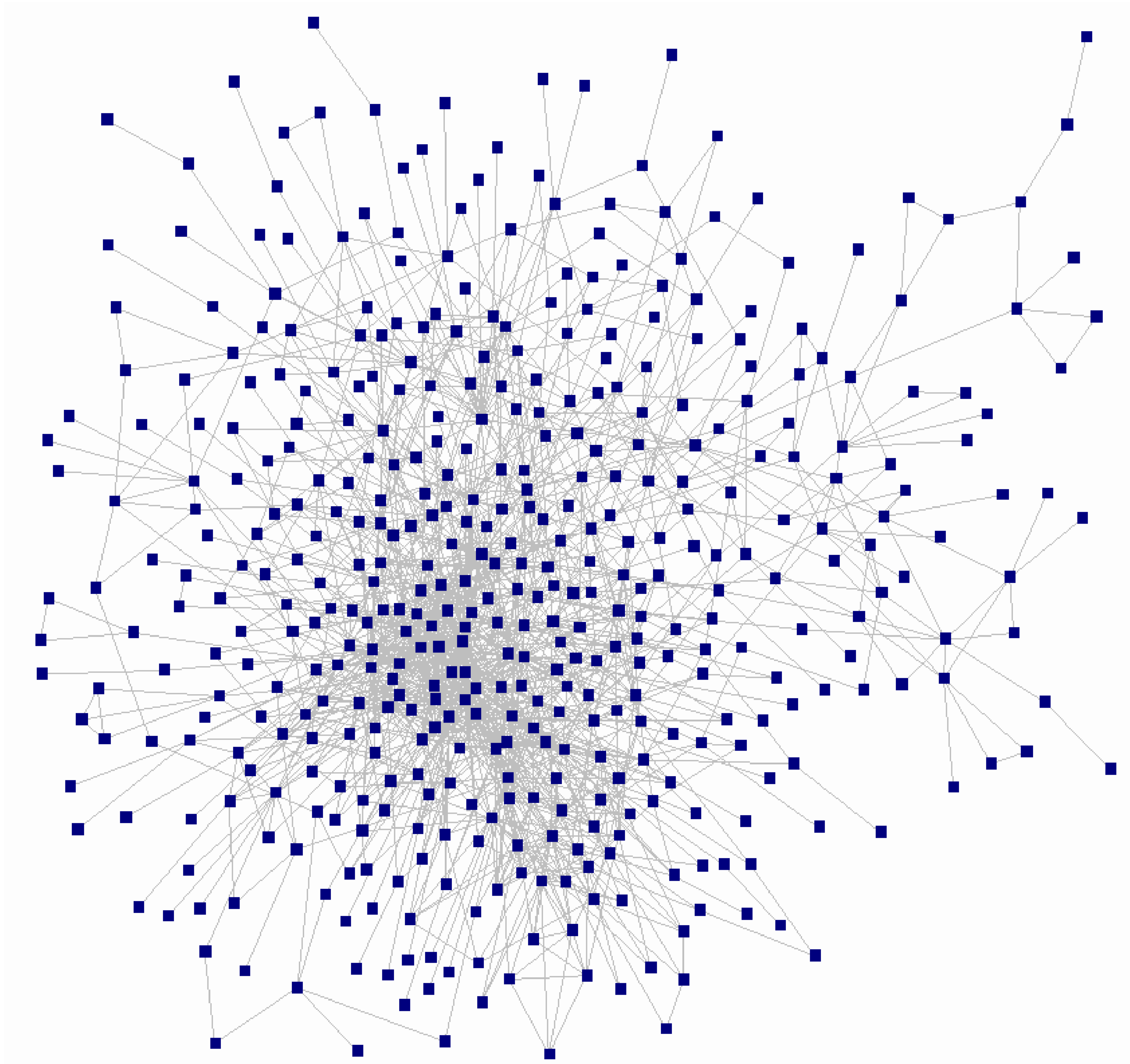
Random graphs: a probabilistic point of view

Laurent Ménard
Modal'X, Université Paris Ouest

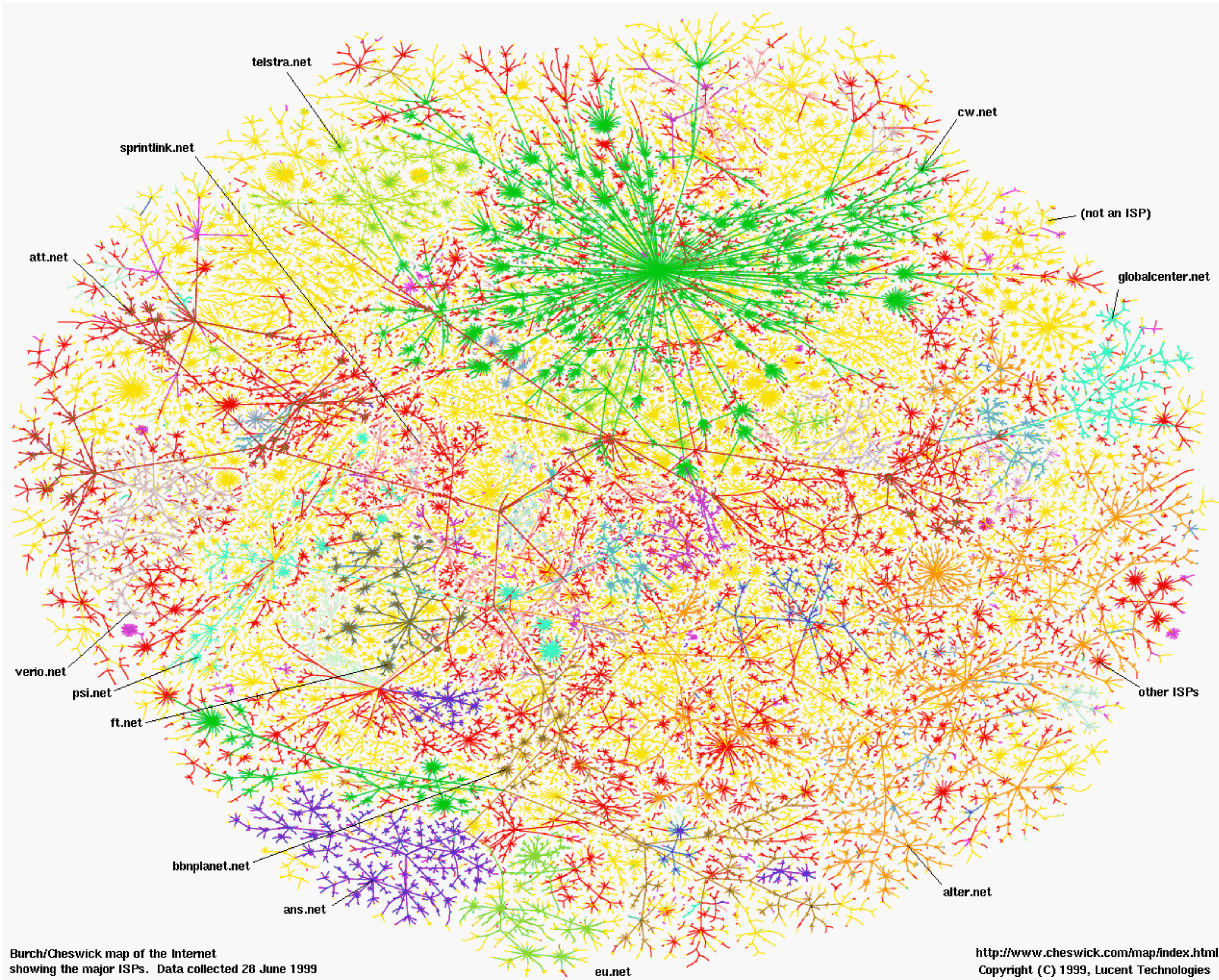
Stats in Paris, November 2013

"Real world" networks



Collaboration graph of mathematicians
[The Erdős number project, 2004]

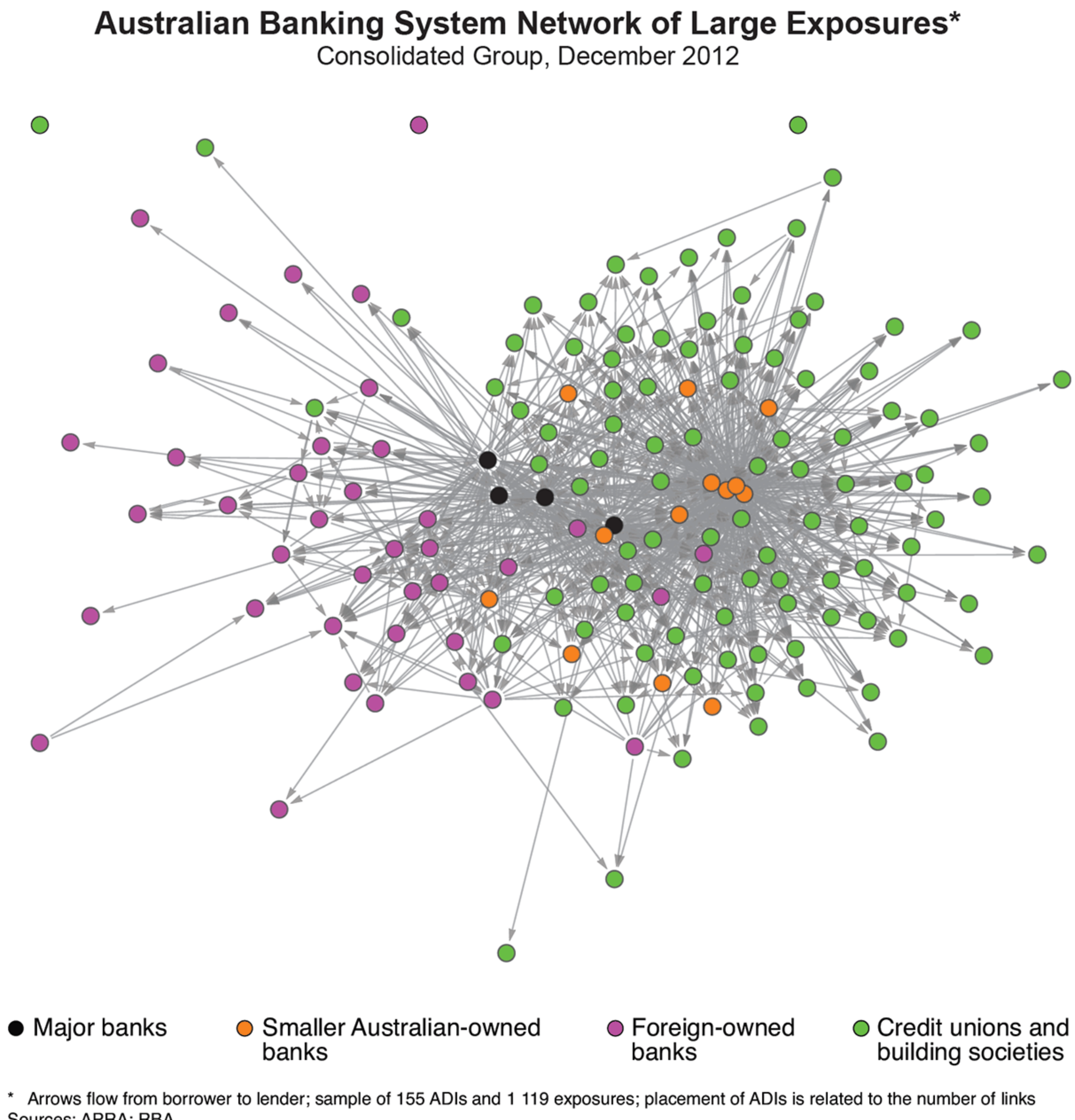
"Real world" networks



The internet topology in 1999

[The internet mapping project]

"Real world" networks



[Tellez 2013]

Outline

What are we looking for ?

Most common properties of "real world networks".

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Statistical mechanics on random graphs

Contagion models, systemic risk, first passage percolation, . . .

Modeling networks: Graph theory

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A (simple, undirected) graph $G = (V, E)$ consists of

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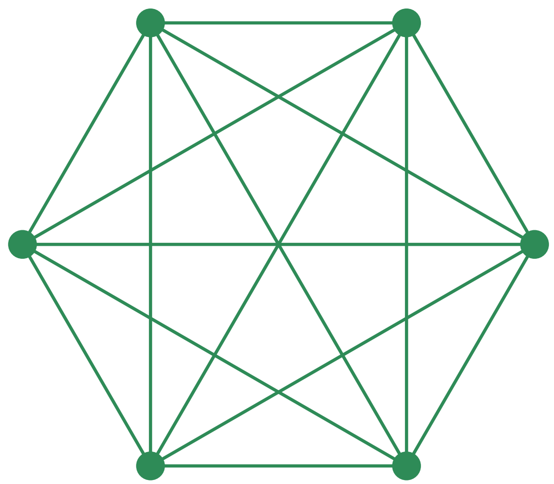

Complete graph with

- 6 vertices
- 15 edges

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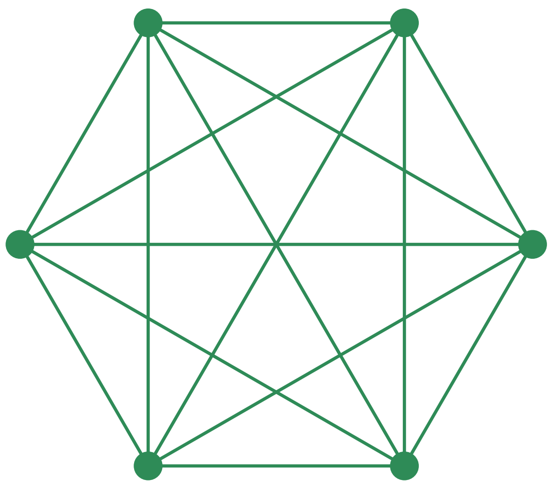

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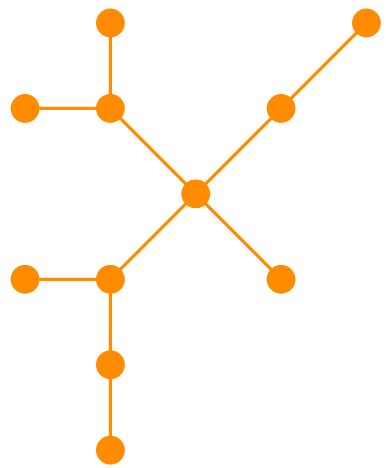

Tree with

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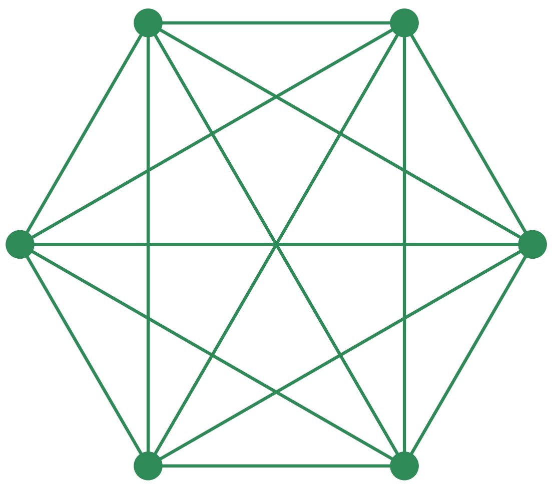

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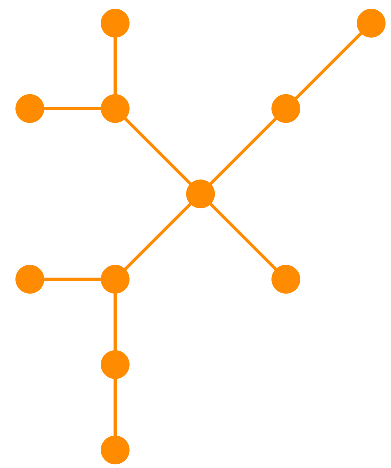

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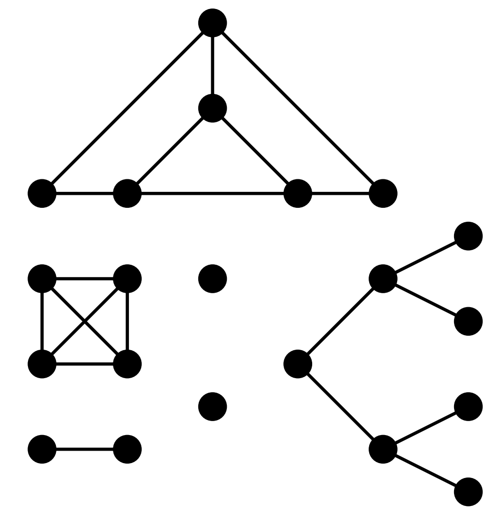
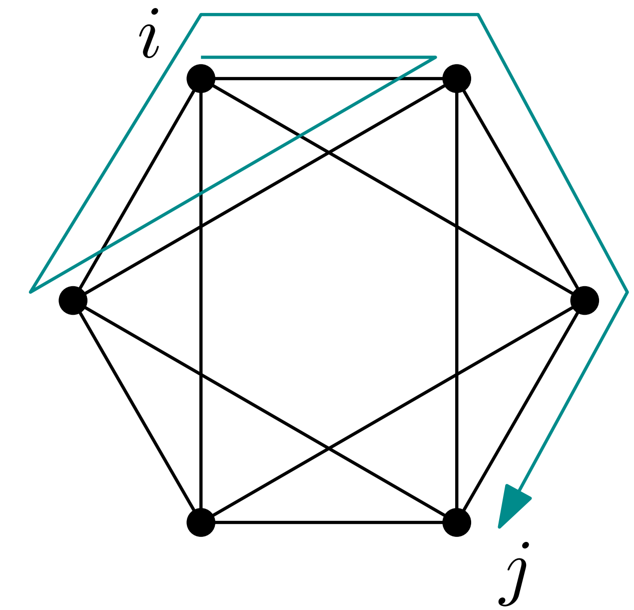


Graph with

- 21 vertices
- 21 edges

Graph theory: some vocabulary

- **Path** from vertex i to vertex j: sequence of edges connecting i to j



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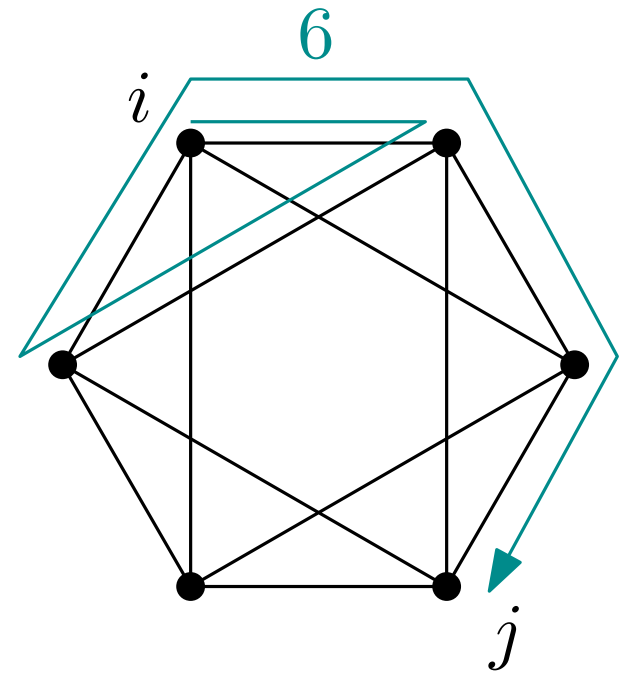

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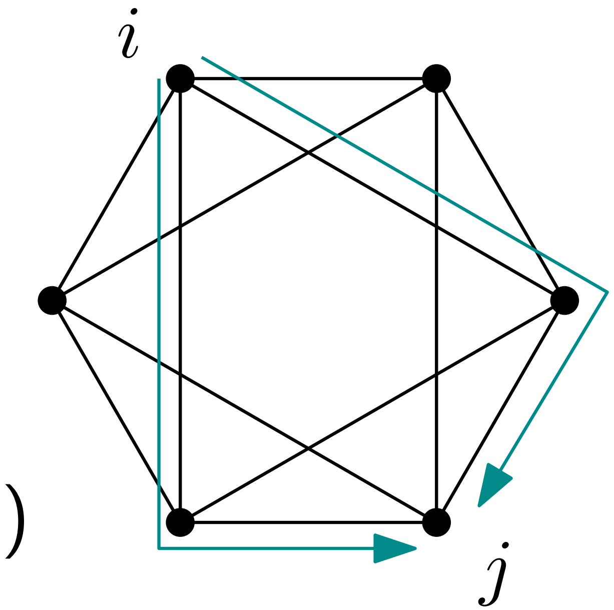

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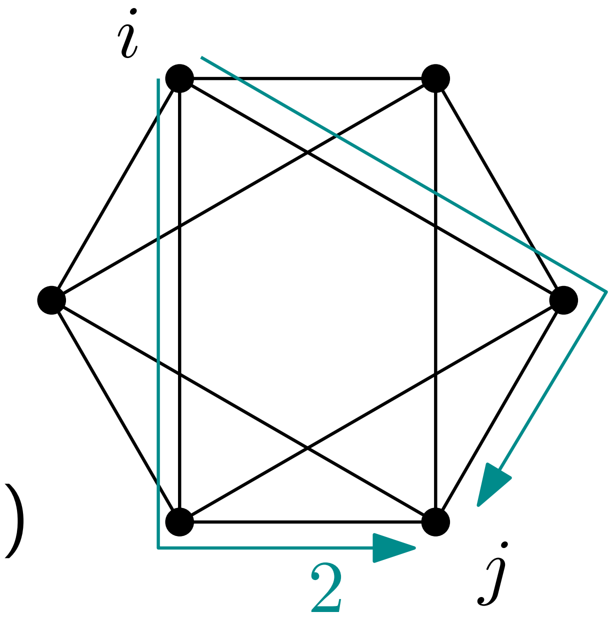

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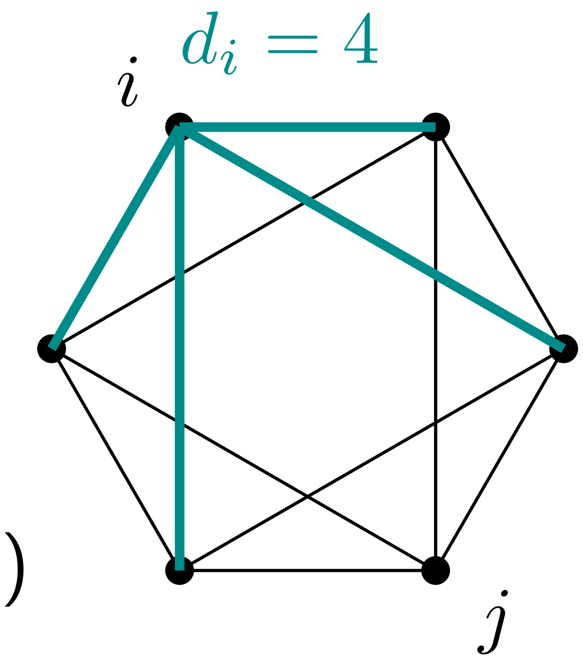

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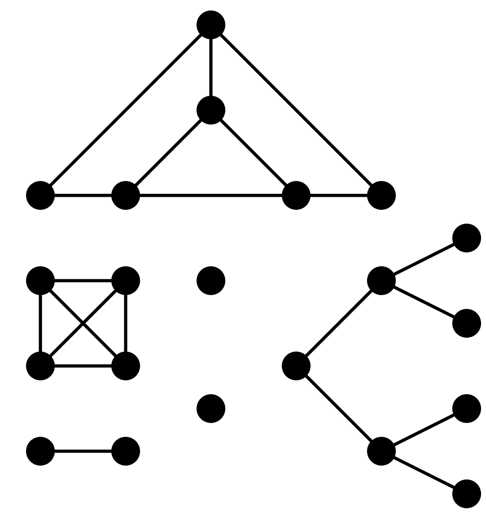

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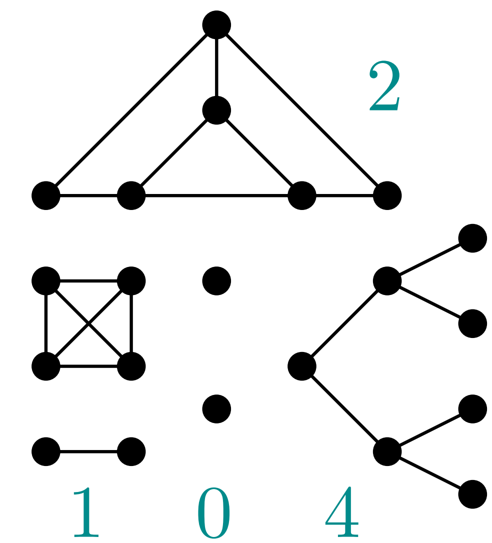

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- Size of the network: n **vertices** (n deterministic and large)
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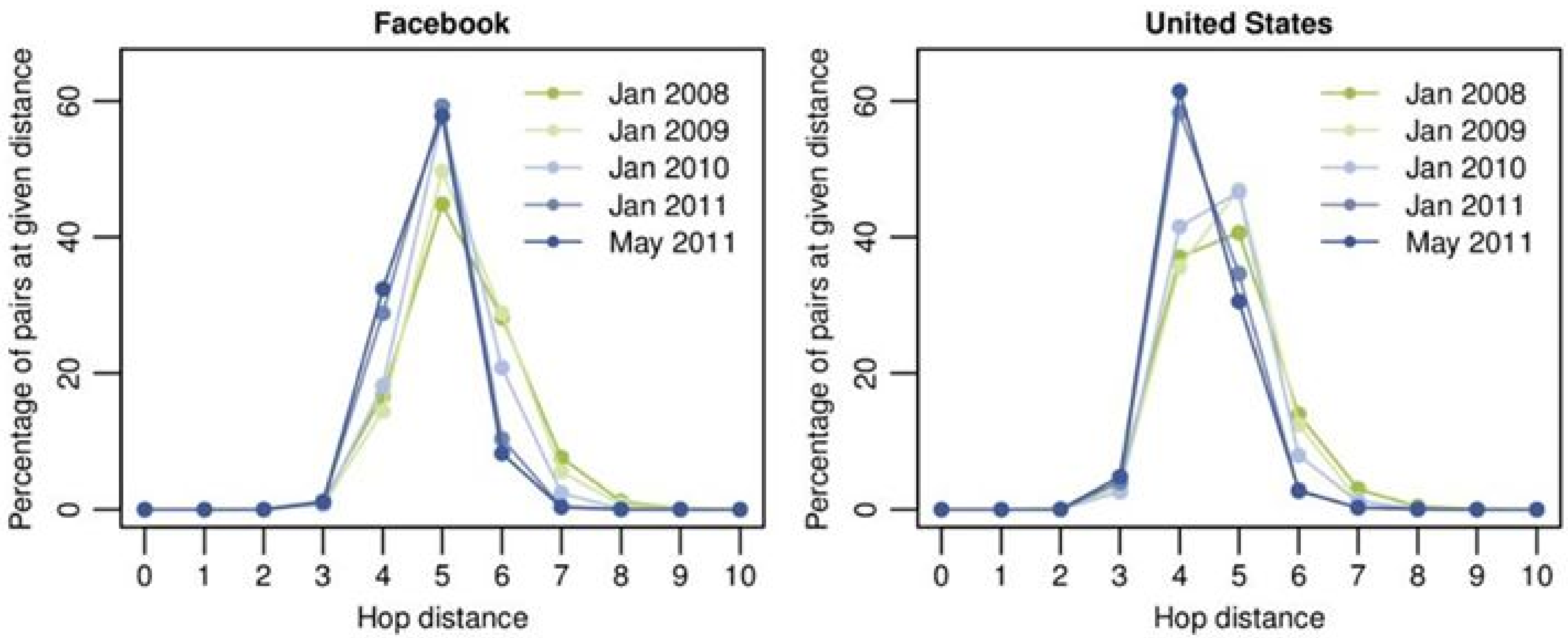
- [The Erdős number project]
  average collaboration distance between two mathematicians $= 7.64$, diameter $= 23$

The small world of Facebook

721 million active users, 69 billion friendship links: average degree $= 191$



Distances in Facebook

[Backstrom, Boldi, Rosa, Ugander and Vigna 2011]

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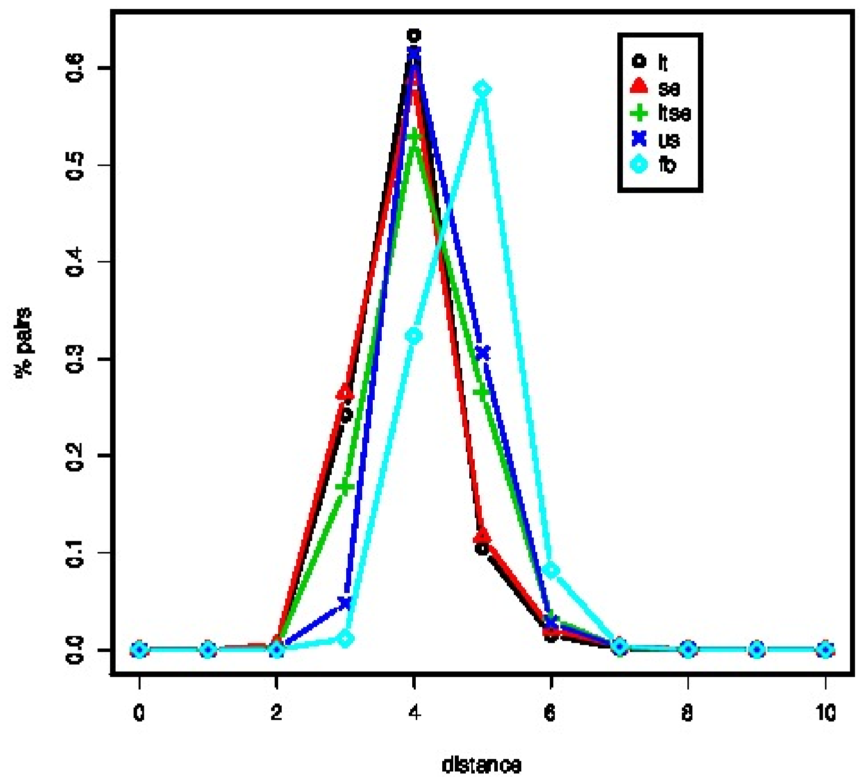

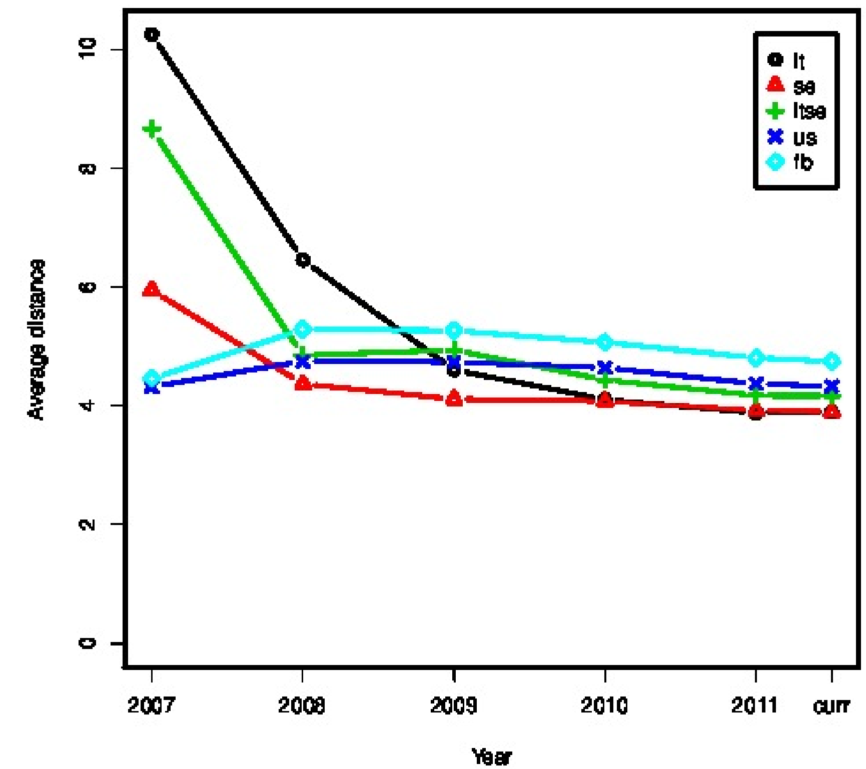

Distances in Facebook in different subgraphs
[Backstrom, Boldi, Rosa, Ugander and Vigna 2011]

The small world effect: mathematical modeling

Two interesting criteria:

Small Diameter:

$$\max_{1 \leqslant i,j \leqslant n} d_{G_n}(i,j) \ll n$$

Small average distance

$$\frac{2}{n(n-1)} \sum_{i,j=1}^{n} d_{G_n}(i,j) \ll n$$

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Both these quantities will grow very slowly with n, often as slowly as $\log n$.

For example:

- $\log(721\,000\,000) \simeq 20$ (Facebook)
- $\log(\log(721\,000\,000)) \simeq 3$
- $\log(10\,000\,000\,000) \simeq 23$

Scale free property

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Degree sequence of a graph G_n with n vertices:

$$\mathbf{d}_n = (d_1(n), \ldots, d_n(n))$$

Degree distribution of G_n: proportion $P_{\mathbf{d}_n}$ of vertices with given degree

$$P_{\mathbf{d}_n}(\{k\}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{d_i(n)=k\}}$$

$$P_{\mathbf{d}_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{d_i(n)}$$

If G_n is a random graph, $P_{\mathbf{d}_n}$ is a (random) probability distribution:
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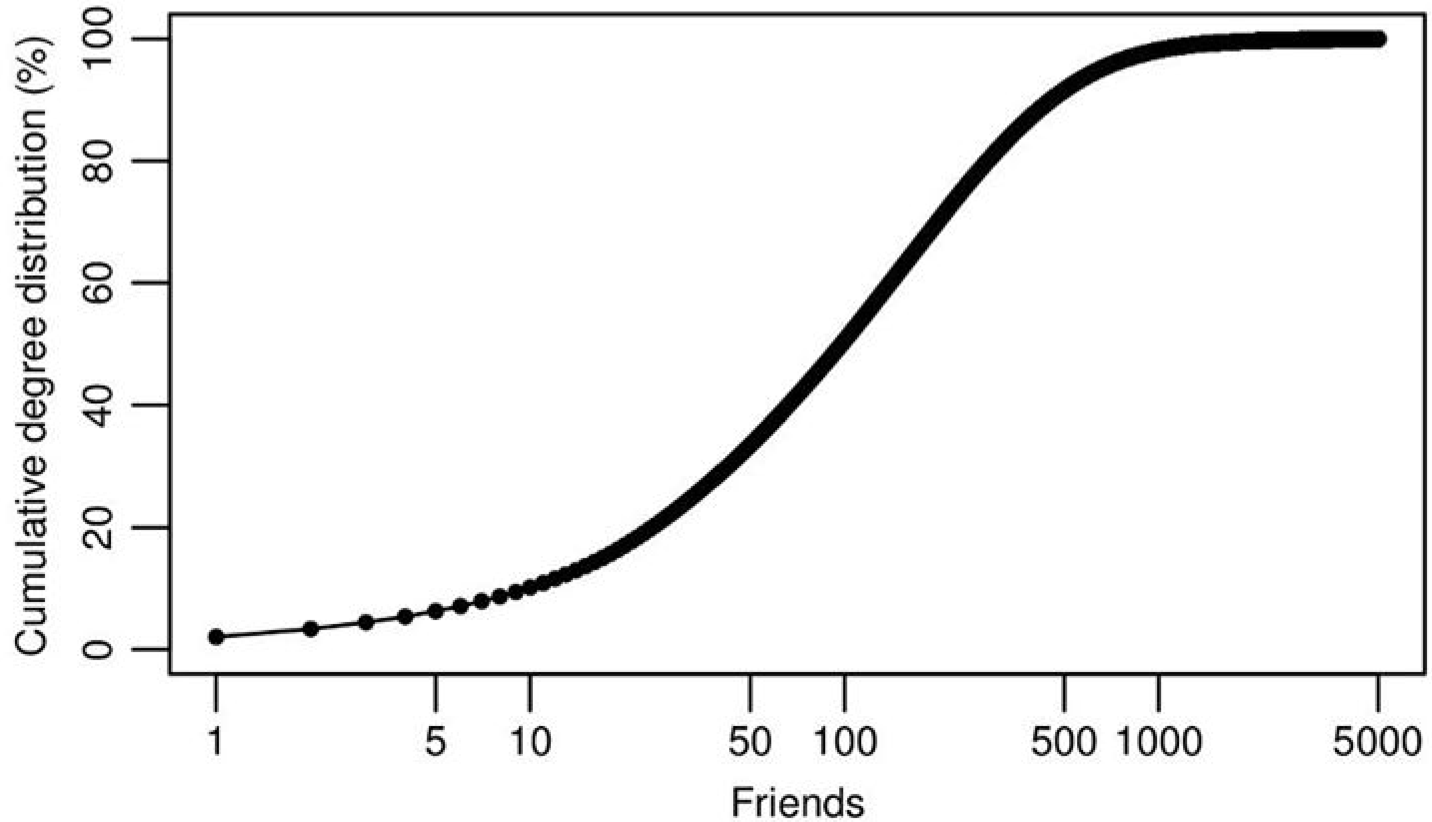
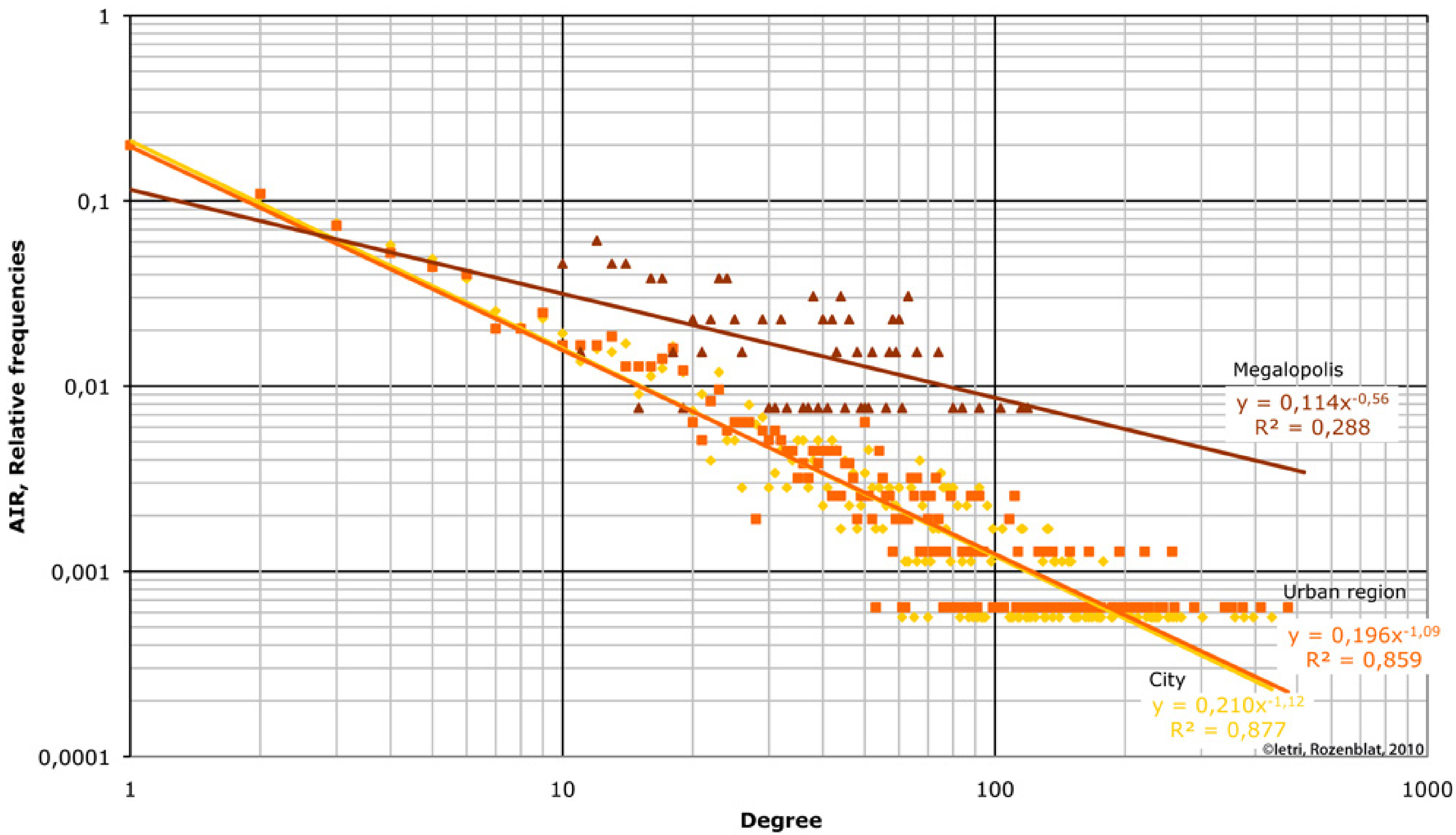
$\boxed{\textbf{Scale free property: } P_{\mathbf{d}_n} \textbf{ "asymptotically has a heavy tail"}}$

Scale free property: Facebook



Cumulative degree distribution in Facebook
[Ugander, Karrer, Backstrom and Marlow 2011]

Scale free property: log-log plots



Degrees in the worldwide air transportation network
[Ducruet, Ietri and Rozenblat 2011]

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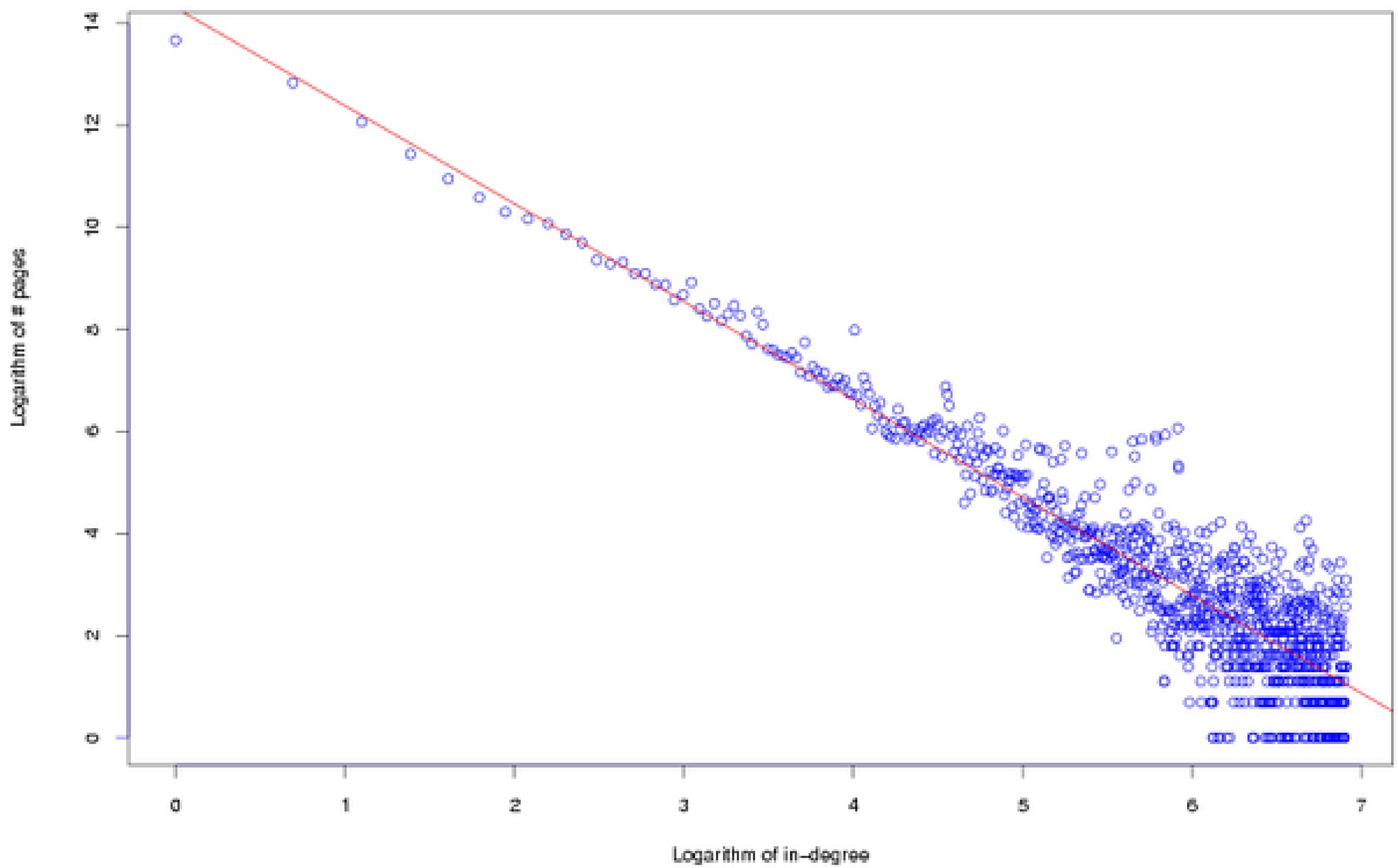

Number of links pointing to webpages in the African Web
[Boldi, Codenotti, Santini, Vigna 2002]

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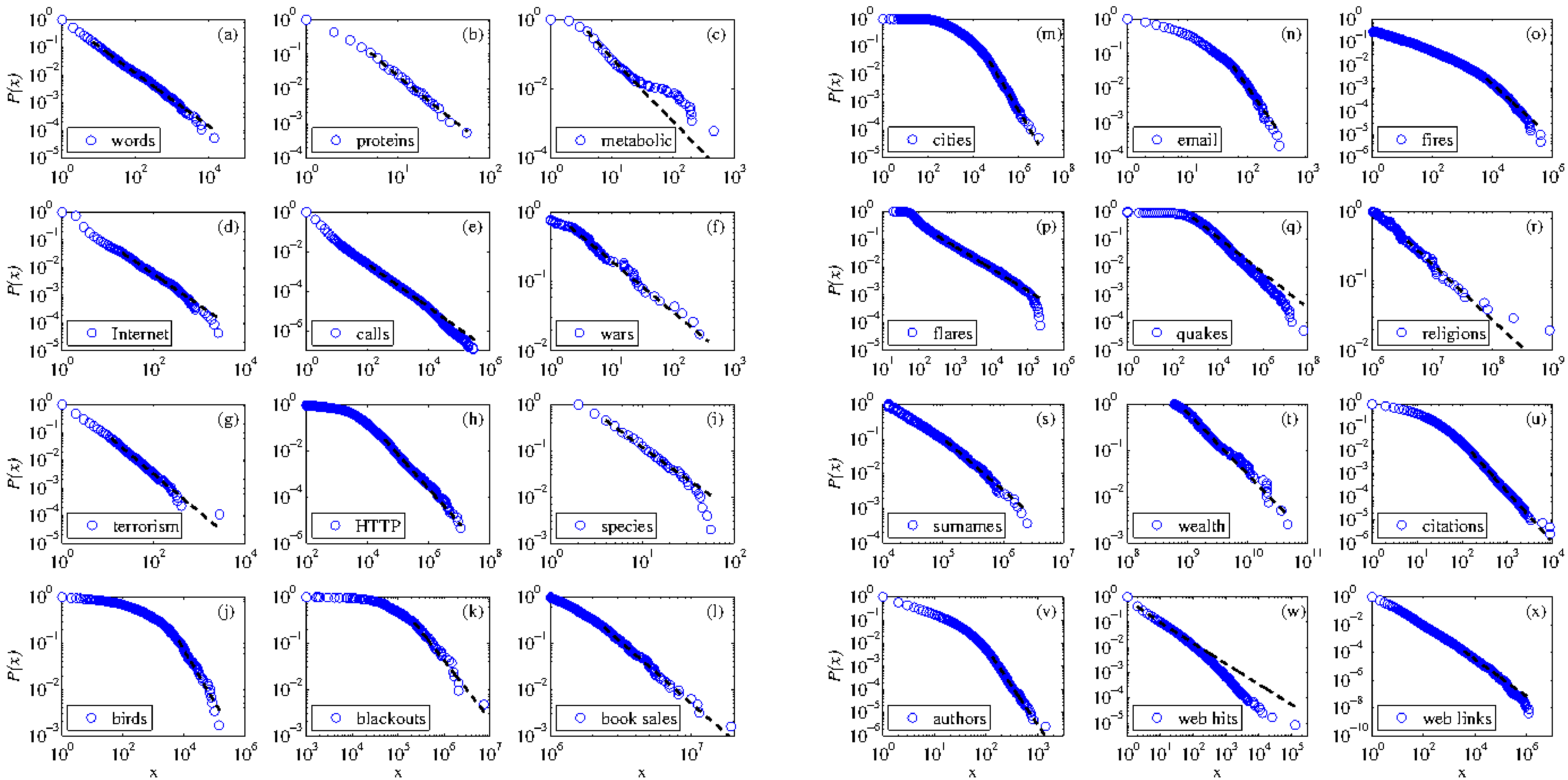

Degree distributions in real world networks
[Clauset, Shalizi and Newman 2007]

Scale free property: mathematical modeling

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Most common example of random variable X with a heavy tail:
power law with exponent $\tau > 1$

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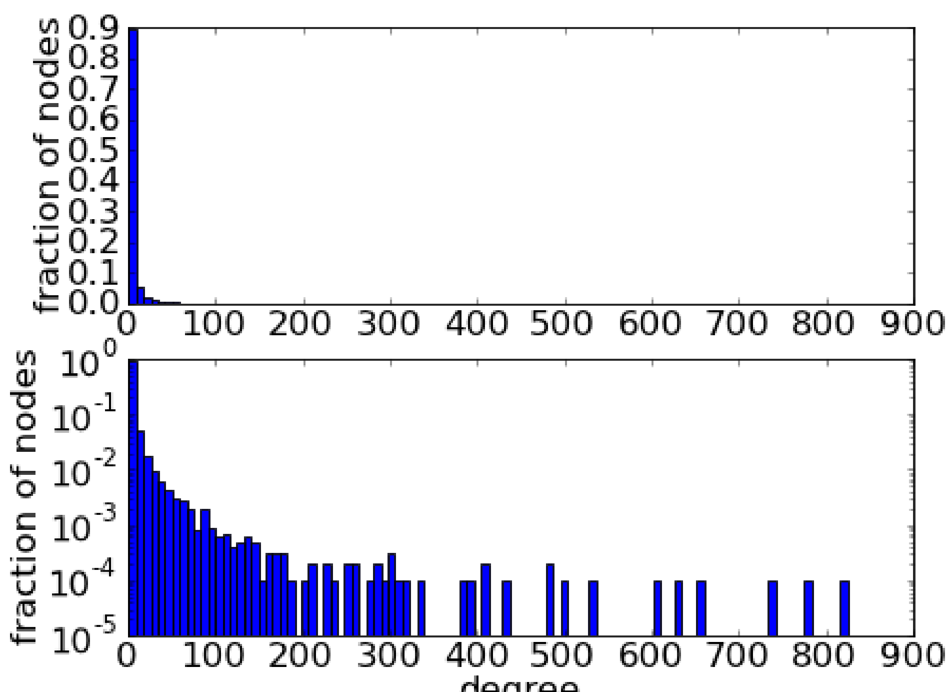

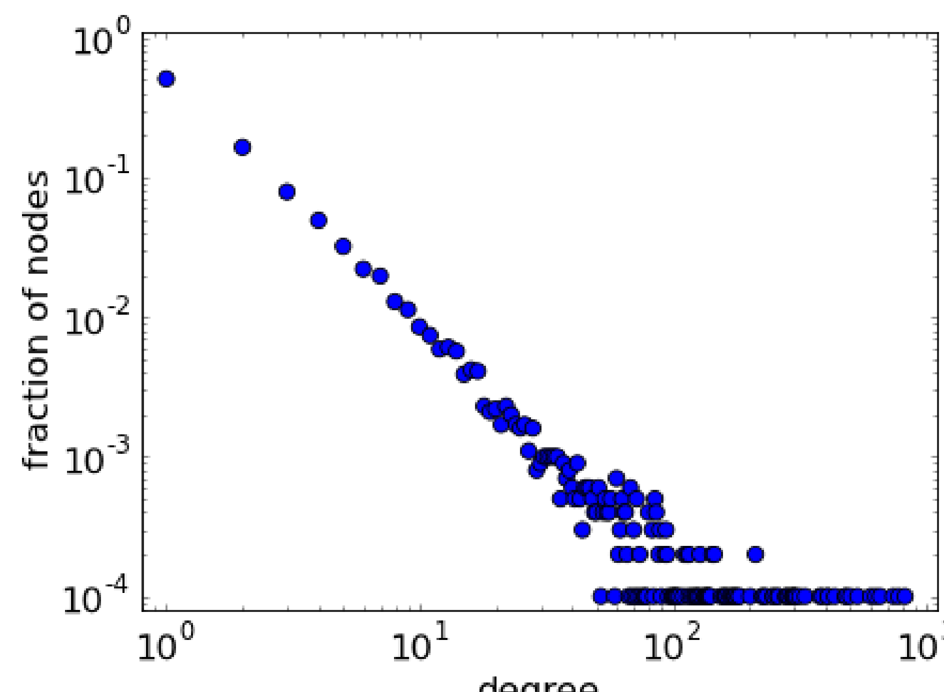

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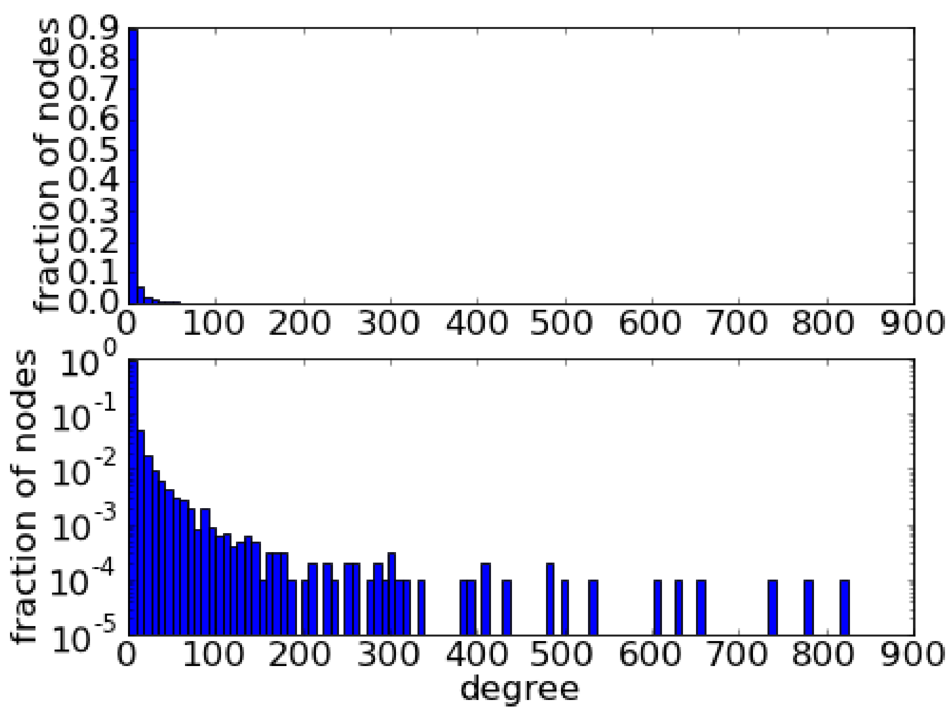
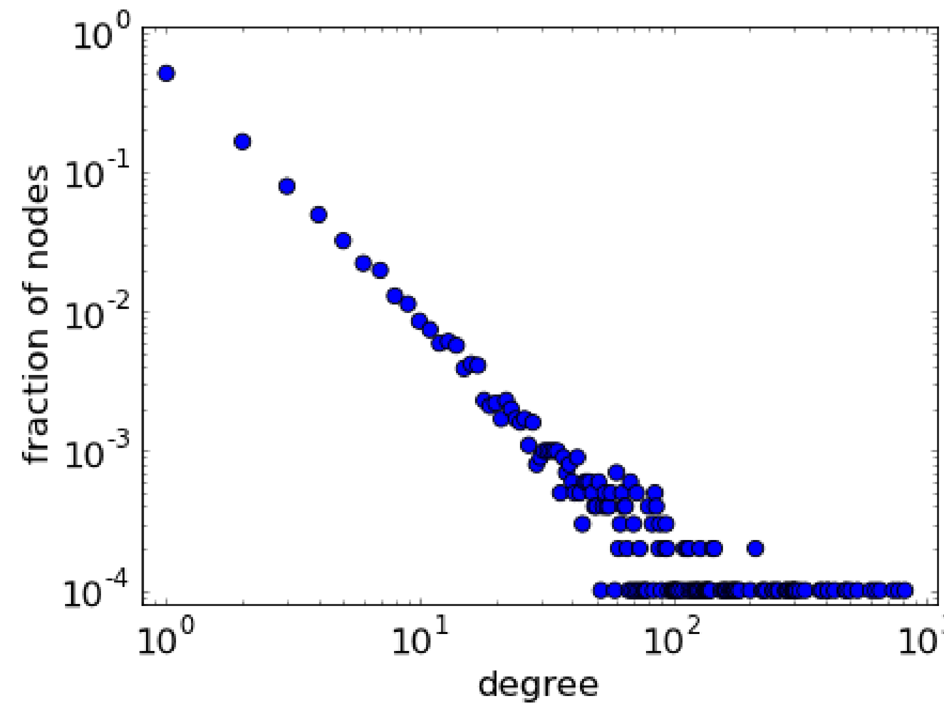
some properties
- no exponential moments
- infinite mean if $\tau \in (1, 2]$
- infinite variance if $\tau \in (1, 3]$
- moments of order $< \tau - 1$





Scale free property and sparsity: technical assumptions

Take a sequence $\mathbf{G} = (G_n)_{n\geq 1}$ of random graphs such that for every n:

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$\boxed{\text{Regularity assumptions}}$ D_n r.v. with law $P_{\mathbf{d}_n}$ and D r.v. with law P

- First moment $\quad E[D_n] \underset{n\to\infty}{\to} E[D] < \infty$
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$\boxed{\text{Scale free}}$ P has a heavy tail (for example, it is a Power Law)

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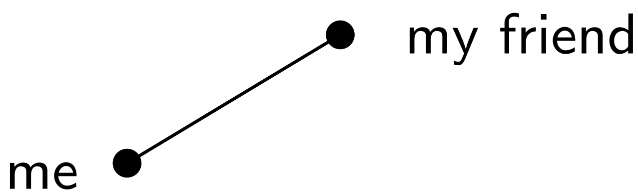

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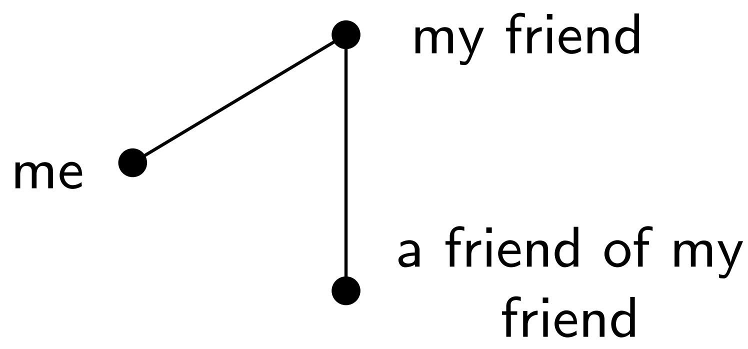

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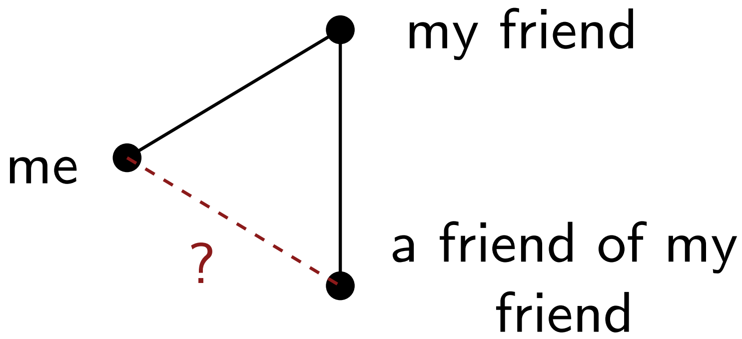

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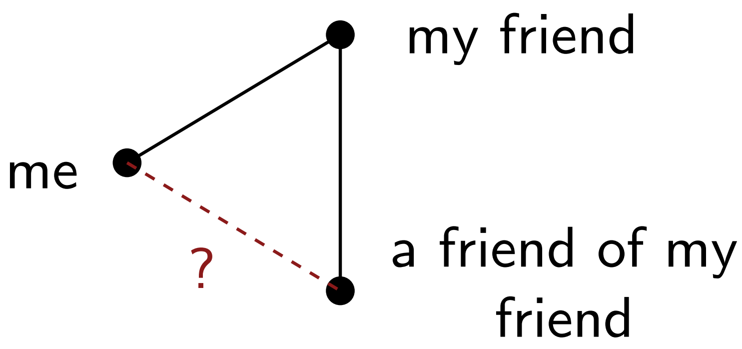
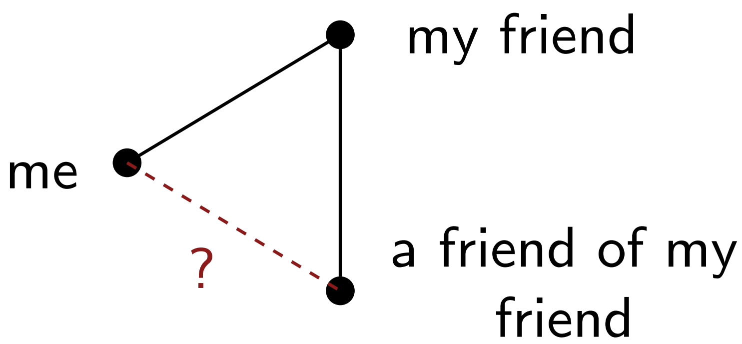


Criterion that compares
the number of triangles to the number
of connected triplets of vertices

Clustering

Measures the network's transitivity: the friends of my friends are more likely to be my friends



Criterion that compares the number of triangles to the number of connected triplets of vertices

Global clustering of a graph G

$$\mathrm{CL}(G) = \frac{3 \times E\,(\text{nb of triangles})}{E\,(\text{nb of connected triplets})}$$

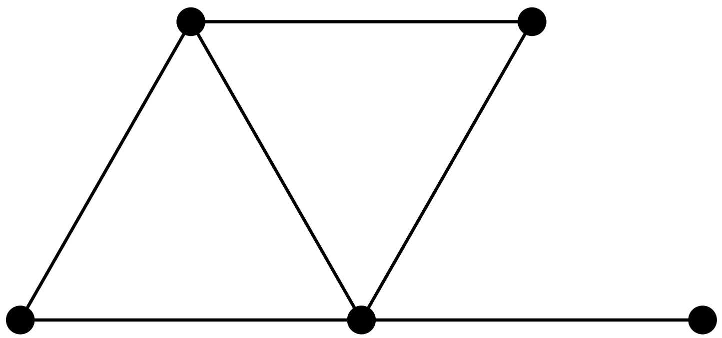
Individual clustering of vertex i

$$\mathrm{CL}_i(G) = \frac{E\,(\text{nb of triangles containing } i)}{E\,(\text{nb of connected triplets centered at } i)}$$

Average clustering of G

$$\overline{\mathrm{CL}}(G) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{CL}_i(G)$$

Clustering: an example



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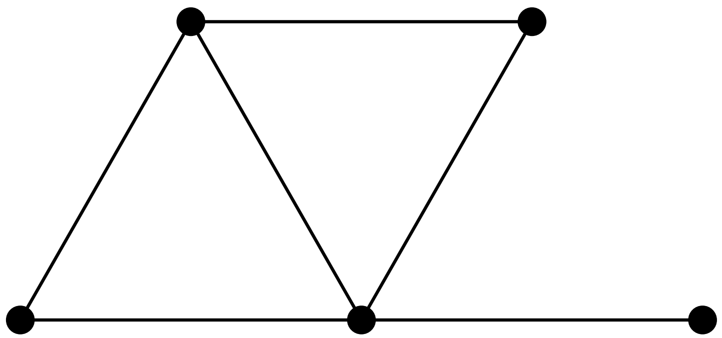
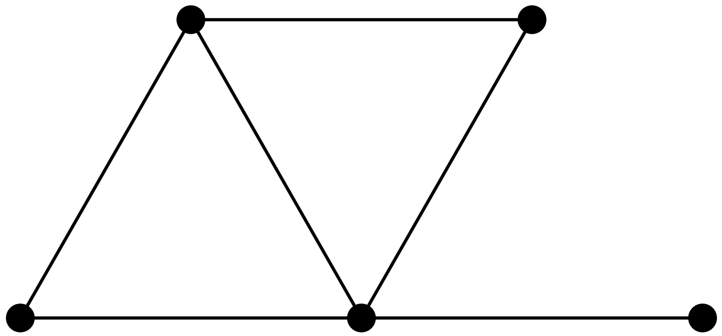


2 trianges

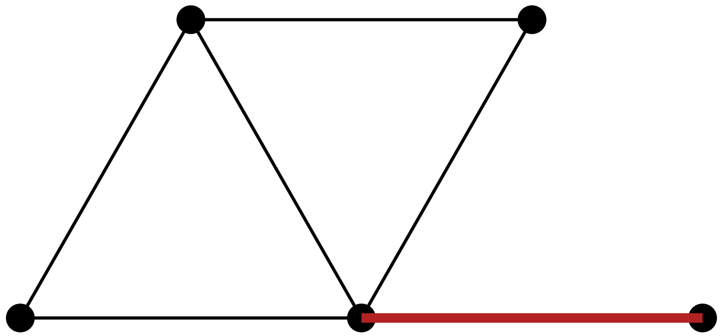
Clustering: an example



2 trianges

10 connected triplets:

Clustering: an example



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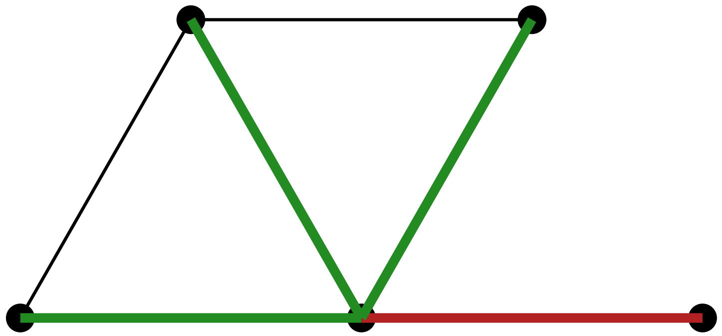
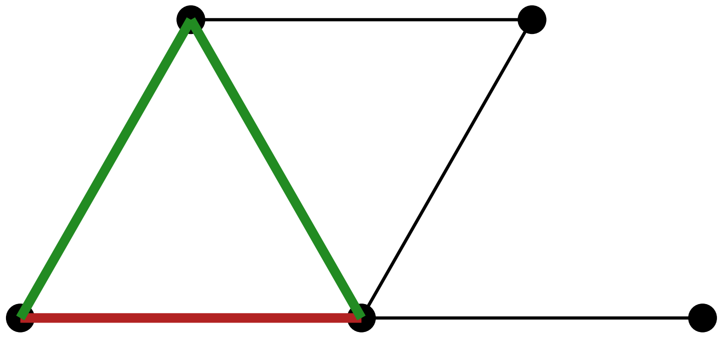


2 trianges

10 connected triplets:

3

Clustering: an example

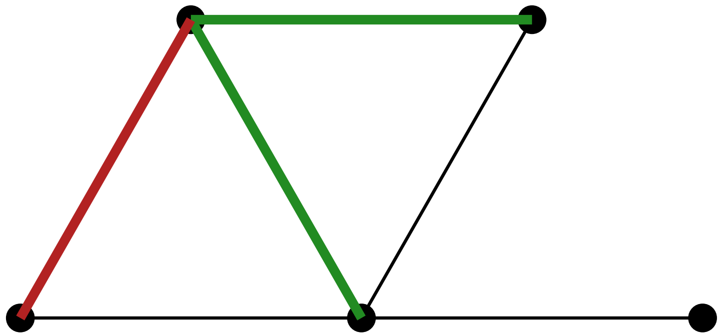


2 trianges

10 connected triplets:

$3 + 2$

Clustering: an example



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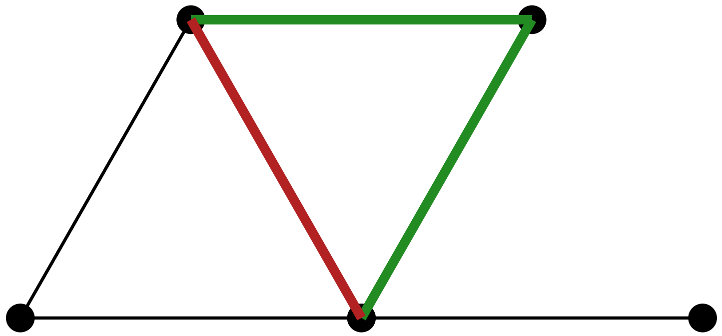

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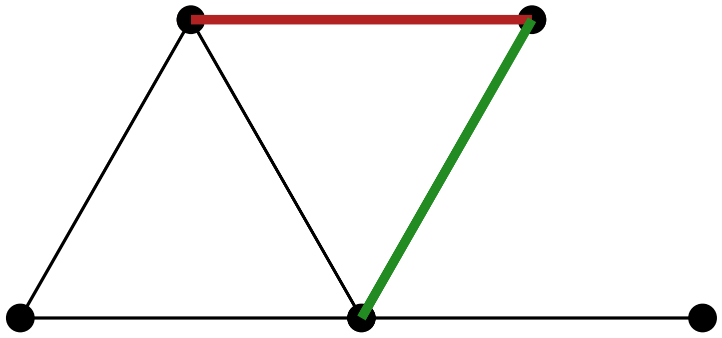

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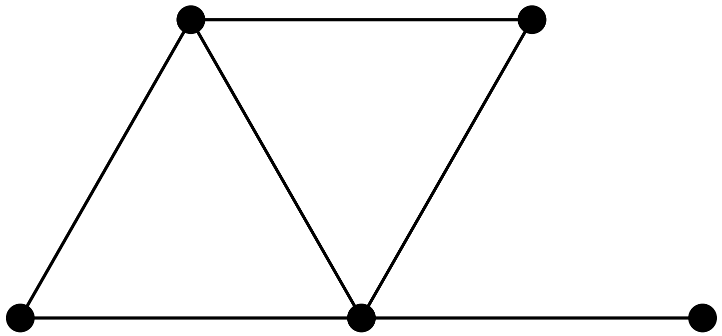

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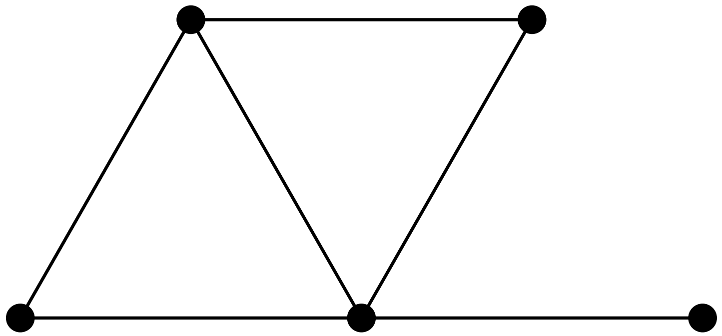
Global clustering coefficient $CL(G) = \frac{3 \times 2}{10} = \frac{3}{5}$

Clustering: an example



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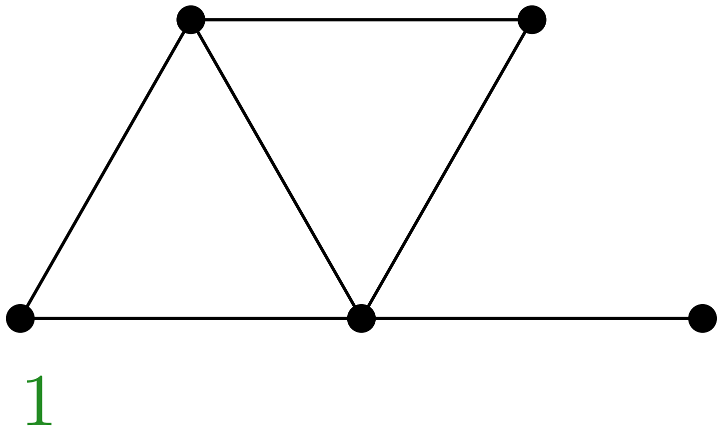

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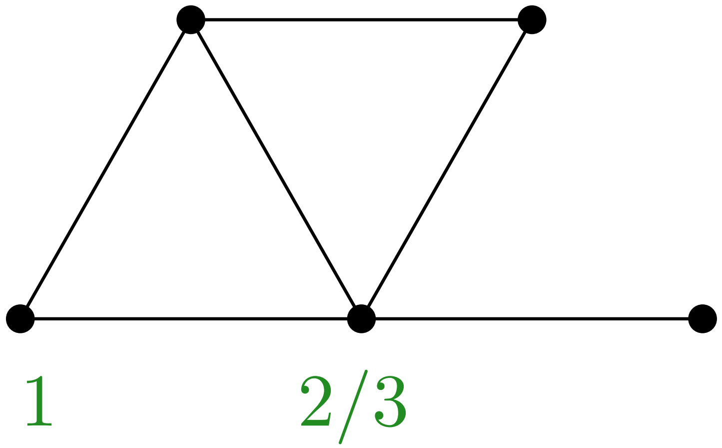

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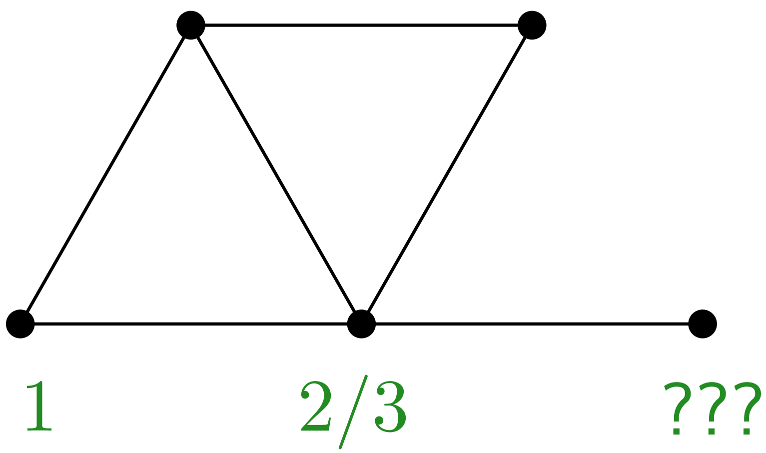

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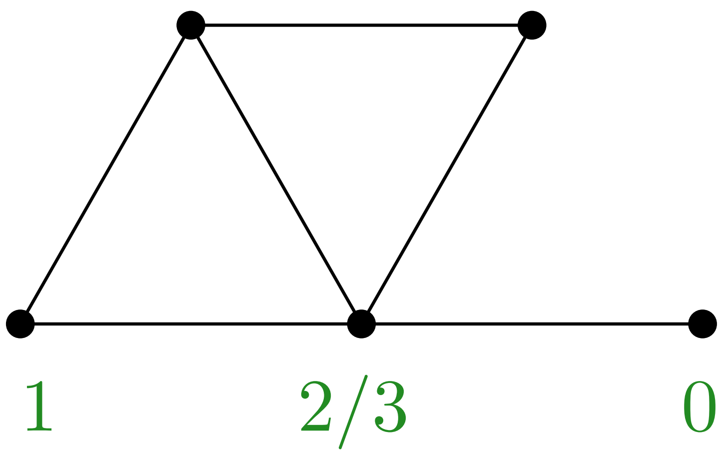

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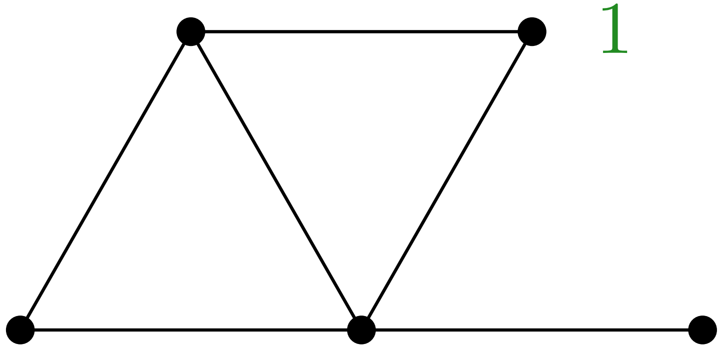

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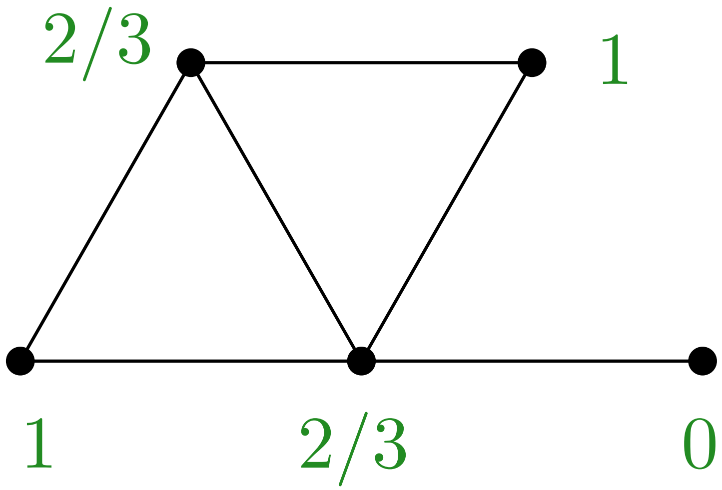

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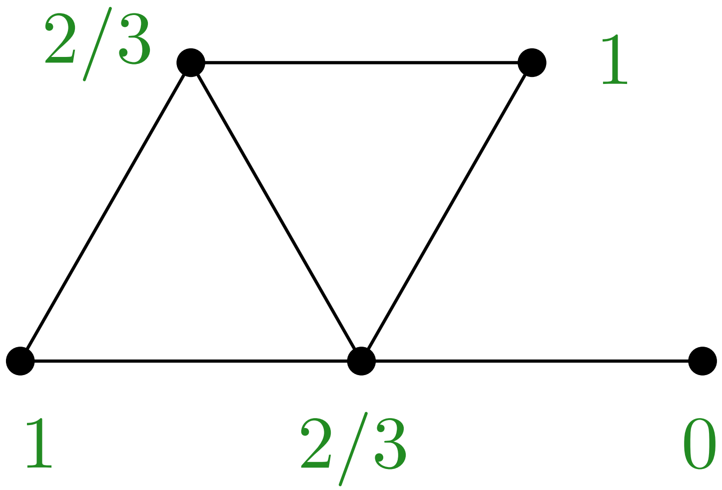

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Different models of random graphs

Different models of random graphs

- **Erdős-Rényi random graph**

  Simplest interesting model

- **Inhomogeneous random graphs**

  Generalisation of Erdős-Rényi random graphs,
  independent edges with inhomogeneous edge occupation probabilities

- **Configuration model**

  Static random graph with prescribed degree sequence

- **Preferential attachment**

  Dynamical model, attachment proportional to degree plus constant

The Erdős-Rényi random graph

The Erdős-Rényi random graph

Origins in [Erdős and Rényi 1959]

$ER(n,p)$

- n vertices
- independant edges
- edge between i and j with probability p

Egalitarian model: every vertex has the same role

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d_i degree of the node i: binomial r.v. with parameters $(n-1, p)$

- If $np \to \infty$, d_i diverges almost surely
- **Sparse graph** when $p = \dfrac{c}{n}$, $c > 0$

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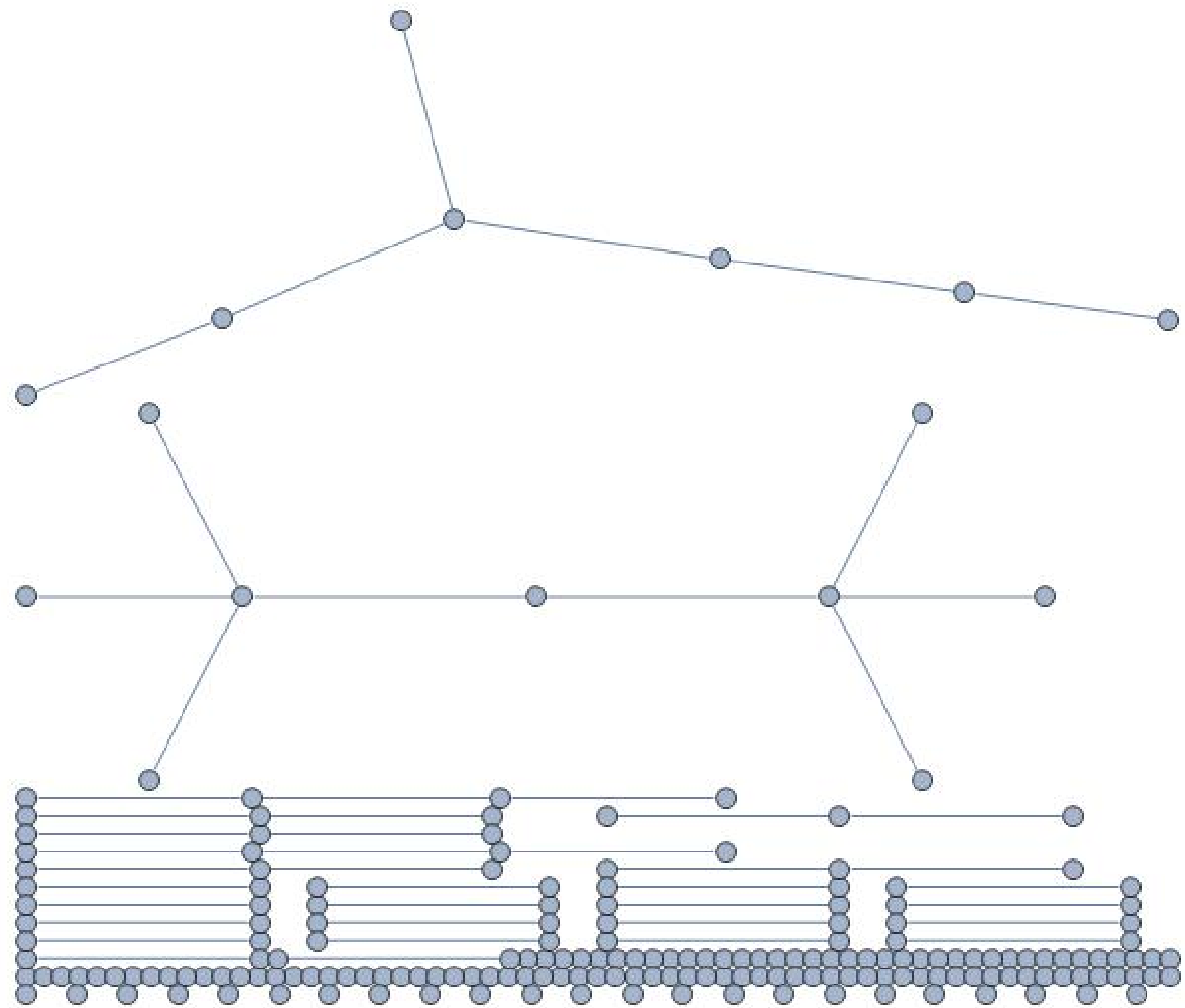
Poisson approximation:

$P_{\mathbf{d}_n}$ converges weakly to a Poisson r.v. with parameter c

$$P_{\mathbf{d}_n}(\{k\}) \underset{n\to\infty}{\to} \mathcal{P}_c(\{k\}) = \frac{c^k}{k!}e^{-c}$$

$\boxed{ER(n,c/n) \textbf{ is not scale free}}$

The Erdős-Rényi random graph: simulations



Erdős-Rényi random graph with 200 vertices and $c = 0.5$

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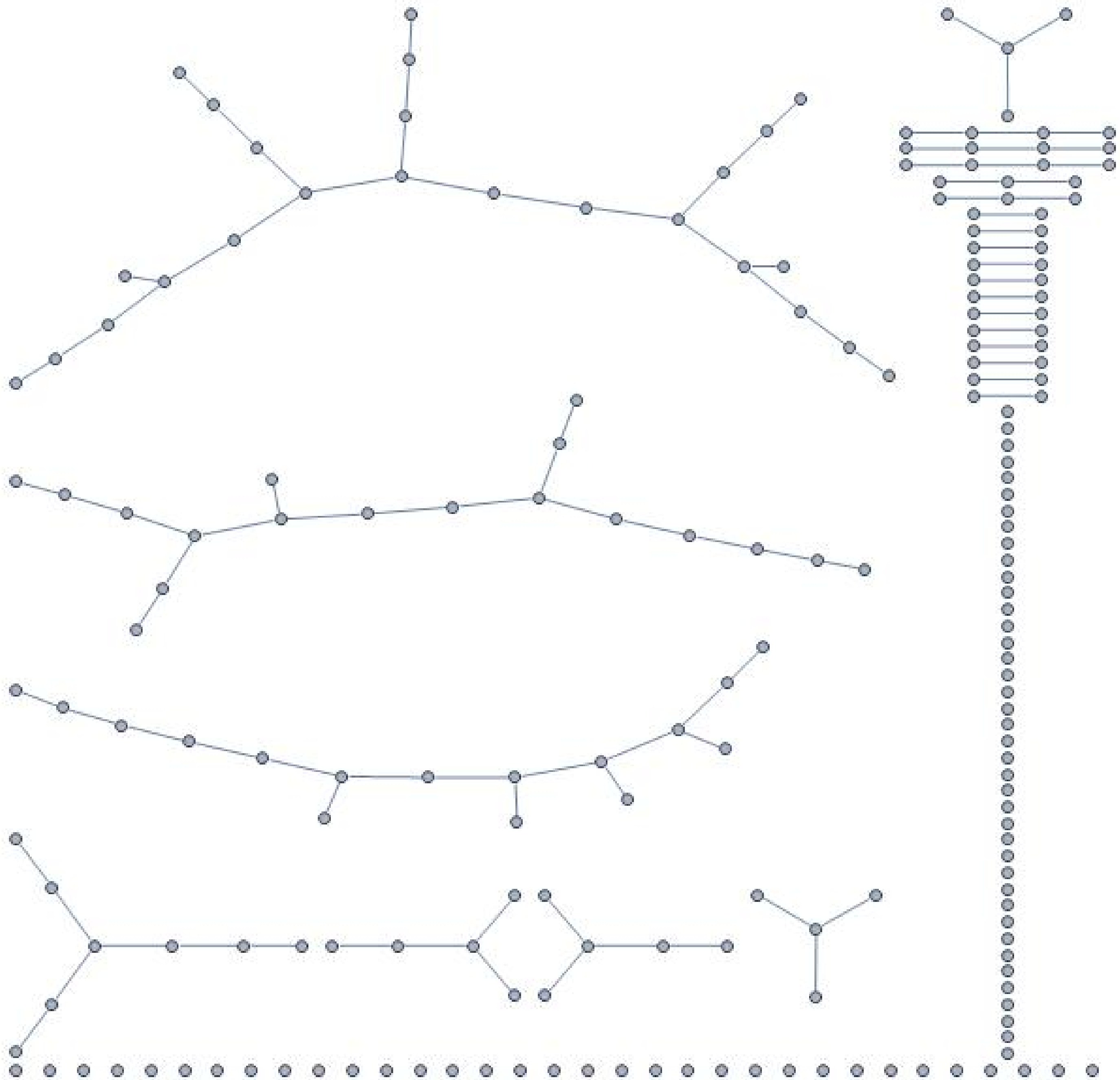
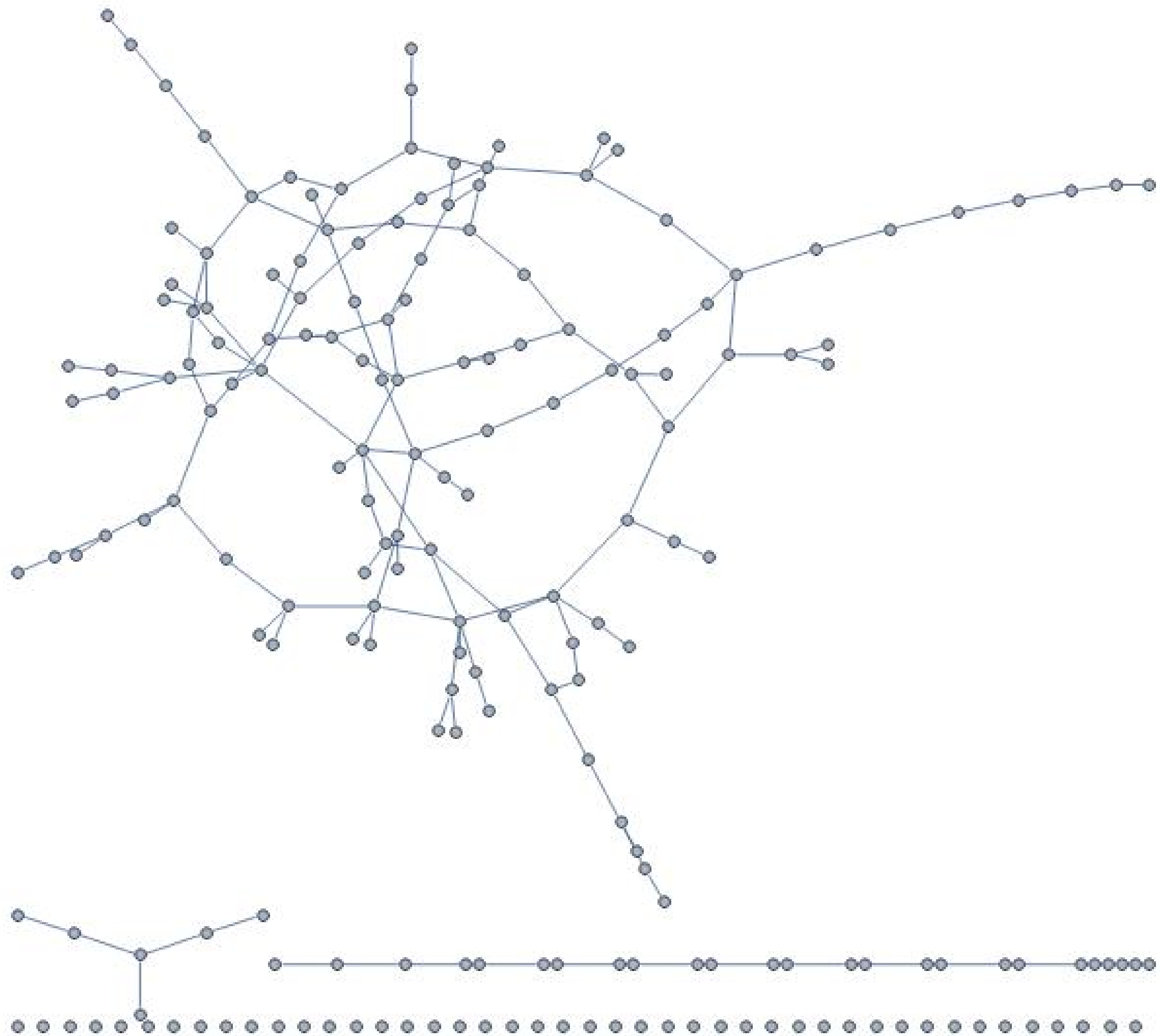


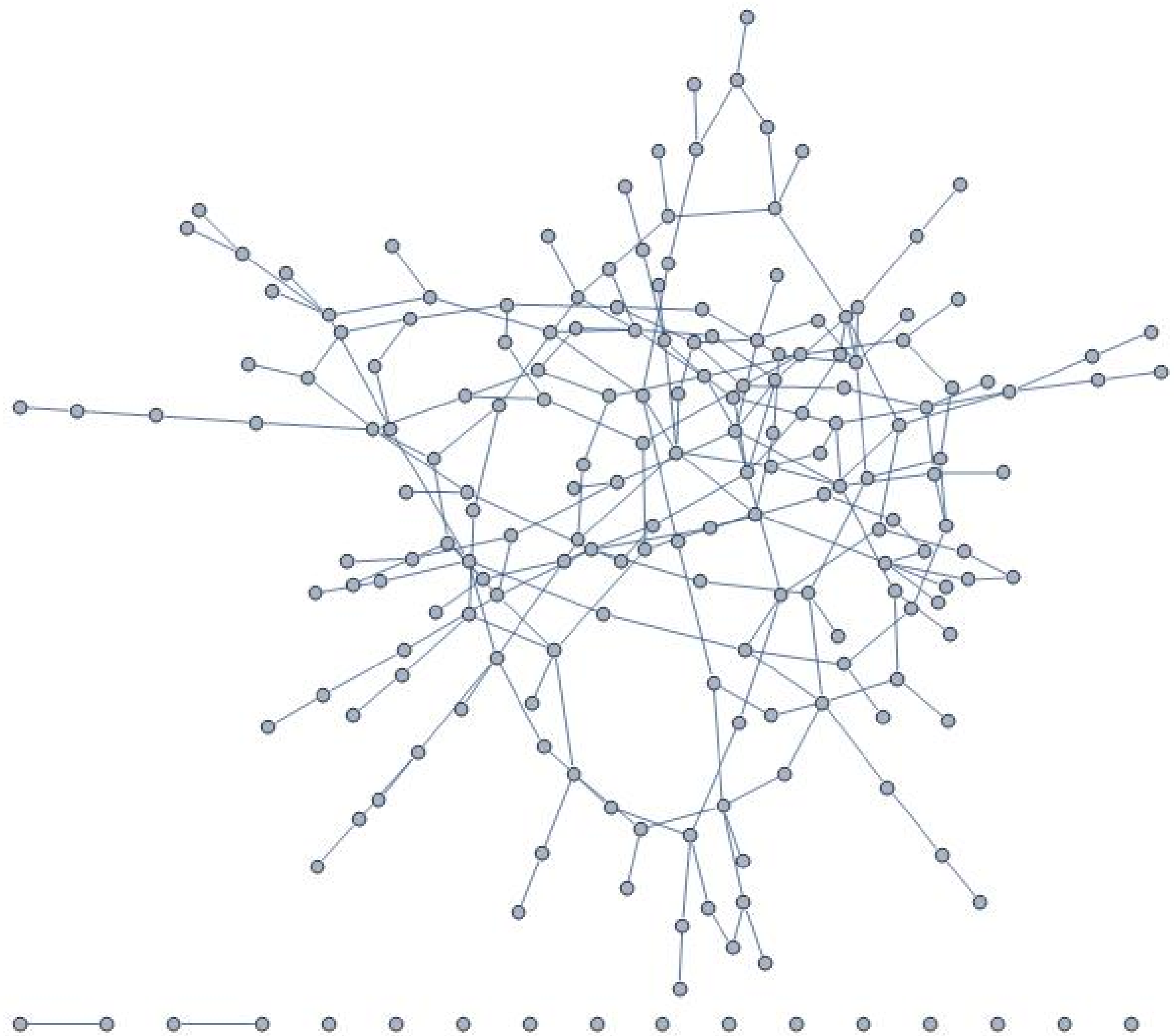
Erdős-Rényi random graph with 200 vertices and $c = 1$

The Erdős-Rényi random graph: simulations



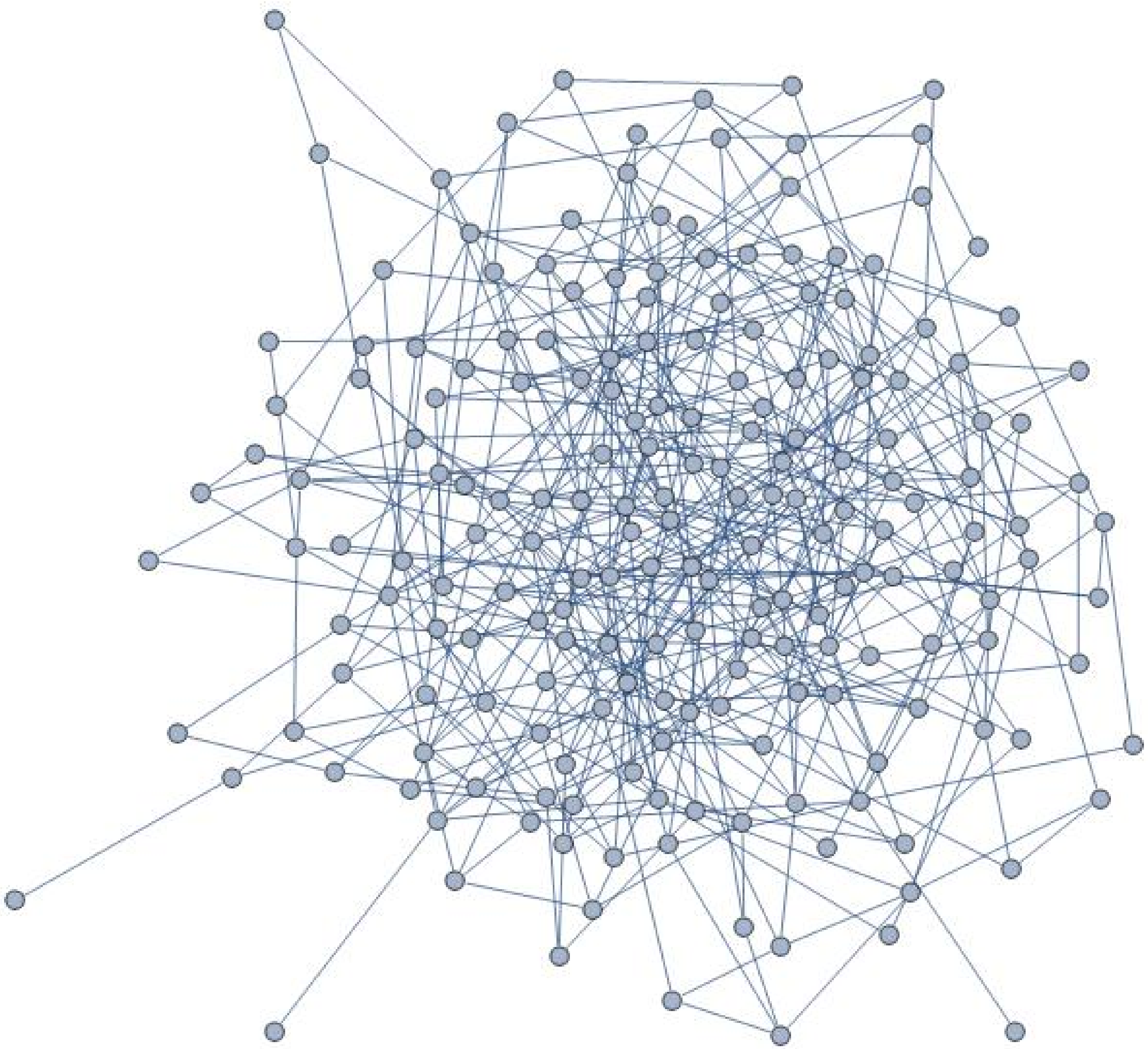
Erdős-Rényi random graph with 200 vertices and $c = 1.5$

The Erdős-Rényi random graph: simulations



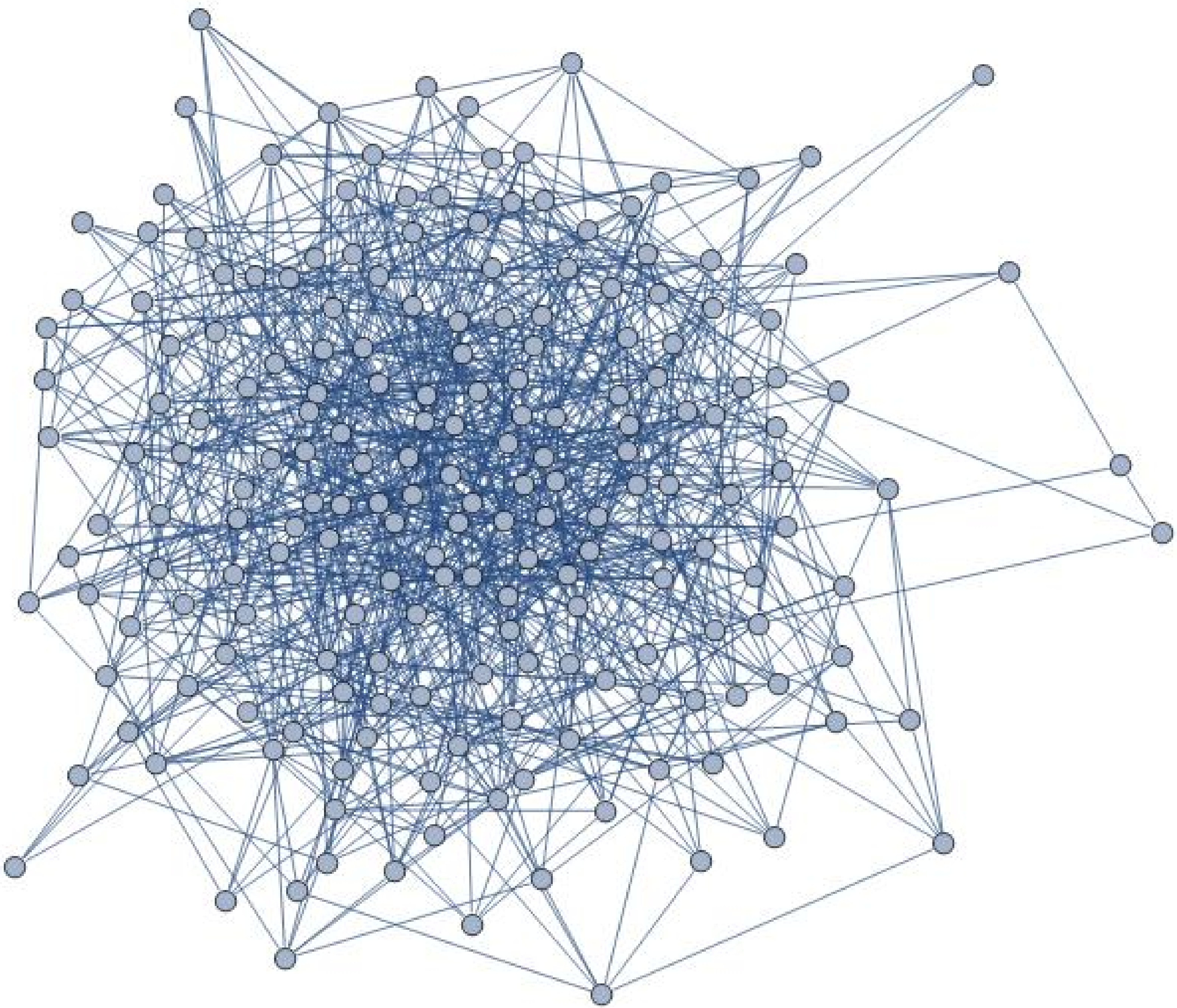
Erdős-Rényi random graph with 200 vertices and $c = 2$

The Erdős-Rényi random graph: simulations



Erdős-Rényi random graph with 200 vertices and $c = 5$

The Erdős-Rényi random graph: simulations



Erdős-Rényi random graph with 200 vertices and $c = 10$

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Clustering coefficient:

$$CL\left(ER(n, c/n)\right) = \frac{3 \times E\left(\text{nb of triangles}\right)}{E\left(\text{nb of connected triplets}\right)} = \frac{3\binom{n}{3}\left(\frac{c}{n}\right)^3}{3\binom{n}{3}\left(\frac{c}{n}\right)^2} = \frac{c}{n}$$

no transitivity

Inhomogeneous random graphs

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Generalisation of Erdős-Rényi random graphs

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Random graphs with given expected degrees:

- **independent** edges
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edge between i and j with probability $p_{i,j} = \dfrac{w_i w_j}{\sum_{k=1}^{n} w_k + w_i w_j}$

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Proper choice of $(w_i)_{1 \leq i \leq n}$:

- unique giant component
- power law degree sequence **scale free**
- diameter of order $\log n$ **small world**
- still has **low clustering**

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D_n r.v. with law $P_{\mathbf{d}_n}$ and D r.v. with law P

- weak convergence: $P_{\mathbf{d}_n}$ converges weakly to P
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$\boxed{\textbf{Scale free}\text{: degree distribution converging to a power law}}$

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Same result: denote resulting (multi)-graph by $CM(\mathbf{d}_n)$

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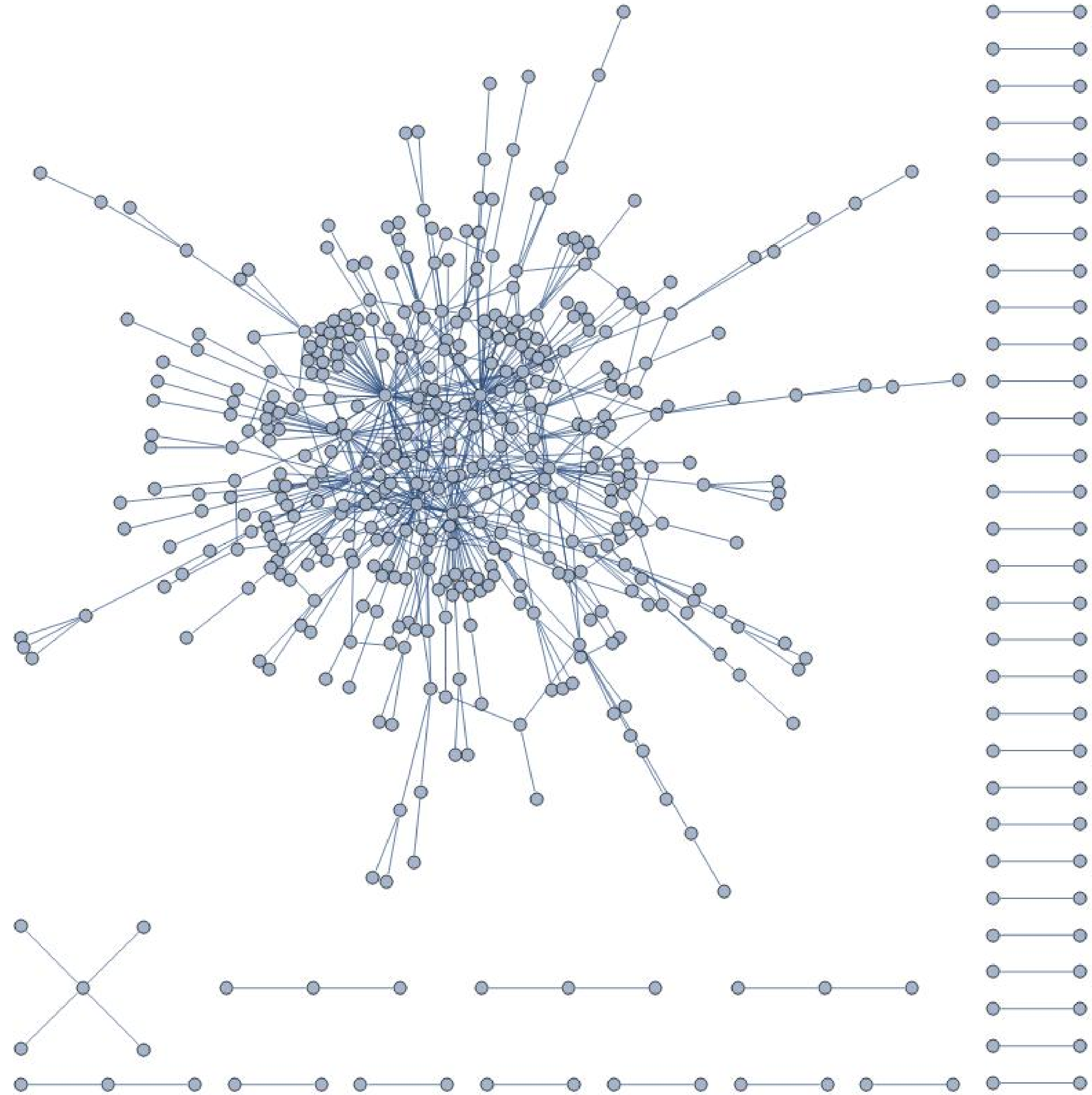
- Second moment regularity assumption:

As $n \to \infty$, the probability that $CM(\mathbf{d}_n)$ is **simple** converges to

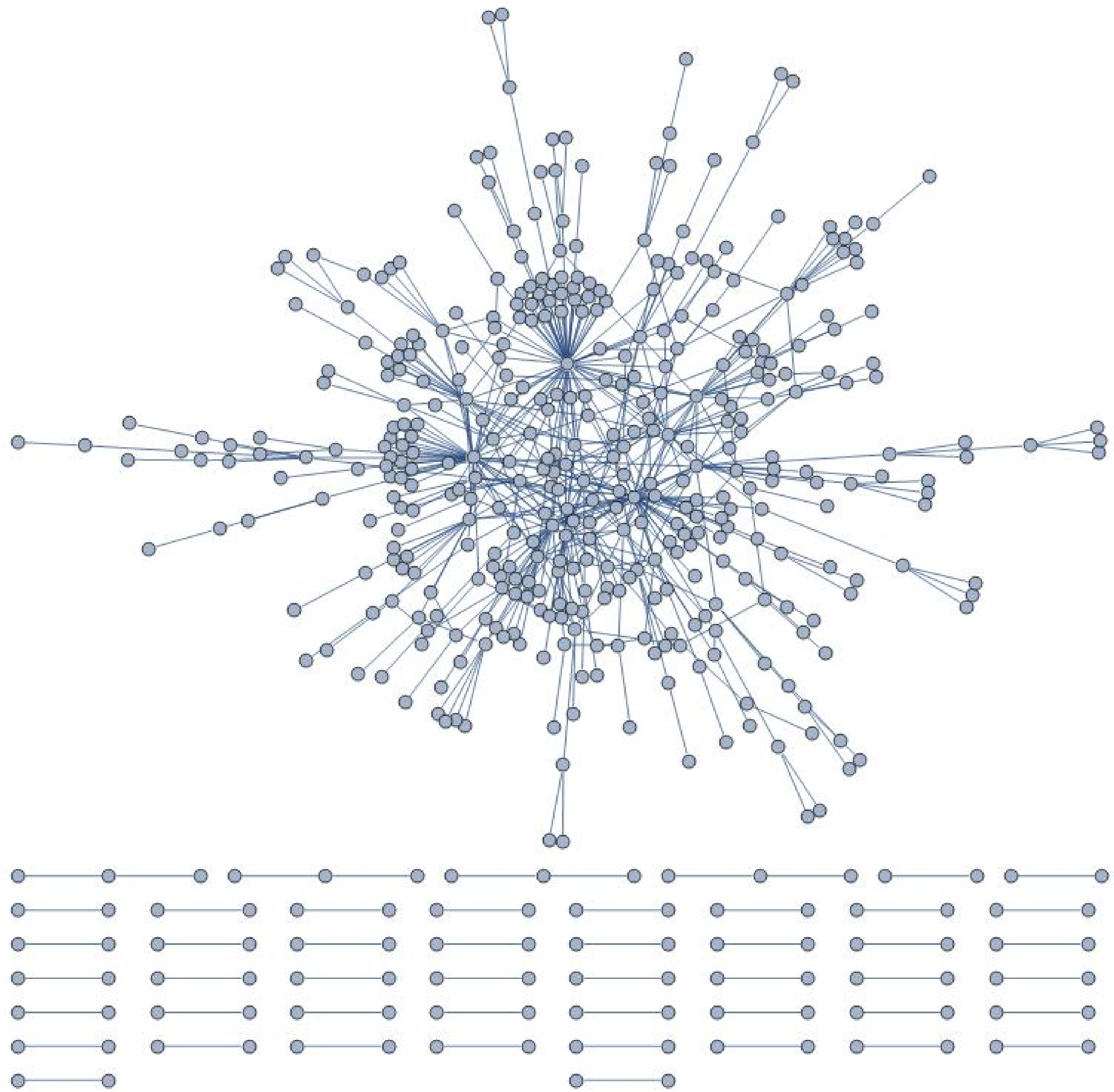
$$e^{-\frac{\nu}{2}-\frac{\nu^2}{4}} \quad \text{where } \nu = \frac{E\left[D(D-1)\right]}{E\left[D\right]}$$

Configuration model: simulations



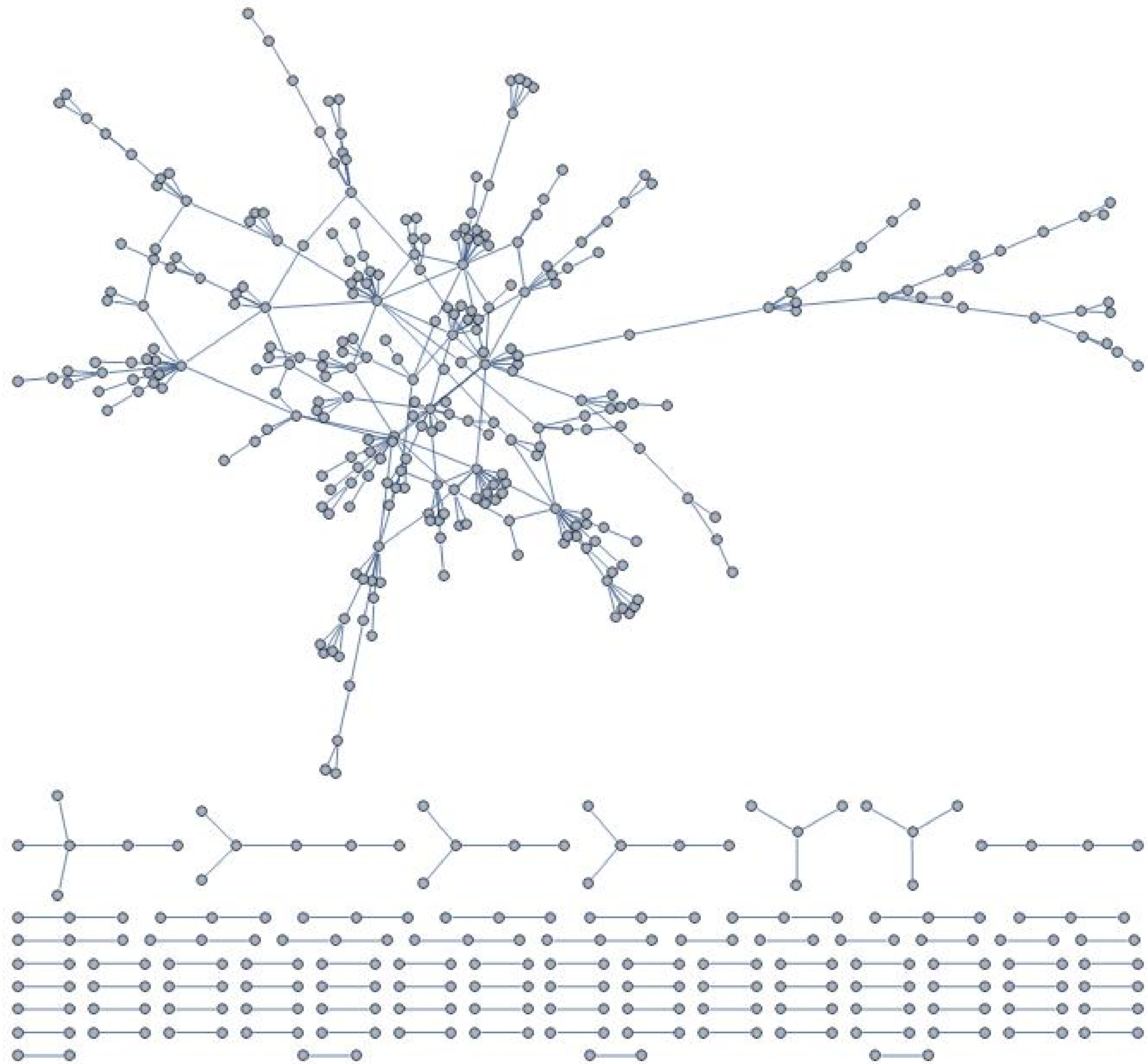
Configuration Model with 500 vertices
and degrees power law with exponent 1.1

Configuration model: simulations



Configuration Model with 500 vertices
and degrees power law with exponent 1.2

Configuration model: simulations



Configuration Model with 500 vertices
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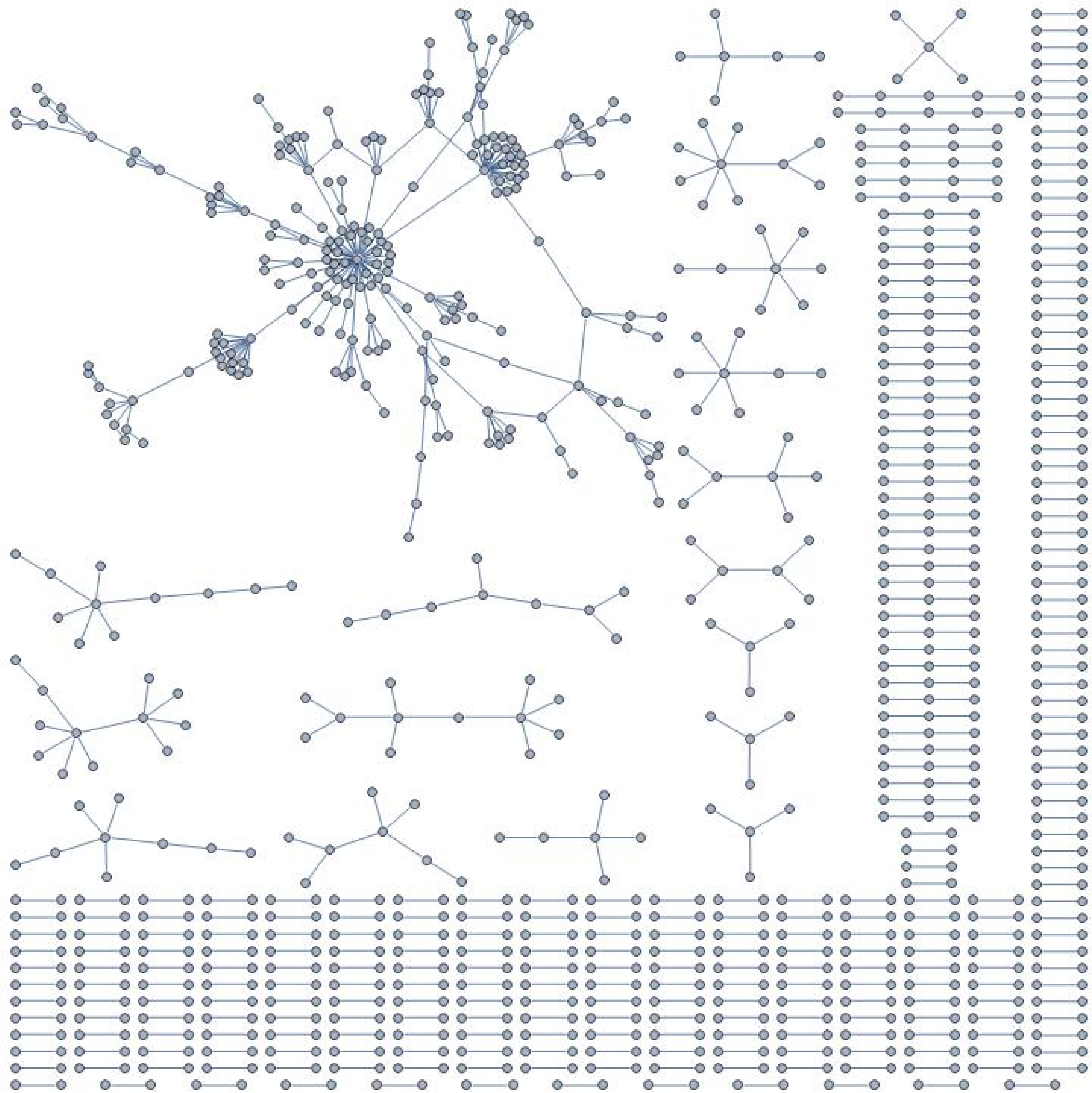
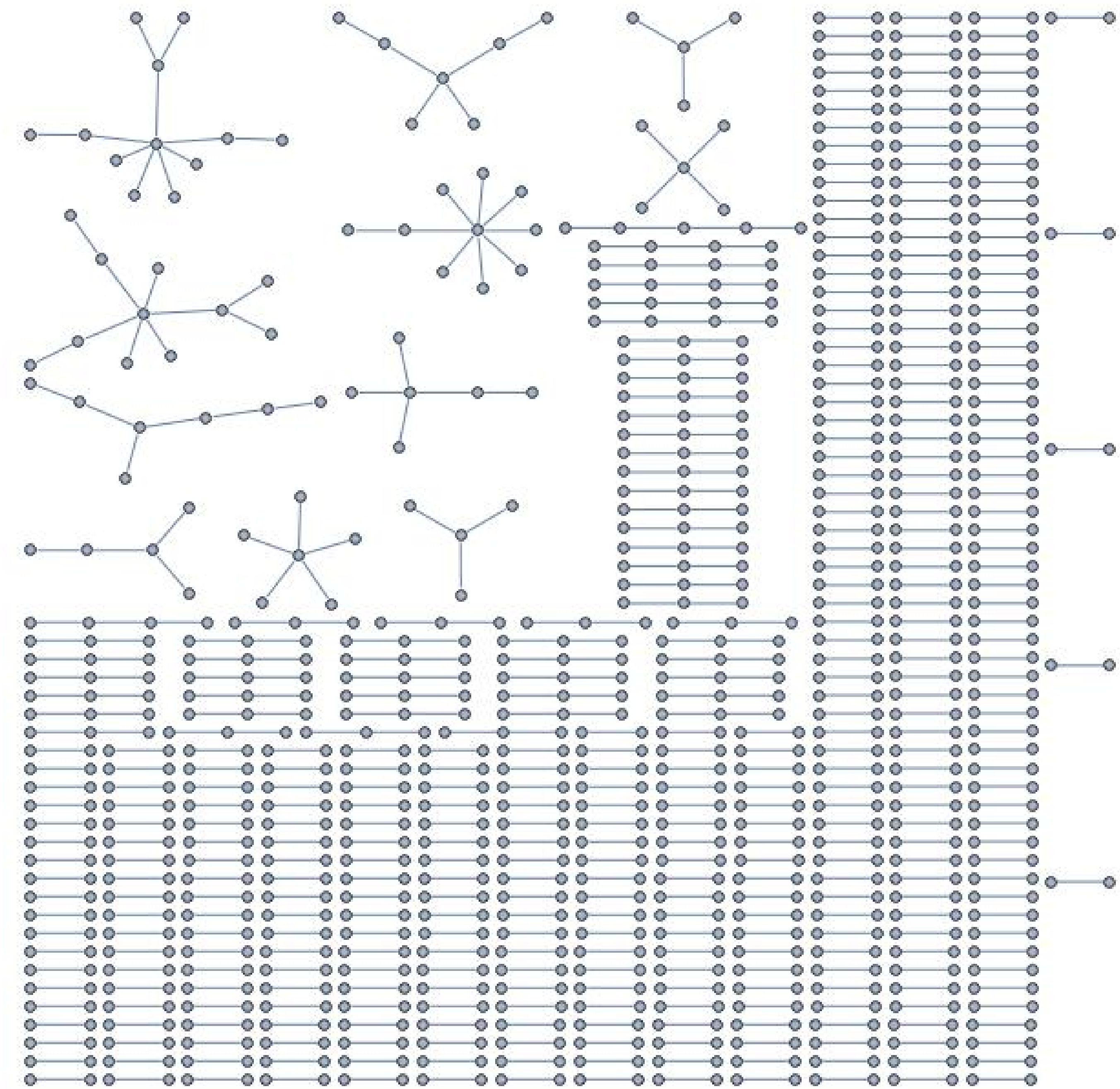
Configuration model: simulations



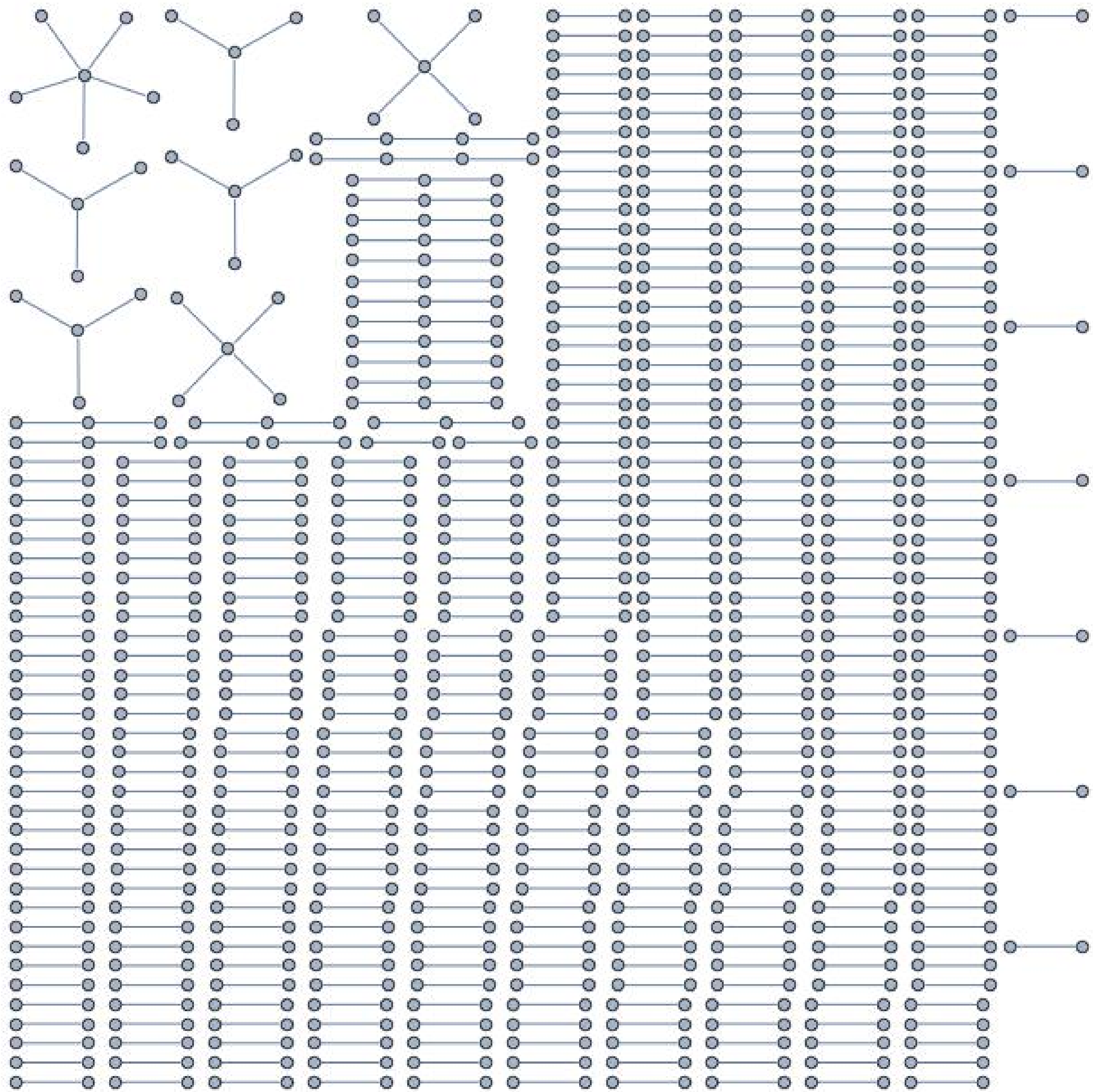
Configuration Model with 1000 vertices
and degrees power law with exponent 2

Configuration model: simulations



Configuration Model with 1000 vertices
and degrees power law with exponent 3

Configuration model: simulations



Configuration Model with 1000 vertices
and degrees power law with exponent 4

Configuration model: properties

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Preferential attachment graphs

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Preferential attachment graphs: construction

Two parameters: $m \in \mathbb{N}$ and $\delta > -m$

At time n, existing graph $PA_n(m, \delta)$ has n vertices and degree sequence $\mathbf{D}(n) = (D_1(n), \ldots, D_n(n))$

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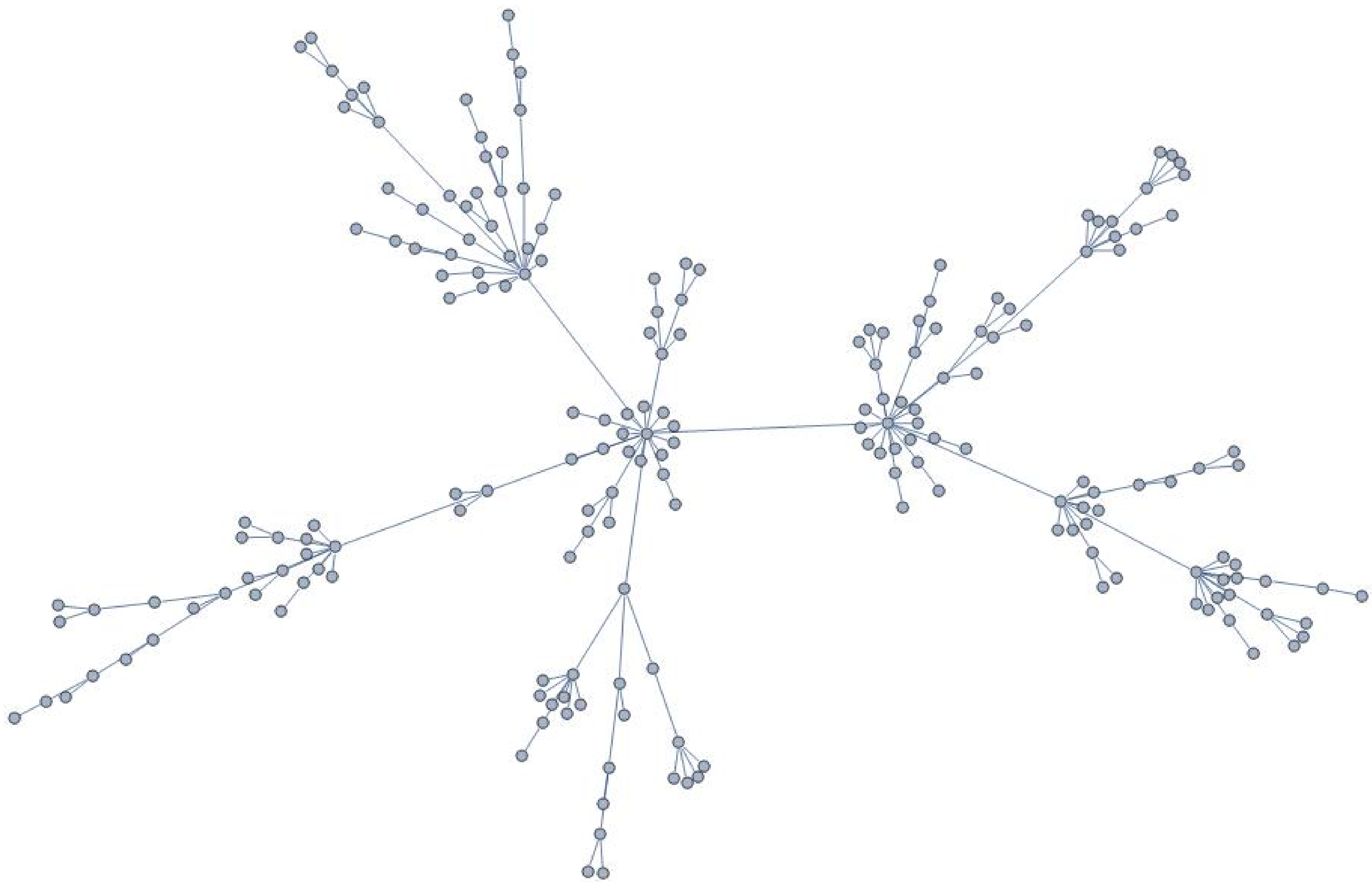
Scale free: power law degree sequence with exponent

$$\tau = 3 + \frac{\delta}{m} > 2$$

Preferential attachment graphs: simulations



Barabási-Albert graph with 200 vertices
each new vertex comes with 1 edge

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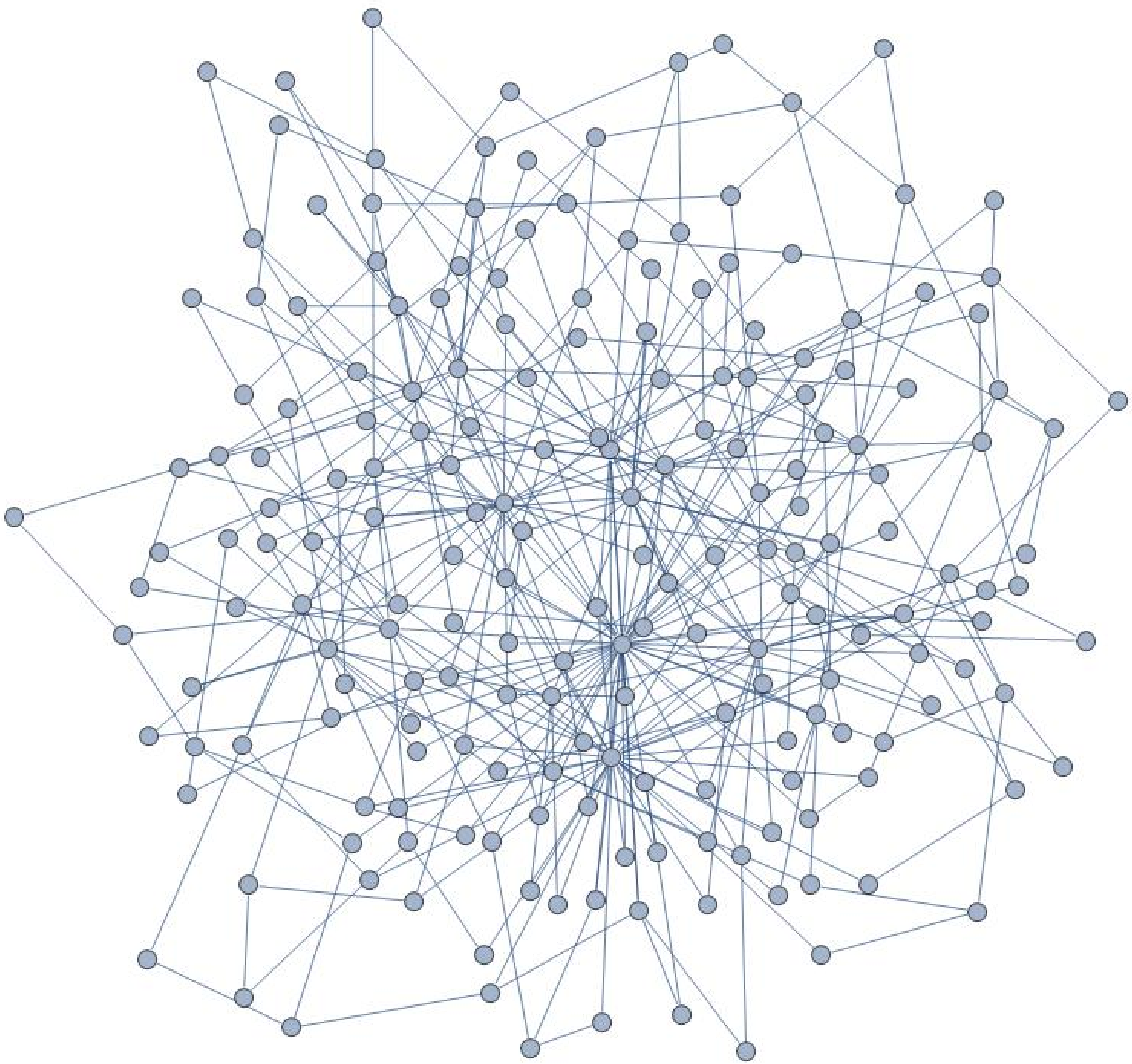
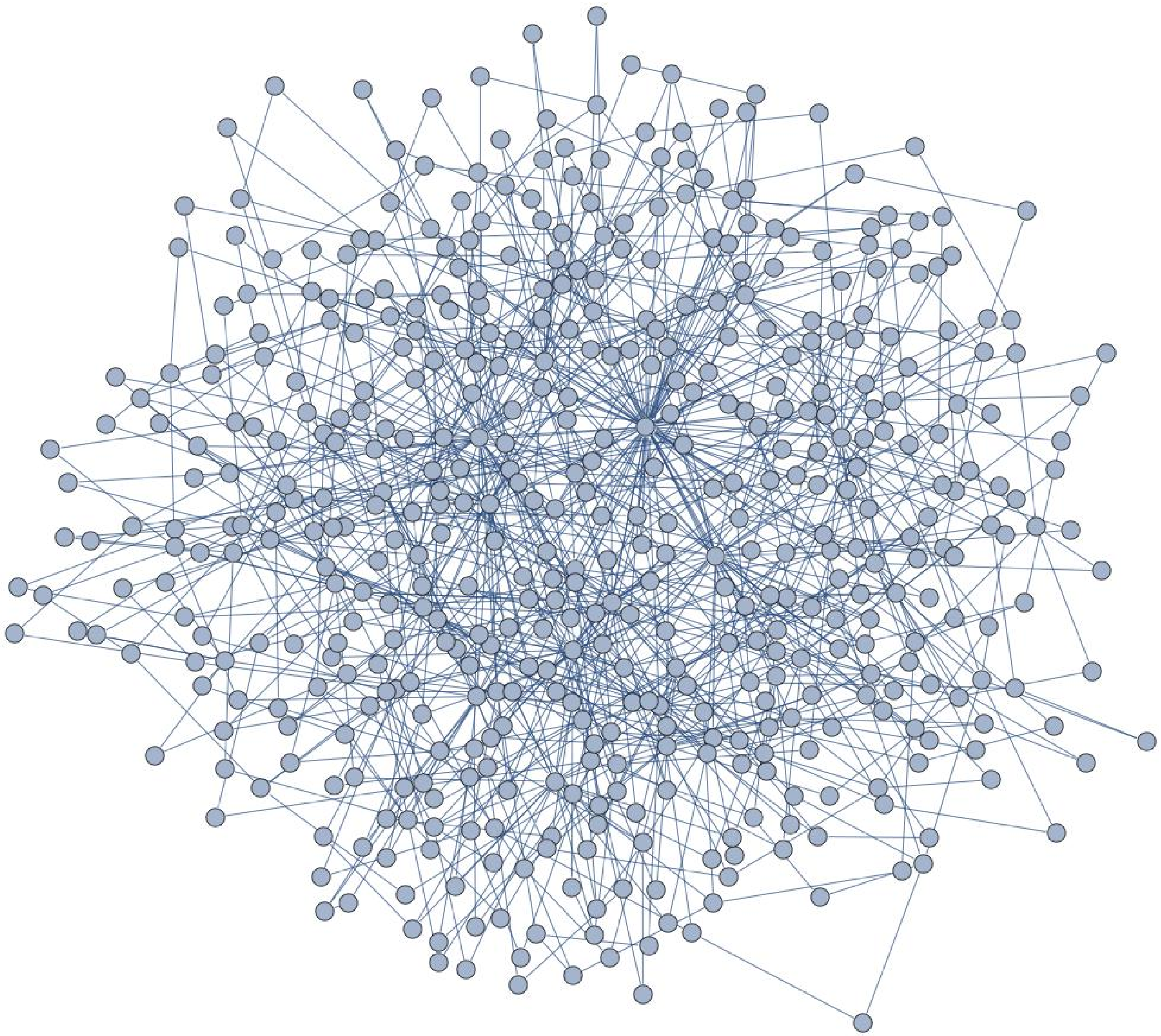


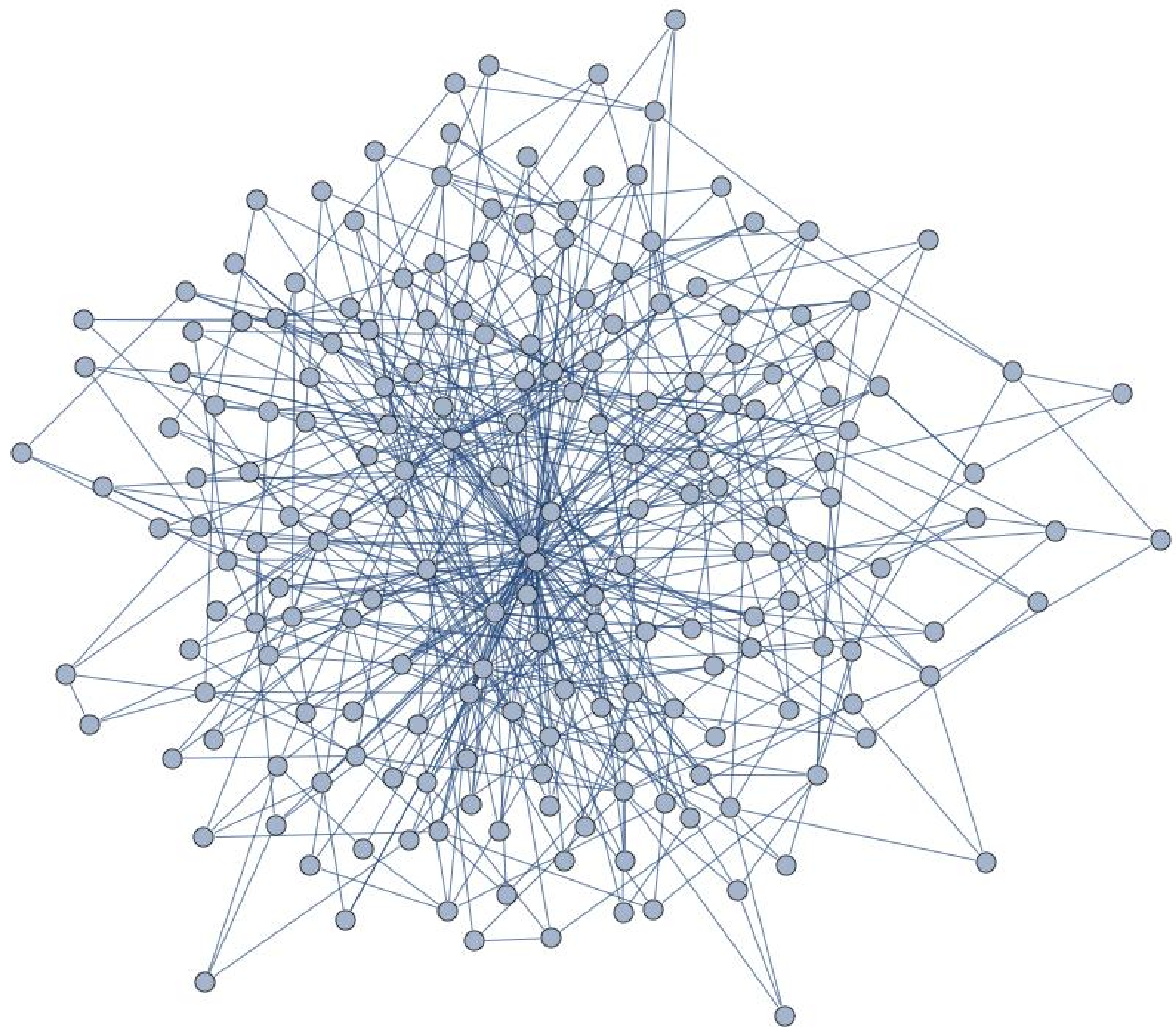
Barabási-Albert graph with 200 vertices
each new vertex comes with 2 edges

Preferential attachment graphs: simulations



Barabási-Albert graph with 500 vertices
each new vertex comes with 2 edges

Preferential attachment graphs: simulations



Barabási-Albert graph with 200 vertices
each new vertex comes with 3 edges

Preferential attachment graphs: other properties

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No rigorous result on clustering, but empirical studies $n^{-3/4}$:
no transitivity

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Models **not locally tree like** are **much harder to deal with**!

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Possible limits: **locally finite graphs**
(graphs with infinitely many vertices, but each vertex has finite degree)

Local weak convergence: formal definition

$$\mathcal{G}^\star = \{\text{locally finite rooted graphs}\}$$

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Local topology on $\mathcal{G}^\star$:

Take $R \in \mathbb{N}$ and $(G, o) \in \mathcal{G}^\star$,
define the subgraph of G inside a radius R around o:

$$\mathrm{Ball}_G(o, R) = \begin{cases} \text{Vertices} \subset \{v \in V(G) \,:\, d_G(o, v) \leqslant R + 1\} \\ \text{Edges} = \{\{v, v'\} \in E(G) \,:\, d_G(o, v) \leqslant R\} \end{cases}$$

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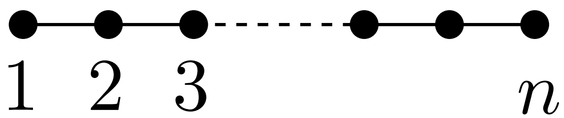
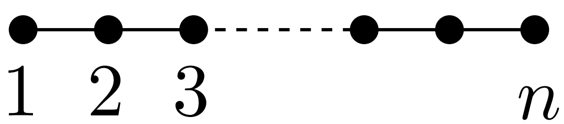


$\longrightarrow (\mathbb{Z}, 0)$

Local weak convergence: simple examples



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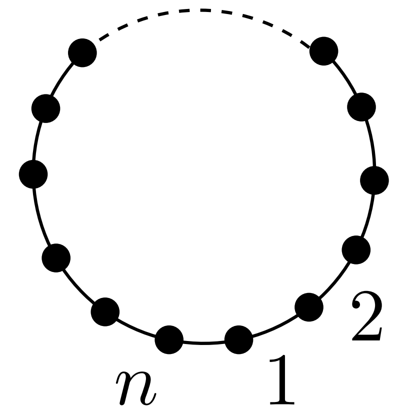

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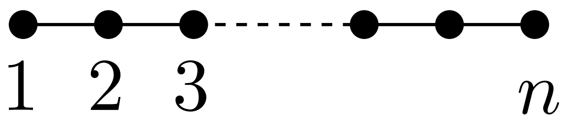

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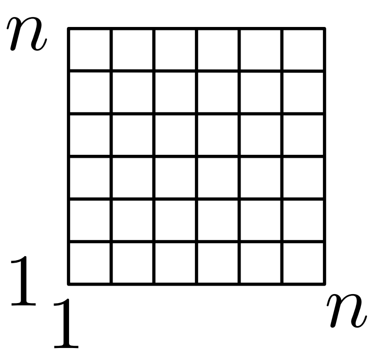

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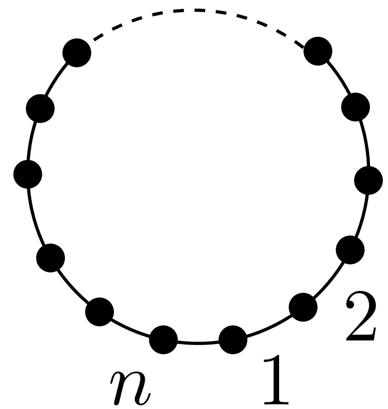

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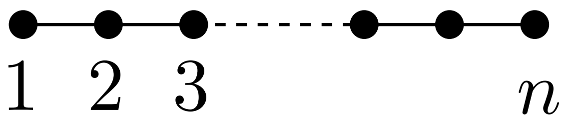

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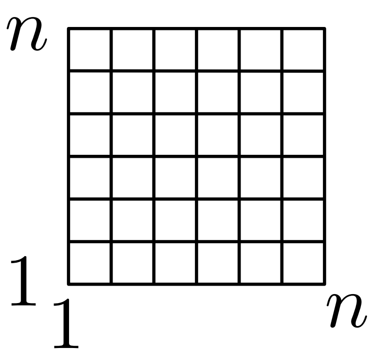

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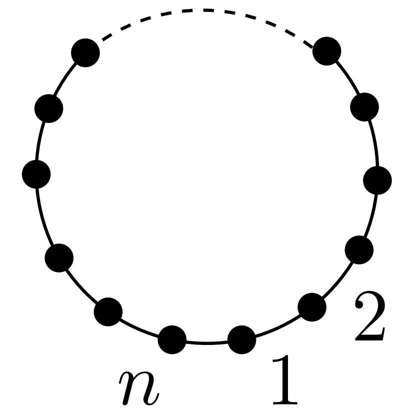

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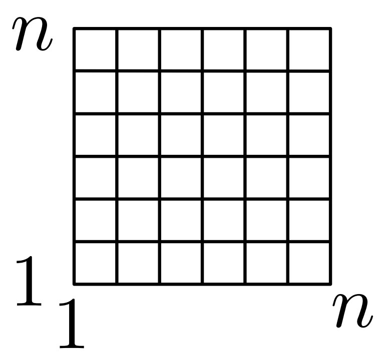

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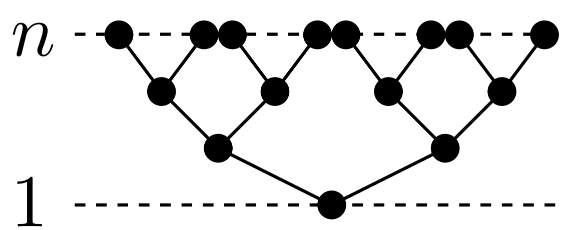
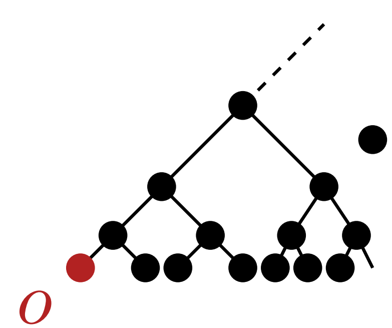
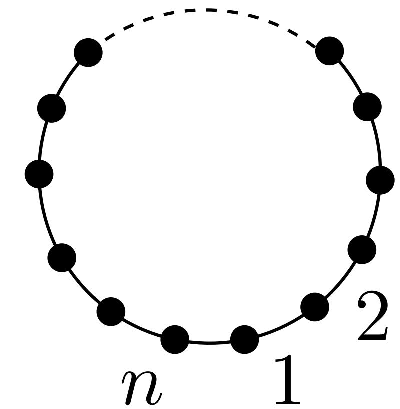
binary tree of height n



$\longrightarrow$

Canopy tree





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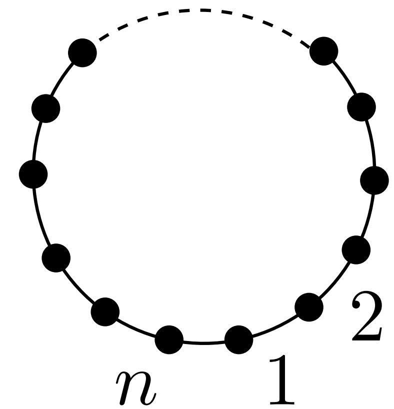

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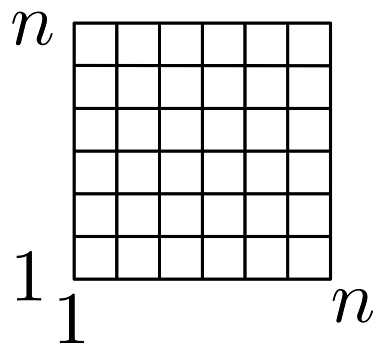

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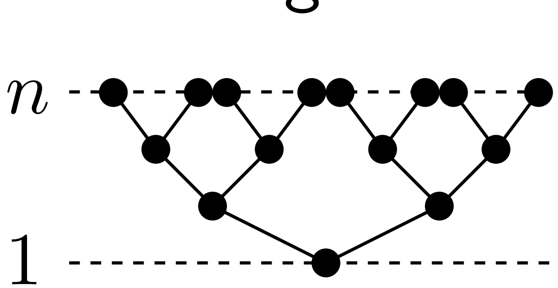

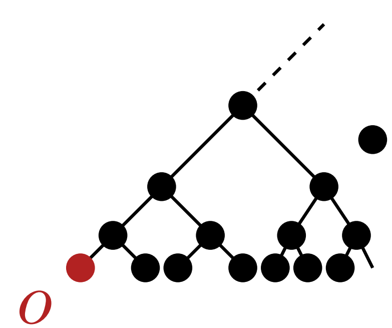

Uniform random tree with n vertices $\longrightarrow$ Skeleton tree

Local weak convergence: simple examples

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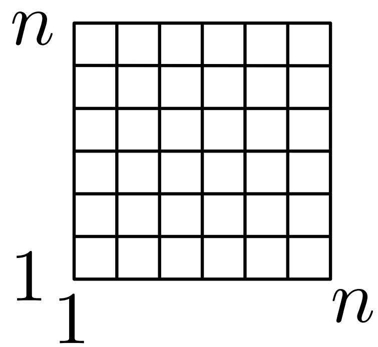

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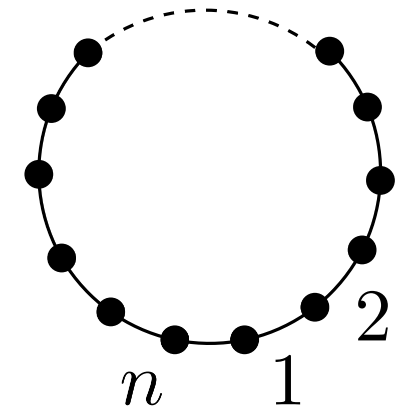

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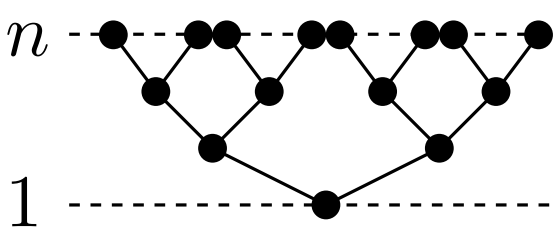

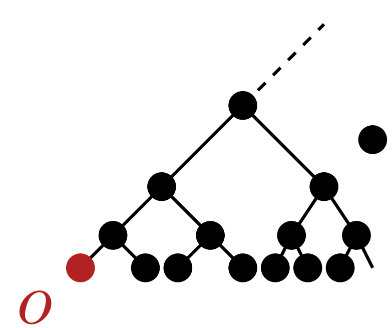

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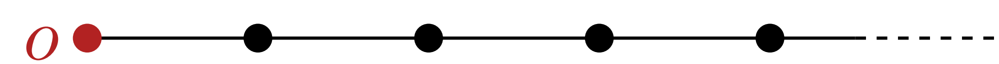

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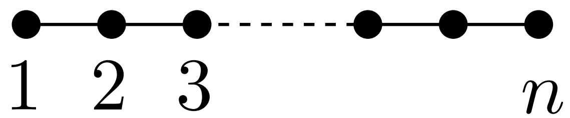

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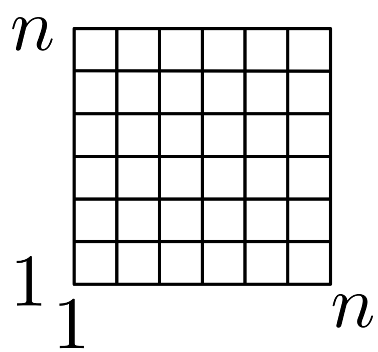

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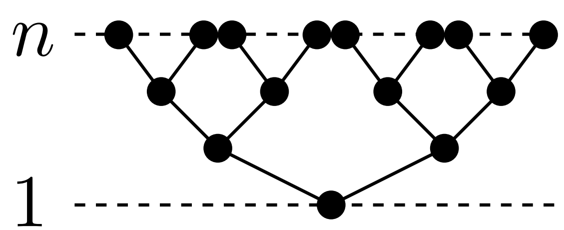
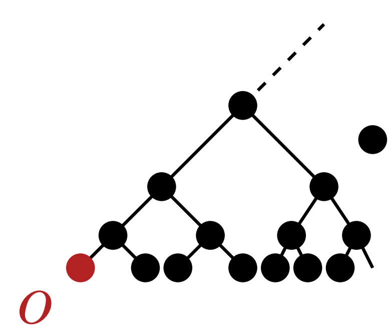
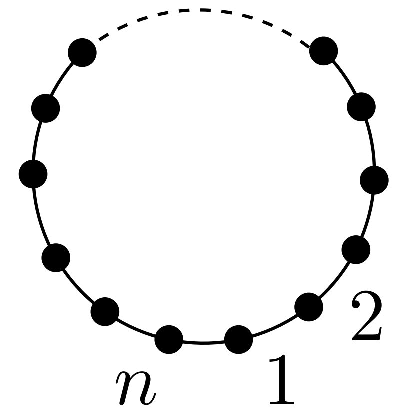




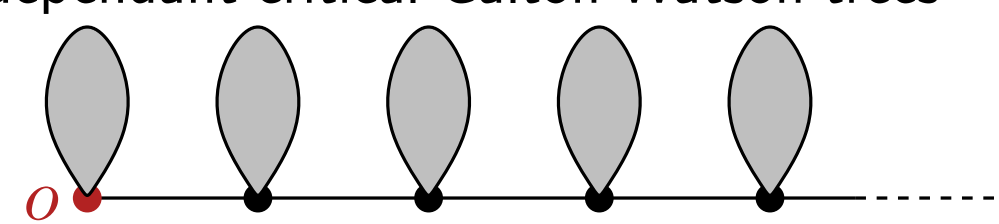


$\longrightarrow (\mathbb{Z}, 0)$

$\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow (\mathbb{Z}^2, 0)$

Uniform random tree with n vertices $\longrightarrow$ Skeleton tree

independant critical Galton Watson trees



Local weak convergence: Erdős-Rényi random graphs

Graph $ER(n, c/n)$,
degree distribution converges to Poisson r.v. $\mathcal{P}(c)$ with parameter c

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- Distances (**very sketchy!**): height of a supercritical Galton-Watson tree conditioned to have n vertices of order $\log n$

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- subtrees issued from first generation vertices are Galton-Watson trees with reproduction law $\hat{P}$, size-biaised version of P:

$$\hat{P}(\{k\}) = \frac{(k+1)P(\{k+1\})}{\sum_{k\geq 0} kP(\{k\})}$$

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$\longrightarrow$ Phase transition: the tree survives *iff* $\dfrac{E[D(D-1)]}{E[D]} > 1$

Other notions of convergence for graphs

[Berger, Borgs, Chayes and Saberi 2013]and [Dereich and Mörters 2013]: preferential attachment graphs are locally tree-like

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Global notions of convergence:

- **Scaling limits**: in **sparse** G_n, typical distances of order $\log n$
    1. consider G_n as the (discrete) **metric space** $(V(G_n), d_{G_n})$
    2. **rescale** the distances by a factor $\log n$
    3. does $(V(G_n), (\log n)^{-1} d_{G_n})$ converge to a limiting continuous random metric space ?
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- **Graphons**: [Borgs, Chayes, Lovasz, Sos and Vesztergombi 2008]
    1. **represent** graphs by functions $[0,1]^2 \to [0,1]$
    2. **metric** on these functions that keeps track of the frequency of **appearance of any finite graph** H in G_n
    3. works for sequences of **dense** graphs

Statistical mechanics on random graphs

Study random models or random evolutions on random graphs: random walks, percolations, ising model, . . .

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- **First passage percolation**

  Crossing an edge has a cost

- **Percolation**

  Robustness under attacks

- **Contagion model**

  Game-theoretic diffusion model

- **Systemic risk**

  Default cascades in interbank networks

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Large random graph $G_n = (V_n, E_n)$

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Now take uniformly at random two vertices $v, v' \in V_n$, define

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Are $\mathcal{W}_n$ and H_n similar to average distance ?

First passage percolation on configuration model

G_n = configuration model with *iid* power law degrees with exponent $\tau > 2$

Edge weights Y_e are *iid* exponential r.v.

[Bhamidi, Hooghiemstra and van der Hofstad 2010]:

There exists $\alpha > 0$ such that for $\tau \neq 3$:

$$\frac{H_n - \alpha \log n}{\sqrt{\alpha \log n}} \xrightarrow{d} \mathcal{N}(0, 1)$$

There exists $\gamma > 0$ such that for $\tau > 3$:

$$\mathcal{W}_n - \gamma \log n \xrightarrow{d} \mathcal{W}_\infty$$

For $\tau \in (2, 3)$:

$$\mathcal{W}_n \xrightarrow{d} \mathcal{W}_\infty$$

Robustness of networks [Bollobás and Riordan 2003]

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Now, for $p \in (0, 1)$, remove the **first** pn edges: **targeted attack**

There exists $0 < p_c < 1$ such that:

- if $p < p_c$, there is a giant component
- if $p > p_c$, there is no giant component

"Large random graphs are vulnerable against targeted attacks"

Contagion models and cascades

Game-theoretic model from [Morris 2000]
graph G, parameter $q \in (0, 1)$

- each vertex chooses between 2 behaviours: ● ■
- **Interaction payoff:**
  - $\longrightarrow$ If two neighbours are ● , they both receive payoff q
  - $\longrightarrow$ If two neighbours are ■ , they both receive payoff $1 - q$
  - $\longrightarrow$ If two neighbours disagree, they both receive 0
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  - → i adopts ■ if $N_i^{\blacksquare} > qd_i$
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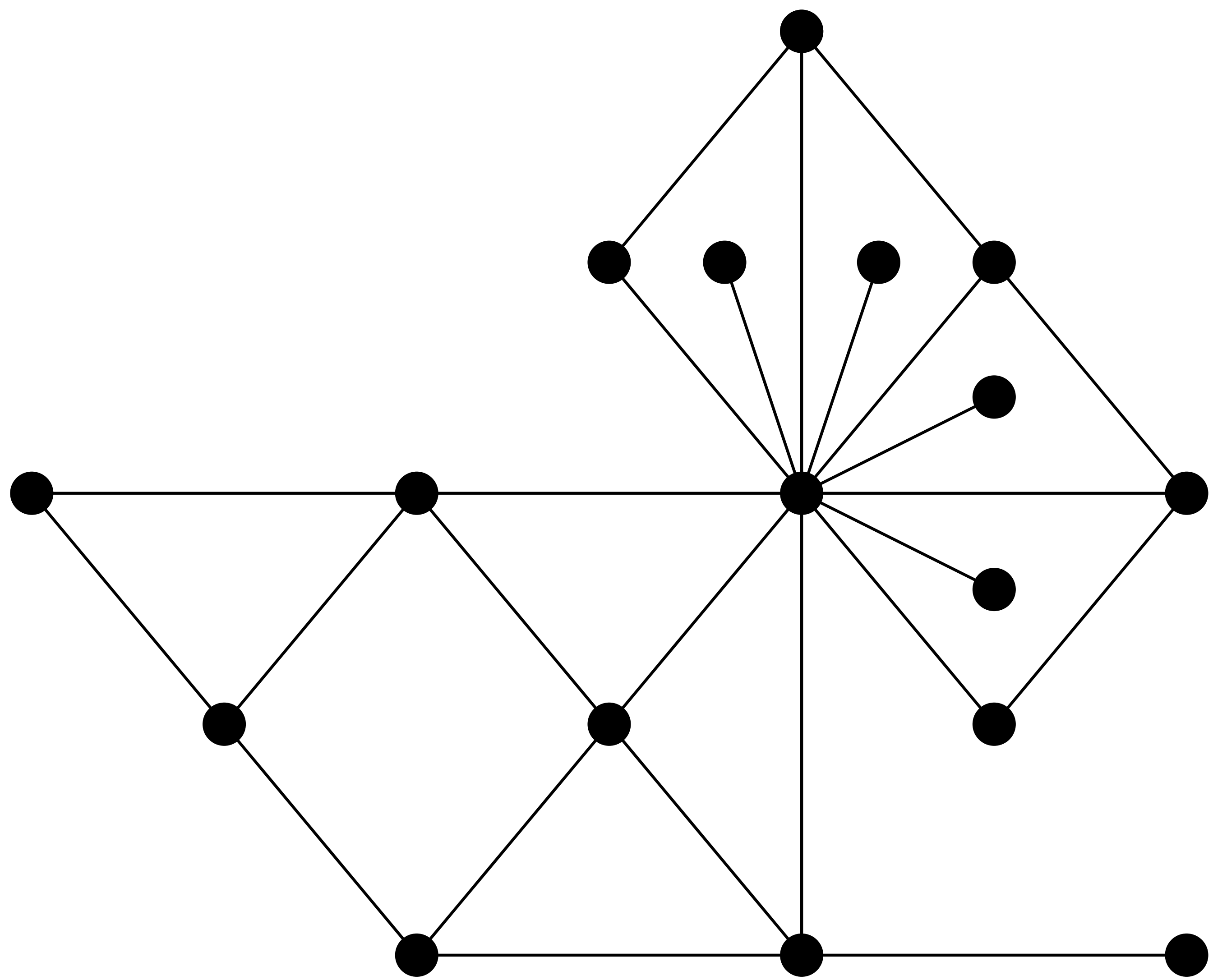
Cascade: Can we convert a macroscopic fraction of the graph to ■ by forcing few vertices to adopt ■ ?

Contagion models and cascades: an example

$q = 1/4$



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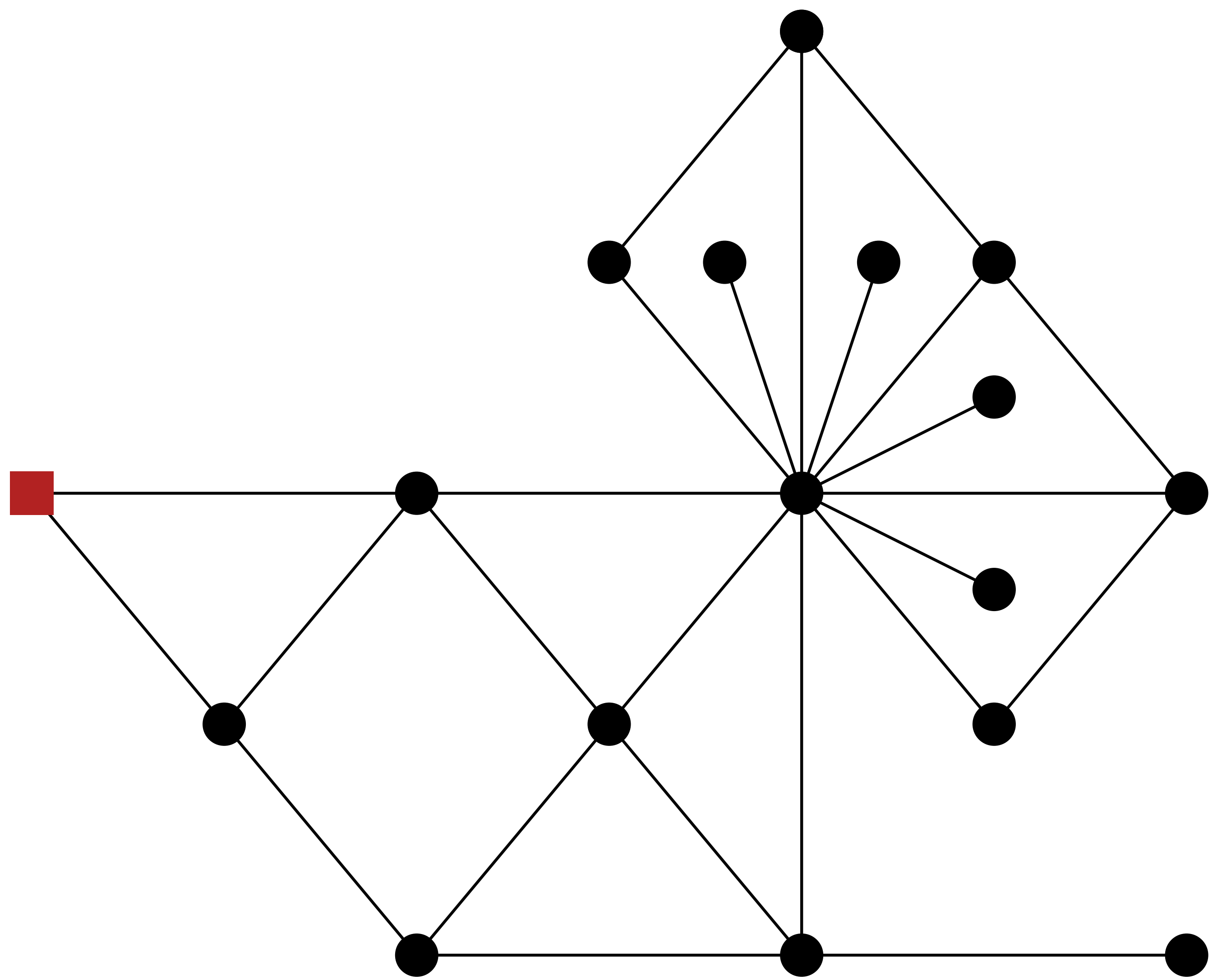

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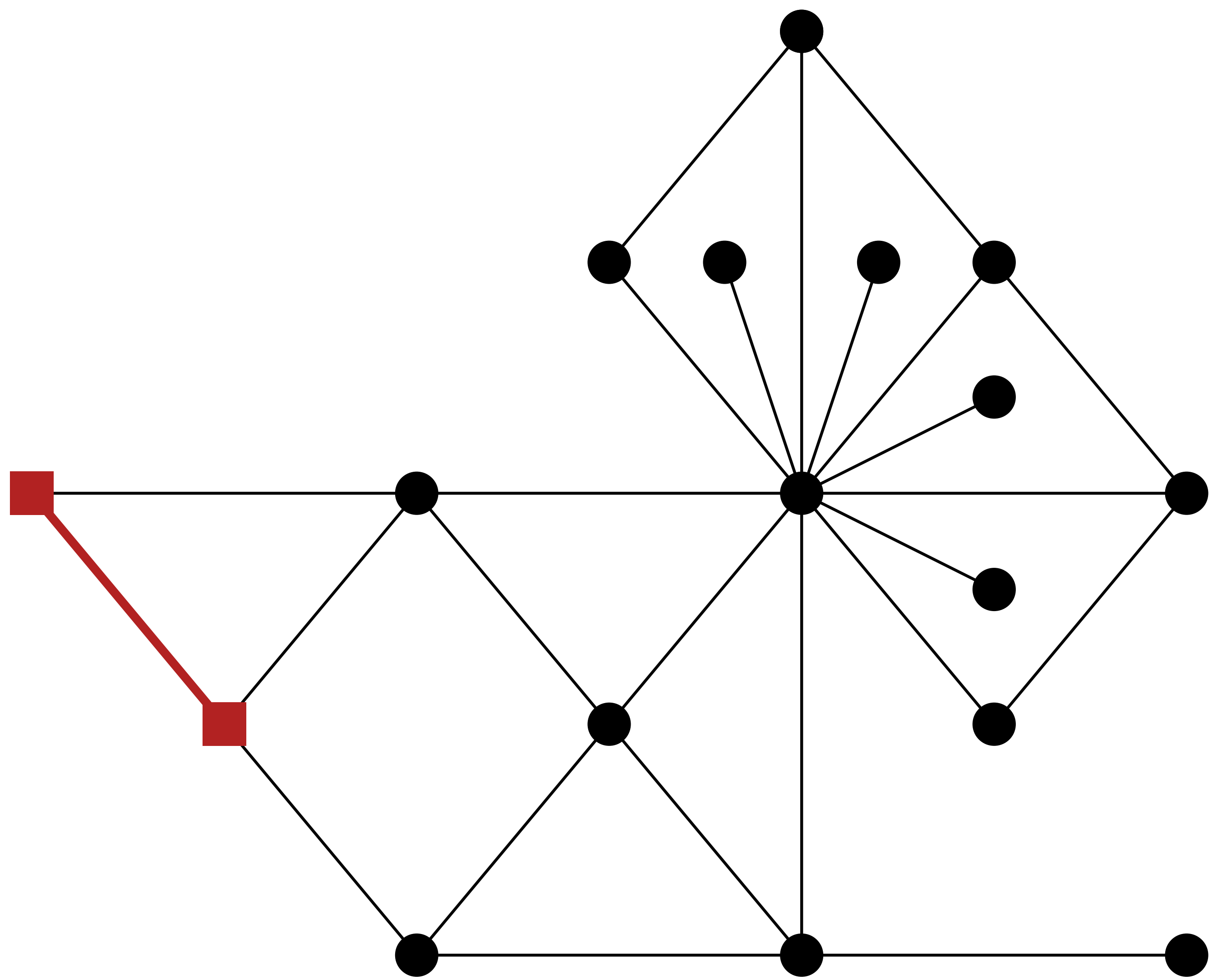

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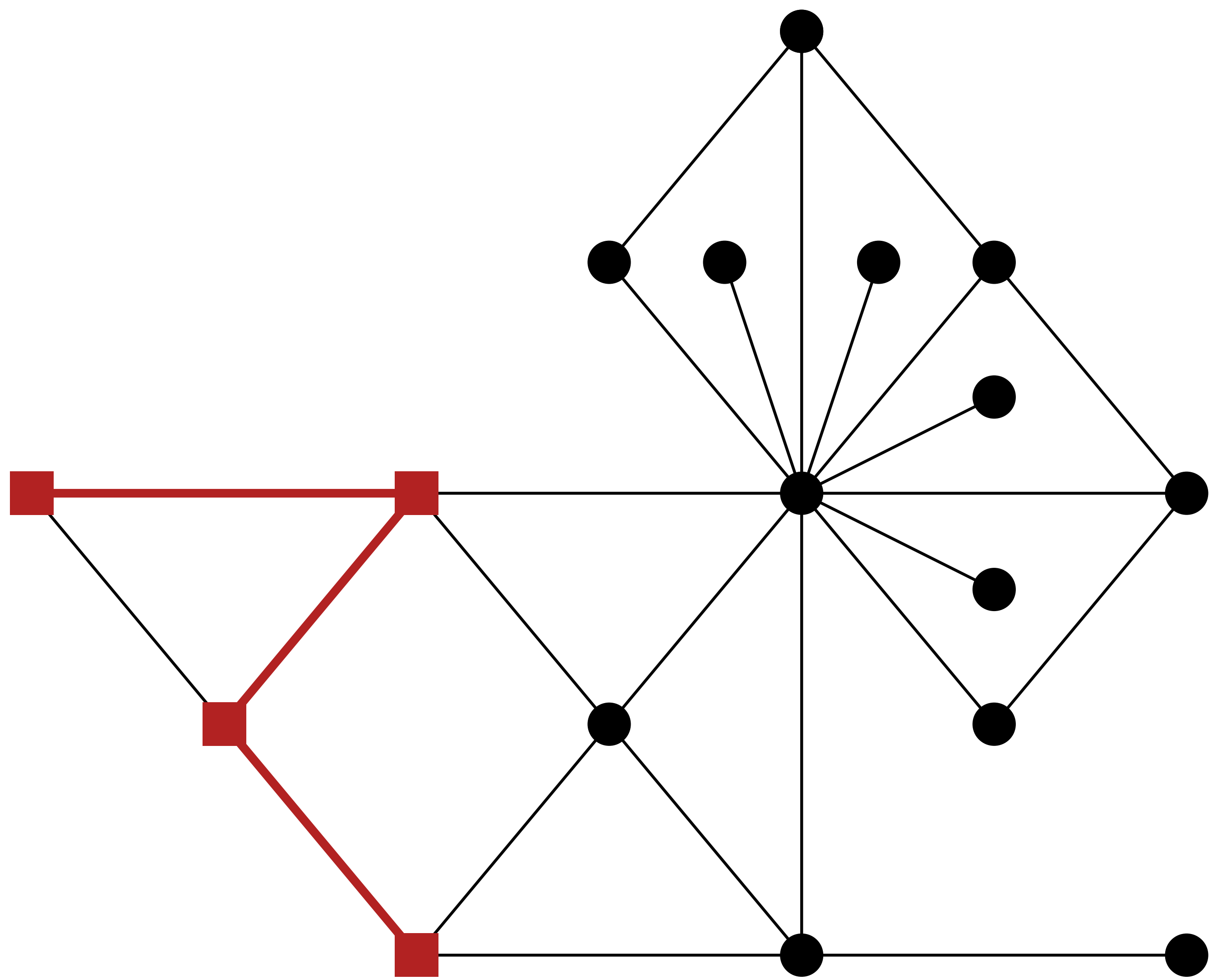

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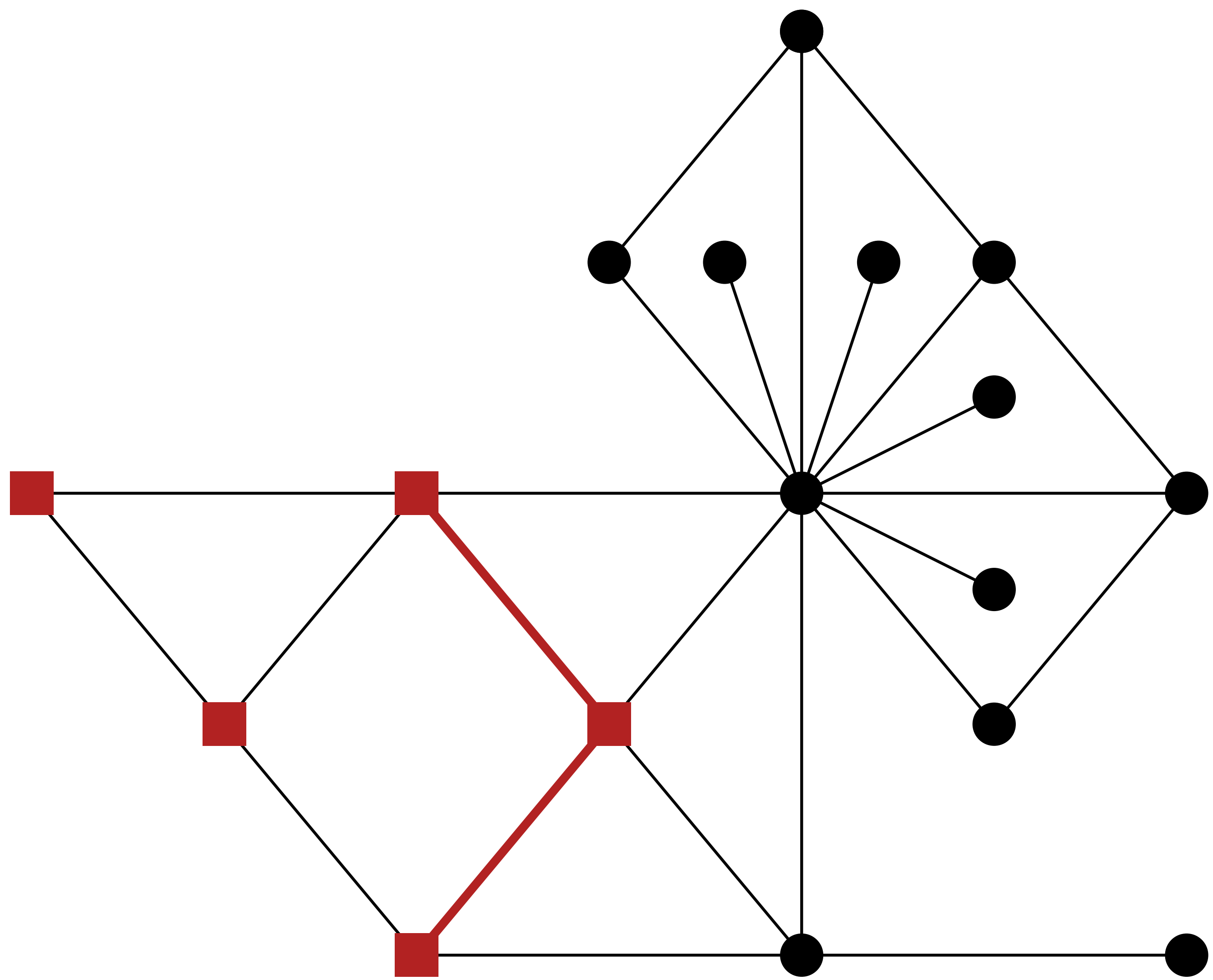

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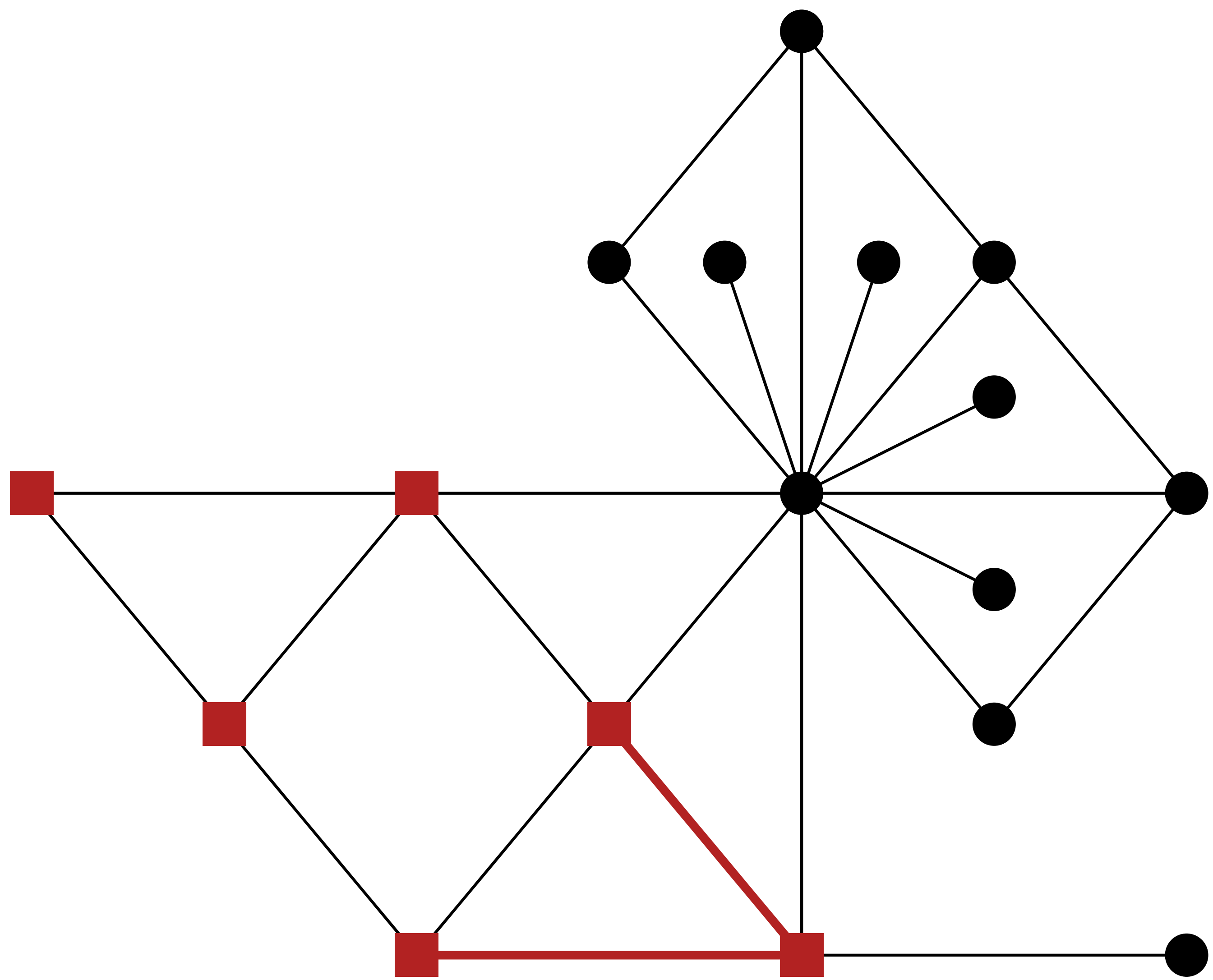

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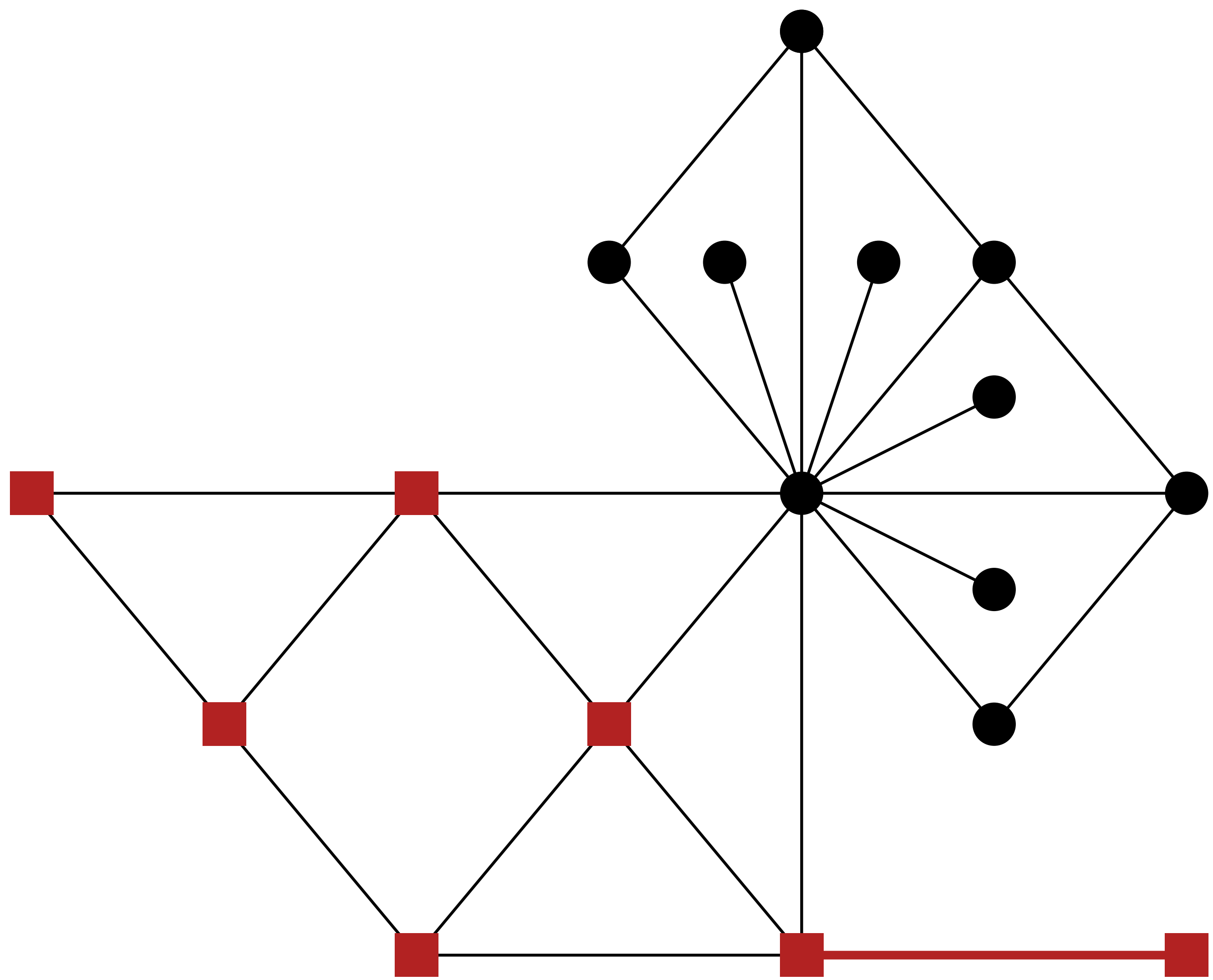

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Let $q_c = \sup\{q : E\left[D(D-1)\mathbf{1}_{\{D<q^{-1}\}}\right] > E[D]\}$:

- If $q < q_c$, for any $v \in P_n(q)$, with high probability $C(v,q) = \mathcal{O}(n)$
- If $q > q_c$, for any $v \in P_n(q)$, with high probability $C(v,q) = o(n)$

Systemic risk

[Cont, Moussa and Bastos e Santos 2010]: Brasilian interbank network

Model for interbank network: **directed** random graph

- each vertex i has a **capital** $c_i > 0$
- weight $E_{i,j} > 0$ on directed edge (i,j): **exposure** of i to j
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Indentify institution posing a systemic risk ?

Thank you for your attention and have a very nice week!