```
> restart;
```

Conjugate parameters s, t and singularity of t at s=1-u:

```
> st := t·sqrt(3 - 2·t); y[c] := 1/12; x[c] := 1/(12·sqrt(3));
```

$$st := t\sqrt{3-2\,t}$$

$$y_c := \frac{1}{12}$$

$$x_c := \frac{1}{36}\sqrt{3} \tag{1.1}$$

```
> with(gfun):
> algt := s^2 - t^2·(3 - 2·t);
```

$$algt := s^2 - t^2\,(3-2\,t) \tag{1.2}$$

```
> allvalues( algeqtoseries(subs(s = 1 - u, algt), u, t, 4));
```

$$\left[-\frac{1}{2}+\frac{4}{9}\,u+\frac{10}{243}\,u^2+\frac{64}{6561}\,u^3+\mathrm{O}(u^4), 1+\frac{1}{3}\sqrt{6}\,\sqrt{u}-\frac{2}{9}\,u-\frac{7}{324}\sqrt{6}\,u^{3/2}+\mathrm{O}(u^2)\right], \left[-\frac{1}{2}+\frac{4}{9}\,u+\frac{10}{243}\,u^2+\frac{64}{6561}\,u^3+\mathrm{O}(u^4), 1-\frac{1}{3}\sqrt{6}\,\sqrt{u}-\frac{2}{9}\,u+\frac{7}{324}\sqrt{6}\,u^{3/2}+\mathrm{O}(u^2)\right] \tag{1.3}$$

t=1 if u=0 and is smaller than 1:

```
> tseru := 1 - 1/3*sqrt(6)*sqrt(u) - 2/9*u + 7/324*sqrt(6)*u^(3/2);
```

$$tseru := 1-\frac{1}{3}\sqrt{6}\,\sqrt{u}-\frac{2}{9}\,u+\frac{7}{324}\sqrt{6}\,u^{3/2} \tag{1.4}$$

For later when s=exp(-lambda·x):

```
> tserlambda := convert(simplify(series(subs(u = 1 - exp(-lambda·x), tseru), x, 2), assume
      = positive), polynom);
```

$$tserlambda := 1-\frac{1}{3}\sqrt{6}\,\sqrt{\lambda}\,\sqrt{x}-\frac{2}{9}\,\lambda\,x+\frac{17}{162}\sqrt{6}\,\lambda^{3/2}\,x^{3/2} \tag{1.5}$$

Phi and kippas (sections 3.1 and 3.3)

```
> phitz := 1 - (1/sqrt(1 - z)·sqrt((3 - 2·t)/t) + sqrt(1 + 3/(1 - z)·(1 - t)/t))^(-2);
```

$$phitz := 1-\frac{1}{\left(\frac{\sqrt{\frac{3-2\,t}{t}}}{\sqrt{1-z}}+\sqrt{1+\frac{3\,(1-t)}{(1-z)\,t}}\right)^2} \tag{2.1}$$

```
> phit0 := simplify(subs(z = 0, phitz));
```

$$phit0 := \frac{3}{4}\,\frac{-4+3\,t}{-3+2\,t} \tag{2.2}$$

> phit20 := simplify(subs(z = phit0, phitz)) assuming 0 < t < 1;

$$phit20 := \frac{8}{9}\,\frac{18-27\,t+10\,t^2}{(-4+3\,t)^2} \tag{2.3}$$

> phiz := simplify(subs(t = 1, phitz));

$$phiz := \frac{1+2\sqrt{1-z}}{\left(1+\sqrt{1-z}\right)^2} \tag{2.4}$$

> $phitrz := 1 - (1-z)\cdot\left(\mathrm{sqrt}\left(1+\frac{t\cdot(1-z)}{3\cdot(1-t)}\right)\sinh\left(r\cdot\mathrm{arccosh}\left(\mathrm{sqrt}\left(\frac{3-2\cdot t}{t}\right)\right)\right) + \cosh\left(r\cdot\mathrm{arccosh}\left(\mathrm{sqrt}\left(\frac{3-2\cdot t}{t}\right)\right)\right)\right)^{-2};$

$$phitrz := 1-(1-z)\Bigg/\left(\sqrt{1+\frac{(1-z)\,t}{3-3\,t}}\,\sinh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}}\right)\right)+\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}}\right)\right)\right)^2 \tag{2.5}$$

> phitr0 := simplify(subs(z = 0, phitrz));

$$phitr0 := \Bigg(3\,\Bigg(-6\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)^2+6+5\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)^2 t-5\,t+2\sqrt{3}\sqrt{\frac{-3+2\,t}{t-1}}\,\sinh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)t-2\sqrt{3}\sqrt{\frac{-3+2\,t}{t-1}}\,\sinh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)\Bigg)\Bigg)\Bigg/\Bigg((t-1)\left(\sqrt{3}\sqrt{\frac{-3+2\,t}{t-1}}\,\sinh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)+3\cosh\left(r\,\mathrm{arccosh}\left(\sqrt{-\frac{-3+2\,t}{t}}\right)\right)\right)^2\Bigg) \tag{2.6}$$

> $Ksz := simplify\left((y_c\cdot t)^2\cdot\left(\frac{phit0}{1-\frac{phit0}{z}}\right)\cdot\left(phiz-\frac{phit0}{z}\cdot phit20\right)\right);$

$$Ksz := \frac{1}{144}\left(t\left(72 - 72z - 108t + 40t^2 + 102tz + 18z\sqrt{-\frac{-3+2t}{t}}\sqrt{\frac{2t+tz-3}{(-1+z)t}}\sqrt{1-z}\,t^2 - 20\sqrt{-\frac{-3+2t}{t}}\sqrt{\frac{2t+tz-3}{(-1+z)t}}\sqrt{1-z}\,t^2 - 24z\sqrt{-\frac{-3+2t}{t}}\sqrt{\frac{2t+tz-3}{(-1+z)t}}\sqrt{1-z}\,t + 24\sqrt{-\frac{-3+2t}{t}}\sqrt{\frac{2t+tz-3}{(-1+z)t}}\sqrt{1-z}\,t - 35t^2z\right)\right) \Big/ \left(\left(\sqrt{-\frac{-3+2t}{t}} + \sqrt{\frac{2t+tz-3}{(-1+z)t}}\sqrt{1-z}\right)^2(-12z + 8tz + 12 - 9t)\right) \quad \textbf{(2.7)}$$

> $simplify(limit(Ksz, z = 1, left));$

$$\frac{1}{144}t^2 \quad \textbf{(2.8)}$$

> $K1z := simplify(subs(t = 1, Ksz));$

$$K1z := \frac{1}{144}\frac{-4 + 5z + 6z\sqrt{1-z} - 4\sqrt{1-z}}{\left(1+\sqrt{1-z}\right)^2(4z-3)} \quad \textbf{(2.9)}$$

> $K1zAlt := \frac{1}{144}\cdot\left(1 - 2\cdot\frac{(1-z)}{(1+2\cdot \mathrm{sqrt}(1-z))\cdot\left(1+\sqrt{1-z}\right)}\right);$

$$K1zAlt := \frac{1}{144} - \frac{1}{72}\frac{1-z}{\left(1+2\sqrt{1-z}\right)\left(1+\sqrt{1-z}\right)} \quad \textbf{(2.10)}$$

> $simplify(K1z - K1zAlt);$

$$0 \quad \textbf{(2.11)}$$

Generating function of nu:

> $nugen := 144\cdot K1zAlt;$

$$nugen := 1 - \frac{2(1-z)}{\left(1+2\sqrt{1-z}\right)\left(1+\sqrt{1-z}\right)} \quad \textbf{(2.12)}$$

Expectation of nu:

> $subs(z = 1, diff(nugen, z));$

$$2 \quad \textbf{(2.13)}$$

Verifying that K(t,1) is T1bullet:

> $T1 := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\,x\,T2};$

$$T1 := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\,x\,T2} \quad \textbf{(2.14)}$$

> $Z1 := subs\left(x = x_c \cdot s, T2 = \frac{y_c \cdot t}{x_c \cdot s} \cdot phit0, T1\right);$

$$Z1 := \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{4}\,\frac{t\,(-4+3\,t)}{-3+2\,t}} \qquad \textbf{(2.15)}$$

> $dt := simplify(diff(st, t))^{-1};$

$$dt := -\frac{1}{3}\,\frac{\sqrt{3-2\,t}}{t-1} \qquad \textbf{(2.16)}$$

> $T1bullet := simplify\left(diff(Z1, t) \cdot x_c \cdot (st)^2 \cdot dt\right)$ assuming $t < 1$;

$$T1bullet := \frac{1}{144}\,t^2 \qquad \textbf{(2.17)}$$

> $phiz;$

$$\frac{1+2\sqrt{1-z}}{\left(1+\sqrt{1-z}\right)^2} \qquad \textbf{(2.18)}$$

Two point function obtained at s=1 by the rho-skeleton (3.3.1 alternative)

> $T1hplus1alt := simplify\left(\frac{1}{(r+1)^3}\left(subs\left(z = 1 - \frac{1}{(r+1)^2}, simplify\left(\left(phiz + z \cdot diff(phiz, z) + \frac{simplify(diff(K1zAlt, z))}{y_c \cdot y_c}\right) \cdot phiz\right)\right)\right), assume = positive\right);$

$$T1hplus1alt := \frac{4}{(r+2)\,(r+3)\,(r+1)} \qquad \textbf{(2.19)}$$

Two point function (section 3.3.2)

Singularity of phitr0 for r fixed and s=x_c*(1-u):

> $phitr0ser := map(simplify, collect(convert(simplify(series(simplify(subs(t = tseru, phitr0)), u, 3)), polynom), u));$

$phitr0ser :=$ **(3.1)**

$$-\frac{1}{4725}\,\frac{1}{(r+1)^2}\left(r\left(252\,r^7 + 2016\,r^6 + 5856\,r^5 + 6912\,r^4 + 2811\,r^3 + 2796\,r^2 + 4057\,r - 550\right)u^2\right) + \frac{1}{2268}\,\frac{r\sqrt{2}\sqrt{3}\left(430 - 73\,r + 1656\,r^3 + 864\,r^4 + 864\,r^2 + 144\,r^5\right)u^{3/2}}{(r+1)^2} - \frac{2}{45}\,\frac{r\left(9\,r^3 + 36\,r^2 + 34\,r - 4\right)u}{(r+1)^2} + \frac{1}{3}\,\frac{r\,(r+2)\sqrt{2}\sqrt{3}\sqrt{u}}{(r+1)^2} + \frac{r\,(r+2)}{(r+1)^2}$$

Singularity of G_(h+1) for h fixed and s=x_c*(1-u):

> $Ghp1ser := map\Bigg(factor, collect\Bigg(convert\Bigg(simplify\Bigg(series\Bigg(simplify\Bigg(subs\Bigg(t = tseru, \left(y_c \cdot t\right)^2$

$$\cdot phit0 \cdot \left(\frac{subs(r = h+1, phitr0ser) - phit20}{1 - \frac{phit0}{subs(r = h, phitr0ser)}} - \frac{subs(r = h, phitr0ser) - phit20}{1 - \frac{phit0}{subs(r = h-1, phitr0ser)}} \right) \Bigg) \Bigg) \Bigg),$$

$$u, 2 \Big) \Big) , polynom \Big) , u \Big) \Big) ;$$

$$Ghp1ser := \frac{1}{2835} \frac{1}{(3+h)(h+2)(h+1)} \left(\sqrt{3} \left(10 h^6 + 120 h^5 + 588 h^4 + 1504 h^3 + 2114 h^2 + 1544 h + 455\right) \sqrt{2}\, u^{3/2}\right) - \frac{1}{90} \frac{\left(h^4 + 8 h^3 + 23 h^2 + 28 h + 15\right) u}{(3+h)(h+2)(h+1)} + \frac{1}{36\,(3+h)(h+2)(h+1)} \quad \textbf{(3.2)}$$

> $Ghser := map(factor, subs(h = h-1, Ghp1ser));$

$$Ghser := \frac{1}{2835} \frac{\sqrt{3}\left(10 h^6 + 60 h^5 + 138 h^4 + 152 h^3 + 80 h^2 + 16 h - 1\right) \sqrt{2}\, u^{3/2}}{(h+2)(h+1)\, h} - \frac{1}{90} \frac{\left(h^4 + 4 h^3 + 5 h^2 + 2 h + 3\right) u}{(h+2)(h+1)\, h} + \frac{1}{36\,(h+2)(h+1)\, h} \quad \textbf{(3.3)}$$

> $simplify\left(\frac{9 \cdot 36}{2835}\right);$

$$\frac{4}{35} \quad \textbf{(3.4)}$$

Two Point Function of 2DQG, s=exp(- lambda x) with x = H^(-4):

> $tser := convert(simplify(series(subs(u = 1 - \exp(-\text{lambda} \cdot x), tseru), x, 2), assume = positive), polynom);$

$$tser := 1 - \frac{1}{3}\sqrt{6}\sqrt{\lambda}\sqrt{x} - \frac{2}{9}\lambda x + \frac{17}{162}\sqrt{6}\,\lambda^{3/2} x^{3/2} \quad \textbf{(3.5)}$$

> $Phi0ser := map(simplify, collect(convert(simplify(series(subs(t = tser, phit0), x, 2), assume = positive), polynom), x));$

$$Phi0ser := \frac{3}{4} + \frac{1}{4}\sqrt{6}\sqrt{\lambda}\sqrt{x} - \frac{5}{6}\lambda x + \frac{79}{216}\sqrt{6}\,\lambda^{3/2} x^{3/2} \quad \textbf{(3.6)}$$

> $Phi20ser := map(simplify, collect(convert(simplify(series(subs(t = tser, phit20), x, 2), assume = positive), polynom), x));$

$$Phi20ser := \frac{8}{9} + \frac{8}{27}\sqrt{6}\sqrt{\lambda}\sqrt{x} - \frac{224}{81}\lambda x + \frac{2476}{729}\sqrt{6}\,\lambda^{3/2} x^{3/2} \quad \textbf{(3.7)}$$

> $PhiH0ser := map\left(simplify, collect\left(convert\left(simplify\left(series\left(subs\left(t = tser, r = h \cdot x^{-\frac{1}{4}}, phitr0\right), x, 2\right), assume = positive\right), polynom\right), x\right)\right):$

> $PhiHmoins10ser := map\left(simplify, collect\left(convert\left(simplify\left(series\left(subs\left(t = tser, r = h \cdot x^{-\frac{1}{4}} - 1, phitr0\right), x, 2\right), assume = positive\right), polynom\right), x\right)\right):$

$$> PhiHplus10ser := map\left(simplify, collect\left(convert\left(simplify\left(series\left(subs\left(t = tser, r = h\cdot x^{-\frac{1}{4}} + 1, phitr0\right), x, 2\right), assume = positive\right), polynom\right), x\right)\right):$$

>

$$> GH := simplify\left(factor\left(convert\left(series\left(simplify\left((y_c\cdot tser)^2\cdot Phi0ser \cdot\left(\frac{PhiHplus10ser - Phi20ser}{1 - \frac{Phi0ser}{PhiH0ser}} - \frac{PhiH0ser - Phi20ser}{1 - \frac{Phi0ser}{PhiHmoins10ser}}\right)\right), x, 1\right), polynom\right)\right)\right);$$

$$GH := \frac{\lambda^{3/4}\, 3^{3/4}\, 2^{3/4}\left(e^{2\,h\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} + 1\right) e^{2\,h\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}}\, x^{3/4}}{9\,e^{6\,h\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} - 27\,e^{4\,h\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} + 27\,e^{2\,h\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} - 9} \quad (3.8)$$

$$> GH0 := limit(GH, \mathrm{lambda} = 0);$$

$$GH0 := \frac{1}{36}\,\frac{x^{3/4}}{h^3} \quad (3.9)$$

$$> simplify\left(subs\left(h = 1, \frac{GH}{GH0}\right)\right);$$

$$\frac{4\,\lambda^{3/4}\, 3^{3/4}\, 2^{3/4}\left(e^{2\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} + 1\right) e^{2\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}}}{e^{6\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} - 3\,e^{4\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} + 3\,e^{2\,2^{1/4}\,3^{1/4}\,\lambda^{1/4}} - 1} \quad (3.10)$$

Scaling limits for horohulls (section 5.2)

x ~ 1/r^2

$$> s1ser := series\left(\exp\left(-\lambda_1\cdot x^2\right), x, 3\right); s2ser := series\left(\exp\left(-\lambda_2\cdot x\right), x, 3\right);$$

$$s1ser := 1 - \lambda_1\, x^2 + O\left(x^4\right)$$

$$s2ser := 1 - \lambda_2\, x + \frac{1}{2}\,\lambda_2^2\, x^2 + O\left(x^3\right) \quad (4.1)$$

$$> tser1 := simplify\left(subs\left(\mathrm{lambda} = \lambda_1, x = x^2, tser\right), assume = positive\right); tser2 := subs\left(\mathrm{lambda} = \lambda_2, tser\right);$$

$$tser1 := 1 - \frac{1}{3}\sqrt{6}\,\sqrt{\lambda_1}\, x - \frac{2}{9}\,\lambda_1\, x^2 + \frac{17}{162}\sqrt{6}\,\lambda_1^{(3/2)}\, x^3$$

$$tser2 := 1 - \frac{1}{3}\sqrt{6}\,\sqrt{\lambda_2}\,\sqrt{x} - \frac{2}{9}\,\lambda_2\, x + \frac{17}{162}\sqrt{6}\,\lambda_2^{(3/2)}\, x^{3/2} \quad (4.2)$$

$$> zser := convert\left(simplify\left(series\left(\frac{s1ser\cdot s2ser}{tser1}, x, 3\right)\right), polynom\right);$$

$$zser := 1 + \left(-\lambda_2 + \frac{1}{3}\sqrt{6}\,\sqrt{\lambda_1}\right) x + \left(\frac{1}{2}\,\lambda_2^2 - \frac{1}{9}\,\lambda_1 - \frac{1}{3}\sqrt{6}\,\sqrt{\lambda_1}\,\lambda_2\right) x^2 \quad (4.3)$$

$$> dPhir := diff(phitrz, z);$$

$$dPhir := 1 \Bigg/ \left(\left(\sqrt{1 + \frac{(1-z)\,t}{3-3\,t}}\ \sinh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right) + \cosh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right) \right)^2 - \left((1 - z)\ \sinh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right) t \right) \Bigg/ \left(\left(\sqrt{1 + \frac{(1-z)\,t}{3-3\,t}}\ \sinh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right) + \cosh\left(r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right) \right)^3 \sqrt{1 + \frac{(1-z)\,t}{3-3\,t}}\ (3 - 3\,t) \right) \right) \tag{4.4}$$

> $A := convert\left(simplify\left(series\left(subs\left(t = tser1, r = x^{-\frac{1}{2}}, r\,\mathrm{arccosh}\left(\sqrt{\frac{3-2\,t}{t}} \right) \right), x, 1 \right), assume = positive \right), polynom \right);$

$$A := 6^{1/4}\,\lambda_1^{(1/4)} \tag{4.5}$$

> $B := convert\left(simplify\left(series\left(simplify\left(subs\left(z = zser, t = tser1, \mathrm{sqrt}\left(1 + \frac{(1-z)\,t}{3-3\,t} \right) \right) \right), x, 1 \right), assume = positive \right), polynom \right);$

$$B := \frac{1}{6}\,\frac{2^{3/4}\,\sqrt{2\,\sqrt{2}\,\sqrt{3}\,\sqrt{\lambda_1} + 3\,\lambda_2}\ 3^{1/4}}{\lambda_1^{(1/4)}} \tag{4.6}$$

> $C := convert\left(simplify\left(series\left(simplify\left(subs\left(z = zser, t = tser1, \frac{t\cdot(1-z)}{3\cdot(1-t)} \right) \right), x, 1 \right), assume = positive \right), polynom \right);$

$$C := -\frac{1}{18}\,\frac{\left(\sqrt{6}\,\sqrt{\lambda_1} - 3\,\lambda_2 \right) \sqrt{6}}{\sqrt{\lambda_1}} \tag{4.7}$$

> $ScalingLimit := simplify\left(\frac{1}{(B\cdot\sinh(A) + \cosh(A))^2} - \frac{C\cdot\sinh(A)}{(B\cdot\sinh(A) + \cosh(A))^3\cdot B} \right):$

> $simplify\left(\left(B - \frac{C}{B} \right)\cdot B \right);$

$$1 \tag{4.8}$$

> $$Paper := \frac{\left(\frac{2}{3}+\frac{\lambda_2}{(6\cdot\lambda_1)^{\frac{1}{2}}}\right)^{-\frac{1}{2}}\cdot\sinh\left((6\cdot\lambda_1)^{\frac{1}{4}}\right)+\cosh\left((6\cdot\lambda_1)^{\frac{1}{4}}\right)}{\left(\left(\frac{2}{3}+\frac{\lambda_2}{(6\cdot\lambda_1)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\cdot\sinh\left((6\cdot\lambda_1)^{\frac{1}{4}}\right)+\cosh\left((6\cdot\lambda_1)^{\frac{1}{4}}\right)\right)^3};$$

$$Paper := \frac{\frac{\sinh\left(6^{1/4}\,\lambda_1^{(1/4)}\right)}{\sqrt{\frac{2}{3}+\frac{1}{6}\,\frac{\lambda_2\sqrt{6}}{\sqrt{\lambda_1}}}}+\cosh\left(6^{1/4}\,\lambda_1^{(1/4)}\right)}{\left(\sqrt{\frac{2}{3}+\frac{1}{6}\,\frac{\lambda_2\sqrt{6}}{\sqrt{\lambda_1}}}\,\sinh\left(6^{1/4}\,\lambda_1^{(1/4)}\right)+\cosh\left(6^{1/4}\,\lambda_1^{(1/4)}\right)\right)^3} \tag{4.9}$$

>