Algebraic number theory
Exercise sheet for chapter 2

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Answers must be submitted by Friday February 12, 14:00

Exercise 1 (5 points)

Is $\frac{3+2\sqrt{6}}{\sqrt{6}-1}$ an algebraic integer ?

Exercise 2 (35 points)

Let $b, c, n \in \mathbb{Z}$, $n \geqslant 2$, and let $P(x) = x^n + bx + c$. Prove that

$$\operatorname{disc} P = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big).$$

Hint : start by proving that $\operatorname{disc} P = (-1)^{n(n-1)/2} n^n \prod_{k=0}^{n-2} \left(\left(1 - \frac{1}{n}\right) \beta \zeta^k b + c \right)$, where $\zeta = e^{2\pi i/(n-1)}$ and $\beta \in \mathbb{C}$ is such that $\beta^{n-1} = -b/n$.

Exercise 3 (60 points)

Let $K = \mathbb{Q}(\alpha)$, where α satisfies $\alpha^3 - \alpha - 4 = 0$.

1. (10 points) Compute the discriminant of $\mathbb{Z}[\alpha]$.

2. (20 points) At this point, what are the possibilities for $\operatorname{disc} K$?

3. (15 points) Prove that $\frac{\alpha^2+\alpha}{2}$ is an algebraic integer.

4. (15 points) What is the ring of integers of K ?