an integral ideal $\mathfrak{a}$ such that $N(\mathfrak{a}) = n$. But we have seen that $h_K = 1$, so $\mathfrak{a}$ is principal: let z be a generator of $\mathfrak{a}$, so that $z \in \mathbb{Z}_K$. Then $n = N(\mathfrak{a}) = N(z\mathbb{Z}_K) = |N_{\mathbb{Q}}^K(z)| = N_{\mathbb{Q}}^K(z)$ is a sum of two squares. □

4.5.2 Another norm equation

Proposition 4.5.3. *The integers of the form $x^2 + xy + 5y^2$ are exactly the positive integers such that for every prime $p \mid n$ such that $p = 2$ or that -19 is not a square mod p, the exponent of p in the factorisation of n is even.*

Proof. The statement suggests to look at $K = \mathbb{Q}(\sqrt{-19})$. Since -19 is squarefree and $-19 \equiv 1 \bmod 4$, we have $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ with $\alpha = \frac{1+\sqrt{-19}}{2}$, and $\operatorname{disc} K = -19$. The norm of a generic element $z = x + y\alpha$ is

$$N_{\mathbb{Q}}^K(z) = (x + \tfrac{y}{2})^2 + 19(\tfrac{y}{2})^2 = x^2 + xy + 5y^2.$$

So the problem again reduces to a norm equation.

As before, a positive integer n is the norm of an integral ideal of $\mathbb{Z}_K$ if and only if every prime that is inert appears with an even exponent.

In order to apply the same method as before, we need to compute the class group of K. Since the signature of K is $(0, 1)$, the Minkowski bound is $M_K = \frac{2}{\pi}\sqrt{19} \approx 2.77 < 3$, so the class group of K is generated by the classes of ideals above 2. Since $-19 \equiv 5 \bmod 19$, the prime 2 is inert, so the unique ideal above 2 is (2) which is principal, so $h_K = 1$.

As before, since $\mathbb{Z}_K$ is a PID, a positive integer is the norm of an integral ideal of K if and only if it is the norm of an element of $\mathbb{Z}_K$.

Since 2 is inert, the condition on p in the Proposition is indeed equivalent to p being inert. □

Remark 4.5.4. The ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a PID, but one can prove that it is not Euclidean![1]

4.5.3 A norm equation with a nontrivial class group

Let us see what happens when the class group is not trivial. Let us take $K = \mathbb{Q}(\sqrt{-23})$. We know from example 7.3.4 that $\mathbb{Z}_K = \mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$, that the norm of $x + y\frac{1+\sqrt{-19}}{2}$ is $x^2 + xy + 6y^2$, that 2 splits in K, say $(2) = \mathfrak{p}_2\mathfrak{p}_2'$, and that $\mathrm{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$ is gnerated by the class of $\mathfrak{p}_2$.

Because $\mathrm{Cl}(K)$ is not trivial, discussing which $n \in \mathbb{N}$ are of the form $x^2 + xy + 6y^2$ is much more difficult in general, so we are going to restrict ourselves to prime powers. So lets us fix a prime $p \in \mathbb{N}$, and study the set of $n \in \mathbb{Z}_{\geqslant 0}$ such that p^n is of the form $x^2 + xy + 6y^2$.

This is the case if and only if there exists an elemnt $\alpha \in \mathbb{Z}_K$ of norm p^n, and the ideal (α) is then an ideal of $\mathbb{Z}_K$ of norm p^n. Conversely, if there exits

[1] To see this, assume on the contrary that d is a Euclidean function, and let $\alpha \in \mathbb{Z}_K \setminus \mathbb{Z}_K^\times$ minimizing d. Then every element of $\mathbb{Z}_K/\alpha\mathbb{Z}_K$ can be represented by an element of $\mathbb{Z}_K^\times \cup \{0\}$, whence $|N_{\mathbb{Q}}^K(\alpha)| = \#(\mathbb{Z}_K/\alpha\mathbb{Z}_K) \leqslant 1 + \#\mathbb{Z}_K^\times = 3$. But that is not possible since 2 and 3 are both inert in K.

a *principal* ideal of $\mathbb{Z}_K$ of norm p^n, then there exists an $\alpha \in \mathbb{Z}_K$ of norm $\pm p^n$; but $N_{\mathbb{Q}}^K(\alpha) > 0$ for all $\alpha \in K^\times$, so the equation $x^2 + xy + 6y^2 = p^n$ has then a solution. To sum up,

$x^2+xy+6y^2 = p^n$ has a solution $\iff \exists$ ideal $\mathfrak{a} \subset \mathbb{Z}_K$ principal and of norm p^n.

Note that the condition $N(\mathfrak{a}) = p^n$ implies that the prime ideals dividing $\mathfrak{a}$ all lie above p. We now distinguish three cases.

- If p is inert in K (example : $p = 5$), then the only prime above p is $\mathfrak{p} = p\mathbb{Z}_K$, which has norm p^2. So there exists an ideal of norm p^n if and only if n is even; besides, this ideal, if it exists, is always principal since $\mathfrak{p} = (p)$ is principal. As a conculsion, p^n is of the form $x^2 + xy + 6y^2$ if and only if n is even.

- If p splits in K, say $p\mathbb{Z}_K = \mathfrak{p}\mathfrak{p}'$, then we have $N(\mathfrak{p}) = N(\mathfrak{p}') = p$, so the primes of norm p^n are exactly the $\mathfrak{p}^a\mathfrak{p}'^b$, where a and b are nonnegative integers such that $a + b = n$. We now distinguish two subcases:

  - If $\mathfrak{p}$ is principal (example: $p = 59$), then so is $\mathfrak{p}'$ since $[\mathfrak{p}'] = [\mathfrak{p}]^{-1}$, so $\mathfrak{p}^a\mathfrak{p}'^b$ is principal for all a and b. As a result, p^n is of the form $x^2 + xy + 6y^2$ for all n.
  - If $\mathfrak{p}$ is not principal (example: $p = 2$), then neither is $\mathfrak{p}'$ since $[\mathfrak{p}'] = [\mathfrak{p}]^{-1}$. Since $\mathrm{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$, $[\mathfrak{p}]$ and $[\mathfrak{p}']$ are inverse classes of order 3 in $\mathrm{Cl}(K)$. Therefore, $\mathfrak{p}^a\mathfrak{p}'^b$ is principal if and only if $a \equiv b \bmod 3$. As a result, p^n is of the form $x^2 + xy + 6y^2$ if and only if the equation
    $$\begin{cases} a + b = n \\ a \equiv b \bmod 3 \end{cases}$$
    has a solution with $a, b \in \mathbb{Z}_{\geqslant 0}$. Now, if $n = 1$, this equation clearly has no solution, whereas if n is even we can take $a = b = n/2$, and finally if n is odd and $\geqslant 3$, we can write $n = 2m + 3$ and take $a = m + 3$ and $b = m$. As a conculsion, p^n is of the form $x^2 + xy + 6y^2$ if and only if $n \neq 1$.

- If p is ramifies in K, say $p\mathbb{Z}_K = \mathfrak{p}^2$, then $p \mid \operatorname{disc} K$ so $p = 23$. There exits an element of norm 23 in $\mathbb{Z}_K$, namely $\sqrt{-23}$, and the ideal generated by this element, which has norm 23, can only be $\mathfrak{p}$. So $\mathfrak{p}$ is principal. Next, an ideal of norm p^n can only be $\mathfrak{p}^n$, which is principal for all n. As a conculsion, 23^n is of the form $x^2 + xy + 6y^2$ for all n.

4.5.4 Mordell equations

A *Mordell equation* is a Diophantine equation of the form

$$y^2 = x^3 + k, \quad x, y \in \mathbb{Z},$$

for some fixed $k \in \mathbb{Z}$. Our plan to study these equations is to factor them in the form

$$x^3 = (y - \sqrt{k})(y + \sqrt{k}),$$

and then hope that the factors on the right hand side must be cubes.

Lemma 4.5.5. *If a and b are coprime integers, and ab is an n-th power, then a and b are both of the form $\pm x^n$.*

Proof. Up to sign, the factorisation of ab is the product of the factorisations of a and b. Since ab is an n-th power, the exponent of every prime is a multiple of n. Since the sets of primes dividing a and b are disjoint, the exponents in their respective factorisation are multiples of n, so a and b are of the form $\pm x^n$. □

Example 4.5.6. Consider the Mordell equation

$$y^2 = x^3 + 16, \quad x, y \in \mathbb{Z}.$$

There are obvious solutions $(0, \pm 4)$. Are there any other ones?

We factor the equation as

$$x^2 = (y - 4)(y + 4).$$

If y is odd, then $a = y-4$ and $b = y+4$ are coprime: any common divisor would have to divide $b - a = 8$, but a and b are odd. By Lemma 4.5.5, a and b are cubes, but they differ by 8. Since cubes get further and further apart, there are no odd cubes that differ by 8. To see this, write the first few odd cubes:

$$\ldots, -27, -1, 1, 27, \ldots$$

and note that after these, the differences are larger than 8.

If y is even, then x is even, so $y^2 = x^3 + 16$ is divisible by 8 and y is divisible by 4: $y = 4y'$. Then $x^3 = 16y'^2 - 16$ is divisible by 16, so x is also a multiple of 4: $x = 4x'$. The equation now becomes

$$y'^2 = 4x'^3 + 1,$$

so y' is odd: $y' = 2y'' + 1$. Simplifying the equation we get

$$x'^3 = y''(y'' + 1).$$

Since y'' and $y'' + 1$ are coprime, they are cubes. Since they differ by 1, we must have $y'' = 0$ or $y'' = -1$. Again, to see this you can write down the small cubes

$$\ldots, -8, -1, 0, 1, 8, \ldots$$

and note that after these, the differences are larger than 1.

This gives $y' = \pm 1$ and hence $y = \pm 4$, so the obvious solutions are the only ones.

Proposition 4.5.7. *Let K be a number field and let $n \geq 1$ be coprime to the class number of K. Let $a, b \in \mathbb{Z}_K$ be such that the ideals (a) and (b) are coprime and such that ab is an n-th power. Then $a = ux^n$ and $b = vy^n$ where $u, v \in \mathbb{Z}_K^\times$ are units and $x, y \in \mathbb{Z}_K$.*

Proof. Since the ideal (ab) is the n-th power of an ideal and (a) is coprime to (b), by the factorisation theorem 3.5.1, (a) and (b) are n-th powers of ideals: $(a) = \mathfrak{a}^n$ and $(b) = \mathfrak{b}^n$. Now n is coprime to the class number, and the n-th power of $\mathfrak{a}$ and $\mathfrak{b}$ are principal, so $\mathfrak{a}$ and $\mathfrak{b}$ are principal: $\mathfrak{a} = (x)$ and $\mathfrak{b} = (y)$. This gives $(a) = (x^n)$ and $(b) = (y^n)$, so that a/x^n and b/y^n are units in $\mathbb{Z}_K$. □

Example 4.5.8. Consider the Mordell equation

$$y^2 = x^3 - 2, \quad x, y \in \mathbb{Z}.$$

Let $K = \mathbb{Q}(\sqrt{-2})$. Since $-2 \equiv 2 \bmod 4$, we have $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-2}]$ and the discriminant of K is $\operatorname{disc} K = -8$. The Minkowski bound is

$$M_K = \frac{4}{\pi}\frac{2!}{2^2}\sqrt{|-8|} \approx 1.8 < 2,$$

so the class group of K is trivial.

Suppose (x, y) is a solution. We have

$$x^3 = (y - \sqrt{-2})(y + \sqrt{-2}).$$

We want to prove that the ideals $(y - \sqrt{-2})\mathbb{Z}_K$ and $(y + \sqrt{-2})\mathbb{Z}_K$ are coprime. Let $\mathfrak{p}$ be a prime dividing both these ideals. Then both $y - \sqrt{-2}$

and $y + \sqrt{-2}$ belong to $\mathfrak{p}$, so their difference also does: $\mathfrak{p}$ divides $2\sqrt{-2}$. Since $N_{\mathbb{Q}}^K(2\sqrt{-2}) = 8$ is a power of 2 and 2 decomposes as $\mathfrak{p}_2^2$ in $\mathbb{Z}_K$, the prime $\mathfrak{p}$ must necessarily be $\mathfrak{p}_2$.

We compute that $\mathfrak{p}_2 = (2, \sqrt{-2})$. In particular, $\sqrt{-2} \in \mathfrak{p}_2$, so we have $y = (y + \sqrt{-2}) - \sqrt{-2} \in \mathfrak{p}_2$. But $y \in \mathbb{Z}$ and $\mathfrak{p}_2 \cap \mathbb{Z} = 2\mathbb{Z}$ (this is just saying that the prime $\mathfrak{p}_2$ lies above 2), so this implies that y is even. The equation then implies that x is also even, but then reducing modulo 4 gives a contradiction. So $\mathfrak{p}$ does not exist, and the ideals $(y - \sqrt{-2})\mathbb{Z}_K$ and $(y + \sqrt{-2})\mathbb{Z}_K$ are indeed coprime.

We have $\mathbb{Z}_K^\times = \{\pm 1\}$ (we will see why in the next chapter), so every element of $\mathbb{Z}_K^\times$ is a cube. By Proposition 4.5.7, $y + \sqrt{-2}$ is thus a cube, say $(a + \sqrt{-2}b)^3$ with $a, b \in \mathbb{Z}$. We have

$$(a + b\sqrt{-2})^3 = (a^3 - 6ab^2) + (3a^2b - 2b^3)\sqrt{-2},$$

giving the equations

$$a(a^2 - 6b^2) = y \text{ and } b(3a^2 - 2b^2) = 1.$$

By the second equation, we must have $b = \pm 1$.

- if $b = 1$, then $3a^2 - 2b^2 = 1$ so $3a^2 = 3$ and $a = \pm 1$, giving the solutions $(x, y) = (3, \pm 5)$.
- if $b = -1$, then $3a^2 - 2b^2 = -1$, so $3a^2 = 1$, which is impossible.

In conclusion, the solutions of the equation are $(x, y) = (3, \pm 5)$.

4.5.5 The regular case of Fermat's last theorem

Using Proposition 4.5.7, Kummer was able to fix Lamé's approach to Fermat's equation. Namely he proved that if $p \geq 3$ is *regular*, that is to say that it does not divide the class number of $\mathbb{Q}(\zeta_p)$ (which happens for all $p \leq 100$ except 37, 59 and 67), then $x^p + y^p = z^p$ has non nontrivial solutions.

Chapter 5

Units

As we saw in the last two chapters, using ideals we can recover a good factorisation theory in number fields. But by going from elements to ideals, we loose something: associate elements of $\mathbb{Z}_K$ generate the same ideal. This motivates the study of the unit group $\mathbb{Z}_K^\times$.

5.1 Units in a domain

In this section, we fix a commutative domain $\mathcal{R}$.

Definition 5.1.1. Let $u \in \mathcal{R}$. We say that u is a *unit* in $\mathcal{R}$ if it is invertible in $\mathcal{R}$, that is to say if there exists $v \in \mathcal{R}$ such that $uv = 1$.

Such a v is then necessarily unique[1], and is denoted by $v = u^{-1}$.

The set of units of $\mathcal{R}$ is denoted by $\mathcal{R}^\times$. It is an Abelian group under multiplication.

Example 5.1.2.

- For $\mathcal{R} = \mathbb{Z}$, we have $\mathcal{R}^\times = \{\pm 1\}$; this explains the term *unit*.
- If $\mathcal{R}$ is actually a field, then $\mathcal{R}^\times = \mathcal{R} \setminus \{0\}$.
- If $\mathcal{R} = k[X]$ is a polynomial ring over a field k, then $\mathcal{R}^\times = k^\times = k \setminus \{0\}$ consists of the nonzero constant polynomials.

[1] Indeed, if we have $uv = uv' = 1$, then multiplying the identity $uv = 1$ by v' yields $v = v'$.

Proposition 5.1.3. *Let $a, b \in \mathcal{R}$. Then the ideals $a\mathcal{R}$ and $b\mathcal{R}$ agree if and only if a and b are* associate *in ring, that is to say if and only if there exists a unit $u \in \mathcal{R}^\times$ such that $b = au$.*

In practicular, $a\mathcal{R} = \mathcal{R}$ if and only if a is a unit.

Proof. First, note that

$$b\mathcal{R} \subseteq a\mathcal{R} \iff b \in a\mathcal{R} \iff \exists u \in \mathcal{R} \colon b = au.$$

So if we have $a\mathcal{R} = b\mathcal{R}$, then there exist $u, v \in \mathcal{R}$ such that $b = au$ and $a = bv$, whence $a(1 - uv) = 0$. If $a \neq 0$, this implies that $uv = 1$ since $\mathcal{R}$ is a domain, so that u and v are units in $\mathcal{R}$; and if $a = 0$, then $b \in b\mathcal{R} = a\mathcal{R} = \{0\}$ so $b = 0$, and a and b are then trivially associate.

Conversely, if $b = au$ with $u \in \mathcal{R}^\times$, then $b\mathcal{R} \subseteq a\mathcal{R}$; but we also have $a = bu^{-1}$ with $u^{-1} \in \mathcal{R}$, so $a\mathcal{R} \subseteq b\mathcal{R}$. □

5.2 Units in $\mathbb{Z}_K$

Definition 5.2.1. Let K be a number field. A *unit* in K is an element of $\mathbb{Z}_K^\times$.

Note that we are slightly twisting the definition of unit here: in principle, we should talk about units in $\mathbb{Z}_K$, not in K, but such is the terminology!

Proposition 5.2.2. *Let K be a number field, and let $\alpha \in K$. The following are equivalent:*

(i) $\alpha \in \mathbb{Z}_K^\times$;

(ii) $(\alpha) = \mathbb{Z}_K$;

(iii) $\alpha \in \mathbb{Z}_K$ *and* $N_{\mathbb{Q}}^K(\alpha) = \pm 1$;

(iv) $\alpha \in \mathbb{Z}_K$ *and the constant term of the minimal polynomial of α is ± 1;*

(v) $\alpha \in \mathbb{Z}_K$ *and* $\alpha^{-1} \in \mathbb{Z}[\alpha]$.

Proof.

- (i) ⇒ (ii). This is a special case of proposition 5.1.3.

- (ii) $\Rightarrow$ (iii). Since $\alpha\mathbb{Z}_K = \mathbb{Z}_K$, we have $\alpha \in \mathbb{Z}_K$. By taking norms we have $|N_{\mathbb{Q}}^K(\alpha)| = N((\alpha)) = 1$.
- (iii) $\Rightarrow$ (iv). $N_{\mathbb{Q}}^K(\alpha) = \pm 1$ is a power of this constant term, and both are integers, so the constant term is also ± 1.
- (iv) $\Rightarrow$ (v). Let $\sum_{i=0}^n a_i x^i$ be the minimal polynomial of α, where $a_0 = \pm 1$ and $a_n = 1$. Write it as

$$\alpha \left(\sum_{i=1}^{n} a_i \alpha^{i-1} \right) = - \pm 1.$$

  This proves that

$$\alpha^{-1} = - \pm \sum_{i=1}^{n} a_i \alpha^{i-1} \in \mathbb{Z}[\alpha],$$

- (v) $\Rightarrow$ (i). Since $\alpha \in \mathbb{Z}_K$ and $\alpha^{-1} \in \mathbb{Z}[\alpha] \subset \mathbb{Z}_K$, we have $\alpha \in \mathbb{Z}_K^\times$.

□

Example 5.2.3.

1. $\phi = \frac{1+\sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5})$ is a unit of norm -1.
2. $\alpha = \frac{3+4i}{5} \in \mathbb{Q}(i)$ has norm 1 but is not a unit, since it is not an algebraic integer.

Corollary 5.2.4. *For all number fields $K \subset L$, we have*

$$\mathbb{Z}_K^\times = K \cap \mathbb{Z}_L^\times.$$

Proof. Since $\mathbb{Z}_K = K \cap \mathbb{Z}_L$, this follows from (iv) of Proposition 5.2.2. □

Corollary 5.2.5. *Let $\mathcal{O} \subset \mathbb{Z}_K$ be an order. Then*

$$\mathcal{O}^\times = \mathcal{O} \cap \mathbb{Z}_K^\times.$$

Proof. The inclusion $\subset$ is clear. The opposite inclusion follows from (v) of Proposition 5.2.2. □

Proposition 5.2.6. *Let K be a number field, let $\mathcal{O}$ be an order in K and let f be the index of $\mathcal{O}$ in $\mathbb{Z}_K$. Then $\mathcal{O}^\times \subset \mathbb{Z}_K^\times$ is a subgroup of finite index, and*

$$[\mathbb{Z}_K^\times \colon \mathcal{O}^\times] \leq \#(\mathbb{Z}_K/f\mathbb{Z}_K)^\times.$$

Proof. Let $\phi \colon \mathbb{Z}_K^\times \to (\mathbb{Z}_K/f\mathbb{Z}_K)^\times$ be the reduction modulo f map. Let $u \in \ker\phi$. Then $u - 1 \in f\mathbb{Z}_K \subset \mathcal{O}$, so $u \in \mathcal{O}$. This proves that

$$\ker\phi \subset \mathcal{O}^\times \subset \mathbb{Z}_K^\times,$$

so that

$$[\mathbb{Z}_K^\times : \mathcal{O}^\times] \leq [\mathbb{Z}_K^\times \colon \ker\phi] \leq \#(\mathbb{Z}_K/f\mathbb{Z}_K)^\times,$$

where the last inequality holds because the induced map

$$\mathbb{Z}_K^\times/\ker\phi \to (\mathbb{Z}_K/f\mathbb{Z}_K)^\times$$

is injective. □

5.3 Roots of unity

We will start by studying the simplest units: the ones that have finite order, that is, the roots of unity.

5.3.1 Roots of unity under complex embeddings

Definition 5.3.1. Let K be a number field of signature (r_1, r_2). We define the *Minkowski space* to be $K_\mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Let $\sigma_1, \ldots, \sigma_{r_1}$ be the real embeddings of K, and let $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ be representatives of the nonreal embeddings of K up to complex conjugation. The *Minkowski embedding*

$$\Sigma \colon K \hookrightarrow K_\mathbb{R}$$

is defined by

$$\Sigma(x) = \big(\sigma_i(x)\big)_{i=1}^{r_1+r_2}.$$

Example 5.3.2.

1. Let $K = \mathbb{Q}(\alpha)$ where $\alpha^2 = 2$. The number field K has signature $(2, 0)$, and the two real embeddings are $\sigma_1 \colon \alpha \mapsto \sqrt{2}$ and $\sigma_2 \colon \alpha \mapsto -\sqrt{2}$. So the Minkowski embedding is

$$\Sigma \colon x + y\alpha \mapsto (x + y\sqrt{2}, x - y\sqrt{2}).$$

2. Let $K = \mathbb{Q}(\beta)$ where $\beta^3 = 2$. The number field K has signature $(1, 1)$, and we can choose the embeddings to be $\sigma_1 \colon \beta \mapsto 2^{1/3}$ and $\sigma_2 \colon \beta \mapsto 2^{1/3}j$ (where $j = \exp(2i\pi/3)$). The third complex embedding is $\sigma_3 = \overline{\sigma_2} \colon \beta \mapsto 2^{1/3}j^2$. We obtain the Minkowski embedding

$$\Sigma \colon x + y\beta + z\beta^2 \mapsto (x + y2^{1/3} + z2^{2/3}, x + y2^{1/3}j + z2^{2/3}j^2).$$

Proposition 5.3.3. *Let K be a number field, and let $B \subset K_{\mathbb{R}}$ be a bounded subset. Then $\Sigma(\mathbb{Z}_K) \cap B$ is finite.*

Proof. Let $x \in \mathbb{Z}_K$ be such that $\Sigma(x) \in B$, so that we have a bound on every complex embedding of x, say $|\sigma(x)| \leq R$ for all $\sigma \colon K \hookrightarrow \mathbb{C}$ (where R depends only on B, not on x). Let m_x be the minimal polynomial of x, whose roots are the $\sigma(x)$ by Corollary 1.3.8. By the expression of the coefficients in terms of the roots, we obtain a bound on the coefficients of m_x ($|a_i| \leq \binom{n}{i} R^{n-i}$ where $n = [K : \mathbb{Q}]$).

So there are finitely many possible characteristic polynomials for elements of $\Sigma(\mathbb{Z}_K) \cap B$, and each of them has at most $[K : \mathbb{Q}]$ roots. So the set $\Sigma(\mathbb{Z}_K) \cap B$ is finite. □

For K be a number field, we write W_K for the group of roots of unity in K (other notations exist, such as μ_K or $\mu_\infty(K)$).

Remark 5.3.4. We have $W_K \subset \mathbb{Z}_K^\times$. Indeed, let $\alpha \in W_K$. Then α has finite order, say n, so α is a root of $x^n - 1 \in \mathbb{Z}[x]$, so $\alpha \in \mathbb{Z}_K$. Moreover the inverse of α is also a root of unity, so $\alpha \in \mathbb{Z}_K^\times$.

Theorem 5.3.5. *Let K be a number field. Then W_K is a finite cyclic group. For all nonzero $x \in \mathbb{Z}_K$, the following are equivalent:*

(i) $x \in W_K$;

(ii) $|\sigma(x)| = 1$ *for every complex embedding σ of K;*

(iii) $|\sigma(x)| \leq 1$ *for every complex embedding σ of K.*

Proof. Let W_2 be the set of nonzero elements of $\mathbb{Z}_K$ satsifying (ii), and let W_3 be those satisfying (iii). We clearly have $W_K \subset W_2 \subset W_3$ since every complex embedding of a root of unity is of the form $\exp(ai\pi/b)$ for some $a, b \in \mathbb{Z}$. By Proposition 5.3.3, W_3 is finite, so W_K is also finite. Since every finite subgroup of the nonzero elements of a field is cyclic, W_K is cyclic.

To show that these sets are equal, let $x \in W_3$. For all integers $n \geq 0$, $x^n \in W_3$, but W_3 is finite, so some of these powers coincide, say $x^n = x^m$ with $n < m$. This gives $x^n(1 - x^{m-n})$, but $x \neq 0$ so $x^{m-n} = 1$, and x is a root of unity. □

Example 5.3.6.

- Let $K = \mathbb{Q}(i, \sqrt{2})$, and let $x = \frac{1+i}{\sqrt{2}}$. Then $x^2 = i$, so $x^4 + 1 = 0$, which proves that x is an algebraic integer. Besides, $|\sigma(x)| = 1$ for every complex embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, so x is a root of unity by theorem 5.3.5. In fact, since $x^4 = -1$, we have $x^8 = 1$, so x is an 8-th root of unity. Since $x^4 \neq 1$, it is actually a *primitive* 8-th root of unity.
- Let $x = \frac{3+4i}{5} \in \mathbb{Q}(i)$. Then $|\sigma(x)| = 1$ for all embeddings of $\mathbb{Q}(i)$ into $\mathbb{C}$, but x is not a root of unity since it is not an algebraic integer.

5.3.2 Bounding the size of W_K

Remark 5.3.7. If K is a number field and $\zeta \in K$ is a primitive n-th root of unity, then the subfield $\mathbb{Q}(\zeta)$ of K is isomorphic to the n-th cyclotomic field.

Example 5.3.8. Let $K = \mathbb{Q}(\alpha)$ where α is a root of the irreducible polynomial $P = x^4 - x + 1$. We compute that $\operatorname{disc}(\mathbb{Z}[\alpha]) = \operatorname{disc}(P) = 229$ is prime, so $\mathbb{Z}[\alpha]$ is maximal and the discriminant of K is $\operatorname{disc} K = 229$. Suppose that K contains a p-th root of unity for some odd prime p. Then K contains the p-th cyclotomic field, which has degree $p-1$ over $\mathbb{Q}$, so $p-1 \leq 4$ and $p \leq 5$. But K is unramified at 3 and 5, so it cannot contain the corresponding cyclotomic fields by Proposition 3.8.1. Similarly, K cannot contain a 4-th root of unity since it is unramified at 2. So $W_K = \{\pm 1\}$.

Proposition 5.3.9. *If K is a number field that admits a real embedding, then $W_K = \{\pm 1\}$.*

Proof. Let $\sigma\colon K \hookrightarrow \mathbb{R}$ be such an embedding, and let ζ be an n-th root of unity in K. Then $\sigma(\zeta)$ is an n-th root of unity in $\mathbb{R}$, so $n \mid 2$. □

Example 5.3.10. If K is an odd degree number field, then $W_K = \{\pm 1\}$.

Proposition 5.3.11. *Let K be a number field, let $\zeta \in K$ be a primitive n-th root of unity and let $\mathfrak{p}$ be a prime ideal such that n is coprime to $\mathfrak{p}$. Then $n \mid N(\mathfrak{p}) - 1$.*

Proof. Let g be the reduction modulo $\mathfrak{p}$ of ζ. If g does not have order n, then $g^d = 1$ for some strict divisor d of n. After replacing d by a multiple if necessary, we may assume that $n = qd$ with q prime. This means that $\zeta^d - 1 \in \mathfrak{p}$, but ζ^d is a q-th root of unity so the norm of $\zeta^d - 1$ is a power of q, and $q \mid n$ is coprime to $N(\mathfrak{p})$, which is impossible.

So there is an element of order n in the group $(\mathbb{Z}_K/\mathfrak{p})^\times$, which has cardinality $N(\mathfrak{p}) - 1$. □

Example 5.3.12. If 2 is unramified in K and there is a prime above 2 of inertial degree 1, then $W_K = \{\pm 1\}$. Indeed, by Proposition 5.3.11 the only possible roots of unity would be ζ_{2^k}, but if $k \geq 2$ then 2 is ramified in $\mathbb{Q}(\zeta_{2^k})$, so this cyclotomic field cannot be contained in K by Proposition 3.8.1.

5.4 Dirichlet's theorem

We will now describe the structure of the full unit group.

Definition 5.4.1. Let K be a number field of signature (r_1, r_2). Let $\sigma_1, \ldots, \sigma_{r_1}$ be the real embeddings of K, and let $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ be representatives of the nonreal embeddings of K up to complex conjugation. For all $1 \leq i \leq r_1 + r_2$, let $n_i = 1$ if σ_i is real and $n_i = 2$ otherwise. The *logarithmic embedding*

$$\mathcal{L} \colon \mathbb{Z}_K^\times \longrightarrow \mathbb{R}^{r_1+r_2}$$

is defined by

$$\mathcal{L}(x) = \big(n_i \log |\sigma_i(x)|\big)_{i=1}^{r_1+r_2}.$$

Theorem 5.4.2 (Dirichlet)**.** *Let K be a number field of signature (r_1, r_2). Let $V \subset \mathbb{R}^{r_1+r_2}$ be the subspace of vectors whose coordinates sum to zero. Then $\mathcal{L}(\mathbb{Z}_K^\times)$ is a lattice in V. As an abstract abelian group, we have*

$$\mathbb{Z}_K^\times \cong W_K \times \mathbb{Z}^{r_1+r_2-1}.$$

Recall that every finitely generated abelian group is isomorphic to $T \times \mathbb{Z}^r$, where T is a finite group and $r \geq 0$ is an integer called the *rank*. The second part of the theorem says that the rank of the unit group of K is $r_1 + r_2 - 1$.

We will also prove this theorem after introducing "geometry of numbers" techniques.

Definition 5.4.3. Let K be a number field of signature (r_1, r_2), and let $r = r_1 + r_2 - 1$. A set of *fundamental units* of K is a $\mathbb{Z}$-basis for the unit group $\mathbb{Z}_K^\times / W_K$, that is, a set of units $\varepsilon_1, \dots, \varepsilon_r \in \mathbb{Z}_K^\times$ such that $(\mathcal{L}(\varepsilon_1), \dots, \mathcal{L}(\varepsilon_r))$ is a $\mathbb{Z}$-basis of the lattice $\mathcal{L}(\mathbb{Z}_K^\times)$. Let $M \in \mathrm{Mat}_{r+1,r}(\mathbb{R})$ be the matrix with columns $\mathcal{L}(\varepsilon_1), \dots, \mathcal{L}(\varepsilon_r)$. Let M' be a matrix obtained by deleting a row of M, which is an $r \times r$ matrix. The *regulator of* K is

$$\mathrm{Reg}_K = |\det M'|.$$

Proposition 5.4.4. *The regulator does not depend on the choice of a set of fundamental units, on the ordering of the complex embeddings, or on the choice of the deleted row.*

Proof. If we change the set of fundamental units, this amounts to multiplying M on the right by a matrix $P \in \mathrm{GL}_r(\mathbb{Z})$, and similarly M' becomes $M'P$. Since $\det P = \pm 1$, this does not change the regulator.

If we permute the rows of M', this amounts to multiplying it on the left by a permutation matrix, which has determinant ± 1. So this does not change the regulator.

If we change the deleted row, since the sum of all the rows in M is zero, this amounts to multiplying M' on the left by a matrix

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ -1 & \cdots & -1 & \cdots & -1 & \\ & & & \ddots & & \\ & & & & 1 & \end{pmatrix}.$$

Such a matrix is block upper triangular, and its determinant is -1. So changing the deleted row does not change the determinant. □

Example 5.4.5. Let K be a real quadratic field, which we see as a subfield of $\mathbb{R}$. By Dirichlet's theorem, the units of K have rank 1. Let ε be a fundamental unit of K. After changing ε into $\pm\varepsilon^{\pm 1}$ if necessary, we may assume that $\varepsilon > 1$. Then $\mathrm{Reg}_K = \log \varepsilon$: if σ denotes the other real embedding of K, then $\sigma(\varepsilon) = N_{\mathbb{Q}}^K(\varepsilon)/\varepsilon = \pm\varepsilon^{-1}$, so that

$$M = \begin{pmatrix} \log \varepsilon \\ \log|\pm\varepsilon^{-1}| \end{pmatrix} = \begin{pmatrix} \log \varepsilon \\ -\log \varepsilon \end{pmatrix}.$$

5.5 The case of quadratic fields

Proposition 5.5.1. *Let K be an imaginary quadratic field of discriminant* $\operatorname{disc} K$*. Then the unit group $\mathbb{Z}_K^\times$ is isomorphic to*

- $\{\pm 1\}$ *if* $\operatorname{disc} K \notin \{-3, -4\}$*;*
- $\mathbb{Z}/6\mathbb{Z}$ *if* $\operatorname{disc} K = -3$*;*
- $\mathbb{Z}/4\mathbb{Z}$ *if* $\operatorname{disc} K = -4$*.*

Proof. By Dirichlet's theorem we have $\mathbb{Z}_K^\times = W_K$. If $W_K = \langle \zeta_n \rangle$ with $n \geq 3$, then K contains the n-th cyclotomic field, and since K is quadratic we have $K = \mathbb{Q}(\zeta_n)$. So we need to determine the quadratic cyclotomic fields. The n-th cyclotomic field has degree $\varphi(n)$, and it is easy to see from the formula for $\varphi(n)$ that we have $\varphi(n) = 2$ if and only if $n \in \{3, 4, 6\}$. Finally $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\zeta_6)$ has discriminant -3, and $\mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$ has discriminant -4. □

Proposition 5.5.2. *Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field ($d > 0$ square-free), seen as a subfield of $\mathbb{R}$. Then $\mathbb{Z}_K^\times \cong \{\pm 1\} \times \mathbb{Z}$. Moreover, there exists a fundamental unit $u \in \mathbb{Z}_K^\times$ such that $u > 1$. Write $u = x + y\sqrt{d}$ with x, y integers or half-integers. Then u is characterized among the elements of $\mathbb{Z}_K^\times$ by any of the following properties:*

(i) $u > 1$ *is the smallest as a real number;*

(ii) $x > 0$ *is the smallest possible;*

(iii) $y > 0$ *is the smallest possible (except for $d = 5$ where there are two units with $y = 1/2$ and the fundamental unit is the one having norm -1).*

Proof. Since the signature of K is $(2, 0)$, we have $W_K = \{\pm 1\}$ by Proposition 5.3.9, and $\mathbb{Z}_K^\times \cong \{\pm 1\} \times \mathbb{Z}$ by Dirichlet's theorem 5.4.2. If u is a fundamental unit, then $u, -u, u^{-1}, -u^{-1}$ are fundamental units and among those there is one in the interval $(1, \infty)$ since $u \neq 1$. Since every other unit in $(1, \infty)$ is of the form u^n for some $n > 0$, u is the smallest one as a real number. Let us prove that x and y are positive: we have $\pm u^{\pm 1} = \pm x \pm y\sqrt{d}$, and the choice giving the largest real value is $x, y > 0$. We now want to see that smallest u is equivalent to smallest x or y. Let $s \in \{\pm 1\}$, and consider units $x + y\sqrt{d} > 1$ of norm s, i.e. satisfying

$$x^2 - dy^2 = s, \quad x, y > 0.$$

We have $x = \sqrt{s + dy^2}$ and $y = \sqrt{\frac{x^2-s}{d}}$, so x is an increasing function of y, y is an increasing function of x, and $x + \sqrt{d}y$ is an increasing function of x and of y. So if we sort the units of fixed norm by increasing x, y or $x + \sqrt{d}y$, the resulting ordering is the same.

- If a fundamental unit has norm 1, then every unit has norm 1 and we are done.
- If a fundamental unit u has norm -1, we need to compare it with units of norm 1 the smallest of which is u^2. But we have $(x + y\sqrt{d})^2 = (x^2 + dy^2) + (2xy)\sqrt{d}$ and we have $x^2 + dy^2 > x$ and $2xy > y$, unless $x = 1/2$. In order to prove the proposition, we only have to eliminate the possibility that $x = 1/2$. If that is the case, then $1 - dz^2 = \pm 4$ where $z = 2y \in \mathbb{Z}_{>0}$, so that $dz^2 = 5$ or -3. Since $dz^2 > 0$ we must have $dz^2 = 5$, so that $d = 5$. If the $d = 5$ case, there are two units with $y = 1/2$, and the unit of norm 1 is the square of the unit of norm -1.

□

Example 5.5.3.

1. $\phi = \frac{1+\sqrt{5}}{2}$ is a fundamental unit in $\mathbb{Q}(\sqrt{5})$. By Examples 5.4.5, the regulator is $\mathrm{Reg}_K = \log(\phi) \approx 0.481$.
2. Let $K = \mathbb{Q}(\sqrt{6})$, so that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{6}]$. In order to find the fundamental unit, we look for solutions of

$$x^2 - 6y^2 = \pm 1, \quad x, y \in \mathbb{Z}_{>0}.$$

   We try successive possible values for y:

   - if $y = 1$, then $x^2 = \pm 1 + 6y^2 = \pm 1 + 6 = 5$ or 7, which is impossible.
   - if $y = 2$, then $x^2 = \pm 1 + 6y^2 = \pm 1 + 24 = 23$ or 25, and $25 = 5^2$, so we found the smallest solution $(5, 2)$ and a fundamental unit $5 + 2\sqrt{6}$. The regulator of K is $\log(5 + 2\sqrt{6}) \approx 2.29$.

   We could have enumerated the values of x, but we would have had to try more values. This will be true in general, since $x \approx \sqrt{d}y$.

3. Let $K = \mathbb{Q}(\sqrt{13})$, so that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ with $\alpha = \frac{1+\sqrt{13}}{2}$. So we need to look at solutions of $x^2 - 13y^2 = \pm 1$ with x, y positive half-integers, or equivalently for solutions of

$$X^2 - 13Y^2 = \pm 4, \quad X, Y \in \mathbb{Z}_{>0}$$

by setting $X = 2x$, $Y = 2y$. We try values of Y:

- if $Y = 1$, then $X^2 = \pm 4 + 13Y^2 = \pm 4 + 13 = 9$ or 17, and $9 = 3^2$ so we find the smallest solution $(3, 1)$ and a fundamental unit $\varepsilon = \frac{3+\sqrt{13}}{2}$, which has norm -1. The regulator of K is $\log \varepsilon \approx 1.19$.

We need to be careful that the result $\frac{X+Y\sqrt{d}}{2}$ is an algebraic integer. But since $X^2 - dY^2 = \pm 4$ and d is odd, X and Y have the same parity so this will always work.

4. Fundamental units of real quadratic fields can be very large! For instance:

- In $\mathbb{Q}(\sqrt{19})$, the fundamental unit is $170 + 39\sqrt{19}$;
- In $\mathbb{Q}(\sqrt{94})$, the fundamental unit is $2143295 + 221064\sqrt{94}$;
- In $\mathbb{Q}(\sqrt{9619})$, the fundamental unit is

$$811190220112488603988085333020463275297114310840237706438446585902266575498241529588046630415138220142 90 + 827099472230816363716635228974328535731023047629801451791438952247858704503541263833471709896096965161\sqrt{9619}.$$

Remark 5.5.4. If you know what continued fractions are: it is also possible to find fundamental units of real quadratic fields by computing the continued fraction expansion of $\sqrt{d}$. This leads to an algorithm that is much faster than the one we saw here, but it is outside the scope of this course.

5.6 The case of cyclotomic fields

Proposition 5.6.1. *Let $n \geq 3$ and $K = \mathbb{Q}(\zeta_n)$. Then we have*

- $W_K \cong \mathbb{Z}/(2n)\mathbb{Z}$ *if n is odd;*
- $W_K \cong \mathbb{Z}/n\mathbb{Z}$ *if n is even.*

Proof. Let $m = \#W_K$. Then the m-th cyclotomic field embeds in K, so that $\varphi(m) \leq \varphi(n)$. But $n \mid m$ since K contains the n-th roots of unity, so $\varphi(m) = \varphi(n)$. Moreover, if k is a multiple of n such that $\varphi(n) = \varphi(k)$, then K embeds in $\mathbb{Q}(\zeta_k)$ and by equality if degrees we have $K = \mathbb{Q}(\zeta_k)$ so K contains a primitive k-th root of unity. So m is the largest multiple of n such that $\varphi(m) = \varphi(n)$. But for all primes p, we have $\varphi(pk) = (p-1)\varphi(k)$ if $p \nmid k$ and $\varphi(pk) = p\varphi(k)$ if $p \mid k$. So $m = 2n$ if n is odd and $m = n$ if n is even. □

Remark 5.6.2. Since the signature of $K = \mathbb{Q}(\zeta_n)$ is $(0, \varphi(n)/2)$, the rank of the unit group $\mathbb{Z}_K^\times$ is $\varphi(n)/2 - 1$.

5.7 The Pell–Fermat equation

Let $d > 1$ be a squarefree integer. The *Pell–Fermat equation* is:

$$x^2 - dy^2 = 1, \quad x, y \in \mathbb{Z}. \tag{5.1}$$

We can immediately reinterpret it as follows: let $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, which is an order in $K = \mathbb{Q}(\sqrt{d})$. Then the solutions of Equation (5.1) are the $(x, y) \in \mathbb{Z}^2$ such that $u = x + y\sqrt{d}$ is a solution of

$$N_{\mathbb{Q}}^K(u) = 1, \quad u \in \mathcal{O}^\times. \tag{5.2}$$

By Proposition 5.2.6, $\mathcal{O}^\times$ has finite index in $\mathbb{Z}_K^\times$, and the index of $\mathcal{O}^\times$ in $\mathbb{Z}_K^\times$ is at most $\#(\mathbb{Z}_K/f\mathbb{Z}_K)^\times$, where f is the index of $\mathcal{O}$ in $\mathbb{Z}_K$. If $f = 1$ then $\mathcal{O}^\times = \mathbb{Z}_K^\times$. If $f = 2$, then 2 is unramified in K, so $\mathbb{Z}_K/f\mathbb{Z}_K$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$ or $\mathbb{F}_4$, so $\#(\mathbb{Z}_K/f\mathbb{Z}_K)^\times \leq 3$. Since $\pm 1 \in \mathcal{O}$, we have

$$\mathcal{O}^\times \cong \{\pm 1\} \times \mathbb{Z} \text{ and } [\mathcal{O}^\times : \mathbb{Z}_K^\times] \leq 3.$$

Let $\varepsilon_0 \in \mathbb{Z}_K^\times$ be a fundamental unit for $\mathbb{Z}_K^\times$. Let $n \geq 1$ be the smallest positive integer such that $\varepsilon_0^n \in \mathcal{O}$, and let $\varepsilon_1 = \varepsilon_0^n$. Then $n \leq 3$, and ε_1 is a fundamental unit for $\mathcal{O}^\times$. Let $\varepsilon_2 = \varepsilon_1^2$ if $N_{\mathbb{Q}}^K(\varepsilon_1) = -1$ and $\varepsilon_2 = \varepsilon_1$ otherwise. Then the solutions of Equation (5.2) are the $\pm\varepsilon_2^n$, $n \in \mathbb{Z}$, and the solutions of Equation (5.1) are the $\pm(x, y)$ where $x + y\sqrt{d} = \varepsilon_2^n$, $n \in \mathbb{Z}$.

Example 5.7.1. Consider the Pell–Fermat equation

$$x^2 - 13y^2 = 1, \quad x, y \in \mathbb{Z}.$$

Let $K = \mathbb{Q}(\sqrt{13})$, so that $\mathbb{Z}_K = \mathbb{Z}[\frac{1+\sqrt{13}}{2}]$, and let $\mathcal{O} = \mathbb{Z}[\sqrt{13}]$. We saw in Example 5.5.3 that $\varepsilon_0 = \frac{3+\sqrt{13}}{2}$ is a fundamental unit of $\mathbb{Z}_K^\times$ and has norm -1. Since $\varepsilon_0 \notin \mathcal{O}$ and $\varepsilon_0^2 = \frac{11+3\sqrt{13}}{2} \notin \mathcal{O}$, the unit $\varepsilon_1 = \varepsilon_0^3 = 18 + 5\sqrt{13}$ is a fundamental unit of $\mathcal{O}^\times$. Since $N_{\mathbb{Q}}^K(\varepsilon_0) = -1$, we have $N_{\mathbb{Q}}^K(\varepsilon_1) = -1$, so $\varepsilon_2 = \varepsilon_1^2 = 649 + 180\sqrt{13}$ is a fundamental unit for the norm 1 subgroup of $\mathcal{O}^\times$. The solutions of that Pell–Fermat equation are the

$$(x, y) \text{ where } x + y\sqrt{13} = \pm(649 + 180\sqrt{13})^n,\ n \in \mathbb{Z}.$$

5.8 Class groups of real quadratic fields

In this example we will see how to use units to compute class groups.

Example 5.8.1. Let $K = \mathbb{Q}(\sqrt{79})$, so that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{79}]$ and the discriminant is $\operatorname{disc} K = 4 \cdot 79$. Let $\alpha = \sqrt{79}$. The signature of K is $(2, 0)$, so the Minkowski bound is $M_K = \frac{2}{4}\sqrt{\operatorname{disc} K} \approx 8.89$. The class group of K is generated by the classes of prime ideals of norm up to 8. We compute the decomposition of the small primes:

- 2 is ramified: $(2) = \mathfrak{p}_2^2$, and $[\mathfrak{p}_2]^2 = 1$.
- $79 \equiv 1 \equiv (\pm 1)^2 \bmod 3$, so 3 splits: $(3) = \mathfrak{p}_3\mathfrak{p}_3'$ where $\mathfrak{p}_3 = (3, \alpha + 2)$ and $\mathfrak{p}_3' = (3, \alpha + 1)$, and $[\mathfrak{p}_3'] = [\mathfrak{p}_3]^{-1}$.
- $79 \equiv 4 \equiv (\pm 2)^2 \bmod 5$, so 5 splits: $(5) = \mathfrak{p}_5\mathfrak{p}_5'$ where $\mathfrak{p}_5 = (5, \alpha + 3)$ and $\mathfrak{p}_5' = (5, \alpha + 2)$, and $[\mathfrak{p}_5'] = [\mathfrak{p}_5]^{-1}$.
- $79 \equiv 2 \equiv (\pm 3)^2 \bmod 7$, so 7 splits: $(7) = \mathfrak{p}_7\mathfrak{p}_7'$ where $\mathfrak{p}_7 = (7, \alpha + 4)$ and $\mathfrak{p}_7' = (7, \alpha + 3)$, and $[\mathfrak{p}_7'] = [\mathfrak{p}_7]^{-1}$.

So $\operatorname{Cl}(K)$ is generated by the classes of $\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_5$ and $\mathfrak{p}_7$.

Let us compute some elements of small norm to try to get relations in the class group. The norm of a generic element $z = x + y\alpha \in \mathbb{Z}_K$ is $x^2 - 79y^2$. With $y = 1$ we try $x = 8, 9, 10$ and we get respective norms $-15, 2, 21$. Let us compute the factorisation of the corresponding elements.

- The ideal $(8 + \alpha)$ has norm $15 = 3 \cdot 5$, so it is the product of a prime of norm 3 and a prime of norm 5. Since $8 + \alpha \equiv 2 + \alpha \bmod 3$ we have $\mathfrak{p}_3 \mid (8 + \alpha)$. Since $8 + \alpha \equiv 3 + \alpha \bmod 5$ we have $\mathfrak{p}_5 \mid (8 + \alpha)$. This gives $(8 + \alpha) = \mathfrak{p}_3\mathfrak{p}_5$, so that $[\mathfrak{p}_5] = [\mathfrak{p}_3]^{-1}$.

- The ideal $(9+\alpha)$ has norm 2, so $(9+\alpha) = \mathfrak{p}_2$ and $[\mathfrak{p}_2] = 1$.
- The ideal $(10+\alpha)$ has norm $21 = 3 \cdot 7$, so it is the product of a prime of norm 3 and a prime of norm 7. Since $10+\alpha \equiv 1+\alpha \bmod 3$, we have $\mathfrak{p}_3' \mid (10+\alpha)$. Since $10+\alpha \equiv 3+\alpha \bmod 7$, we have $\mathfrak{p}_7' \mid (10+\alpha)$. This gives $(10+\alpha) = \mathfrak{p}_3'\mathfrak{p}_7'$, so that $[\mathfrak{p}_3'][\mathfrak{p}_7'] = 1$ and $[\mathfrak{p}_7] = [\mathfrak{p}_3]^{-1}$.

We still need to find an ideal class of finite order. After a few trials, we obtain the element $17+2\alpha$, which has norm $-27 = -3^3$. Since $17+2\alpha \equiv 2+2\alpha \equiv 2(1+\alpha) \bmod 3$, we have $\mathfrak{p}_3' \mid (17+2\alpha)$ and $\mathfrak{p}_3 \nmid (17+2\alpha)$, so that $(17+2\alpha) = (\mathfrak{p}_3')^3$. This gives $[\mathfrak{p}_3]^3 = 1$.

With this we know that $\mathrm{Cl}(K) \cong 1$ or $\mathbb{Z}/3\mathbb{Z}$, and distinguishing between these cases is equivalent to deciding whether $\mathfrak{p}_3$ is principal. If we were in an imaginary quadratic field, we could easily determine every element of norm 3. Here, because there are infinitely many units, there could be infinitely many elements of norm ± 3! However, up to multiplication by a unit, there are still finitely many. In order to determine these elements, we need to first compute a fundamental unit[2] of K. We need to find the smallest solution to

$$x^2 - 79y^2 = \pm 1, \quad x, y \in \mathbb{Z}.$$

After trying successive values of y, we find that $u = 80 + 9\sqrt{79}$ is a fundamental unit, and has norm 1.

Let us try to find an element $z = x + y\alpha$ of norm ± 3. By reducing modulo 4:

$$x^2 - 79y^2 \equiv x^2 + y^2 \bmod 4,$$

we see that 3 cannot be the norm of an element in $\mathbb{Z}_K$. However, we cannot rule out -3 using congruences. Suppose z has norm 3. After multiplying z by some power of u, we can assume that

$$u^{-1/2} < z < u^{1/2}.$$

Since $z = x + y\alpha$ has norm -3, we have $3/z = -x + y\alpha$. But we also have the inequality

$$3u^{-1/2} < 3/z < 3u^{1/2}.$$

[2] Actually, any unit of infinite order would do, but using a fundamental one gives better bounds and hence saves us some work.

Summing these inequalities we get

$$4u^{-1/2} < 2y\alpha < 4u^{1/2}, \text{ i.e. } \frac{2}{\alpha u^{1/2}} < y < \frac{2u^{1/2}}{\alpha}.$$

We compute $\frac{2}{\alpha u^{1/2}} \approx 0.017$ and $\frac{2u^{1/2}}{\alpha} \approx 2.85$. So it is enough to try $y = 1$ and $y = 2$, and we easily see that none of these work. So $\mathbb{Z}_K$ does not contain any element of norm ± 3, the ideal $\mathfrak{p}_3$ is not principal, and finally

$$\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}.$$

Chapter 6

Geometry of numbers

This chapter is not examinable.

6.1 Lattices

Proposition 6.1.1. *Let $n \geq 1$, V a real vector space of dimension n and let $L \subset V$ be a subgroup. The following are equivalent:*

(i) There exists a $\mathbb{Z}$-basis $b_1, \ldots, b_r$ of L with $(b_1, \ldots, b_r)$ linearly independent over $\mathbb{R}$;

(ii) Every $\mathbb{Z}$-basis of L is linearly independent over $\mathbb{R}$;

(iii) L is discrete;

(iv) For all compact subsets $K \subset V$, the intersection $K \cap L$ is finite.

Proof.

- (i) $\Rightarrow$ (ii): Two $\mathbb{Z}$-bases differ by multiplication by an invertible matrix.
- (ii) $\Rightarrow$ (iii): Let $y_m = \sum_{i=1}^{r} x_i^{(m)} b_i$ be a sequence in L ($x_i^{(m)} \in \mathbb{Z}$), and assume that it has a limit: $y_m \to_{m\to\infty} y \in L$. Since (b_i) are linearly independent, each $(x_i^{(m)})_m$ converges, as $m \to \infty$, to some $x_i \in \mathbb{R}$. But since $x_i^{(m)} \in \mathbb{Z}$, the sequences $(x_i^{(m)})_m$ are eventually constant, so y_m is eventually constant. Hence L is discrete.
- (iii) $\Rightarrow$ (iv): $K \cap L$ is compact and discrete, hence finite.

- (iv) $\Rightarrow$ (i): Let $X = \{b_1, \ldots, b_r\}$ be a maximal $\mathbb{R}$-linearly independent subset of L (it is finite since V has finite dimension n). Let $L' = \{\sum_i x_i b_i \colon x_i \in \mathbb{Z}\}$ be the group generated by X. We will first prove that L' has finite index in L. Let $K = \{\sum_i x_i b_i \colon x_i \in [0,1]\}$, which is compact. Then $K \cap L$ is finite, say of cardinality M, so let $d = M!$.

  Now let $x \in L$ be arbitrary. We can write $x = \sum_i x_i b_i$ where $x_i \in \mathbb{R}$. For all $m \geq 1$, define $y_m = \sum_i (mx_i - \lfloor mx_i \rfloor) b_i$, which belongs to K. Since $mx \in L$ and $\sum_i \lfloor mx_i \rfloor b_i \in L' \subset L$, we have $y_m \in L$. Since $M = \#K \cap L$, as m ranges over $0, \ldots, M$, some of the y_m must coincide, say

  $$y_{m_1} = y_{m_2}.$$

  For all i we then have

  $$m_1 x_i - \lfloor m_1 x_i \rfloor = m_2 x_i - \lfloor m_2 x_i \rfloor,$$

  which we rewrite as

  $$x_i = \frac{\lfloor m_1 x_i \rfloor - \lfloor m_2 x_i \rfloor}{m_1 - m_2} \in \frac{1}{d}\mathbb{Z},$$

  since $|m_1 - m_2| \leq M$. We have proved that

  $$L' \subset L \subset \frac{1}{d} L',$$

  so that L' has finite index in L. Now by construction, L' has a $\mathbb{Z}$-basis that is $\mathbb{R}$-linearly independent, and since it has finite index in L we can obtain a $\mathbb{Z}$-basis of L by multiplying the $\mathbb{Z}$-basis of L' by a matrix with nonzero determinant, so that $\mathbb{Z}$-basis is also linearly independent over $\mathbb{R}$.

□

Remark 6.1.2. We must have $r \leq n$: a linearly independent set in V can have at most n elements.

Example 6.1.3.

- $L = \mathbb{Z}(1,2) + \mathbb{Z}(\pi, 0) \subset \mathbb{R}^2$ satisfies the conditions of Proposition 6.1.1.
- $L = \mathbb{Z} + \mathbb{Z}\sqrt{2} \subset \mathbb{R}$ does not satisfy the conditions of Proposition 6.1.1.

Proposition 6.1.4. *Let $n \geq 1$, V a real vector space of dimension n, let $L \subset V$ be a subgroup satisfying the equivalent conditions of Proposition 6.1.1, and let $b_1, \ldots, b_r$ be a $\mathbb{Z}$-basis of L. The following are equivalent:*

(i) $r = n$;

(ii) There exists a compact subset $K \subset V$ such that for all $x \in V$, there exists $y \in L$ such that $x - y \in K$.

Proof.

- (i) $\Rightarrow$ (ii): Let $K = \{\sum_{i=1}^n x_i b_i \colon x_i \in [0,1]\}$. Let $x \in V$, and write $x = \sum_{i=1}^n x_i b_i$ with $x_i \in \mathbb{R}$. Let $y = \sum_{i=1}^n \lfloor x_i \rfloor b_i \in \mathbb{Z}$. Then $x - y = \sum_{i=1}^n (x_i - \lfloor x_i \rfloor) b_i \in K$.
- (ii) $\Rightarrow$ (i): Let $W \subset V$ be the real vector space generated by L. Since $b_1, \ldots, b_r$ is a basis of W, we want to prove that $W = V$. Let $x \in V$. For all $m \geq 1$, $mx \in V$, so there exists $y_m \in L$ and $k_m \in K$ such that $mx = y_m + k_m$. We write this as

  $$x = \frac{y_m}{m} + \frac{k_m}{m}.$$

  Since K is compact, $k_m/m \to 0$ when $m \to \infty$. Since $y_m/m = x - k_m/m$, $y_m/m \to x$ when $m \to \infty$. But $y_m/m \in W$ and W is closed, so x belongs to W. This proves that $V = W$, so that $r = n$.

□

Remark 6.1.5. The set K constructed in the proof of Proposition 6.1.4 is called a *fundamental parallelotope* of L.

A subgroup $L \subset V$ satisfying the properties of Propositions 6.1.1 and 6.1.4 is a *lattice*.

Example 6.1.6.

- $L = (1, \sqrt{2})\mathbb{Z} + (\sqrt{3}, 0)\mathbb{Z} \subset \mathbb{R}^2$ is a lattice.
- $L = (1, 2)\mathbb{Z} + (\pi, 2\pi)\mathbb{Z} \subset \mathbb{R}^2$ is not a lattice.
- $L = (1, -1)\mathbb{Z} \subset \mathbb{R}^2$ is not a lattice in $\mathbb{R}^2$, but it is a lattice in $V = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$.

We will admit that there is a good notion of "volume" of subsets of $\mathbb{R}^n$, such that the volume of the unit cube $[0,1]^n \subset \mathbb{R}^n$ is 1, and such that any linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ multiplies volumes by $|\det f|$.

Definition 6.1.7. Let $n \geq 1$, $V = \mathbb{R}^n$ and L be a lattice in V. Let $b_1, \ldots, b_n$ be a $\mathbb{Z}$-basis of L and let

$$K = \Big\{ \sum_{i=1}^{n} x_i b_i \mid x_i \in [0,1] \Big\}$$

be a fundamental parallelotope. The *covolume* $\operatorname{covol}(L)$ *of* L is the volume of K.

Proposition 6.1.8. *Let $n \geq 1$, $V = \mathbb{R}^n$ and L be a lattice in V. Let $b_1, \ldots, b_n$ be a $\mathbb{Z}$-basis of L, let A be the matrix with columns (b_i), and let G be the matrix with (i,j)-th coefficient $\langle b_i, b_j \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Then*

$$\operatorname{covol}(L) = |\det A| = (\det G)^{1/2}.$$

In particular, the covolume of L does not depend on the choice of a $\mathbb{Z}$-basis.

Proof. Let $C = [0,1]^n$ be the unit cube and let K be the fundamental parallelotope corresponding to the basis (b_i). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map sending the element e_i of the standard basis of $\mathbb{R}^n$ to b_i. Then A is the matrix of f, so $|\det A| = |\det f|$. Moreover, $K = f(C)$, so the volume of K is $|\det A|$ times the volume of C, which is 1.

The matrix G is exactly $A^t \cdot A$, so $\det G = (\det A)^2$.

If we change the $\mathbb{Z}$-basis, we replace A with AP where $P \in \mathrm{GL}_n(\mathbb{Z})$, so $\det P = \pm 1$ and the covolume does not change. □

Corollary 6.1.9. *Let $L' \subset L$ be lattices in $\mathbb{R}^n$. Then L' has finite index in L, and*

$$\operatorname{covol}(L') = [L\colon L'] \operatorname{covol}(L).$$

Proof. The index is finite because L and L' have the same rank n. We can obtain a basis of L' from a basis of L by multiplying it on the right by a matrix with determinant $[L\colon L']$. □

6.2 Minkowski's theorem

Lemma 6.2.1 (Blichfeldt)**.** *Let $L \subset \mathbb{R}^n$ be a lattice and let S be a subset*[1] *such that*

$$\mathrm{vol}(S) > \mathrm{covol}(L).$$

Then there exists distinct elements $x, y \in S$ such that $x - y \in L$.

Proof. Let (b_i) be a $\mathbb{Z}$-basis of L and $K = \{\sum_i x_i b_i \colon x_i \in [0,1]\}$ be the corresponding fundamental parallelotope. Let $f : \mathbb{R}^n \to K$ be the map defined by $f(\sum_i x_i b_i) = \sum_i (x_i - \lfloor x_i \rfloor) b_i$. By construction, for all $x \in \mathbb{R}^n$ we have $f(x) - x \in L$.

Now assume that the restriction of f to S is injective. Since f is a piecewise translation, $\mathrm{vol}(f(S)) = \mathrm{vol}(S) > \mathrm{vol}(K)$. But $f(S) \subset K$, so that is impossible.

So f is not injective: there exists distinct elements $x, y \in S$ such that $f(x) = f(y)$. But then $x - y = x - f(x) + f(y) - y \in L$. □

Definition 6.2.2. A subset $S \subset \mathbb{R}^n$ is called

- *convex* if for all $x, y \in S$, the line segment $[x, y]$ is contained in S, i.e. for all $t \in [0,1]$, $tx + (1-t)y \in S$.
- *symmetric* if for all $x \in S$, we have $-x \in S$.

Note that any nonempty convex symmetric subset always contains the lattice point $0 = (x + (-x))/2$.

Theorem 6.2.3 (Minkowski)**.** *Let $L \subset \mathbb{R}^n$ be a lattice, and let S be a convex symmetric subset such that*

$$\mathrm{vol}(S) > 2^n \,\mathrm{covol}(L).$$

Then there exists a nonzero vector $v \in S \cap L$.

Proof. Let $S' = \frac{1}{2}S$, which has volume $\mathrm{vol}(S)/2^n > \mathrm{covol}(L)$. By Blichfeldt's Lemma 6.2.1, there exists distinct elements $x, y \in S'$ such that $v = x - y \in L$. But $v = \frac{2x + (-2y)}{2}$, and $2x$ and $2y$ are in S, so $v \in S$. □

[1] The correct hypothesis on S is that it is Lebesgue measurable.

Corollary 6.2.4. *Let $L \subset \mathbb{R}^n$ be a lattice, and let S be a compact convex symmetric subset such that*

$$\mathrm{vol}(S) \geq 2^n \,\mathrm{covol}(L).$$

Then there exists a nonzero vector $v \in S \cap L$.

Proof. For all $m \geq 1$, let $S_m = (1 + 1/m)S$, which has volume strictly greater than $\mathrm{covol}(L)$. By Minkowski's Theorem 6.2.3, there exists nonzero vectors $v_m \in S_m \cap L$. Write $v_m = (1 + 1/m)s_m$ with $s_m \in S$. Since S is compact, there exists a subsequence $s_{\phi(m)}$ which converges to an element $s \in S$. But then $v_{\phi(m)} = (1 + 1/\phi(m))s_{\phi(m)}$ also converges to $s \in S$. Every $v_{\phi(m)}$ is in $L \setminus \{0\}$, which is closed, so the limit s is also a nonzero element of L. □

Remark 6.2.5. The Corollary is false without the compactness hypothesis: take $L = \mathbb{Z}^2 \subset \mathbb{R}^2$ and S the open square with sides of length 2.

6.3 Applications to number theory

Proposition 6.3.1. *Let K be a number field of signature (r_1, r_2) and let $\Sigma : K \hookrightarrow K_{\mathbb{R}} \cong \mathbb{R}^{r_1+2r_2}$ be the Minkowski embedding of K. Let $\mathcal{O}$ be an order in K and let $L = \Sigma(\mathcal{O})$. Then L is a lattice in $K_{\mathbb{R}}$ and*

$$\mathrm{covol}(L) = 2^{-r_2} |\operatorname{disc}(\mathcal{O})|^{1/2}.$$

Proof. By Proposition 5.3.3 and since $\mathcal{O}$ has rank $n = r_1 + 2r_2$, L is a lattice in $K_{\mathbb{R}}$. Let $b_1, \ldots, b_n$ be a $\mathbb{Z}$-basis of $\mathcal{O}$, so that by definition we have $\operatorname{disc}(\mathcal{O}) = \det\big(\mathrm{Tr}_{\mathbb{Q}}^K(b_i b_j)\big)$. Let $\sigma_1, \ldots, \sigma_{r_1+r_2}$ be representatives of the complex embeddings of K up to conjugacy. For $1 \leq k \leq r_1 + r_2$, let $n_k = 1$ if σ_k is real and $n_k = 2$ otherwise. By Corollary 1.3.8 we have

$$\mathrm{Tr}_{\mathbb{Q}}^K(b_i b_j) = \sum_k n_k \,\mathrm{Re}(\sigma_k(b_i)\sigma_k(b_j)).$$

We want to relate this to

$$\langle \Sigma(b_i), \Sigma(b_j) \rangle = \sum_k \mathrm{Re}(\sigma_k(b_i)\overline{\sigma_k(b_j)}).$$

Let $C : K_{\mathbb{R}} \to K_{\mathbb{R}}$ be the linear map defined by $C(x)_k = n_k \overline{x_k}$ for all $1 \leq k \leq r_1 + r_2$. Then $|\det C| = 2^{r_2}$, and we have

$$\mathrm{Tr}_{\mathbb{Q}}^K(b_i b_j) = \langle \Sigma(b_i), C(\Sigma(b_j)) \rangle,$$

so that by Proposition 6.1.8, we have

$$\operatorname{disc}(\mathcal{O}) = |\det C| \operatorname{covol}(L) = 2^{r_2} \operatorname{covol}(L).$$

□

Corollary 6.3.2. *Let K be a number field of signature (r_1, r_2) and discriminant $\operatorname{disc} K$ and let $\Sigma : K \hookrightarrow K_{\mathbb{R}} \cong \mathbb{R}^{r_1+2r_2}$ be the Minkowski embedding of K. Let I be a fractional ideal in K and let $L = \Sigma(I)$. Then L is a lattice in $K_{\mathbb{R}}$ and*

$$\operatorname{covol}(L) = 2^{-r_2} |\operatorname{disc} K|^{1/2} N(I).$$

Proof. Since for all $a \in \mathbb{Q}$ we have $N(aI) = a^n N(I)$ and $\operatorname{covol}(aL) = a^n \operatorname{covol}(L)$, it is enough to prove it for integral ideals. If I is an integral ideal then $N(I) = [\mathbb{Z}_K : I]$, so the result follows from Corollary 6.1.9 and Proposition 6.3.1. □

Lemma 6.3.3. *Let $V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^{r_1+2r_2}$, and let*

$$S_t = \{((x_i)_i, (z_j)_j) \in V \mid \sum |x_i| + 2 \sum |z_j| \leq t\}.$$

Then

$$\operatorname{vol}(S_t) = 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2} \frac{t^n}{n!}.$$

Proof. Omitted. □

We now prove Minkowski's theorem, which we restate here for convenience.

Theorem 6.3.4 (Minkowski)**.** *Let K be a number field of signature (r_1, r_2) and degree $n = r_1 + 2r_2$. Let*

$$M_K = \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^{r_2} \sqrt{|\operatorname{disc} K|}.$$

Then every ideal class is represented by an integral ideal of norm at most M_K.

Proof. Let Σ be the Minkowski embedding of K, let I be a fractional ideal of K and let $L = \Sigma(I^{-1})$. Let $t > 0$ be such that

$$\operatorname{vol}(S_t) = 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2} \frac{t^n}{n!} = 2^{n-r_2} |\operatorname{disc} K|^{1/2} N(I^{-1}) = 2^n \operatorname{covol}(L),$$

which we can rewrite as

$$t^n = n! \left(\frac{4}{\pi}\right)^{r_2} |\operatorname{disc} K|^{1/2} N(I)^{-1}.$$

By Minkowski's theorem (Corollary 6.2.4) there exists a nonzero vector $y \in S_t \cap L$, which we write $y = \Sigma(x)$ with $x \in I^{-1}$.

By definition of S_t we have

$$\sum_\sigma |\sigma(x)| \leq t,$$

so by the inequality of arithmetic and geometric means we have

$$|N_{\mathbb{Q}}^K(x)| = \prod_\sigma |\sigma(x)| \leq \left(\frac{t}{n}\right)^n = \frac{t^n}{n^n}.$$

Let $J = xI$, which is a fractional ideal in the same ideal class as I. Since $x \in I^{-1}$, we have $J = xI \subset II^{-1} = \mathbb{Z}_K$, so J is an integral ideal. In addition we have

$$N(J) = |N_{\mathbb{Q}}^K(x)| N(I) \leq \frac{t^n}{n^n} N(I) = M_K,$$

which proves the theorem. □

Definition 6.3.5. Let K be a number field of signature (r_1, r_2) and $K_{\mathbb{R}} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. We endow $K_{\mathbb{R}}$ with the structure of a ring by coordinatewise multiplication. Let $x \in K_{\mathbb{R}}$. Then multiplication by x induces an $\mathbb{R}$-linear endomorphism of $K_{\mathbb{R}}$, denoted by

$$\begin{array}{rccc} \mu_x \colon & K_{\mathbb{R}} & \longrightarrow & K_{\mathbb{R}} \\ & y & \longmapsto & xy. \end{array}$$

We define $N_{\mathbb{R}}^{K_{\mathbb{R}}}(x) = \det(\mu_x)$.

Lemma 6.3.6. *With notations as in Definition 6.3.5, let $x = ((x_i), (z_j)) \in K_{\mathbb{R}}$. Then*

$$N_{\mathbb{R}}^{K_{\mathbb{R}}}(x) = \prod_i x_i \times \prod_j |z_j|^2.$$

In particular, for all $x \in K$, we have $N_{\mathbb{Q}}^K(x) = N_{\mathbb{R}}^{K_{\mathbb{R}}}(\Sigma(x))$.

Proof. It suffices to prove it on each $\mathbb{R}$ or $\mathbb{C}$ factor. For $x \in \mathbb{R}$, μ_x is the 1×1 matrix (x). For $z = x + iy \in \mathbb{C}$, on the basis $(1, i)$ the matrix of the endomorphism μ_z is $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, which has determinant $x^2 + y^2 = |z|^2$. The second claim is a restatement of Corollary 1.3.8. □

We now prove Dirichlet's theorem, which we restate here for convenience.

Theorem 6.3.7 (Dirichlet)**.** *Let K be a number field of signature (r_1, r_2). Let $V \subset \mathbb{R}^{r_1+r_2}$ be the subspace of vectors whose coordinates sum to zero. Then $\mathcal{L}(\mathbb{Z}_K^\times)$ is a lattice in V. As an abstract abelian group, we have*

$$\mathbb{Z}_K^\times \cong W_K \times \mathbb{Z}^{r_1+r_2-1}.$$

Proof. We first prove that $L = \mathcal{L}(\mathbb{Z}_K^\times)$ is a subgroup of V: this follows from the fact that every unit has norm ± 1 (Proposition 5.2.2) and Corollary 1.3.8.

To prove that L is a lattice in V, we will prove that

a) $L \cap B$ is finite for all compact subsets $B \subset V$, and

b) there exists a compact subset $C \subset V$ such that for all $v \in V$, there exists $x \in L$ such that $v - x \in C$.

By Propositions 6.1.1 and 6.1.4, this is equivalent to L being a lattice.

a) In B, every coordinate is bounded, so by Proposition 5.3.3, the intersection with L is finite.

b) Let $K_{\mathbb{R}}^1 = \{x \in K_{\mathbb{R}} \mid N_{\mathbb{R}}^{K_{\mathbb{R}}}(x) = 1\}$, which contains $\Sigma(\mathbb{Z}_K^\times)$. By taking exponentials, it is enough to construct a set $C' \subset K_{\mathbb{R}}^1$ such that for all $x \in K_{\mathbb{R}}^1$, there exists $u \in \mathbb{Z}_K^\times$ such that $x\Sigma(u)^{-1} \in C'$.

Let $L' = \Sigma(\mathbb{Z}_K)$, and let S be a compact convex symetric subset of $K_{\mathbb{R}}$ such that

$$\operatorname{vol}(S) \geq 2^n \operatorname{covol}(L').$$

Let $R = N_{\mathbb{R}}^{K_{\mathbb{R}}}(S) \cap \mathbb{Z}$, which is finite. For all $r \in R$, define

- $S_r = \{x \in S \mid N_{\mathbb{R}}^{K_{\mathbb{R}}}(x) = r\}$;
- Y_r a set of representatives of the elements of norm r in $\mathbb{Z}_K$ up to multiplication by units.

Then S_r is compact, and Y_r is finite since two element differ by a unit if and only if they generate the same ideal and there are finitely many integral ideals of norm r. So $C_r = \Sigma(Y_r)^{-1}S_r = \{\Sigma(y)^{-1}x \colon y \in Y_r, x \in S_r\}$ is compact.

Let $C' = \bigcup_{r\in R} C_r$, which is compact. We claim that C' works.

Let $x \in K^1_{\mathbb{R}}$. Since $\operatorname{covol}(xL') = \operatorname{covol}(L')$, by Minkowski's theorem (Corollary 6.2.4) there exists $y \in S \cap xL'$. Let us write $y = x \cdot \Sigma(a)$ with $a \in \mathbb{Z}_K$. We have $r = N_{\mathbb{R}}^{K_{\mathbb{R}}}(y) = N_{\mathbb{R}}^{K}(a)$ since $N_{\mathbb{R}}^{K_{\mathbb{R}}}(x) = 1$, so $r \in R$ and $y \in S_r$. By definition of Y_r, there exists a unit $u \in \mathbb{Z}_K^\times$ such that $au \in Y_r$. But now $x\Sigma(u)^{-1} = \Sigma(au)^{-1}y \in Y_r^{-1}S_r \in C'$, as claimed.

This proves that $L \cong \mathbb{Z}^r$, where $r = \dim V = r_1 + r_2 - 1$. By Theorem 5.3.5, we have

$$\ker \mathcal{L} = W_K,$$

which is precisely the torsion subgroup of $\mathbb{Z}_K^\times$. By the structure theory of abelian groups, we get

$$\mathbb{Z}_K^\times \cong W_K \times \mathbb{Z}^r$$

as claimed. □

Chapter 7

Summary of methods and examples

7.1 Discriminant and ring of integers

In summary, to compute the ring of integers in a number field $K = \mathbb{Q}(\alpha)$ given by the minimal polynomial $P \in \mathbb{Z}[x]$ of α:

- Compute the discriminant of $\mathbb{Z}[\alpha]$, i.e. the **discriminant of** P, and find its **factorisation into prime numbers**.
- The discriminant $\operatorname{disc} K$ of K and the index f of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$ satisfy $\operatorname{disc}(\mathbb{Z}[\alpha]) = f^2 \operatorname{disc} K$. In particular every prime that ramifies in K must divide $\operatorname{disc}(\mathbb{Z}[\alpha])$.
- If a prime divides $\operatorname{disc}(\mathbb{Z}[\alpha])$ with **exponent** 1, then $\mathbb{Z}[\alpha]$ is p-maximal.
- If the minimal polynomial of α is **Eisenstein at** p, then $\mathbb{Z}[\alpha]$ is p-maximal.
- The order $\mathbb{Z}[\alpha]$ is often not maximal. In this case, you need to find an element $x \in \mathbb{Z}_K \setminus \mathbb{Z}[\alpha]$ and examine the larger order $\mathbb{Z}[\alpha, x]$. You are not supposed to know a general method to find x.

7.2 Factorisation

In summary, to compute factorisations in a number field $K = \mathbb{Q}(\alpha)$ given by the minimal polynomial $P \in \mathbb{Z}[x]$ of α:

- To find the decomposition of a prime number p (equivalently the factorisation of the ideal $p\mathbb{Z}_K$) when $\mathbb{Z}[\alpha]$ is maximal at p, **factor P modulo p into irreducible polynomials in $\mathbb{F}_p[x]$**, and apply Theorem 3.7.1.
- To find the factorisation of an integral ideal I, for instance of the form (β) with $\beta \in \mathbb{Z}_K$, first **compute the norm $N(I) \in \mathbb{Z}$ and factor this norm** into prime numbers.
- For each prime p dividing $N(I)$ with exponent e, find the decomposition of p into prime ideals, compute the possible products of these primes that have norm p^e, then find out which product really divides I.
- Testing whether a prime ideal $\mathfrak{p}$ divides (β) is equivalent to testing whether the image of β under the reduction modulo $\mathfrak{p}$ map $\mathbb{Z}_K \to \mathbb{Z}_K/\mathfrak{p}$ is zero.
- To compute the **reduction modulo $\mathfrak{p} = (p, \phi(\alpha))$** of an element β, when $\mathbb{Z}[\alpha]$ is p-maximal and ϕ is an irreducible factor of $P \bmod p$: write β as a polynomial in α[1]: $\beta = R(\alpha)$ for some $R \in \mathbb{Z}[x]$. Then you can obtain the reduction modulo $\mathfrak{p}$ of β by reducing every coefficient modulo p, and then dividing the resulting polynomial by ϕ in $\mathbb{F}_p[x]$.

7.3 Class group and units

We first study a number of examples of computations of class groups. The general method is to first compute the Minkowski bound for the field under consideration, then study the prime ideals up to that bound, and then study their products. We will focus mostly on quadratic fields.

Example 7.3.1. Let $K = \mathbb{Q}(i)$, so that $\mathbb{Z}_K = \mathbb{Z}[i]$ and $\operatorname{disc} K = -4$. We already know that $\mathbb{Z}_K$ is a Euclidean domain, hence a PID, but let's reprove

[1] If $\beta \notin \mathbb{Z}[\alpha]$, write $\beta = \beta'/d$ with $\beta' \in \mathbb{Z}[\alpha]$ and d coprime to p; this is always possible if $\mathbb{Z}[\alpha]$ is p-maximal. Then d has an inverse u modulo p ($du \equiv 1 \bmod p$), the reduction map sends $1/d$ to u.

it using our new methods. The Minkowski bound is $M_K \approx 1.27 < 2$, so by Corollary 4.4.4, the class group of K is trivial; in other words, $\mathbb{Z}_K$ is a PID (and hence a UFD) in this case.

Example 7.3.2. Let $K = \mathbb{Q}(\sqrt{437})$. Since 437 is squarefree and $437 \equiv 1 \bmod 4$, we have $\mathbb{Z}_K = \mathbb{Z}[\frac{1+\sqrt{437}}{2}]$ and $\operatorname{disc} K = 437$. The Minkowski bound is $M_K \approx 10.45 < 11$. By Corollary 4.4.4, $\mathrm{Cl}(K)$ is generated by the classes of the prime ideals of norm at most 10. Since $437 \equiv 5 \bmod 8$, by Theorem 3.9.3 the prime 2 is inert in K. So the only prime ideal above 2 is (2), which is principal, so it does not contribute to the class group. Since $437 \equiv 2 \bmod 3$ which is not a square, by Theorem 3.9.1 the prime 3 is inert in K. Again the prime (3) is principal. We similarly compute that 5 and 7 are inert in K. So again, the class group of K is trivial.

What you should remember from this example is that you can ignore inert primes when computing the class group.

Now let us complete the study of our examples $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-23})$.

Example 7.3.3. Let $K = \mathbb{Q}(\sqrt{-5})$. We saw that $\mathbb{Z}_K = \mathbb{Z}[\sqrt{-5}]$ and $\operatorname{disc} K = -20$. The Minkowski bound is $M_K \approx 2.85 < 3$. By Corollary 4.4.4, $\mathrm{Cl}(K)$ is generated by the classes of the prime ideals of norm at most 2. As we saw in Example 4.3.3, the unique ideal $\mathfrak{p}_2$ above 2 has norm 2 and is not principal, but its square is principal. We conclude that $\mathrm{Cl}(K) \cong \mathbb{Z}/2\mathbb{Z}$, our first nontrivial class group!

What you should remember from this example is that it is easy to get an upper bound on the order of totally ramified primes in the class group.

Example 7.3.4. Let $K = \mathbb{Q}(\sqrt{-23})$. We have $\mathbb{Z}_K = \mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$ and $\operatorname{disc} K = -23$. The Minkowski bound is $M_K \approx 3.05 < 4$. The class group of K is generated by the classes of prime ideals of norm at most 3. We have seen that the prime 2 splits in K: $(2) = \mathfrak{p}_2\mathfrak{p}_2'$. This implies that $[\mathfrak{p}_2'] = [\mathfrak{p}_2]^{-1}$, so it suffices to consider one of them. We saw that $\mathfrak{p}_2$ and $\mathfrak{p}_2^2$ are not principal but that $\mathfrak{p}_2^3$ is. This proves that $[\mathfrak{p}_2]$ has order 3. Since $-23 \equiv 1 \bmod 3$ is a square, the prime 3 also splits in K: $(3) = \mathfrak{p}_3\mathfrak{p}_3'$, and $[\mathfrak{p}_3] = [\mathfrak{p}_3']^{-1}$. Now we can conclude by two different methods.

1. Recall that the norm of a generic element $z = x + \frac{1+\sqrt{-23}}{2}y \in \mathbb{Z}_K$ is $x^2 + xy + 6y^2$. Taking $(x, y) = (0, 1)$, we get an element $z \in \mathbb{Z}_K$ of

norm 6. We factor the ideal $(z) = \mathfrak{q}\mathfrak{q}'$ where $N(\mathfrak{q}) = 2$ and $N(\mathfrak{q}') = 3$. In the class group we have $[\mathfrak{q}'] = [\mathfrak{q}]^{-1}$, so that $[\mathfrak{p}_3]$ belongs to the group generated by $[\mathfrak{p}_2]$. This proves that $\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}$.

2. By Minkowski's Theorem 4.4.1, every ideal class is represented by an integral ideal of norm at most 3. The only such ideals are $\mathbb{Z}_K, \mathfrak{p}_2, \mathfrak{p}_2', \mathfrak{p}_3$, and $\mathfrak{p}_3'$, so we have $h_K \leq 5$. But we already exhibited an element $[\mathfrak{p}_2]$ of order 3 so h_K is a multiple of 3, so $\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}$.

What you should remember from this example is that the splitting of primes gives relations in the class group for free, and elements of small norm in $\mathbb{Z}_K$ provide the additional relations.

Example 7.3.5. Let $K = \mathbb{Q}(\sqrt{10})$. Since 10 is squarefree and $10 \equiv 2 \bmod 4$, by Theorem 2.4.2 we have $\mathbb{Z}_K = \mathbb{Z}[\sqrt{10}]$ and $\operatorname{disc} K = 40$. The Minkowski bound is $M_K \approx 3.16 < 4$. Since $2 \mid \operatorname{disc} K$, the prime 2 ramifies in K: $(2) = \mathfrak{p}_2^2$. Since $N(\mathfrak{p}_2) = 2$, if $\mathfrak{p}_2$ were principal then a generator $z = x + \sqrt{10}y \in \mathbb{Z}_K$ would have to satisfy

$$x^2 - 10y^2 = \pm 2.$$

By reducing modulo 5, we see that x would be a square root of $\pm 2 \bmod 5$. But only $0, 1$ and 4 are squares modulo 5, so z cannot exist. Hence $\mathfrak{p}_2$ is not principal, and $[\mathfrak{p}_2]$ has order 2 since $[\mathfrak{p}_2]^2 = [(2)] = 1$. Since $10 \equiv 1 \bmod 3$ is a square, the prime 3 splits in K: $(3) = \mathfrak{p}_3\mathfrak{p}_3'$. We have $[\mathfrak{p}_3] = [\mathfrak{p}_3']^{-1}$. We find another relation by looking for elements of small norm: taking $(x, y) = (2, 1)$ gives an element z of norm -6. We factor the ideal $(z) = \mathfrak{p}_2\mathfrak{p}_3''$, where $\mathfrak{p}_3''$ is a prime of norm 3, so it is either $\mathfrak{p}_3$ or $\mathfrak{p}_3'$. This gives $[\mathfrak{p}_3''] = [\mathfrak{p}_2]^{-1} = [\mathfrak{p}_2]$. Again, we can conclude by two methods.

1. By Minkowski's Theorem 4.4.1, every ideal class is represented by an integral ideal of norm at most 3. The only such ideals are $\mathbb{Z}_K, \mathfrak{p}_2, \mathfrak{p}_3$, and $\mathfrak{p}_3'$, but $[\mathfrak{p}_2]$ is the same as the class of one of the primes above 3, so $h_K \leq 3$. Since we already know an element $[\mathfrak{p}_2]$ of order 2, h_K is a multiple of 2, so $h_K = 2$ and $\mathrm{Cl}(K) \cong \mathbb{Z}/2\mathbb{Z}$.

2. We have $\langle [\mathfrak{p}_3] \rangle = \langle [\mathfrak{p}_3'] \rangle = \langle [\mathfrak{p}_3''] \rangle = \langle [\mathfrak{p}_2] \rangle$. Since the class group is generated by the classes of prime ideals above 2 and 3, we obtain $\mathrm{Cl}(K) = \langle [\mathfrak{p}_2] \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

What you should remember from this example is that you can sometimes use congruences to prove that certain ideals are not principal.

In summary, to compute a class group:

- First, compute the **Minkowski bound**, and list the prime ideals up to that bound by decomposing prime integers.
- The **decomposition of primes** provides relations in $\mathrm{Cl}(K)$ for free.
- **Elements of $\mathbb{Z}_K$ of small norm** provide additional relations.
- In imaginary quadratic fields, you can test whether an ideal is principal by computing bounds on the coordinates of a possible generator.
- You can often prove that an ideal is not principal using congruences.
- In real quadratic fields, after computing a fundamental unit, you can determine all elements of $\mathbb{Z}_K$ of a given norm up to multiplication by a unit. This allows you to test whether an ideal is principal.
- Conclude using Minkowski's theorem and the group structure.

7.4 Complete examples

Here is an example of computations with a number field. It is meant to illustrate pretty much everything that has been seen in this course, and represents the upper limit of what can be done without the help of a computer so do not be alarmed by its length! Also, if you feel that you could have performed these computations by yourself (with a reasonable amount of intermediate questions) this means that you have understood algebraic number theory very well.

Example 7.4.1. Let $P(x) = x^3 + 6x + 6 \in \mathbb{Z}[x]$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $P(x)$. Since $P(x)$ is Eisenstein at 2 (and also at 3), it is irreducible over $\mathbb{Z}$ and over $\mathbb{Q}$, so K is a well-defined number field of degree $[K : \mathbb{Q}] = 3$.

In order to compute the ring of integers $\mathbb{Z}_K$ of K, let us have a look at the order $\mathcal{O} = \mathbb{Z}[\alpha]$. Its discriminant is

$$\operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} P(x) = -4.6^3 - 27.6^2$$

by theorem 2.3.12. We want it in factored form (it is more interesting this way, because we can then get information about ramification), so we factor

and compute that

$$\operatorname{disc}\mathbb{Z}[\alpha] = -6^2(2^2 6 + 3^3) = -6^2 3(2^3 + 3^2) = -2^2 3^3 17.$$

Next, we know that $\operatorname{disc}\mathbb{Z}[\alpha] = f^2 \operatorname{disc} K$, where $f = [\mathbb{Z}_K : \mathbb{Z}[\alpha]] \in \mathbb{N}$ is the index of $\mathbb{Z}[\alpha]$. This tells us that $f \in \{1, 2, 3, 6\}$, so that the order $\mathbb{Z}[\alpha]$ is maximal at every prime except possibly at 2 and at 3. Besides, since the primes that ramify in K are the ones that divide $\operatorname{disc} K$, we see that the set of ramified primes is a subset of $\{2, 3, 17\}$. In fact, we can be more accurate: since $\operatorname{disc}\mathbb{Z}[\alpha]$ differs from $\operatorname{disc} K$ by a square and since the exponents of 3 and 17 in $\operatorname{disc}\mathbb{Z}[\alpha]$ are odd, both 3 and 17 do ramify in K. On the other hand, we do not know yet whether 2 ramifies (the factor 2^2 in $\operatorname{disc}\mathbb{Z}[\alpha]$ could come either from $\operatorname{disc} K$, in which case 2 would ramify, or from f^2, in which case 2 would not ramify).

To determine $\mathbb{Z}_K$, we must discover whether $\mathbb{Z}[\alpha]$ is maximal at 2 and 3 or not. In general, we have not seen how to do that, but in this particular case, our only weapon, theorem 3.8.4, applies, and tells us that since $P(x)$ is Eisenstein at 2, the order $\mathbb{Z}[\alpha]$ is maximal at 2 (i.e $2 \nmid f$), and similarly for 3. As a result, we have $f = 1$ and $\mathbb{Z}_K = \mathbb{Z}[\alpha]$. In particular, $\operatorname{disc} K = -2^2 3^3 17$, so 2 does ramifies in K.

Let us now compute the class group of K. The signature of K is (r_1, r_2) with $r_1 + 2r_2 = [K : \mathbb{Q}] = 3$, so the only possibilities are $(3, 0)$ and $(1, 1)$. This corresponds to $P(x)$ having 3 real roots, vs. $P(x)$ having 1 real root and 1 conjugate pair of complex roots (note that in the former case, K would be totally real). To find out what the signature of K actually is, we have two possibilities: studying the function $P(x)$ to see if it has 1 or 3 real zeroes, or using the fact that the sign of $\operatorname{disc} K$ is $(-1)^{r_2}$ (proposition 2.3.13). The latter is much faster of course, and allows us to effortlessly see that the signature of K is in fact $(1, 1)$ (so in particular $P(x)$ has 1 real root and 1 conjugate pair of complex roots). As a result, the Minkowski bound for K is

$$M_K = \frac{3!}{3^3}\left(\frac{4}{\pi}\right)^1 \sqrt{2^2 3^3 17} = 12.123\ldots,$$

which tells us that that $\mathrm{Cl}(K)$ is generated by the prime ideals above 2,3,5,7 and 11.

Let us determine these prime ideals. We are of course going to use theorem 3.7.1, which means that we will have to compute the factorisation of $P(x)$ modulo these primes. Now, a polynomial of degree 3 is irreducible over

a field if and only if it has no root in this field, so a table of values of $P(x)$ at small integers will be useful[2]. Here it is:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(n)$	-149	-82	-39	-14	-1	6	13	26	51	94	161

Actually, we do not really need that for $p = 2$ and 3, since we already know by theorem 3.8.4 that these primes are totally ramified in K. Indeed, for both of them we have $P(x) \equiv x^3 \bmod p$, so that

$$2\mathbb{Z}_K = (2, \alpha)^3 = \mathfrak{p}_2^3$$

and

$$3\mathbb{Z}_K = (3, \alpha)^3 = \mathfrak{p}_3^3.$$

Next, we observe that although the values of n in the table represent the whole of $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ (there is even plenty of overlap), none of the values of $P(n)$ is divisible by 5; this means that $P(x)$ has no root mod 5. It is thus is irreducible mod 5, so that 5 is inert in K, i.e.

$$5\mathbb{Z}_K = \mathfrak{p}_{5^3}$$

(here and in what follows, we denote prime ideals by $\mathfrak{p}_N, \mathfrak{p}'_N, \ldots$, where N is their norm). Next, we observe that 7 divides $P(-2)$, but none of the other values of P in our table. Since the values of n in this table cover $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$, this means that -2 is the only root of $P(x)$ mod 7. In fact, we compute by Euclidian division that $P(x) \equiv (x+2)(x^2 - 2x + 3)$ mod 7, and that the quadratic factor is irreducible over $\mathbb{F}_7$ (because it does not have any root, since it does not vanish at -2). In fact, we can save ourselves the trouble of this irreducibility check: we know that 7 does not ramify in K since it does not divide $\operatorname{disc} K$, so theorem 3.7.1 tells us that $P(x)$ is squarefree mod 7, which shows that the quadratic factor is irreducible (since its only possible root over $\mathbb{F}_7$ would be -2). Anyway, we deduce from theorem 3.7.1 that the decomposition of 7 in K is

$$7\mathbb{Z}_K = (7, \alpha + 2)(7, \alpha^2 - 2\alpha + 3) = \mathfrak{p}_7\mathfrak{p}'_{7^2}.$$

[2]It will be even more useful when we will be looking for relations in the class group, cf. infra.

Finally, since the values of n in our table cover $\mathbb{F}_{11}$ and since none of the $P(n)$ is divisible by 11, we have that 11 is inert in K,

$$11\mathbb{Z}_K = \mathfrak{p}_{11^3}.$$

Now that we have decomposed 2, 3, 5, 7 and 11 in K, we can apply theorem 4.4.1, which tells us that $\mathrm{Cl}(K)$ is generated by the classes of $\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_{5^3}, \mathfrak{p}_7, \mathfrak{p}'_{7^2}$ and $\mathfrak{p}_{11^3}$ (in fact we could throw away $\mathfrak{p}_{5^3}, \mathfrak{p}'_{7^2}$ and $\mathfrak{p}_{11^3}$ because their norms exceed the Minkowski bound, but let us pretend we failed to notice that). The above decompositions also give us for free some relations satisfied by these classes, namely

$$[\mathfrak{p}_2]^3 = [2\mathbb{Z}_K] = 1,$$

$$[\mathfrak{p}_3]^3 = [3\mathbb{Z}_K] = 1,$$

$$[\mathfrak{p}_{5^3}] = [5\mathbb{Z}_K] = 1,$$

$$[\mathfrak{p}_7].[\mathfrak{p}'_{7^2}] = [7\mathbb{Z}_K] = 1,$$

$$[\mathfrak{p}_{11^3}] = [11\mathbb{Z}_K] = 1.$$

In particular, we see that $\mathrm{Cl}(K)$ is in fact generated by $[\mathfrak{p}_2]$, $[\mathfrak{p}_3]$ and $[\mathfrak{p}_7]$ only.

There must be extra relations between $[\mathfrak{p}_2]$, $[\mathfrak{p}_3]$ and $[\mathfrak{p}_7]$ (otherwise the class group would be infinite, which would contradict corollary 4.4.2). To find them, we look for elements of small norm. Indeed, if the only prime numbers dividing the norm of some $\beta \in \mathbb{Z}_K$ are 2,3 and 7, then theorem 3.4.4 tells us that the only prime ideals in the factorisation of the ideal (β) are $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_7$ and $\mathfrak{p}'_{7^2}$, so we have found a relation between the classes of these ideals. To find such elements β, we use our table of values of $P(x)$ again.

For instance, we see that $P(0) = 6$, which tells us that the norm of α is ± 6 (here we are using the fact that the constant term of the characteristic polynomial of a matrix is, up to sign, the determinant of this matrix), so the ideal (α) must factor as a prime of norm 2 times a prime of norm 3. As $\mathfrak{p}_2$ (resp. $\mathfrak{p}_3$) is the only prime of norm 2 (resp. 3), we therefore have

$$(\alpha) = \mathfrak{p}_2\mathfrak{p}_3.$$

Indeed, we can check (although is is not necessary of course) that

$$\mathfrak{p}_2\mathfrak{p}_3 = (2, \alpha)(3, \alpha) = (6, 3\alpha, 2\alpha, \alpha^2) = (6, \alpha, \alpha^2) = (6, \alpha),$$

and $N((\alpha)) = |N_{\mathbb{Q}}^K(\alpha)| = 6$ so $6 \in (\alpha)$ by proposition 3.4.3 (another way to see this is simply to write $6 = \alpha(-\alpha^2 - 6)$), whence $(6, \alpha) = (\alpha)$. Anyway, we have found the relation

$$[\mathfrak{p}_2][\mathfrak{p}_3] = [(\alpha)] = 1$$

in $\mathrm{Cl}(K)$.

Similarly, we see in the table that $\alpha + 2$ has norm ± 14, so $(\alpha + 2)$ factors as a prime of norm 2 times a prime of norm 7, whence

$$(\alpha + 2) = \mathfrak{p}_2\mathfrak{p}_7$$

since $\mathfrak{p}_7$ is the only prime of norm 7 (the norm of $\mathfrak{p}_7'$ being 7^2, not 7). Therefore, we have found the relation

$$[\mathfrak{p}_2][\mathfrak{p}_7] = [(\alpha + 2)] = 1.$$

(incidentally, this shows that there exists $\beta \in K^\times$ such that $\mathfrak{p}_{7^2}' = \beta\mathfrak{p}_2$, and also $\gamma \in K^\times$ such that $\mathfrak{p}_7 = \gamma\mathfrak{p}_3$. Of course, such β and γ cannot lie in $\mathbb{Z}_K$.)

As a result, $\mathrm{Cl}(K)$ is generated by the class of $\mathfrak{p}_2$ alone. As $[\mathfrak{p}_2]^3 = 1$, we have two possibilities; either $[\mathfrak{p}_2] = 1$ (i.e $\mathfrak{p}_2$ is principal), and then $\mathrm{Cl}(K)$ is trivial, or $[\mathfrak{p}_2] \neq 1$ (i.e $\mathfrak{p}_2$ is not principal), and then $\mathrm{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$.

We are going to prove that $\mathfrak{p}_2$ is actually not principal. A possibility for this would be to write down the norm of a generic element of $\mathbb{Z}_K$ in terms of $[K : \mathbb{Q}]$ indeterminates, and prove that this norm can never be ± 2. However, this would lead to a horrible homogeneous expression of degree 3 in 3 variables, so this approach would be very tedious, if not intractable. We thus need another method.

If $\mathfrak{p}_2$ were principal, say $\mathfrak{p}_2 = (\beta)$ for some $\beta \in \mathbb{Z}_K$, then we would have $(2) = \mathfrak{p}_2^3 = (\beta^3)$, so that $v = \beta^3/2$ would be a unit. If we could prove that, for all $v \in \mathbb{Z}_K^\times$, the equation $\beta^3 = 2v$ has no solution $\beta \in K$, then we could conclude that $\mathfrak{p}_2$ is not principal. Unfortunately, this means considering infinitely many cases : indeed, the rank of $\mathbb{Z}_K^\times$ is 1 according to Dirichlet's theorem 5.4.2, so $\mathbb{Z}_K^\times$ is infinite. To be precise, the only roots of unity in K are ± 1 since K is not totally complex, so $\mathbb{Z}_K^\times \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

However, we see that $\mathbb{Z}_K^\times/(\mathbb{Z}_K^\times)^3 \simeq \frac{\mathbb{Z}/2\mathbb{Z}}{3(\mathbb{Z}/2\mathbb{Z})} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \simeq \mathbb{Z}/3\mathbb{Z}$, so if u is any unit which is not the cube of a unit, then its image in $\mathbb{Z}_K^\times/(\mathbb{Z}_K^\times)^3$ generates this quotient, so every unit is of the form $u^i w^3$ for some $w \in \mathbb{Z}_K^\times$ and some unique $i \in \{0, 1, 2\}$. Thus, if $\mathfrak{p}_2$ were principal, we could write $\beta^3 = 2v = 2u^i w^3$,

whence $2u^i = (\beta w^{-1})^3$ so either 2, $2u$ or $2u^2$ would be a cube in $\mathbb{Z}_K$. We are going to prove that none of these three possibilities can occur. Note how we are led to studying the units of K in order to reduce the study of its class group from infinitely many cases to finitely many cases.

We need to find a unit u which is not the cube of a unit. In the table of values of $P(x)$, we spot that $P(-1) = -1$, which shows that $u = \alpha + 1$ is a unit. To prove that it is not the cube of a unit, we have two possibilities: reduce u modulo a prime ideal and prove that it is not a cube in the quotient, or write u in terms of a fundamental unit of K.

As an example of the first method, we can try to prove that the image of u mod $\mathfrak{p}_7$ is not a cube in $\mathbb{Z}_K/\mathfrak{p}_7 \simeq \mathbb{F}_7$ (we have skipped $\mathfrak{p}_2$ and $\mathfrak{p}_3$ because everybody is a cube in $\mathbb{F}_2$ and in $\mathbb{F}_3$, and we have skipped $\mathfrak{p}_{5^3}$ because we prefer to stay away from[3] $\mathbb{F}_{5^3}$). Unfortunately, $\alpha+2 \in \mathfrak{p}_7$, so α reduces to -2 mod $\mathfrak{p}_7$, so $u = \alpha + 1$ reduces mod $\mathfrak{p}_7$ to -1 which is a cube in $\mathbb{Z}_K/\mathfrak{p}_7 \simeq \mathbb{F}_7$, so we cannot conclude anything. Let us try another prime. We do not want to work in $\mathbb{F}_{7^2}$ nor in $\mathbb{F}_{11^3}$, so let us try to find a new prime of inertial degree 1. We see in the table that $P(-3)$, $P(1)$ and $P(2)$ are all divisible by 13, which means that $P(x) \equiv (x+3)(x-1)(x-2)$ mod 13. As a result, 13 splits completely,

$$13\mathbb{Z}_K = (13, \alpha + 3)(13, \alpha - 1)(13, \alpha - 2) = \mathfrak{p}_{13}\mathfrak{p}'_{13}\mathfrak{p}''_{13}.$$

The image of α modulo these primes is -3, 1 and 2 respectively, so the image of u is -2, 2 and 3 respectively. This time, we are in luck: -2 is not a cube in $\mathbb{F}_{13}$, so u is not a cube in $\mathbb{Z}_K$. (In fact, neither -2 nor 2 nor 3 are cubes in $\mathbb{F}_{13}$, so we have three distinct proofs of the fact that u is not a cube).

As a example of the second method, we can use the method from assignment 5 to prove that u is actually a fundamental unit of K; in particular, it is not a cube (nor a square, nor a fifth power, ...).

Anyway, we now have that if $\mathfrak{p}_2$ were principal, then either 2, $2u$ or $2u^2$ would be a cube in $\mathbb{Z}_K$. But we have already seen that $u \equiv -1$ mod $\mathfrak{p}_7$, so 2, $2u$ and $2u^2$ reduce to 2, -2 and 2 mod $\mathfrak{p}_7$. But neither 2 nor -2 is a cube in $\mathbb{F}_7$, so we finally conclude that $\mathfrak{p}_2$ is not principal, and that $\mathrm{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$ is a cyclic group of order 3 generated by $[\mathfrak{p}_2]$. Phew!

[3]Actually it is not so hard to work in $\mathbb{F}_{5^3}$, it is even pretty easy; but I don't want to scare you!

Example 7.4.2. Here is another example with almost the same polynomial, to show how delicate an invariant the class group is, and also to illustrate more how to play with units.

Let $P(x) = x^3 - 6x + 6 \in \mathbb{Z}[x]$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $P(x)$.

Just as in the previous example, we find that $P(x)$ is irreducible over $\mathbb{Q}$ because it is Eisenstein at 2 (and also at 3), so K is a well-defined number field. We compute that $\operatorname{disc} P(x) = -2^2 3^3$, and since $P(x)$ is Eisenstein at 2 and 3, we again conclude that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and $\operatorname{disc} K = -2^2 3^3$.

Besides, the sign of $\operatorname{disc} K$ shows that the signature of K is again $(1, 1)$, so the Minkowski bound has the same shape as before. But this time the discriminant is much smaller, so the Minkowski bound is

$$\frac{3!}{3^3}\left(\frac{4}{\pi}\right)^1 \sqrt{2^2 3^3} = 2.94\ldots$$

only. This means that $\mathrm{Cl}(K)$ is generated by the primes above 2 only. But we know by Eisenstein's criterion 3.8.4 that 2 is totally ramified in K,

$$2\mathbb{Z}_K = \mathfrak{p}_2^3,$$

so $\mathrm{Cl}(K)$ is cyclic and generated by the class of $\mathfrak{p}_2$. If we make a table of values of $P(x)$ at small integers, we spot that $P(2) = 2$, which tells us that $N_{\mathbb{Q}}^K(\alpha - 2) = \pm 2$, so that the norm of the ideal $(\alpha - 2)$ is 2 by proposition 3.4.3. Since $\mathfrak{p}_2$ is the only prime of norm 2, this means that $\mathfrak{p}_2 = (\alpha - 2)$. In particular, $\mathfrak{p}_2$ is principal, so the class group of K is trivial, i.e. $\mathbb{Z}_K$ is a PID this time.

The relation $\mathfrak{p}_2 = (\alpha - 2)$ also tells us that $(2) = ((\alpha - 2)^3)$, so $u = \frac{(\alpha-2)^3}{2} = -3\alpha^2 + 9\alpha - 7$ is a unit. As in the previous example, we have $\mathbb{Z}_K^\times = \{\pm\varepsilon^n,\ n \in \mathbb{Z}\}$ for some fundamental unit ε, so could our unit u be a fundamental unit (i.e. one of $\pm\varepsilon^{\pm 1}$)? Let $\sigma\colon K \hookrightarrow \mathbb{R}$ be the unique real embedding of K in $\mathbb{R}$. The real root of $P(x)$ is $-2.85\cdots = \sigma(\alpha)$, so the image of $\sigma(u) = -56.95\ldots$, whereas the method from assignment sheet 5 merely tells us that there exists a fundamental unit ε of K such that $\sigma(\varepsilon) > 2.76\ldots$. Therefore, all that we can say is that $u = -\varepsilon^n$ for some $n \in \{1, 2, 3\}$. In fact, $56.95\ldots$ is so much larger than $2.76\ldots$ that we can legitimately suspect that our unit u is in fact **not** a fundamental unit.

Actually, if we have made a table of values of $P(x)$, we spot that $P(1) = 1$, which means that $N_{\mathbb{Q}}^K(\alpha - 1) = \pm 1$, so that $u' = \alpha - 1$ is also a unit. But

$\sigma(u') = -3.85\ldots$, so we must have $u' = -\varepsilon$, so u' is also a fundamental unit. This confirms our doubts: since $u \neq \pm\varepsilon^{\pm 1}$ (this is obvious under σ), u is not a fundamental unit. In fact, by computing the powers of u', we find that $u = u'^3 = -\varepsilon^3$.

But this means that $2 = \left(\frac{\alpha-2}{u'}\right)^3 = (\alpha^2+\alpha-4)^3$ is a cube in K, i.e. that K contains a subfield isomorphic to $\mathbb{Q}(\sqrt[3]{2})$. Actually, we have $[K:\mathbb{Q}] = 3 = [\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]$ (because x^3-2, being Eisenstein at 2, is irreducible over $\mathbb{Q}$), so that K is in fact isomorphic to $\mathbb{Q}(\sqrt[3]{2})$, an isomorphism being given explicitly by

$$\begin{array}{ccc} \mathbb{Q}(\sqrt[3]{2}) & \xrightarrow{\sim} & K \\ \displaystyle\sum_{j=0}^{2} \lambda_j (\sqrt[3]{2})^j & \longmapsto & \displaystyle\sum_{j=0}^{2} \lambda_j \beta^j \end{array}$$

where $\beta = \alpha^2+\alpha-4$ and the λ_j lie in $\mathbb{Q}$.

If we want, we can compute the reverse isomorphism; this amounts to expressing α in terms of β (which must be possible because the above description of the isomorphism implies that β is a primitive element of K). In general, this kind of rewriting process can be done by computing the powers of β as polynomials in α, and by performing linear algebra over $\mathbb{Q}$. For example, in our case, we know that α must be a polynomial in β of degree at most 2 (since we are in a field of degree 3), so we compute that $\beta^2 = -\alpha^2 - 2\alpha + 4$, and we try to write α as a linear combination of 1, β and β^2. We find that $\alpha = -\beta^2 - \beta$, which means that the reverse isomorphism is

$$\begin{array}{ccc} K & \xrightarrow{\sim} & \mathbb{Q}(\sqrt[3]{2}) \\ \displaystyle\sum_{j=0}^{2} \lambda_j \alpha^j & \longmapsto & \displaystyle\sum_{j=0}^{2} \lambda_j (-\sqrt[3]{2}^2 - \sqrt[3]{2})^j. \end{array}$$

By looking at the image of $\varepsilon = 1-\alpha$ under this isomorphism, we can infer that $\sqrt[3]{2}^2 + \sqrt[3]{2} + 1$ is a fundamental unit in $\mathbb{Q}(\sqrt[3]{2})$, so that

$$\text{Regulator}(K) = \text{Regulator}\big(\mathbb{Q}(\sqrt[3]{2})\big) = \log(\sqrt[3]{2}^2 + \sqrt[3]{2} + 1) = 1.347\cdots.$$