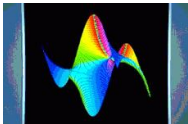


# Autour de l'évaluation numérique des fonctions D-finies

Marc MEZZAROBBA

Projet ALGORITHMS

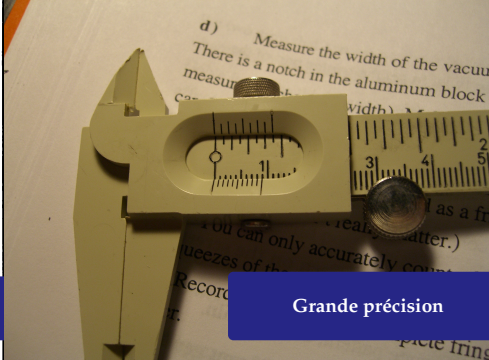


INRIA Paris Rocquencourt

Soutenance de thèse, 27 octobre 2011

0.9900	0.42345	08779	18527	0.83850	80695	55370
01	.42336	70387	10965	.83855	04104	51134
02	.42328	32076	37097	.83859	27429	63383
03	.42319	93846	98665	.83863	50670	92932
04	.42311	55698	97410	.83867	73828	40594
0.9905	0.42303	17632	35074	0.83871	96902	07183
06	.42294	79647	13396	.83876	19891	93512
07	.42286	41743	34116	.83880	42798	00397
08	.42278	03920	98971	.83884	65620	28651
09	.42269	66180	09698	.83888	88358	79088
0.9910	0.42261	28520	68035	0.83893	11013	52524
11	.42252	90942	75717	.83897	33584	49774
12	.42244	53446	54478	.83901	56071	71651
13	.42236	16031	46054	.83905	78475	18972
14	.42227	78698	12177	.83910	00794	92552
0.9915	0.42219	41446	34579	0.83914	23030	93207
16	.42211	04276	1			
17	.42202	67187	5			
18	.42194	30180	5			
19	.42185	93255	2			
0.9920	0.42177	56411	51354	0.83935	32955	31151

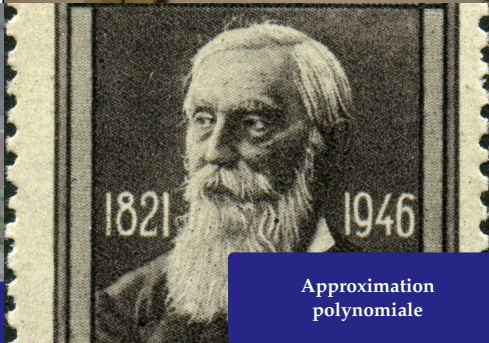
Introduction



Grande précision



Bornes



Approximation polynomiale

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## Introduction

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Calcul formel ?

# Calcul formel ?



approché  
précision fixée

# Calcul formel ?



approche  
précision fixée



calcul exact

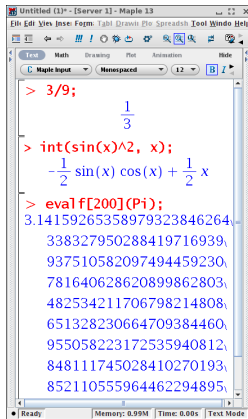
# Calcul formel ?



approché  
précision fixée



calcul exact



calcul exact  
et précision  
arbitraire

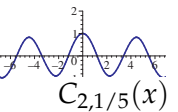
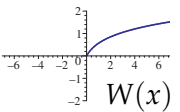
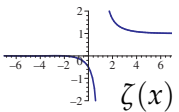
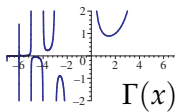
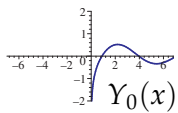
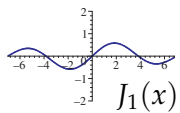
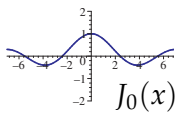
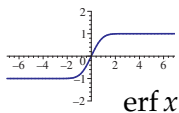
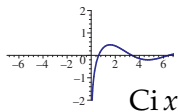
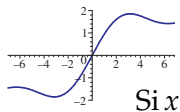
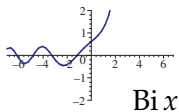
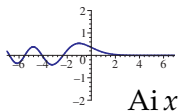
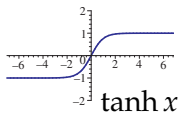
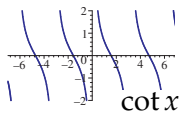
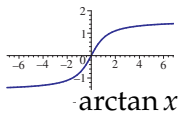
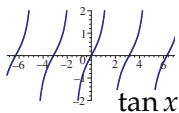
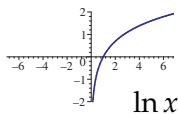
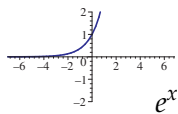
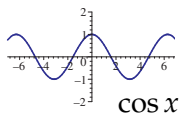
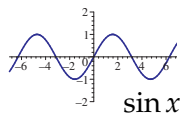
# Efficacité, complexité

$$\begin{array}{r} \leq n \text{ chiffres} \\ \longleftarrow \phantom{1} \phantom{2} \phantom{3} \phantom{4} \phantom{5} \phantom{6} \\ 1 \ 2 \ 3 \\ \times \quad 4 \ 5 \ 6 \\ \hline 7 \ 3 \ 8 \\ 6 \ 1 \ 5 \\ 4 \ 9 \ 2 \\ \hline 5 \ 6 \ 0 \ 8 \ 8 \\ \longleftarrow \phantom{1} \phantom{2} \phantom{3} \phantom{4} \phantom{5} \phantom{6} \\ \leq 2n \text{ chiffres} \end{array} \quad \begin{array}{l} \uparrow \\ \leq n \text{ lignes} \\ \downarrow \end{array}$$

- ▶ Multiplication de l'école primaire :  $O(n^2)$  opérations
- ▶ On peut faire mieux ! (Karatsuba, 1963)
- ▶ On peut multiplier deux nombres de  $n$  chiffres en  $O(n \log n \log \log n)$  opérations (Schönhage-Strassen 1971)



# Fonctions élémentaires, fonctions spéciales



# Fonctions D-finies

Une fonction  $y(z) : \mathbb{C} \rightarrow \mathbb{C}$  est **D-finie** (holonome) si elle est solution d'une équation différentielle linéaire (homogène) à coefficients polynomiaux :

$$a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z].$$

- ▶ La suite des coefficients du développement de Taylor d'une fonction D-finie satisfait une **réurrence** linéaire à coefficients polynomiaux.

**Exemple :**  $y(z) = \sin z$

$$y''(z) + y(z) = 0 \quad y(0) = 0, \quad y'(0) = 1$$

# Fonctions D-finies

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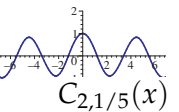
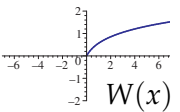
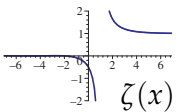
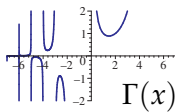
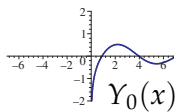
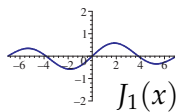
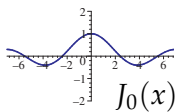
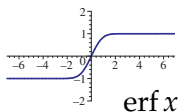
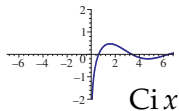
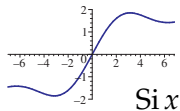
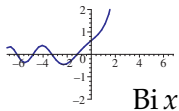
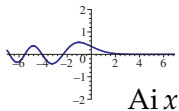
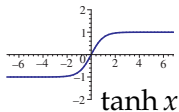
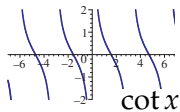
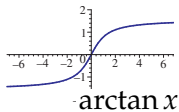
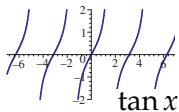
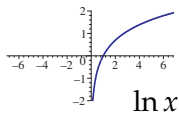
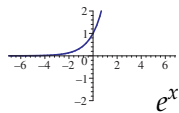
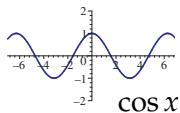
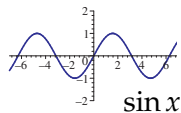
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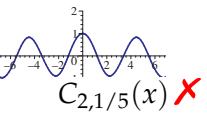
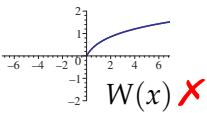
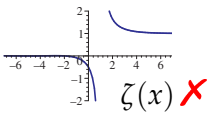
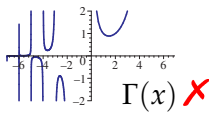
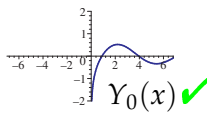
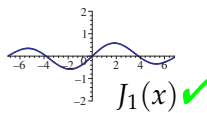
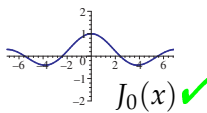
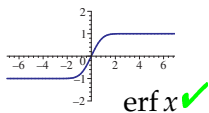
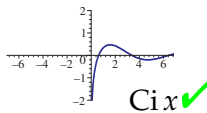
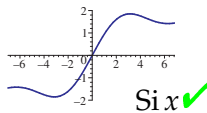
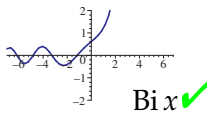
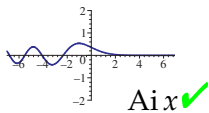
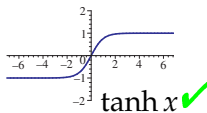
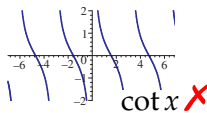
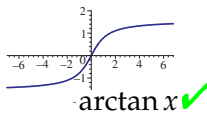
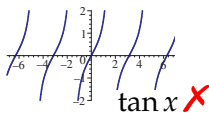
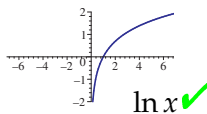
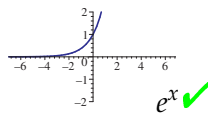
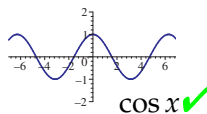
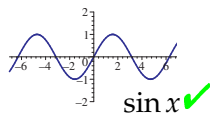
**Exemple :**  $y(z) = K_0(z)$  (fonction de Bessel modifiée)

$$z y''(z) + y'(z) - z y(z) = 0$$

# Fonctions élémentaires, fonctions spéciales

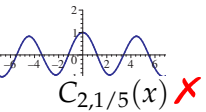
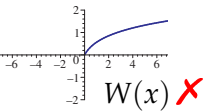
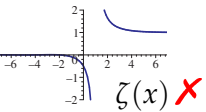
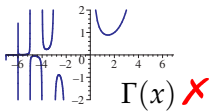
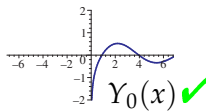
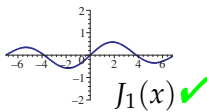
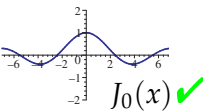
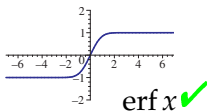
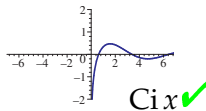
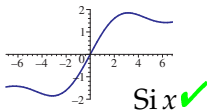
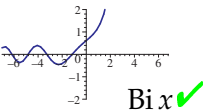
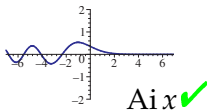
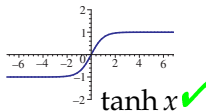
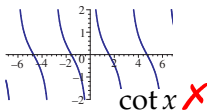
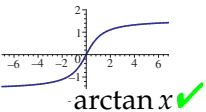
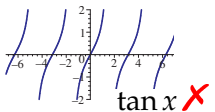
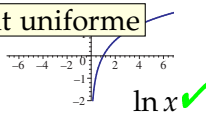
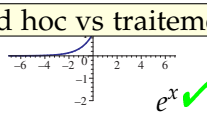
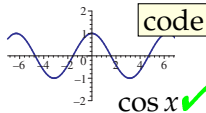
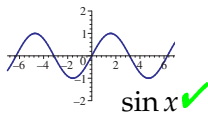


# Fonctions élémentaires, fonctions spéciales



# Fonctions élémentaires, fonctions spéciales

code ad hoc vs traitement uniforme



# Un dictionnaire des fonctions D-finies

Dynamic Dictionary of Mathematical Functions - Iceweasel

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Wikipedia.com

Home Glossary

## Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on [Mathematical Functions](#), with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Contents rendering [link](#)

**Select a special function from the list**

- [Help](#) on selecting and configuring the mathematical rendering
- [DDMF developers](#) list
- [Motivation](#) of the project
- The [inverse cosecant](#)  $\operatorname{arccsc}(x)$
- The [inverse cosine](#)  $\operatorname{arccos}(x)$
- The [inverse cotangent](#)  $\operatorname{arccot}(x)$
- The [inverse hyperbolic cosecant](#)  $\operatorname{arcsch}(x)$
- The [Airy function of the first kind](#)  $\operatorname{Ai}(x)$
- The [inverse secant](#)  $\operatorname{arcsec}(x)$
- The [inverse sine](#)  $\operatorname{arcsin}(x)$
- The [inverse tangent](#)  $\operatorname{arctan}(x)$
- The [Airy function \(of the second kind\)](#)  $\operatorname{Bi}(x)$
- The [hyperbolic cosine integral](#)  $\operatorname{Chi}(x)$
- The [cosine integral](#)  $\operatorname{Ci}(x)$
- The [cosine](#)  $\cos(x)$
- The [exponential integral](#)  $\operatorname{Ei}(x)$
- The [error function](#)  $\operatorname{erf}(x)$
- The [complementary error function](#)  $\operatorname{erfc}(x)$
- The [imaginary error function](#)  $\operatorname{erfi}(x)$

jsMath

Done Proxy: None zotero

# Un dictionnaire des fonctions D-finies

<http://ddmf.msr-inria.inria.fr>

Dynamic Dictionary of Mathematical Functions

Home Glossary

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Contents [rendering](#) [link](#)

**Select a special function from the list**

Benoit, Chyzak, Darrasse, Gerhold, M. & Salvy (2010)

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- [Motivation of the project](#)

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- The [imaginary error function](#)  $\operatorname{erfi}(x)$

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# Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st\_name=AiyAi&parameters={ }

[W] wikipedia (en)

[01] Loading...

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## The Special Function $Ai(x)$

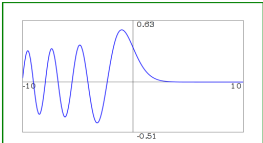
### 1. Differential equation rendering [link](#)

The function  $Ai(x)$  satisfies

$$\frac{d^2}{dx^2}y(x) - xy(x) = 0$$

with initial values  $y(0) = 1/3 \frac{\sqrt[3]{3}}{\Gamma(2/3)}$ ,  $(y')(0) = -1/2 \frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$ . [metadata](#)

### 2. Plot of $Ai(x)$



Done Proxy: None | zotero

jsMath

# Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st\_name=AiyA&parameters={}

[01] Loading...

Home Glossary

## The Special Function Ai(x)

**Données :**  
EDL à coeff. polynomiaux  
+ conditions initiales  
(fonction D-finie)

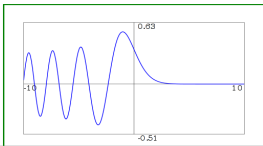
1. Differential equation

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2. Plot of  $Ai(x)$



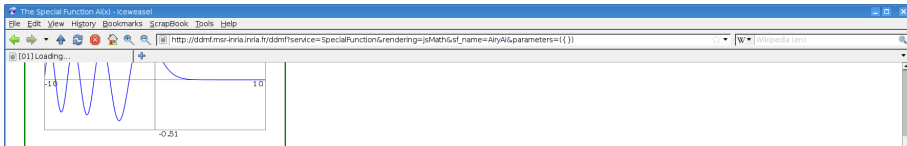
rendering [link](#)

[metadata](#)

jsMath

Done Proxy: None zotero

# Un dictionnaire des fonctions D-finies



min =

max =

## 3. Numerical Evaluation

$$\text{Ai}(1/4 + 1/4i) \approx 0.28881085 - 0.06285935i.$$

(Below, path may be either a point  $z$  or a broken-line path  $[z_1, z_2, \dots, z_n]$  along which to perform analytic continuation of the solution of the defining differential equation. Each  $z_k$  should be of the form  $x + y*i$ .) [metadata](#)

path =

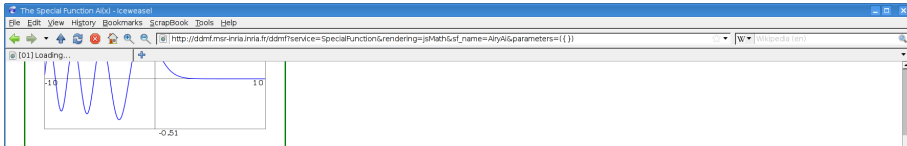
precision =

## 4. Taylor expansion of $\text{Ai}(x)$ at 0

- Expansion of  $\text{Ai}(x)$  at 0:

$$\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

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# Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

File Edit View History Bookmarks ScrapBook Tools Help

http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st\_name=AiryAi&parameters={}

[01] Loading...

min =  max =

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$$\text{Ai}(1/4 + 1/4i) \approx 0.28881085 - 0.06285935i.$$

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path =  precision =

### 4. Taylor expansion of $\text{Ai}(x)$ at 0

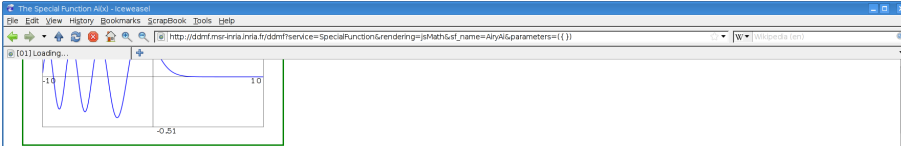
- Expansion of  $\text{AiryAi}$  at 0:

$$\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

me jsMath

Done Proxy: None zotero

# Un dictionnaire des fonctions D-finies



min =  max =

### 3. Numerical Evaluation

$Ai(1/4 + 1/4i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point  $z$  or a broken-line path  $[z_1, z_2, \dots, z_n]$  along which to perform analytic [metadata](#) continuation of the solution of the defining differential equation. Each  $z_k$  should be of the form  $x + y*i$ .)

path =  precision =

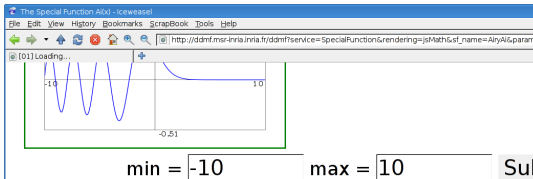
### 4. Taylor expansion of $Ai(x)$ at 0

- Expansion of  $AiryAi$  at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

mejsMath

# Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
- ▶ résultats garantis
- ▶ précision arbitraire
- ▶ efficace

## 3. Numerical Evaluation

$Ai(1/4 + 1/4i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point  $z$  or a broken-line path  $[z_1, z_2, \dots, z_n]$  along which to perform analytic continuation of the solution of the defining differential equation. Each  $z_k$  should be of the form  $x + y*i$ .) [metadata](#)

path = `1/4+1/4*i` precision = `80` `Submit Query`

## 4. Taylor expansion of $Ai(x)$ at 0

- Expansion of  $AiryAi$  at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

# Un dictionnaire des fonctions D-finies

▶ à partir de l'équa. diff.

▶ résultats garantis

▶ précision arbitraire

▶ efficace

36861749378392647020710083742 – 0.062859346556545730232761436943988956545624961055148330

form analytic [metadata](#)

orm x + y\*i.)

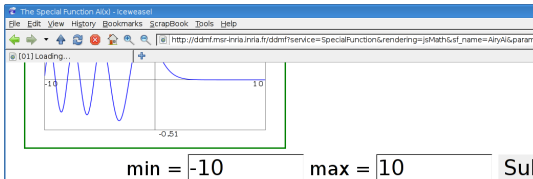
[metadata](#)

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Done Proxy: None zotero



# Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
- ▶ résultats garantis
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## 3. Numerical Evaluation

$Ai(1/4 + 1/4i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

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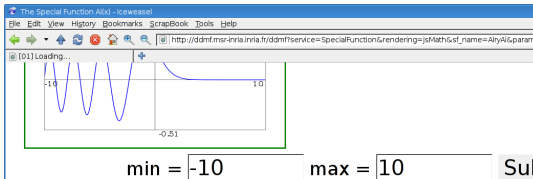
path = `1/4+1/4*i` precision = `80` `Submit Query`

## 4. Taylor expansion of $Ai(x)$ at 0

- Expansion of  $AiryAi$  at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

# Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
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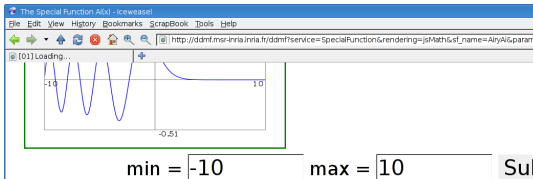
path =  $1/4+1/4*i$  precision = 80 Submit Query

## 4. Taylor expansion of $Ai(x)$ at 0

- Expansion of  $AiryAi$  at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{9} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

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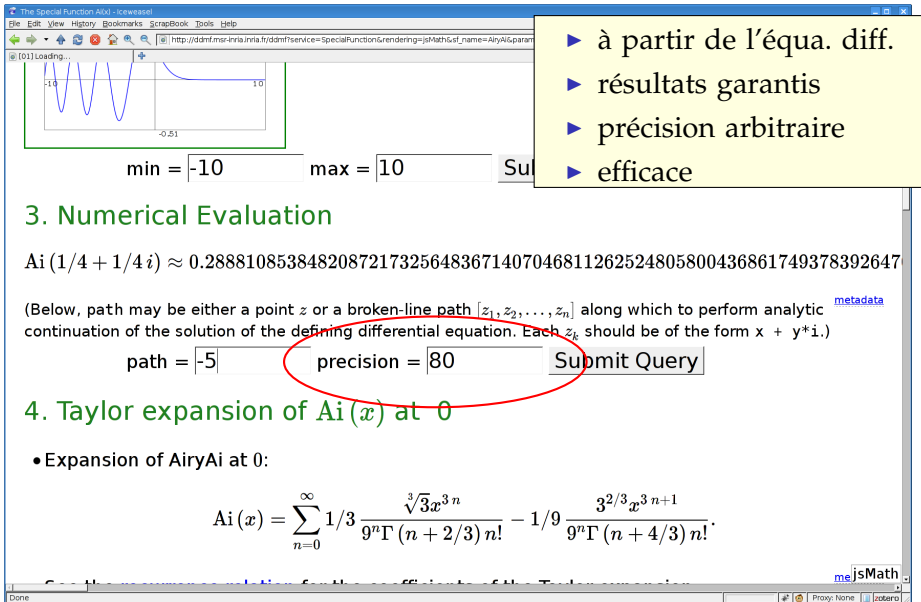
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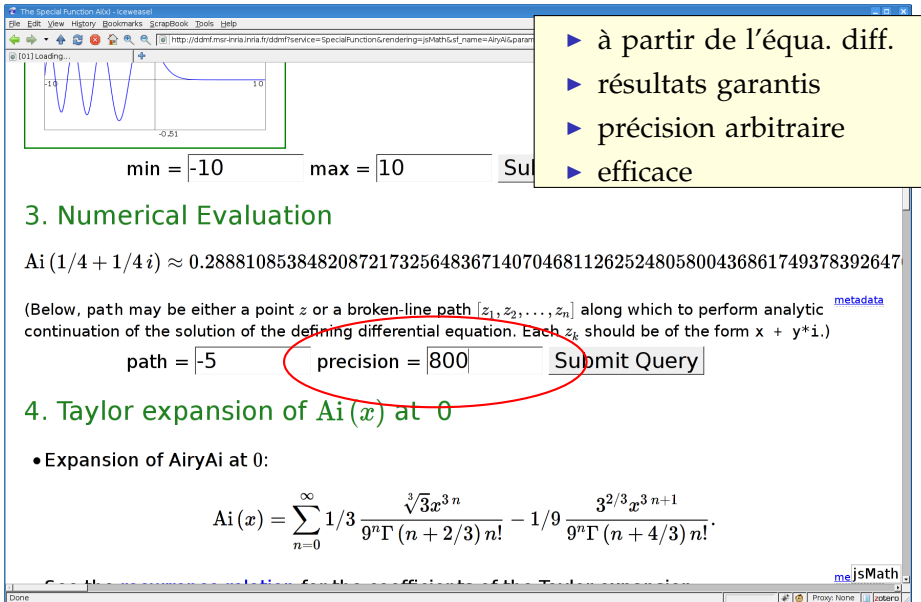
path =  precision =

### 4. Taylor expansion of $Ai(x)$ at 0

- Expansion of AiryAi at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

# Un dictionnaire des fonctions D-finies



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min =  max =

### 3. Numerical Evaluation

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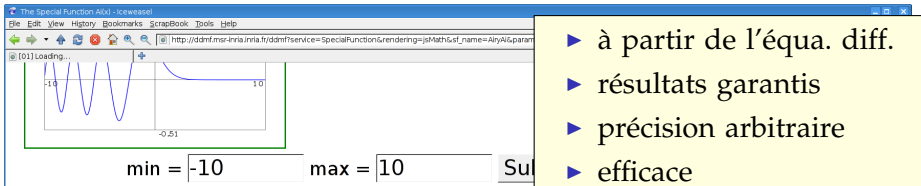
path =  precision =

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$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

# Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
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- ▶ précision arbitraire
- ▶ efficace

## 3. Numerical Evaluation

$Ai(-5) \approx 0.350761009024114319788016327696742221484443250893087208211128178049911192682$

(Below, path may be either a point  $z$  or a broken-line path  $[z_1, z_2, \dots, z_n]$  along which to perform analytic continuation of the solution of the defining differential equation. Each  $z_k$  should be of the form  $x + y*i$ .) [metadata](#)

path =  precision =

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# NumGfun



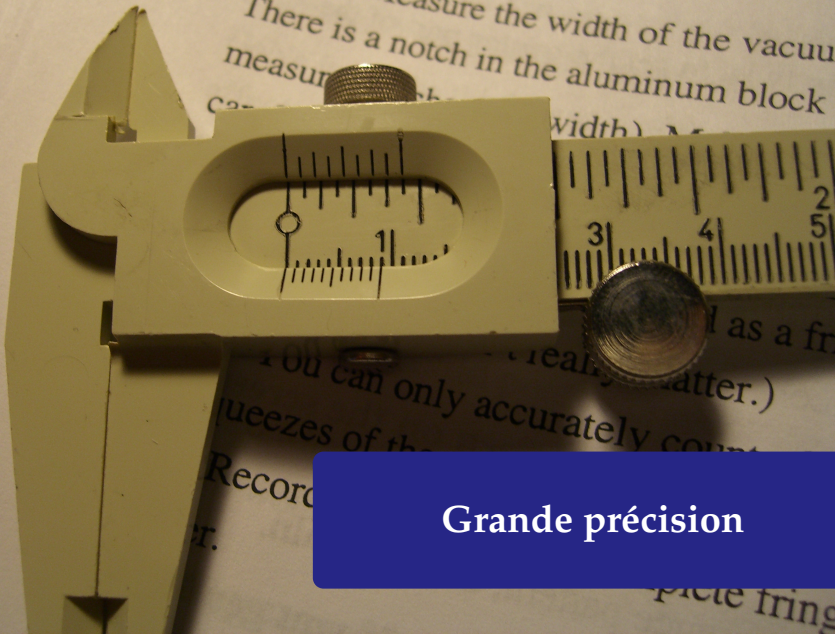
<http://algo.inria.fr/libraries/> (LGPL)



B. Salvy and P. Zimmermann. Gfun : a Maple package for the manipulation of generating and holonomic functions in one variable. ACM TOMS, 1994.



M. Mezzarobba. NumGfun : a Package for Numerical and Analytic Computation with D-finite functions. ISSAC 2010.



**Grande précision**



## Fonctions de Heun doublement confluentes

```
> diffeq := {diff(diff(y(z),z),z)+(2*z^3-  
z^2*a-2*z-a)/((z+1)^2*(z-1)^2)*diff(y(z)  
,z)+(z^2*b+z*c+2*z*a+d)*y(z)/((z-1)^3*  
(z+1)^3),y(0)=1,(D(y))(0)=0};
```

# Fonctions de Heun doublement confluentes

```
> diffeq := {diff(diff(y(z),z),z)+(2*z^3-  
z^2*a-2*z-a)/((z+1)^2*(z-1)^2)*diff(y(z)  
,z)+(z^2*b+z*c+2*z*a+d)*y(z)/((z-1)^3*  
(z+1)^3),y(0)=1,(D(y))(0)=0};
```

$$\text{diffeq} := \left\{ \frac{d^2}{dz^2} y(z) + \frac{(2z^3 - z^2 a - 2z - a) \left( \frac{d}{dz} y(z) \right)}{(z+1)^2 (z-1)^2} \right. \\ \left. + \frac{(z^2 b + z c + 2z a + d) y(z)}{(z-1)^3 (z+1)^3}, y(0) = 1, D(y)(0) = 0 \right\}$$

```
>
```

# Fonctions de Heun doublement confluentes

```
> diffeq := {diff(diff(y(z),z),z)+(2*z^3-  
z^2*a-2*z-a)/((z+1)^2*(z-1)^2)*diff(y(z)  
,z)+(z^2*b+z*c+2*z*a+d)*y(z)/((z-1)^3*  
(z+1)^3),y(0)=1,(D(y))(0)=0};
```

$$\text{diffeq} := \left\{ \frac{d^2}{dz^2} y(z) + \frac{(2z^3 - z^2 a - 2z - a) \left( \frac{d}{dz} y(z) \right)}{(z+1)^2 (z-1)^2} \right. \\ \left. + \frac{(z^2 b + z c + 2z a + d) y(z)}{(z-1)^3 (z+1)^3}, y(0) = 1, D(y)(0) = 0 \right\}$$

```
> a, b, c, d := 1, 1/3, 1/2, 3;
```

# Fonctions de Heun doublement confluentes

```
> diffeq := {diff(diff(y(z),z),z)+(2*z^3-  
z^2*a-2*z-a)/((z+1)^2*(z-1)^2)*diff(y(z),  
z)+(z^2*b+z*c+2*z*a+d)*y(z)/((z-1)^3*  
(z+1)^3),y(0)=1,(D(y))(0)=0};
```

$$\text{diffeq} := \left\{ \frac{d^2}{dz^2} y(z) + \frac{(2z^3 - z^2 a - 2z - a) \left( \frac{d}{dz} y(z) \right)}{(z+1)^2 (z-1)^2} + \frac{(z^2 b + z c + 2z a + d) y(z)}{(z-1)^3 (z+1)^3}, y(0) = 1, D(y)(0) = 0 \right\}$$

```
> a, b, c, d := 1, 1/3, 1/2, 3;
```

$$a, b, c, d := 1, \frac{1}{3}, \frac{1}{2}, 3$$

```
>
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[>
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z)):
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[>
```



## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[> myHeunD(1/3, 50);
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075  
[>
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075  
[>
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075  
[> myHeunD(1/3, 2000);
```

## Questions de précision

```
[> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069  
[> myHeunD := diffeqtoproc(diffeq, y(z));  
[> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075  
[> myHeunD(1/3, 2000);
```

(1.3 s plus tard...)

## Questions de précision

```
> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069
```

```
> myHeunD := diffeqtoproc(diffeq, y(z)):
```

```
> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075
```

```
> myHeunD(1/3, 2000);  
1.237157447563952539180078314058210003954474030520747249\  
77368122339910479272634279104260366917046868224326693\  
22058740005957868869065637255063771378117634825003548\  
.....  
96170152380808246265230916158732964496323766777357428\  
28214335810166903875586333320334746574757060060591160\  
33361999970684428816250827723506800809
```

```
>
```

## Questions de précision

```
> evalf[51](HeunD(a, b, c, d, 1/3));  
1.23715744756395253918007831405821000395447403052069
```

```
> myHeunD := diffeqtoproc(diffeq, y(z)):
```

```
> myHeunD(1/3, 50);  
1.23715744756395253918007831405821000395447403052075
```

```
> myHeunD(1/3, 2000);  
1.237157447563952539180078314058210003954474030520747249\  
77368122339910479272634279104260366917046868224326693\  
22058740005957868869065637255063771378117634825003548\  
00002007 1017 10000001000070101 10120020020 1000200 1\  
96170152380808246265230916158732964496323766777357428\  
28214335810166903875586333320334746574757060060591160\  
33361999970684428816250827723506800809
```

code plus général = moins de bugs !

## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));
```



## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763  
[>
```

## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763  
[> myHeunD(-0.9, 9);
```

## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763  
[> myHeunD(-0.9, 9);  
2.695836219  
[>
```

## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763  
[> myHeunD(-0.9, 9);  
2.695836219  
[>
```

## Évaluation au voisinage d'une singularité

```
[> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763  
[> myHeunD(-0.9, 9);  
2.695836219  
[> evalf(HeunD(a, b, c, d, -0.99));
```

# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
>
```

# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
> myHeunD(-0.99);
```

# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
> myHeunD(-0.99);  
4.6775585280
```

```
>
```



# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
> myHeunD(-0.99);  
4.6775585280
```

```
> myHeunD(-0.99, 500);
```

# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));
```

```
2.695836763
```

```
> myHeunD(-0.9, 9);
```

```
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));
```

```
Warning, breaking after 2000 terms, the series  
is not converging
```

```
undefined
```

```
> myHeunD(-0.99);
```

```
4.6775585280
```

```
> myHeunD(-0.99, 500);
```

(6.1 s plus tard...)

# Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
> myHeunD(-0.99);  
4.6775585280
```

```
> myHeunD(-0.99, 500);  
4.677558527966890481646371616414130565650323560409922037\  
.....  
89542201276207762696563032189351846152496641167932588\  
4660460023972873078881
```

```
>
```

## Évaluation au voisinage d'une singularité

```
> evalf(HeunD(a, b, c, d, -0.9));  
2.695836763
```

```
> myHeunD(-0.9, 9);  
2.695836219
```

```
> evalf(HeunD(a, b, c, d, -0.99));  
Warning, breaking after 2000 terms, the series  
is not converging  
undefined
```

```
> myHeunD(-0.99);  
4.6775585280
```

```
> myHeunD(-0.99, 500);  
4.677558527966890481646371616414130565650323560409922037\  
89542201276207762696563000100051010150100011107000588\  
4660460023972873078881
```

pas d'instabilité numérique  
(on paie en temps de calcul)

```
>
```

## Un exemple aléatoire

```
[> diffeq := random_diffeq(3, 2);
```

## Un exemple aléatoire

```
> diffeq := random_diffeq(3, 2);
```

$$\begin{aligned} \text{diffeq} := & \left\{ \left( \frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left( -\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & \left. \left. - \frac{1}{12} z^2 \right) \left( \frac{d}{dz} y(z) \right) + \left( -\frac{43}{60} + \frac{49}{60} z \right. \right. \\ & \left. \left. + \frac{11}{30} z^2 \right) \left( \frac{d^2}{dz^2} y(z) \right) + \left( -\frac{7}{12} + \frac{17}{30} z \right. \right. \\ & \left. \left. - \frac{3}{5} z^2 \right) \left( \frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \right. \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

## Un exemple aléatoire

```
> diffeq := random_diffeq(3, 2);
```

$$\begin{aligned} \text{diffeq} := & \left\{ \left( \frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left( -\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & \left. \left. - \frac{1}{12} z^2 \right) \left( \frac{d}{dz} y(z) \right) + \left( -\frac{43}{60} + \frac{49}{60} z \right. \right. \\ & \left. \left. + \frac{11}{30} z^2 \right) \left( \frac{d^2}{dz^2} y(z) \right) + \left( -\frac{7}{12} + \frac{17}{30} z \right. \right. \\ & \left. \left. - \frac{3}{5} z^2 \right) \left( \frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \right. \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

## Un exemple aléatoire

```
> diffeq := random_diffeq(3, 2);
```

$$\begin{aligned} \text{diffeq} := & \left\{ \left( \frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left( -\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & \left. \left. - \frac{1}{12} z^2 \right) \left( \frac{d}{dz} y(z) \right) + \left( -\frac{43}{60} + \frac{49}{60} z \right. \right. \\ & \left. \left. + \frac{11}{30} z^2 \right) \left( \frac{d^2}{dz^2} y(z) \right) + \left( -\frac{7}{12} + \frac{17}{30} z \right. \right. \\ & \left. \left. - \frac{3}{5} z^2 \right) \left( \frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \right. \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

```
0.0448555748776784313189330814759311548663  
+ 0.0199048983021280530504789772581099788282 I
```



## Grande précision

```
[> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

## Grande précision






```
[> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

(29 min plus tard...)

# Grande précision

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);  
0.033253281257567506772459381920024394391065961347292863\  
13611785593075654371610784719859620906805710762776061\  
65993844793918297941976188620650536691082179149605904\  
31080482988558239935175505111768194891591740446771304\  
74730251896359727561534310095807343639273056518962333\  
97217595138842309884016425632431029577130431472108646\  
95485154767624024297343851584414126056237771911489680\  
.....  
97933258259972366466573219602501650218139747781157348\  
78322628655747195818205282428148240800376913561455564\  
29598794491231828039584256430669932365880956101719727\  
33806130243940574539991121877851105270752378138422728\  
76176859592508040781771637205060431902227437673286901\  
71292574098466950906705927590030494460150099288210121\  
868701569
```

# Historique

-  Schroepel (1972) – Points particuliers
-  Brent (1976) – Fonctions particulières, points quelconques
-  Chudnovsky & Chudnovsky (1986-1988) – Méthode générale, esquisse points singuliers réguliers
-  van der Hoeven (1999, 2001) – Algorithme complet avec bornes
-  Ce travail – Mise en pratique : implémentation, améliorations d'efficacité, bornes optimales

# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

0 multiplication rapide

1 scindage binaire

2 prolongement analytique

3 *bit burst*

# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

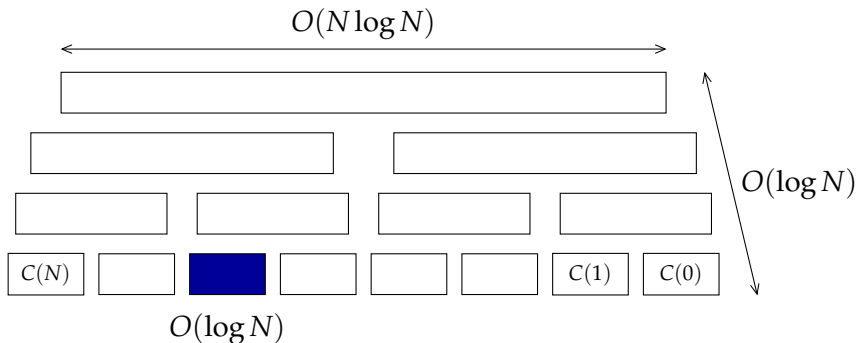
0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

Dans le disque de convergence d'un développement de Taylor :  
sommer efficacement la série (récurrence)



# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

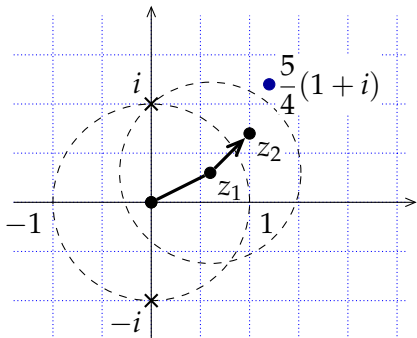
0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

Évaluation hors du disque de convergence :  
transporter les conditions initiales



$$\arctan\left(\frac{5}{4}(1+i)\right) = ?$$

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,570\dots+0,220\dots i \\ 0 & 0,728\dots-0,206\dots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0,365\dots+0,329\dots i \\ 0 & 0,751\dots-0,079\dots i \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$

# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

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Points donnés à grande précision :

prolongement analytique... à l'intérieur du disque de conv<sup>ce</sup>

$$z_0 = 10_2 \rightarrow z_1 = 10,1_2$$

$$\rightarrow z_2 = 10,101_2$$

$$\sin(e) = \sin(2,718\dots) = ?$$

$$\rightarrow z_3 = 10,1011011_2$$

$$\rightarrow z_4 = 10,101101110010100_2$$

$$\rightarrow \dots$$

$$\rightarrow z = 10.101101110010100110000\dots_2 \simeq e$$



# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

0 multiplication rapide

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2 prolongement analytique

3 *bit burst*

## Théorème (Chudnovsky<sup>2</sup>)

À  $z$  fixé, on peut calculer  $y(z)$  à  $2^{-n}$  près en

$$O\left(M\left(n \cdot (\log n)^3\right)\right)$$

opérations binaires.

# Algorithme de Chudnovsky et Chudnovsky

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## Théorème (Chudnovsky<sup>2</sup>, van der Hoeven)

À  $z$  fixé, on peut calculer  $y(z)$  à  $2^{-n}$  près en

$$O\left(M\left(n \cdot \cancel{(\log n)^3} (\log n)^2 \cdot \log \log n\right)\right)$$

opérations binaires.

# Algorithme de Chudnovsky et Chudnovsky

## Idées principales

0 multiplication rapide

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## Théorème (Chudnovsky<sup>2</sup>, van der Hoeven, M.)

À  $z$  fixé, on peut calculer  $y(z)$  à  $2^{-n}$  près en

$$O\left(M\left(n \cdot \cancel{(\log n)^3} (\log n)^2 \cdot \cancel{\log \log n}\right)\right)$$

opérations binaires.

# Améliorations

## L'algorithme de NumGfun

- ▶ Suivi détaillé des erreurs
  - ▶ Précision des calculs intermédiaires
  - ▶ Bornes fines pour les troncatures de séries
- ▶ « Facteur constant » (dépendance en l'équation)
  - ▶ Structure des matrices de récurrence
  - ▶ Calcul simultané efficace de plusieurs dérivées
- ▶ Évaluation aux points singuliers réguliers simplifiée
  - ▶ Méthode de Heffter-Poole simplifiée « à la Ore »
  - ▶ Algorithme de scindage binaire explicite
  - ▶ Traitement plus efficace des « logarithmes »

## Évaluation aux points singuliers réguliers

```
[ > diffeq := diffeqtohomdiffeq(  
    holexprtodiffeq(  
        arctan(z), y(z)), y(z));
```

# Évaluation aux points singuliers réguliers

```
> diffeq := diffeqtohomdiffeq(  
  holexprtodiffeq(  
    arctan(z), y(z)), y(z));
```

$$diffeq := \left\{ -2z \left( \frac{d}{dz} y(z) \right) + (-1 - z^2) \left( \frac{d^2}{dz^2} y(z) \right), y(0) = 0, \right.$$

$$\left. D(y)(0) = 1 \right\}$$

```
>
```

# Évaluation aux points singuliers réguliers

```
> diffeq := diffeqtohomdiffeq(  
  holexprtodiffeq(  
    arctan(z), y(z)), y(z));
```

$$\text{diffeq} := \left\{ -2z \left( \frac{d}{dz} y(z) \right) + (-1 - z^2) \left( \frac{d^2}{dz^2} y(z) \right), y(0) = 0, \right. \\ \left. D(y)(0) = 1 \right\}$$

```
> subs(z=z-I, evaldiffeq(diffeq, y(z),  
  [0, I], ord=3));
```

# Évaluation aux points singuliers réguliers

```
> diffeq := diffeqtohomdiffeq(  
  holexprtodiffeq(  
    arctan(z), y(z)), y(z));
```

$$\text{diffeq} := \left\{ -2z \left( \frac{d}{dz} y(z) \right) + (-1 - z^2) \left( \frac{d^2}{dz^2} y(z) \right), y(0) = 0, \right. \\ \left. D(y)(0) = 1 \right\}$$

```
> subs(z=z-I, evaldiffeq(diffeq, y(z),  
  [0, I], ord=3));
```

$$(-0.5000000000 I) \left( \ln(z-I) + \frac{1}{2} I (z-I) - \frac{1}{8} (z-I)^2 \right) \\ + (0.7853981634 + 0.3465735903 I)$$

```
>
```



II  
MEILEN  
BIS  
BERLIN






**Bornes**

# Motivation (I) : évaluation numérique

$$\sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{N-1} y_n z^n + \underbrace{\sum_{n=N}^{\infty} y_n z^n}_{?}$$

Calculer les ordres de troncature et autres bornes ?

-  Chudnovsky & Chudnovsky — Ordres de grandeur uniquement
-  van der Hoeven (1999, 2001, 2003) — Bornes données par la formule de Cauchy
-  Ce travail – Bornes asymptotiquement optimales

## Motivation (II) : suites récurrentes

### Permutations de Baxter

(OEIS A001181)

- ▶  $(n+2)(n+3)B_n = (7n^2 + 7n - 2)B_{n-1} + 8(n-1)(n-2)B_{n-2}$ ,  
 $B_0 = B_1 = 1$
- ▶  $B_n \leq 2,9 \cdot 8^n$

### Formule de Chudnovsky et Chudnovsky pour $\pi$

- ▶  $\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} t_k$   
où  $t_k = \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k}}$
- ▶  $\left| \frac{640320^{3/2}}{12\pi} - \sum_{k=0}^{n-1} t_k \right| \leq 10^6 (2,3n^3 + 13,6n^2 + 25n + 13,6) \alpha^n$   
où  $\alpha = \frac{1}{151931373056000} \simeq 0,66 \cdot 10^{-14}$

## Bornes « fines »

Entrée Récurrence + conditions initiales

$$\{p_s(n) y_{n+s} + \dots + p_0(n) y_n = 0, \quad y_0 = \dots\}$$

Sortie  $|y_n| \leq n!^{p/q} \alpha^n \varphi(n)$

avec  $\varphi$  sous-exponentielle, i.e.  $\varphi(n) = e^{o(n)}$

- ▶ borne correcte
- ▶ pour des conditions initiales génériques :  
 $p/q$  et  $\alpha$  optimaux (voire  $\varphi(n) = n^{O(1)}$ )

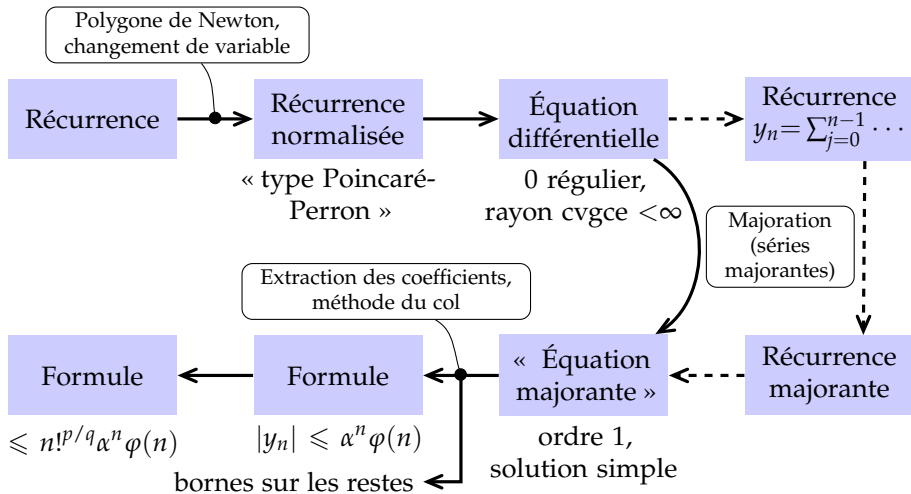
### Théorème

On peut calculer  $p/q$ ,  $\alpha$ ,  $\varphi$  remplissant ces conditions.



M. Mezzarobba and B. Salvy. Effective bounds for P-recursive sequences.  
JSC, 2010.

# Démarche



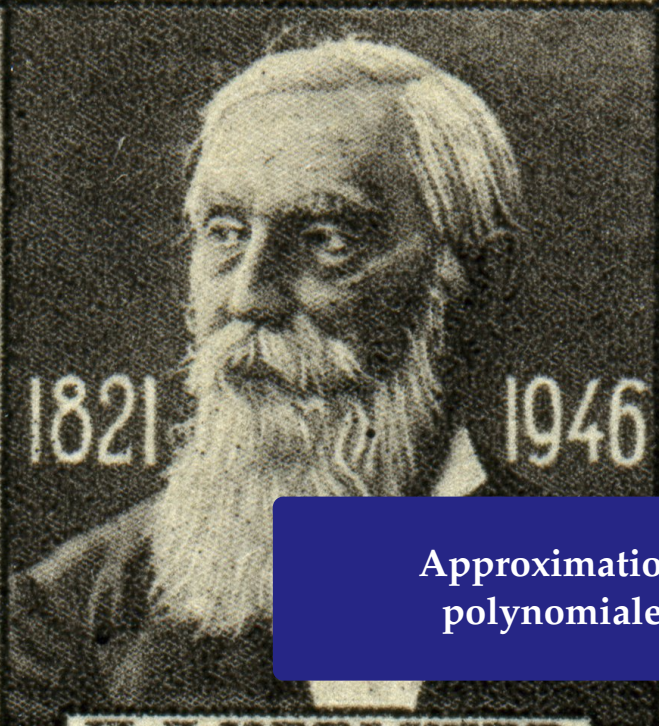
# Influence sur l'évaluation numérique

	$\frac{\operatorname{arccot}(z)}{(z^2-1)(z^2+5)} @ \frac{1}{2}$	$\psi(1/2)$	$\arctan \frac{9}{10}$	$\arctan \frac{99}{100}$
$10^{-10}$	64/27	40/23	336/164	4238/1496
$10^{-100}$	380/321	342/313	2338/2108	25210/21848
$10^{-1000}$	3392/3307	3336/3293	22050/21754	231844/227810

	$\frac{\exp(1/(1-z))}{(1-z)} @ \frac{1}{2}$	$\operatorname{Bi}(\frac{1}{1-z}) @ \frac{1}{2}$	$\operatorname{Ai}(\frac{1}{1-z}) @ \frac{3}{4}$	$\operatorname{Ai}(\frac{1}{1-z}) @ \frac{7}{8}$
$10^{-10}$	70/54	148/56	1558/77	23818/215
$10^{-100}$	418/387	664/416	3430/879	29258/2025
$10^{-1000}$	3568/3490	4700/3645	16284/8372	69594/18529

	$e^{-100}$	$\operatorname{erf}^2(1)$	$\operatorname{erf}(10)$	$\operatorname{erf}(100)$
$10^{-10}$	298/291	60/33	628/574	54492/54388
$10^{-100}$	456/450	190/163	936/894	54904/54800
$10^{-1000}$	1406/1402	1036/1011	2828/2800	58870/58772

nombre de termes calculé / nombre de termes minimal nécessaire



Approximation  
polynomiale

# Évaluations répétées

```
[> deq := holexprtodiffeq(AiryAi(z), y(z)):
```

```
[>
```



## Évaluations répétées

```
[> deq := holerprtodiffeq(AiryAi(z), y(z)):  
[> myAi := diffeqtoproc(deq, y(z),  
                        prec=12, disks=[[0,6]]):
```

## Évaluations répétées

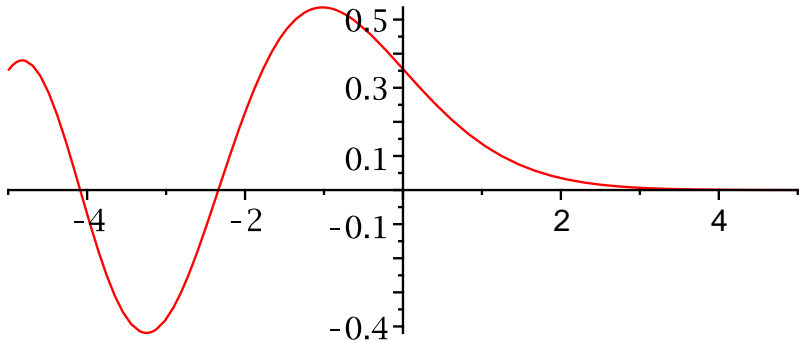
```
[> deq := hosexprtodiffeq(AiryAi(z), y(z)):
[> myAi := diffeqtoproc(deq, y(z),
                        prec=12, disks=[[0,6]]):
[>
```

## Évaluations répétées

```
[> deq := holerprtodiffeq(AiryAi(z), y(z)):
[> myAi := diffeqtoproc(deq, y(z),
                        prec=12, disks=[[0,6]]):
[> plot(myAi, -5..5);
```

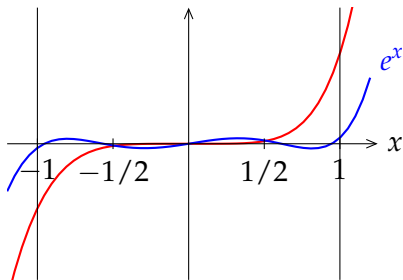
# Évaluations répétées

```
[> deq := hoalexprtodiffeq(AiryAi(z), y(z)):  
[> myAi := diffeqttoproc(deq, y(z),  
                           prec=12, disks=[[0,6]]):  
[> plot(myAi, -5..5);  
[>
```



# Séries de Taylor et séries de Tchebycheff

$$e^x - \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4\right)$$










$$e^x - (1,2660T_0(x) + 1,1303T_1(x) + 0,2714T_2(x) + 0,0443T_3(x) + 0,0054T_4(x))$$

## Approximation quasi-minimax

Pour toute fonction  $f$  suffisamment régulière,

$$\|f - p_d\|_\infty \leq \left( \frac{4}{\pi^2} \log(d+1) + 4 \right) \|f - p_d^*\|_\infty$$

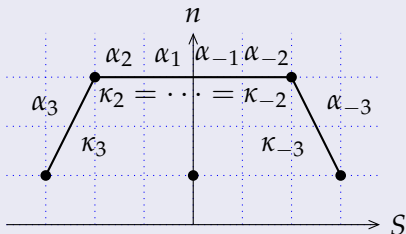
# Historique

- ▶ Calcul des coefficients de Tchebycheff
  -  Lánzos (1938) – méthode  $\tau$
  -  Clenshaw (1957) – calcul itératif à la Miller
- ▶ Récurrence
  -  Fox & Parker (1968) – petits ordres, lien avec Clenshaw
  -  Paszkowski (1975) – cas général
  -  Geddes (1977), Rebillard (1998), Benoit & Salvy (2009) – calcul formel
- ▶ Calcul par intervalles sur les séries de Tchebycheff
  -  Kaucher & Miranker (1984) – ultra-arithmétique
  -  Brisebarre & Joldeş (2010) – ChebModels

# Séries de Tchebycheff D-finies

## Difficultés

- ▶ Solutions (toujours) divergentes
- ▶ Conditions initiales ?
- ▶ Certification du résultat



## Démarche

1. Calcul des coefficients
2. Validation séparée



A. Benoit, M. Joldes and M. Mezzarobba. Rigorous uniform approximation of D-finite functions using Chebyshev expansions. In preparation.

# Calcul des coefficients à la Miller

## Exemple

$$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n T_n(x)$$

$$c_{n+1} + 2n c_n - c_{n-1} = 0$$

$$u_0 \approx -4,40 \cdot 10^{81}$$

$$u_1 \approx 1,96 \cdot 10^{81}$$


$$u_2 \approx -4,72 \cdot 10^{80}$$

⋮

$$u_{50} \approx 1,02 \cdot 10^2$$

$$u_{51} = 1$$

$$u_{52} = 0$$

$$c_n := u_n / S$$


$$c_0 \approx 1,27$$

$$c_1 \approx -5,65 \cdot 10^{-1}$$

$$c_2 \approx 1,36 \cdot 10^{-1}$$

⋮

$$c_{50} \approx 2,93 \cdot 10^{-80}$$

$$c_{51} \approx 2,88 \cdot 10^{-82}$$

$$c_{52} \approx 2,77 \cdot 10^{-84}$$

---

$$S = \sum_{n=-50}^{50} u_n T_n(0) \approx -3,48 \cdot 10^{81}$$



# Calcul des coefficients

Complexité **linéaire** en l'indice de départ  $N$ .

## Proposition (sous hypothèses simplificatrices)

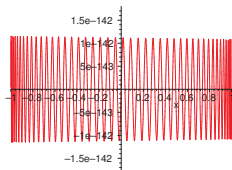
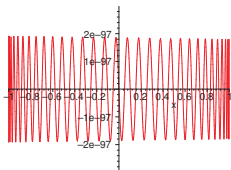
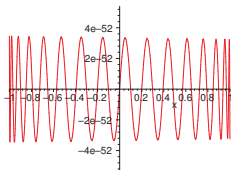
L'erreur sur les coefficients calculés

$$\max_{n=0}^N |c_n^{[N]} - c_n|$$

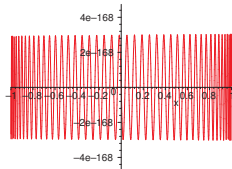
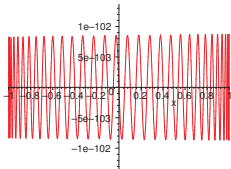
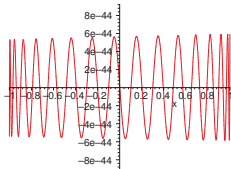
décroît **exponentiellement** avec  $N$ .

# Qualité des polynômes calculés

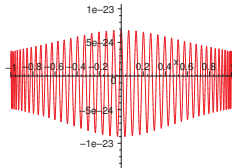
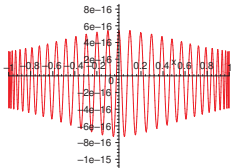
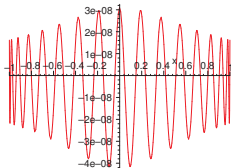
$$\frac{e^{x/2}}{\sqrt{x+16}}$$



$$\frac{3 \cos x - \sin x}{2}$$



$$e^{1/(1+2x^2)}$$



degré = 30

degré = 60

degré = 90

# Validation

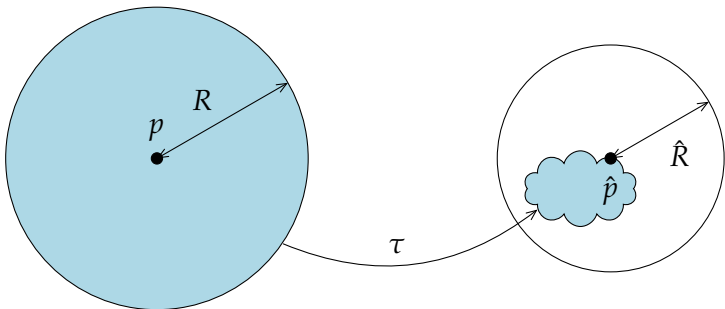
**Entrée** Opérateur différentiel, conditions initiales,  
polynôme  $p$  de degré  $d$ , précision  $\varepsilon$

**Sortie**  $R$  tel que  $\|y - p\|_\infty \leq R = O(\sqrt{d} (\|y^{(r-1)} - p^{(r-1)}\|_\infty + \varepsilon))$

$$\tau(y) := \left( x \mapsto y_0 + \int_0^x \frac{a(t)}{b(t)} y(t) dt \right)$$

$$\|\tau(f) - \tau(g)\|_\infty \leq \gamma \|f - g\|_\infty$$

$\gamma < 1$



# Validation

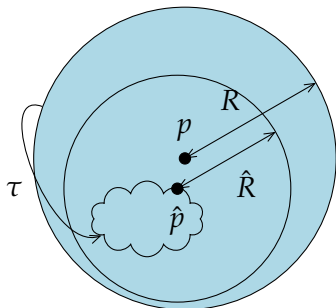
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$$\|p - \hat{p}\|_\infty + \hat{R} \leq R$$

# Validation

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## Algorithme

- ▶ Prendre  $i$  assez grand
- ▶ Calculer  $p_i \approx \tau^i(p)$
- ▶ Renvoyer  $R \geq \frac{\|p - p_i\|_\infty + (\text{erreurs})}{1 - \gamma_i}$

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- ▶ Prendre  $i$  assez grand
- ▶ Calculer  $p_i \approx \tau^i(p)$  O(d) ops
- ▶ Renvoyer  $R \geq \frac{\|p - p_i\|_\infty + (\text{erreurs})}{1 - \gamma_i}$  O(d) ops

## Qualité des bornes validées

$\frac{e^{x/2}}{\sqrt{x+16}}$	4,8	0,58	0,57
$\frac{3 \cos x - \sin x}{2}$	3,1	3,7	4,1
$e^{1/(1+2x^2)}$	0,57	0,56	0,56
	degré = 30	degré = 60	degré = 90

$$\text{Qualité} : \log_{10} \frac{B}{\|y - p\|_{\infty}}$$



Perspectives





## NumGfun et grande précision

- ▶ NumGfun 1.0
- ▶ Applications de l'évaluation aux points singuliers réguliers
- ▶ Moins de dépendance à Maple
- ▶ Meilleures performances, questions théoriques associées



## Approximation polynomiale garantie

- ▶ Combiner les deux résultats
- ▶ Séries de Gegenbauer
- ▶ Équations non-linéaires, complexité des méthodes symboliques-intervalles



## Calcul formel pour l'arithmétique des ordinateurs

- ▶ Analyse automatique des propriétés des fonctions
- ▶ Génération automatique de code C



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**Merci !**



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# Crédits

Source des photos :

- ▶ *Tables of the error function and its derivative*. US National Bureau of Standards, 1954 (domaine public)
- ▶ [http://commons.wikimedia.org/wiki/File:Calipers\\_in\\_physics\\_lab.jpg](http://commons.wikimedia.org/wiki/File:Calipers_in_physics_lab.jpg) (par User:Falcorian, Creative Commons Attribution-Share Alike)
- ▶ [http://en.wikipedia.org/wiki/File:Stamp\\_of\\_USSR\\_1047.jpg](http://en.wikipedia.org/wiki/File:Stamp_of_USSR_1047.jpg) (par User:Butko, repris de <http://kolekzioner.net/modules/smartsection/item.php?itemid=211>, domaine public)
- ▶ <http://en.wikipedia.org/wiki/File:Meilenstein-hohen-neuendorf.jpg> (par User:Saxo, domaine public)

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