Algebraic number theory
Exercise sheet for chapter 3

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Answers must be submitted by Friday February 26, 14:00

Exercise 1 (70 points)

Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 - 5\alpha + 5 = 0$.

1. (10 points) Prove that the ring of integers of K is $\mathbb{Z}[\alpha]$.
2. (5 points) Which primes $p \in \mathbb{N}$ ramify in K ?
3. (21 points) For $n \in \mathbb{N}$, $n \leqslant 7$, compute explicitly the decomposition of $n\mathbb{Z}_K$ as a product of prime ideals.
4. (8 points) Prove that the prime(s) above 5 are principal, and find explicitly a generator for them.
5. (16 points) List the ideals $\mathfrak{a}$ of $\mathbb{Z}_K$ such that $N(\mathfrak{a}) \leqslant 7$.
6. (10 points) Compute and factor explicitly the different of K.

Exercise 2 (30 points)

Let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive 90^{th} root of 1.

1. (3 points) What is the degree of K ?
2. (5 points) Which primes $p \in \mathbb{N}$ ramify in K ?
3. (12 points) For $p = 2, 3, 5, 7$, describe[1] how p decomposes in K.

[1] By this, I mean say how many primes there are above p, and what their ramification index and inertial degree are. Unlike in the previous exercise, you are **NOT** required to compute these primes explicitly.

4. (2 points) Give an example of a prime $p \in \mathbb{N}$ which splits completely in K.
5. (8 points) Does there exist a prime $p \in \mathbb{N}$ which is inert in K ?

UNASSESSED QUESTION

The next question is not worth any points. I still recommend you to try to solve it, for practice. Correction will be available online, just like for the marked questions.

Exercise 3

Let K be a number field of degree n. Prove that if there exists a prime $p < n$ which splits completely in K, then $\mathbb{Z}_K$ is not of the form $\mathbb{Z}[\alpha]$ for any $\alpha \in K$.