Algebraic number theory
Exercise sheet for chapter 5

Nicolas Mascot (n.a.v.mascot@warwick.ac.uk)
Aurel Page (a.r.page@warwick.ac.uk)
TA: Pedro Lemos (lemos.pj@gmail.com)

Version: March 2, 2017

Answers must be submitted by Friday March 18, 14:00

Exercise 1 (30 points)**.** Let $K = \mathbb{Q}(\sqrt{21})$.

1. (5 points) Let $u = 55 + 12\sqrt{21}$. Prove that $u \in \mathbb{Z}_K^\times$.
2. (15 points) Is u a fundamental unit of $\mathbb{Z}_K^\times$?
3. (10 points) Prove that for all $v \in \mathbb{Z}_K^\times$, $N_{\mathbb{Q}}^K(v) = 1$.

Exercise 2 (70 points)**.** Let K be a number field of degree 3 such that $\operatorname{disc} K < 0$.

1. (5 points) Prove that the signature of K is $(1, 1)$.
2. (10 points) From now on, we use the unique real embedding of K to view it as a subfield of $\mathbb{R}$. Prove that there exists $\varepsilon \in K$ such that $\mathbb{Z}_K^\times = \{\pm\varepsilon^n,\ n \in \mathbb{Z}\}$. Why can we assume that $\varepsilon > 1$? We make this assumption from now on.
3. (10 points) Express the regulator of K in terms of ε.
4. (15 points) Prove that ε is a primitive element for K, and deduce that the minimal polynomial of ε factors as $(x - \varepsilon)(x - u^{-1}e^{i\theta})(x - u^{-1}e^{-i\theta})$ for some $\theta \in \mathbb{R}$, where $u = \sqrt{\varepsilon}$.
5. (15 points) Given that

$$\left(\frac{u^3 + u^{-3}}{2} - \cos\theta\right)^2 \sin^2\theta < \frac{u^6}{4} + \frac{3}{2}$$

for all $\theta \in \mathbb{R}$ (you are **NOT** required to prove this), prove that

$$\varepsilon > \sqrt[3]{\frac{|\operatorname{disc} K|}{4} - 6}.$$

Hint: Prove that

$$\operatorname{disc} \mathbb{Z}[\varepsilon] = -16 \left(\frac{u^3 + u^{-3}}{2} - \cos\theta \right)^2 \sin^2\theta.$$

6. (15 points) Application: given that $\sqrt[3]{151/4} \approx 3.354$ and that the complex roots of $x^3 - 5x + 5 = 0$ are $-2.627\dots$ and $1.314\dots \pm 0.421\dots i$, find a fundamental unit for $K = \mathbb{Q}(\alpha)$, where $\alpha^3 - 5\alpha + 5 = 0$.

   Hint: Prove first that the decomposition of 5 *in* $\mathbb{Z}_K$ *is* $5\mathbb{Z}_K = (\alpha)^3$, *use this to find a nontrivial unit in* K, *and prove that this unit is a fundamental unit.*

UNASSESSED QUESTIONS

The next questions are not worth any points. I still recommend you to try to solve them, for practice. Correction will be available online, just like for the marked questions.

Exercise 3. What are the possible values of $\#W_K$ for K a number field of degree 4? Give an example for each possible value.

Exercise 4 (The battle of Hastings)**.** *"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (...) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."*

How many troops does this fictional historical text[1] suggest Harold II had at the battle of Hastings?

[1] Cf. problem no. 129 in *Amusement in Mathematics* (H.E. Dundeney, 1917).