> *restart*;

The series U in the paramatrization

Positivity of U

First we establish that U is the derivative of w tQ1: This is Equation (8) of the paper:

> $wUnu := \frac{1}{32}\,\frac{1}{(-1+2U)^2\nu^3}\left((U\nu+U-2)\,U\left(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu\right)\right);$

$$wUnu := \frac{1}{32\,(-1+2U)^2\nu^3}\left((U\nu+U-2)\,U\left(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu\right)\right) \tag{1.1.1}$$

> $Q1Unu := \frac{1}{2}\left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu)\,U\,(\nu+1)\right)\Big/\left((8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu)\,\nu\right);$

$$Q1Unu := \left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu)\,U\,(\nu+1)\right)\Big/\left(2\,(8U^3\nu^2+16U^3\nu-11U^2\nu^2+8U^3-24U^2\nu+4U\nu^2-13U^2+14U\nu+6U-4\nu)\,\nu\right) \tag{1.1.2}$$

> $wUnu\cdot Q1Unu;$

$$\frac{1}{64\,(-1+2U)^2\nu^4}\left((U\nu+U-2)\,U^2\left(6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu\right)(\nu+1)\right) \tag{1.1.3}$$

> $collect\left((6U^3\nu^2+12U^3\nu-8U^2\nu^2+6U^3-16U^2\nu+3U\nu^2-8U^2+7U\nu+4U-2\nu), U, factor\right);$

$$6(\nu+1)^2U^3-8(\nu+1)^2U^2+(3\nu+4)(\nu+1)U-2\nu \tag{1.1.4}$$

> $factor(simplify(diff(wUnu, U)));$

$$\frac{1}{8\,(-1+2U)^3\nu^3}\left((4U^3\nu^2+8U^3\nu-3U^2\nu^2+4U^3-12U^2\nu-9U^2+6U\nu+6U-2)\,(3U^2\nu+3U^2-3U\nu-3U+\nu)\right) \tag{1.1.5}$$

> $simplify\left(\frac{diff(wUnu\cdot Q1Unu, U)}{diff(wUnu, U)}\right);$

$$\frac{U(\nu+1)}{2\nu} \tag{1.1.6}$$

Radius of Convergence of U

The radius of convergence of U is one of the roots of the discriminant of the algebraic equation of U:

> *algU* := *numer*(*wUnu* − *w*);

$$algU := 8 U^5 \nu^3 + 24 U^5 \nu^2 - 11 U^4 \nu^3 + 24 U^5 \nu - 51 U^4 \nu^2 + 4 U^3 \nu^3 - 128 U^2 \nu^3 w + 8 U^5 - 69 U^4 \nu + 40 U^3 \nu^2 + 128 U \nu^3 w - 29 U^4 + 68 U^3 \nu - 12 U^2 \nu^2 - 32 w \nu^3 + 32 U^3 - 32 U^2 \nu - 12 U^2 + 8 U \nu \quad \textbf{(1.2.1)}$$

> *dis* := *factor*(*discrim*(*algU*, *U*));

$$dis := -4096 (\nu - 1) (\nu - 3)^2 (\nu + 1)^6 (27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36) (131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23) \nu^2 \quad \textbf{(1.2.2)}$$

We have two factors, one of degree 2 and one of degree 3 :

> $P2 := 27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36;$

$$P2 := 27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36 \quad \textbf{(1.2.3)}$$

> $P1 := 131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23;$

$$P1 := 131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23 \quad \textbf{(1.2.4)}$$

Possible nu's where the two have a common root (to find nu_c)

> *factor*(*resultant*(*P1*, *P2*, *w*));

$$4194304 \nu^{12} (13573 \nu^4 - 54292 \nu^3 + 69811 \nu^2 - 31038 \nu + 67482) (7 \nu^2 - 14 \nu + 6)^3 (\nu + 1)^4 (\nu - 3)^4 \quad \textbf{(1.2.1.1)}$$

First factor has no positive root and is not relevant for us:

> *evalf*(*solve*((13573 ν^4 − 54292 ν^3 + 69811 ν^2 − 31038 ν + 67482), nu));

$$-0.145791810 + 0.9404920565 I, 2.145791810 - 0.9404920565 I, -0.145791810 - 0.9404920565 I, 2.145791810 + 0.9404920565 I \quad \textbf{(1.2.1.2)}$$

nu=3 is solution but it gives a negative common root for rho:

> *solve*(*subs*(nu = 3, *algr2*), *w*);

Second last factor will give nu_c and another candidate:

> *solve*(6 − 14 ν + 7 ν^2, nu);

(1.2.1.3)

$$1+\frac{\sqrt{7}}{7},\ 1-\frac{\sqrt{7}}{7} \tag{1.2.1.3}$$

The root with a - gives negative common root for rho:

> $factor\left(simplify\left(subs\left(\text{nu}=1-\frac{1}{7}\sqrt{7}, P1\right)\right)\right);$

$$\frac{1}{661624362}\left(\left(3553\sqrt{7}-9415\right)\left(5609520\,w\sqrt{7}-95551488\,w^2-698005\sqrt{7}+12340944\,w-1878268\right)\left(864\,w+55+25\sqrt{7}\right)\right) \tag{1.2.1.4}$$

> $factor\left(simplify\left(subs\left(\text{nu}=1-\frac{1}{7}\sqrt{7}, P2\right)\right)\right);$

$$\frac{\left(8\sqrt{7}-23\right)\left(864\,w+55+25\sqrt{7}\right)\left(-3456\,w+77+35\sqrt{7}\right)}{1323} \tag{1.2.1.5}$$

This leaves nu_c, the common root is rho_nu_c

> $factor\left(subs\left(\text{nu}=1+\frac{1}{7}\sqrt{7}, P1\right)\right); evalf(solve(\%, w));$

$$-\frac{1}{661624362}\left(\left(9415+3553\sqrt{7}\right)\left(5609520\,w\sqrt{7}+95551488\,w^2-698005\sqrt{7}-12340944\,w+1878268\right)\left(-864\,w-55+25\sqrt{7}\right)\right)$$

$$0.01289789674,\ -0.01308431164+0.01259678620\,I,\ -0.01308431164-0.01259678620\,I \tag{1.2.1.6}$$

> $factor\left(subs\left(\text{nu}=1+\frac{1}{7}\sqrt{7}, P2\right)\right); evalf(solve(\%, w));$

$$-\frac{\left(23+8\sqrt{7}\right)\left(-864\,w-55+25\sqrt{7}\right)\left(3456\,w-77+35\sqrt{7}\right)}{1323}$$

$$0.01289789674,\ -0.00451426385 \tag{1.2.1.7}$$

> $\rho_c := solve\left(-864\,w-55+25\sqrt{7}, w\right); \nu_c := 1+\frac{1}{7}\sqrt{7};$

$$\rho_c := -\frac{55}{864}+\frac{25\sqrt{7}}{864}$$

$$\nu_c := 1+\frac{\sqrt{7}}{7} \tag{1.2.1.8}$$

Plots of positive roots of the discriminant of algU and exact expressions for rho_nu

P2 has real roots fo nu =< 3

> $factor(discrim(P2, w));$

$$-27648\,\nu^2\,(\nu+1)^3\,(\nu-3)^3 \tag{1.2.2.1}$$

P1 has three real roots for nu >= 3 and double real roots for nu=1,3,$1+2\sqrt{2}$

(1.2.2.2)

$$1+2\sqrt{2} \qquad (1.2.2.2)$$

> *factor*(*discrim*(*P1*, *w*)); *solve*(%); *evalf*(%);

$$82556485632\,\nu^{18}\,(\nu-1)^2\,(\nu^2-2\,\nu-7)^2\,(\nu+1)^3\,(\nu-3)^3$$

$$1, 1, -1, -1, -1, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+2\sqrt{2}, 1-2\sqrt{2}$$

$$1., 1., -1., -1., -1., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3.828427124, -1.828427124, 3.828427124, -1.828427124 \qquad (1.2.2.3)$$

Plots of the positive roots of P2 (red) and P1 (blue):

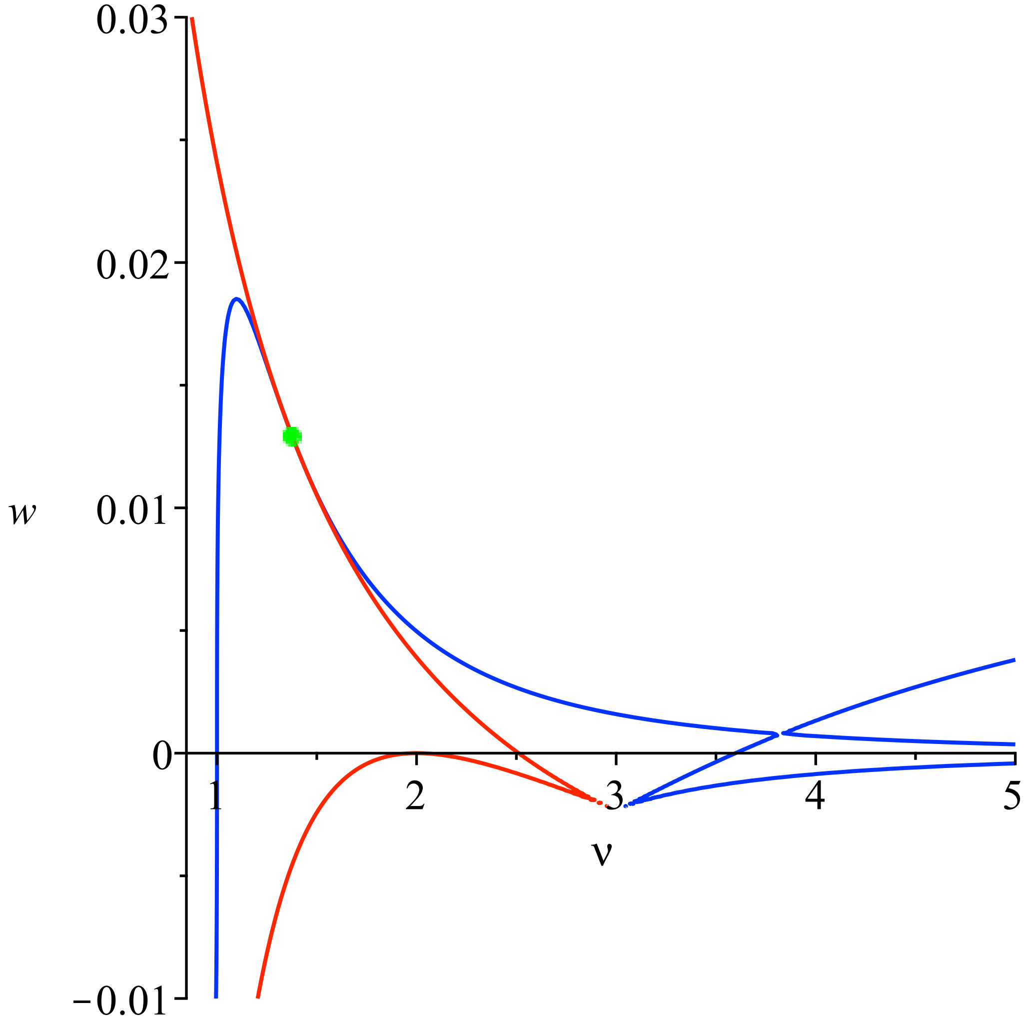
> with(plots, implicitplot) :

> *plotrho2* := *implicitplot*(*P2*, nu = 0 ..5, *w* =-0.01 ..0.03, *numpoints* = 100000, *color* = *red*) :

> *plotrho1* := *implicitplot*(*P1*, nu = 0 ..5, *w* =-0.01 ..0.03, *numpoints* = 100000, *color* = *blue*) :

> *critpoint* := *plot*$\left(\langle\langle\nu_c\rangle|\langle\rho_c\rangle\rangle\right)$, *style* = *point*, *symbol* = *solidcircle*, *color* = *green*, *symbolsize* = 15) :

> *plots*[*display*]({*plotrho2*, *plotrho1*, *critpoint*});



Roots of P2

> w21, w22 := solve(P2, w);

$$w21, w22 := \frac{1}{576\,\nu^3}\left(-9\,\nu^3 + 27\,\nu^2 + \sqrt{-3\,\nu^6 + 18\,\nu^5 - 9\,\nu^4 - 84\,\nu^3 + 27\,\nu^2 + 162\,\nu + 81} - 9\,\nu - 9\right), -\frac{1}{576\,\nu^3}\left(9\,\nu^3 - 27\,\nu^2 + \sqrt{-3\,\nu^6 + 18\,\nu^5 - 9\,\nu^4 - 84\,\nu^3 + 27\,\nu^2 + 162\,\nu + 81} + 9\,\nu + 9\right) \quad \mathbf{(1.2.2.4)}$$

> factor(−9 ν − 9 − 9 ν^3 + 27 ν^2); factor(27 ν^2 − 84 ν^3 − 9 ν^4 + 18 ν^5 − 3 ν^6 + 162 ν + 81);

$$-9\,(\nu - 1)\,(\nu^2 - 2\,\nu - 1)$$

$$-3\,(\nu + 1)^3\,(\nu - 3)^3 \quad \mathbf{(1.2.2.5)}$$

Roots of P2:

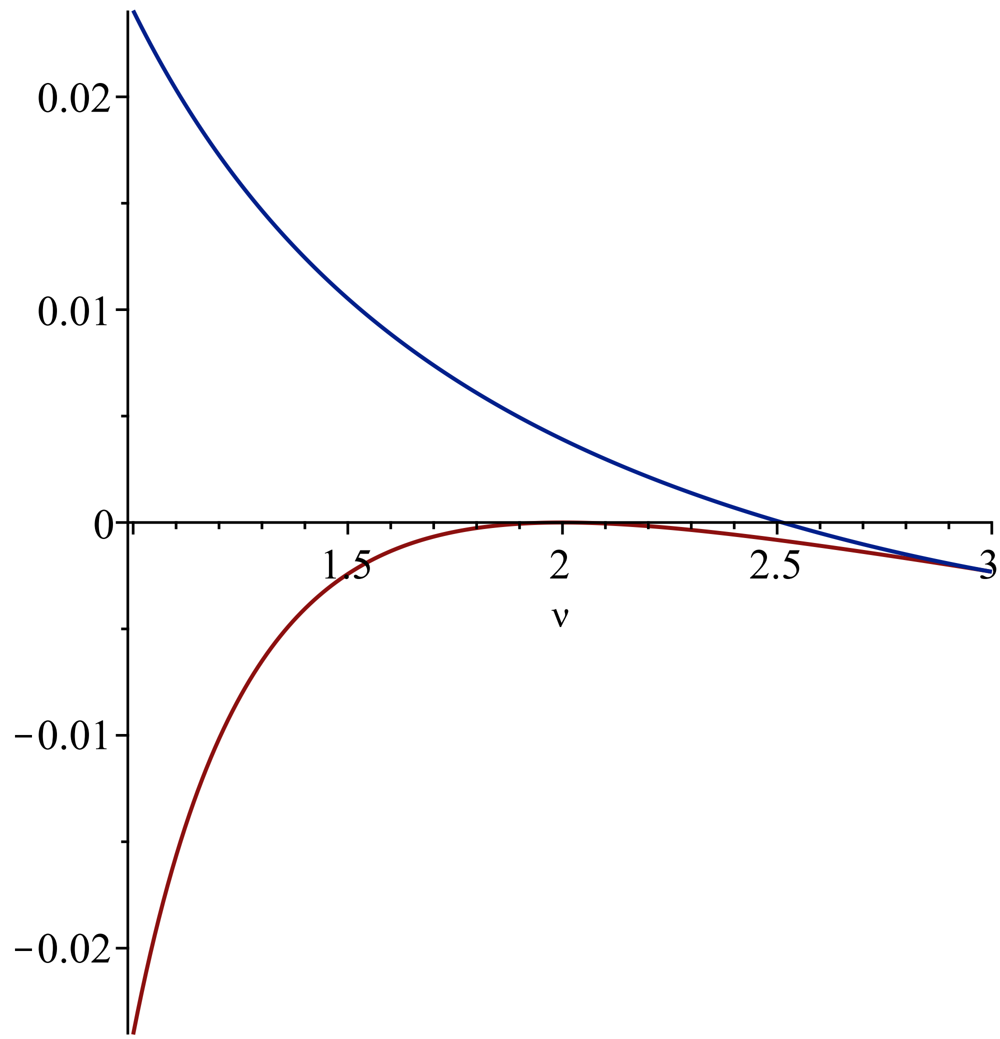
> w21 :=

$$\frac{1}{576}\,\frac{1}{\nu^3}\Big(-9\,(\mathrm{nu}-1)\cdot\big(\nu^2-2\cdot\mathrm{nu}-1\big)+(\mathrm{nu}+1)\cdot(3-\mathrm{nu})\cdot\sqrt{3\cdot(\mathrm{nu}+1)\cdot(3-\mathrm{nu})}\Big):$$

$$w22 := \frac{1}{576}\,\frac{1}{\nu^3}\Big(-9\,(\mathrm{nu}-1)\cdot\big(\nu^2-2\cdot\mathrm{nu}-1\big)-(\mathrm{nu}+1)\cdot(3-\mathrm{nu})\cdot\sqrt{3\cdot(\mathrm{nu}+1)\cdot(3-\mathrm{nu})}\Big):$$

The positive one is w21

> *plot*({*w21*, *w22*}, nu = 1 ..3);



w22 is always negative

> *solve*(*w22*, nu);

$$2,\ 1-\frac{4\sqrt{7}}{7} \qquad \textbf{(1.2.2.6)}$$

An exact expression for the roots of P1 (denoted rho11, rho12, rho13)

> *Delta0* := *factor*(*coeff*(*P1*, *w*, 2)^2 − 3·*coeff*(*P1*, *w*, 3)·*coeff*(*P1*, *w*, 1));

$$\Delta 0 := 331776\,\nu^{12}\left(9\,\nu^4-36\,\nu^3-74\,\nu^2+220\,\nu+393\right)(\nu-1)^2 \qquad \textbf{(1.2.2.7)}$$

> $Delta1 := factor\left(2\cdot coeff(P1, w, 2)^3 - 9\cdot coeff(P1, w, 3)\cdot coeff(P1, w, 2)\cdot coeff(P1, w, 1) + 27\cdot coeff(P1, w, 3)^2\cdot coeff(P1, w, 0)\right);$

$$\Delta 1 := -382205952\, \nu^{18}\, (\nu - 1)\left(27\nu^8 - 216\nu^7 + 180\nu^6 + 1944\nu^5 - 2398\nu^4 - 7400\nu^3 + 1844\nu^2 + 18600\nu + 20187\right) \quad \textbf{(1.2.2.8)}$$

> $factor\left(\Delta 1^2 - 4\cdot \Delta 0^3\right);$

$$-38294359833110460235776\, \nu^{36}\, (\nu - 1)^2 \left(\nu^2 - 2\nu - 7\right)^2 (\nu + 1)^3 (\nu - 3)^3 \quad \textbf{(1.2.2.9)}$$

> sqrt(38294359833110460235776);

$$195689447424 \quad \textbf{(1.2.2.10)}$$

> $Delta2 := \mathrm{sqrt}(38294359833110460235776)\cdot \nu^{18}\cdot(\mathrm{nu} - 1)\cdot\left(\nu^2 - 2\nu - 7\right)(\nu + 1)(\nu - 3)\cdot \mathrm{sqrt}((\mathrm{nu} + 1)\cdot(3 - \mathrm{nu}));$

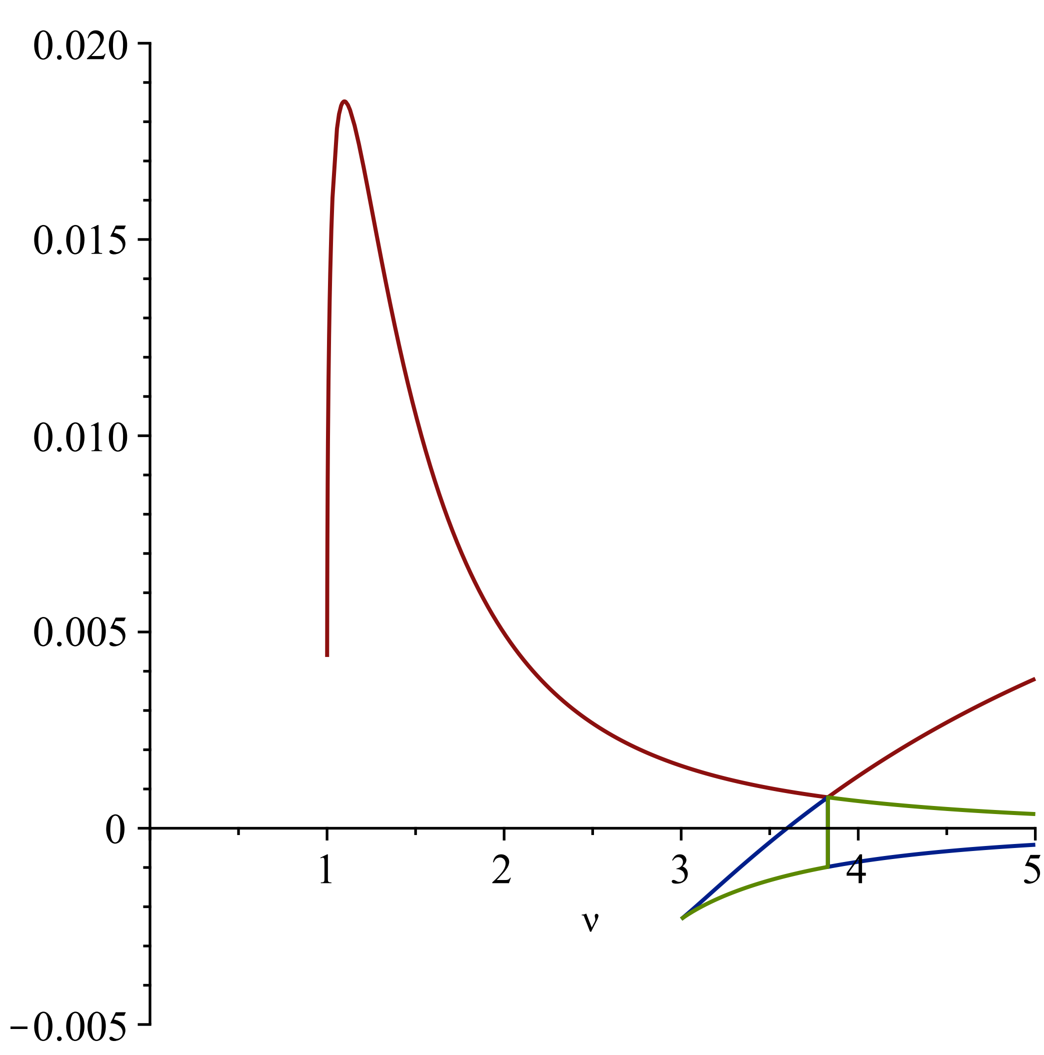
$$\Delta 2 := 195689447424\, \nu^{18}\, (\nu - 1)\left(\nu^2 - 2\nu - 7\right)(\nu + 1)(\nu - 3)\sqrt{(\nu + 1)(3 - \nu)} \quad \textbf{(1.2.2.11)}$$

> $Cm := factor\left(\frac{(Delta1 - Delta2)}{2}\right) : Cp := \frac{(Delta1 + Delta2)}{2} :$

> $rho11 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot\left(-coeff(P1, w, 2) + \mathrm{root}(-\mathrm{Cp}, 3) + \mathrm{root}(-\mathrm{Cm}, 3)\right) :$

$rho12 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot\left(-coeff(P1, w, 2) + \left(\frac{-1 + \mathrm{sqrt}(3)\cdot I}{2}\cdot root(-Cp, 3) + \frac{-1 - \mathrm{sqrt}(3)\cdot I}{2}\,\mathrm{root}(-\mathrm{Cm}, 3)\right)\right) : rho13 := \frac{1}{3\cdot coeff(P1, w, 3)}\cdot\left(-coeff(P1, w, 2) + \left(\frac{-1 - \mathrm{sqrt}(3)\cdot I}{2}\cdot root(-Cp, 3) + \frac{-1 + \mathrm{sqrt}(3)\cdot I}{2}\,\mathrm{root}(-\mathrm{Cm}, 3)\right)\right) :$

> $plot(\{rho11, rho12, rho13\}, \mathrm{nu} = 0\,..5, view = [0\,..5, -0.005\,..0.02]);$



Computation of U(rho)

The caracteristic equation:

> $PhiU := \frac{U}{wUnu}$; $eqUrho := factor(numer(PhiU - U\cdot diff(PhiU, U)))$;

$$PhiU := \left(32\,(-1+2\,U)^2\,\nu^3\right) \Big/ \left((U\,\nu+U-2)\,\left(8\,U^3\,\nu^2+16\,U^3\,\nu-11\,U^2\,\nu^2+8\,U^3-24\,U^2\,\nu+4\,U\,\nu^2-13\,U^2+14\,U\,\nu+6\,U-4\,\nu\right)\right)$$

$$eqUrho := 128\,(-1+2\,U)\,\nu^3\,\left(4\,U^3\,\nu^2+8\,U^3\,\nu-3\,U^2\,\nu^2+4\,U^3-12\,U^2\,\nu-9\,U^2+6\,U\,\nu+6\,U-2\right)\,\left(3\,U^2\,\nu+3\,U^2-3\,U\,\nu-3\,U+\nu\right) \qquad \textbf{(1.3.1)}$$

> $eqUrho3 := 4\,U^3\,\nu^2+8\,U^3\,\nu-3\,U^2\,\nu^2+4\,U^3-12\,U^2\,\nu-9\,U^2+6\,U\,\nu+6\,U-2$:
> $eqUrho2 := \left(3\,U^2\,\nu+3\,U^2-3\,U\,\nu-3\,U+\nu\right)$:

> $collect(eqUrho3, U, factor)$;

$$-2+4(\nu+1)^2 U^3 - 3(\nu+3)(\nu+1)U^2 + (6\nu+6)U \quad \textbf{(1.3.2)}$$

> *solve*(*eqUrho2*, *U*);

$$\frac{3\nu+3+\sqrt{-3\nu^2+6\nu+9}}{6(\nu+1)}, -\frac{-3\nu-3+\sqrt{-3\nu^2+6\nu+9}}{6(\nu+1)} \quad \textbf{(1.3.3)}$$

> *factor*$(-3\nu^2+6\nu+9)$;

$$-3(\nu+1)(\nu-3) \quad \textbf{(1.3.4)}$$

> *factor*(*subs*(nu = 1 + sqrt(7)/7, *eqUrho2*)); *fsolve*(%);

$$-\frac{(14+\sqrt{7})(-9U+4+\sqrt{7})(9U-5+\sqrt{7})}{189}$$

$$0.2615831877, 0.7384168123 \quad \textbf{(1.3.5)}$$

U=1/2 is a problem only when nu=1 or 3 which we will deal with later :

> *factor*(*resultant*(*eqUrho3*, 2 *U* − 1, *U*)); *factor*(*resultant*(*eqUrho2*, 2 *U* − 1, *U*));

$$2(\nu-1)(\nu-3)$$

$$\nu-3 \quad \textbf{(1.3.6)}$$

> *factor*(*resultant*(*eqUrho3*, *eqUrho2*, *U*)); *solve*(%);

$$(7\nu^2-14\nu+6)(\nu-3)^2(\nu+1)^3$$

$$3, 3, -1, -1, -1, 1+\frac{\sqrt{7}}{7}, 1-\frac{\sqrt{7}}{7} \quad \textbf{(1.3.7)}$$

For nu=3, the common root is not the smallest positive root and U(rho) = 1/8 is a root of the factor of degree 3

> *factor*(*subs*(nu = 3, *eqUrho3*)); *factor*(*subs*(nu = 3, *eqUrho2*))

$$2(8U-1)(-1+2U)^2$$

$$3(-1+2U)^2 \quad \textbf{(1.3.8)}$$

At nu_c the common root is U(rho)

> *factor*(*subs*(nu = 1 + $\frac{1}{7}\sqrt{7}$, *eqUrho3*)); *fsolve*(%); *factor*(*subs*(nu = 1 + $\frac{1}{7}\sqrt{7}$, *eqUrho2*)); *fsolve*(%);

$$-\frac{(29+4\sqrt{7})(18U\sqrt{7}-324U^2-7\sqrt{7}+315U-91)(9U-5+\sqrt{7})}{5103}$$

$$0.2615831877$$

$$-\frac{(14+\sqrt{7})(-9U+4+\sqrt{7})(9U-5+\sqrt{7})}{189}$$

$$0.2615831877, 0.7384168123 \quad \textbf{(1.3.9)}$$

At nu=1-sqrt(7)/7, the common root is not the smallest and U(rho) is a root of the factor of degree 2

```
> factor(subs(nu = 1 - 1/7 √7, eqUrho3)); fsolve(%); factor(subs(nu = 1 - 1/7 √7,
      eqUrho2)); fsolve(%);
```

$$\frac{\left(-29+4\sqrt{7}\right)\left(18\,U\sqrt{7}+324\,U^{2}-7\sqrt{7}-315\,U+91\right)\left(-9\,U+5+\sqrt{7}\right)}{5103}$$

$$0.8495279234$$

$$\frac{\left(-14+\sqrt{7}\right)\left(9\,U-4+\sqrt{7}\right)\left(-9\,U+5+\sqrt{7}\right)}{189}$$

$$0.1504720766,\ 0.8495279234 \qquad \textbf{(1.3.10)}$$

At nu=1, the right factor is also the factor of degree 2

```
> factor(subs(nu = 1, eqUrho3)); fsolve(%); factor(subs(nu = 1, eqUrho2)); fsolve(%);
```

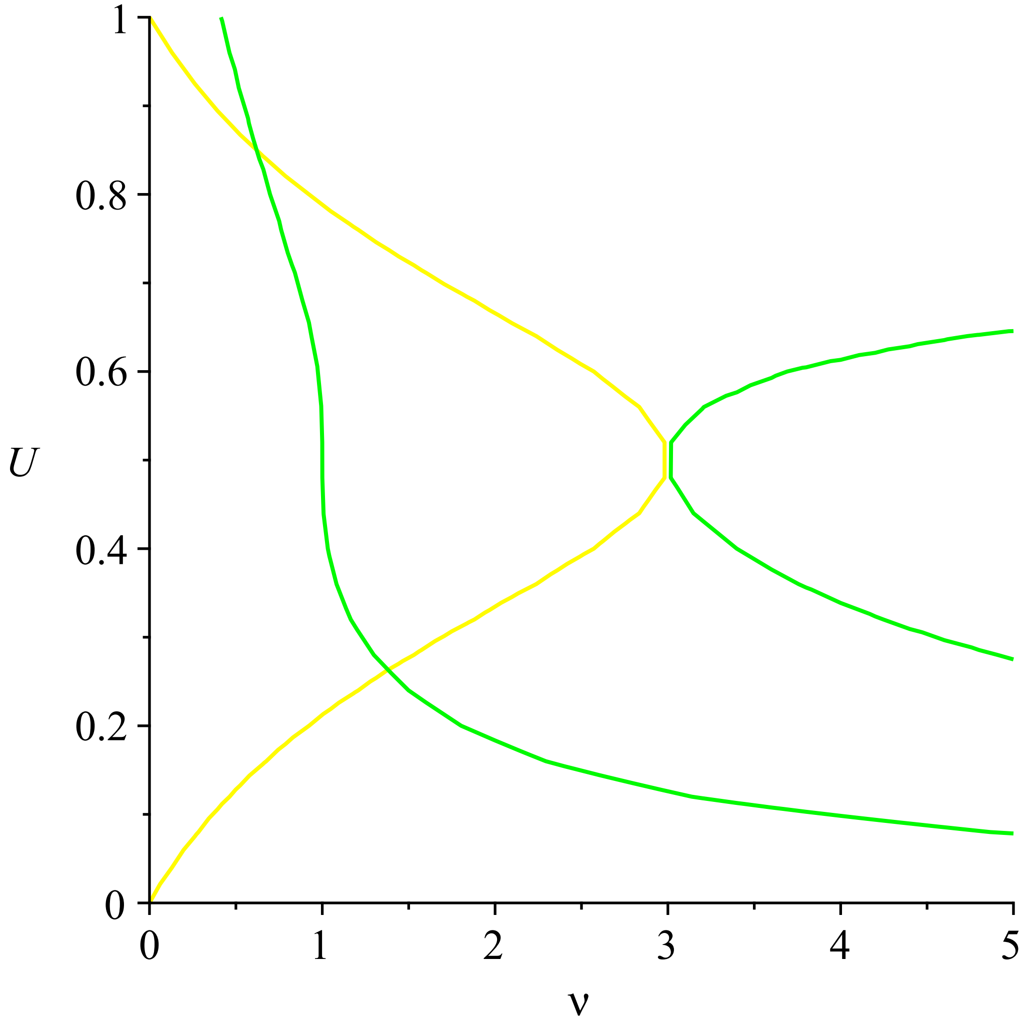
$$2\left(-1+2\,U\right)^{3}$$

$$0.5000000000,\ 0.5000000000,\ 0.5000000000$$

$$6\,U^{2}-6\,U+1$$

$$0.2113248654,\ 0.7886751346 \qquad \textbf{(1.3.11)}$$

```
> plotUrho2 := implicitplot(eqUrho2, nu = 0 ..5, U = 0 ..1, color = yellow) :
> plotUrho3 := implicitplot(eqUrho3, nu = 0 ..5, U = 0 ..1, color = green) :
> plots[display]({plotUrho2, plotUrho3});
```



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Unique dominant singularity

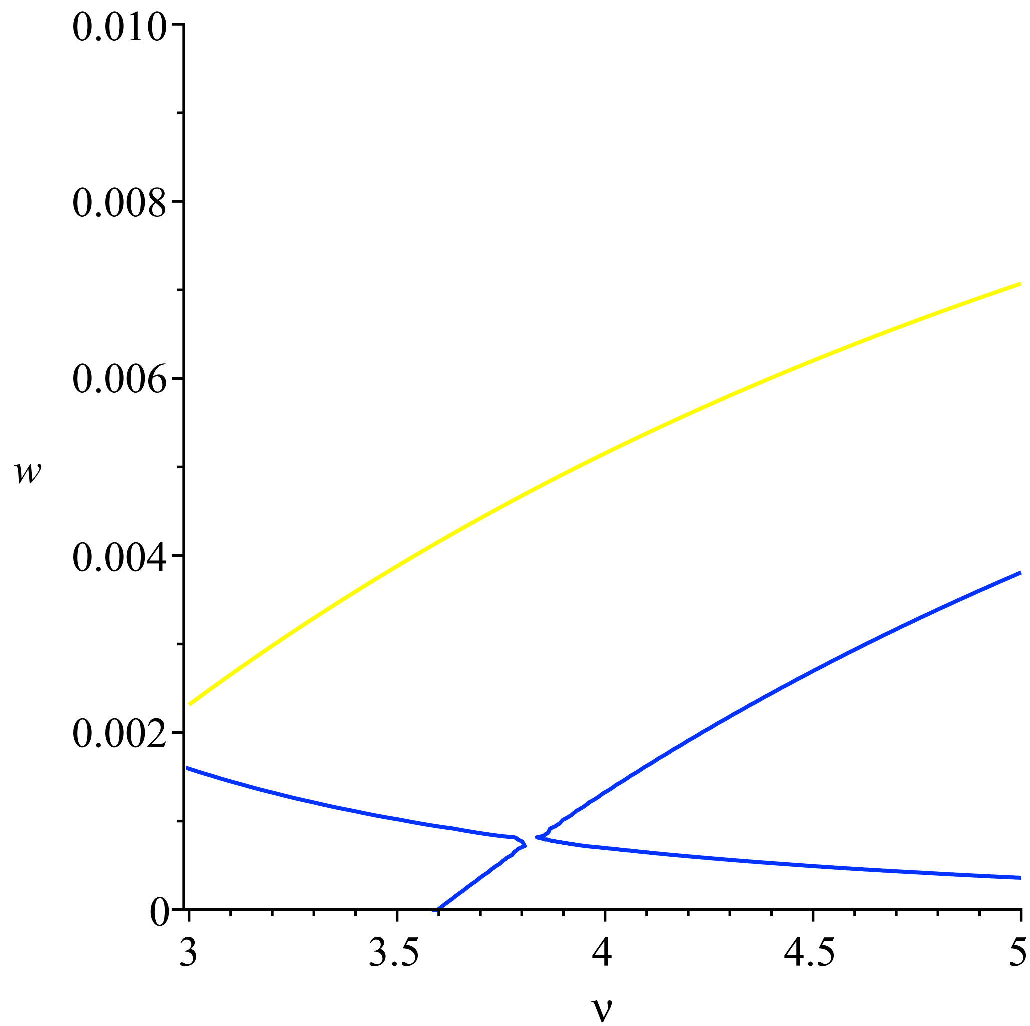
Imaginary roots of P2 for nu >= 3

When nu>3, P2 has two imaginary roots, we check that their modulus is not a root of P1 and therefore are not rho_nu

> $w2mod := factor\left(simplify\left(\frac{subs(w=0, P2)}{coeff(P2, w, 2)}\right)\right);$

$$w2mod := \frac{(7\nu^2 - 14\nu - 9)(-2+\nu)^2}{27648\,\nu^4} \tag{1.4.1.1}$$

> $plotw2mod := plot(\mathrm{sqrt}(w2mod), \mathrm{nu} = 3..5, color = yellow):$
> $plots[display](\{plotrho1, plotw2mod\}, view = [3..5, 0..0.01]);$



The modulus w2mod is increasing after nu=3 and tis oo large

> $factor(diff(w2mod, \mathrm{nu})); evalf(solve(\%));$

$$\frac{(-2+\nu)(7\nu^2 - 11\nu - 12)}{4608\,\nu^5}$$

$$2.,\ 2.312682738,\ -0.7412541663 \tag{1.4.1.2}$$

Root w22 for nu < nu_c

The radius of convergence of U is w21 for nu =< nu_c and w22 is negative. We check if w21 = -w22 for these values of nu:

> $factor(simplify(w21 + w22)); solve(\%); evalf(\%); evalf(\nu_c);$

$$-\frac{(\nu-1)\,(\nu^2-2\,\nu-1)}{32\,\nu^3}$$

$$1,\ 1+\sqrt{2},\ 1-\sqrt{2}$$

$$1.,\ 2.414213562,\ -0.414213562$$

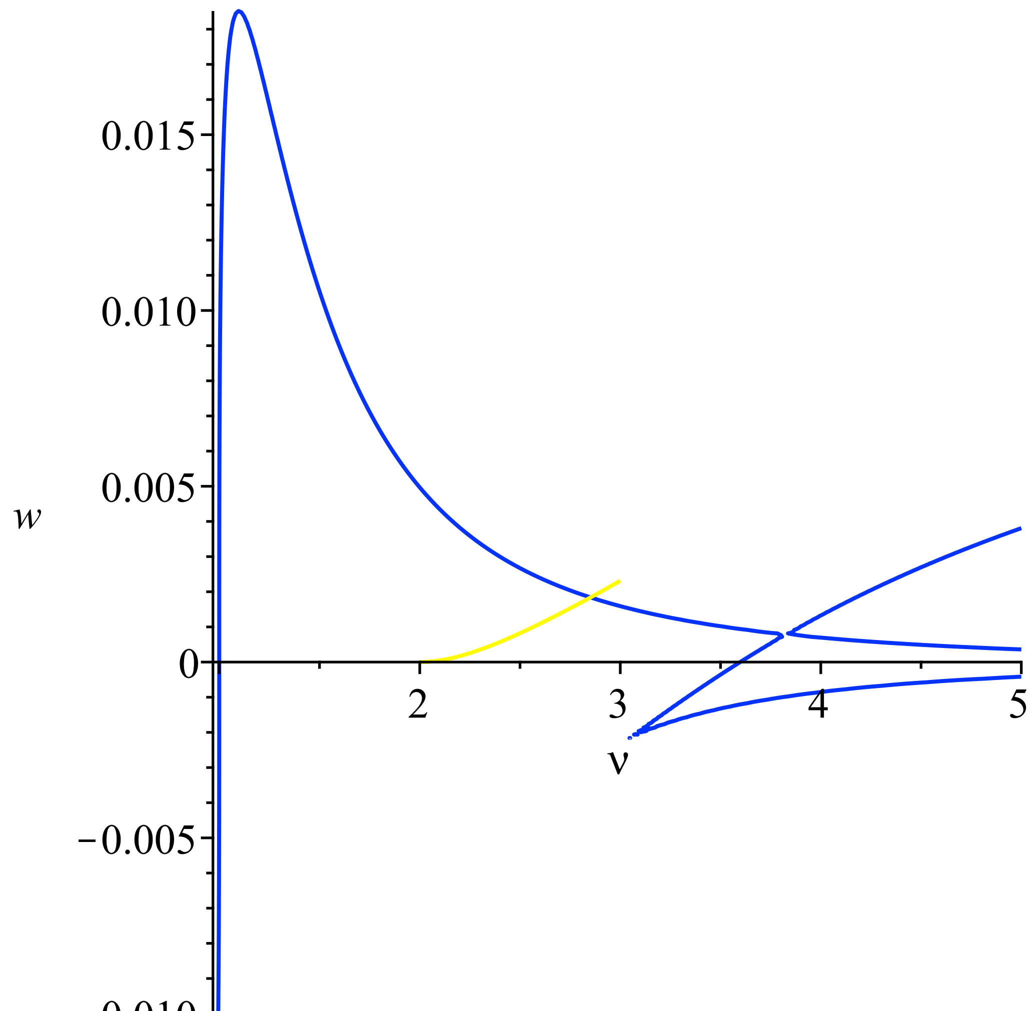
$$1.377964473 \qquad \textbf{(1.4.2.1)}$$

Only possibility is nu=1, which is not a problem since it corresponds to uniform triangulations, and the result is known to be true in this case.

Root w22 for nu_c < nu < 3

The radius of convergence of U is a root of algrho for nu > nu_c and w22 is negative. We check if w22 = -rho for these values of nu:

```
> plots[display]({plotrho1, plot(-w22, nu = 2 ..3, color = yellow)});
```



There is a candidate !

```
> with(algcurves) : puiseux(algU, w = w22, U, 0);
```

$$\left\{ \frac{\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}+3\,\nu+3}{6\,\nu+6} \right. \quad \textbf{(1.4.3.1)}$$

$$+ 1 \Big/ \left(21\,\nu^4 - 45\,\nu^2 - 6\,\nu\right.$$

$$\left. + 18\right)$$

$$\left(\left(\left(\left(w\right.\right.\right.\right.$$

$$- \frac{1}{576\,\nu^3}\left(-9\,(\nu-1)\,\left(\nu^2-2\,\nu-1\right) - (\nu+1)\,(3\right.$$

$$\left.\left. - \nu)\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\right)\right)\left(21\,\nu^4 - 45\,\nu^2 - 6\,\nu + 18\right)\Big) \Big/ \Big($$

$$1/2$$

$$\left. -8\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu^3 + 72\,\nu^4 - 72\,\nu^3\right)\Big) \quad \Big($$

$$\left. -8\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu^3 + 72\,\nu^4 - 72\,\nu^3\right)\Bigg), RootOf\Big(\Big(48\,\nu^2 + 96\,\nu$$

$$+ 48\Big)\,\_Z^3 + \Big(16\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu$$

$$+ 16\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)} - 18\,\nu^2 - 144\,\nu - 126\Big)\,\_Z^2$$

$$+ \Big(2\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu - 34\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)} - 18\,\nu^2$$

$$+ 72\,\nu + 90\Big)\,\_Z - 5\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu$$

$$+ 13\sqrt{3}\sqrt{-(\nu+1)(\nu-3)} + 3\nu^2 + 6\nu - 33)\}$$

Only one branch is singular and the value at w=w22 is

> $Uw22sing := \dfrac{3\nu + 3 + \sqrt{3}\sqrt{-(\nu+1)(\nu-3)}}{6\nu+6};$

$$Uw22sing := \frac{\sqrt{3}\sqrt{-(\nu+1)(\nu-3)} + 3\nu + 3}{6\nu+6} \qquad \textbf{(1.4.3.2)}$$

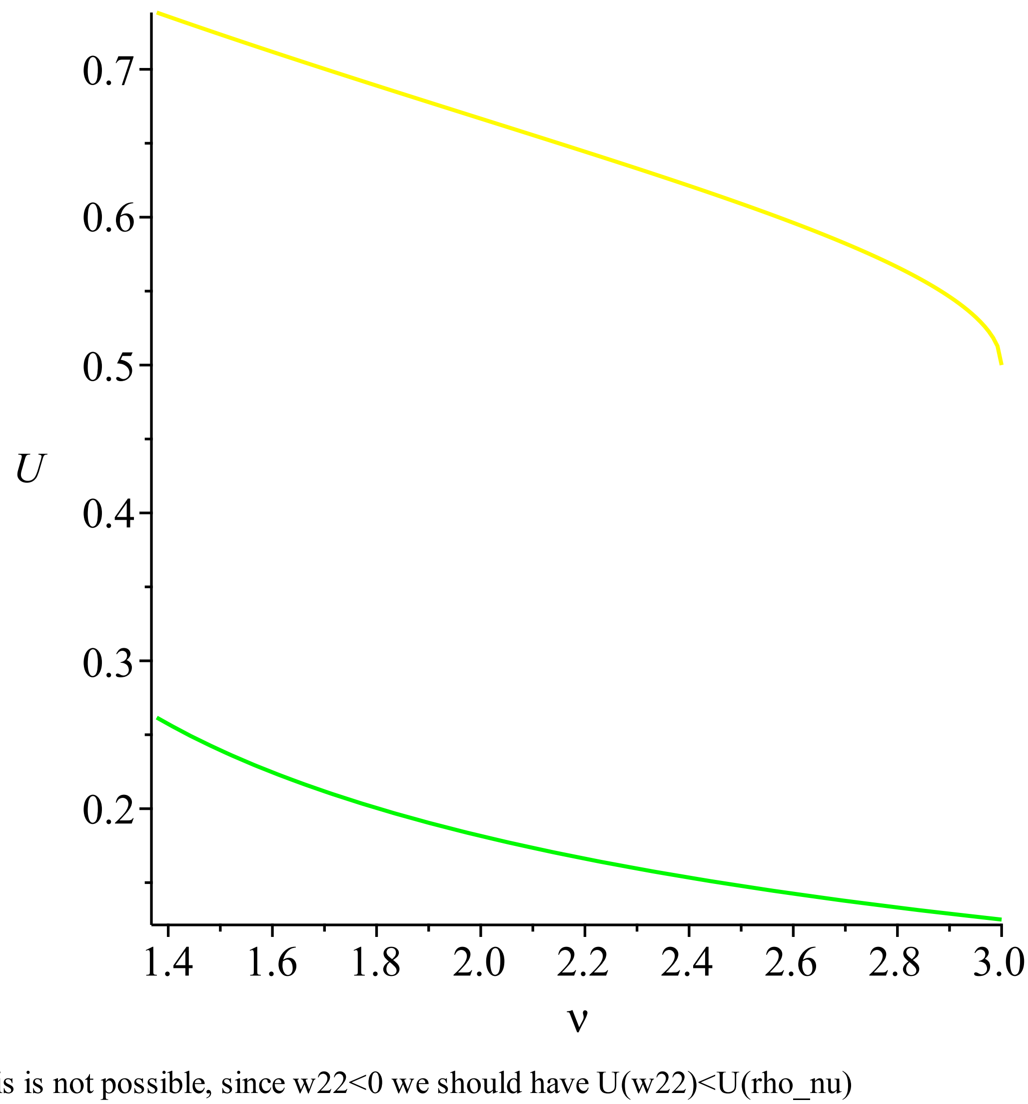
We have to compare this with U(rho_c) who is solution of eqUrho3

>

> $plotUrho3 := implicitplot\left(eqUrho3, \mathrm{nu} = \nu_c..3, U = 0..0.5, color = green\right):$

> $plotUw22sing := plot\left(Uw22sing, \mathrm{nu} = \nu_c..3, color = yellow\right):$

> $plots[display](\{plotUrho3, plotUw22sing\});$

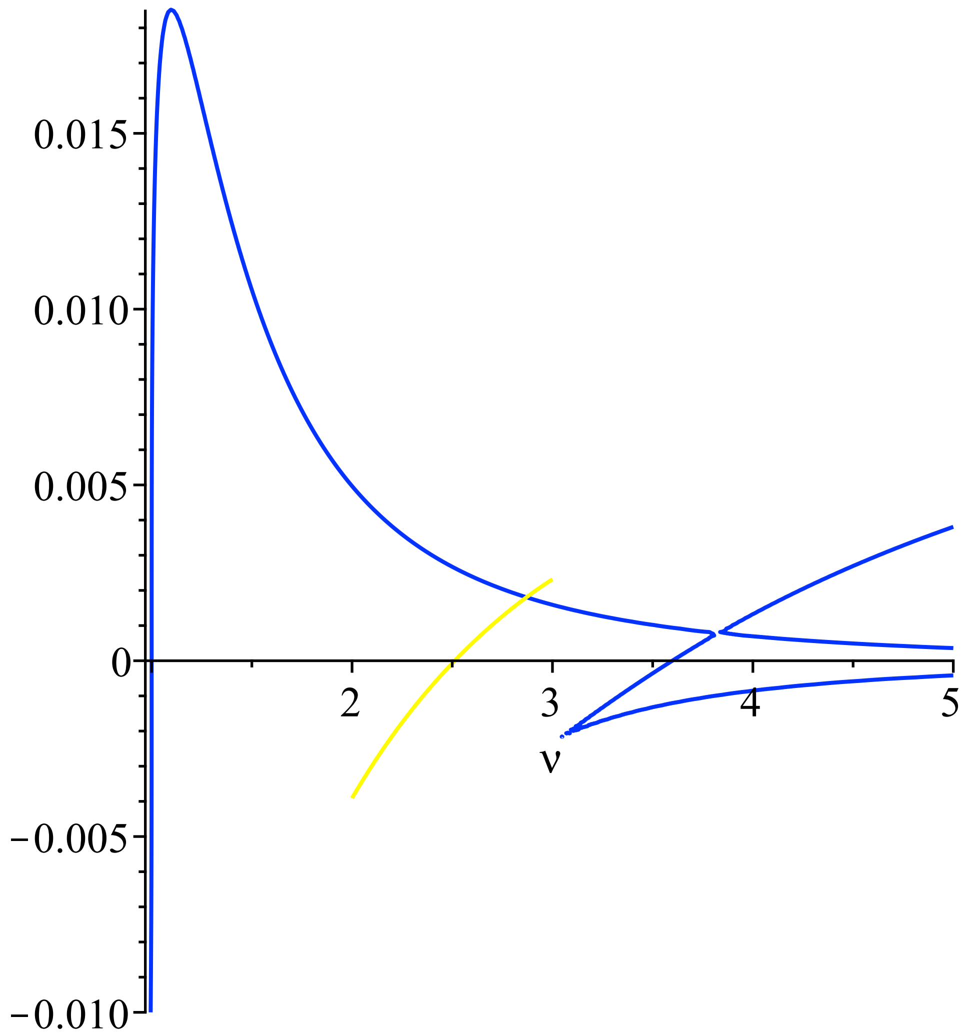


This is not possible, since w22<0 we should have U(w22)<U(rho_nu)

Root w21 for nu_c < nu < 3

This is a lot like the previous case: the radius of convergence of U is a root of algrho for nu > nu_c and w21 can be negative. We check if w21 = -rho for these values of nu:

> *plots*[*display*]({*plotrho1*, *plot*(−*w21*, nu = 2 ..3, *color* = *yellow*)});



There is also a candidate.

> *puiseux*(*algU*, *w* = *w21*, *U*, 0);

$$\left\{ \frac{-\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}+3\,\nu+3}{6\,\nu+6} \right. \qquad \textbf{(1.4.4.1)}$$

$$+\,1 \big/ \left(21\,\nu^{4}-45\,\nu^{2}-6\,\nu\right.$$

$$+ 18)$$

$$\Bigg(\Bigg(\Bigg(\Bigg(w$$

$$- \frac{1}{576\,\nu^3}\big(-9\,(\nu-1)\,(\nu^2-2\,\nu-1)+(\nu+1)\,(3$$

$$-\nu)\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\big)\Big)\,(21\,\nu^4-45\,\nu^2-6\,\nu+18)\Big)\Big/$$

$$\big(8\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu^3+72\,\nu^4-72\,\nu^3\big)\Big)$$

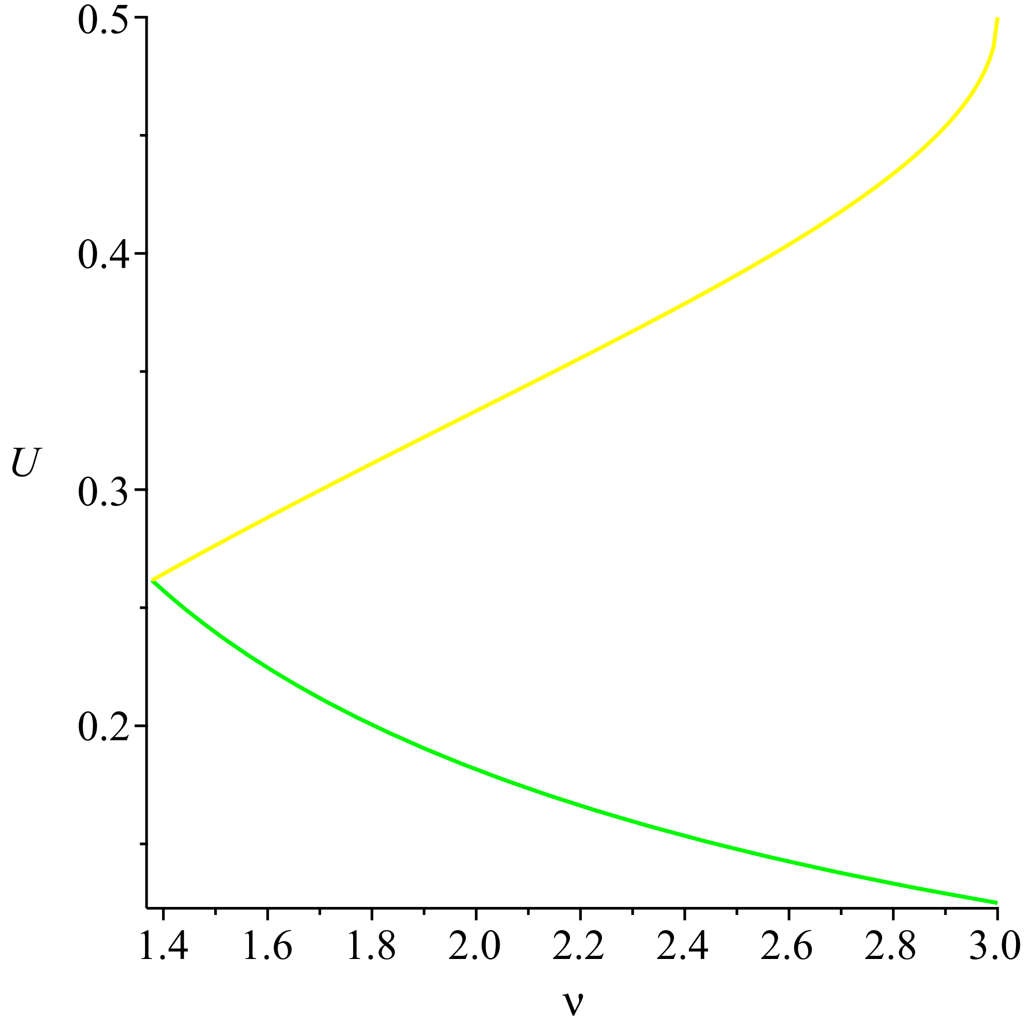
$$^{1/2}$$

$$\big(8\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu^3+72\,\nu^4-72\,\nu^3\big)\Bigg), RootOf\Big(\big(48\,\nu^2$$
$$+96\,\nu+48\big)\,\_Z^3+\big(-16\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu$$
$$-16\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}-18\,\nu^2-144\,\nu-126\big)\,\_Z^2+\big($$
$$-2\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu+34\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}-18\,\nu^2$$
$$+72\,\nu+90\big)\,\_Z+5\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}\,\nu$$
$$-13\,\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}+3\,\nu^2+6\,\nu-33\Big)\}$$

> $Uw21sing := \dfrac{3+3\,\nu-\sqrt{3}\,\sqrt{-(\nu+1)\,(\nu-3)}}{6\,\nu+6}$:

> $plotUw21sing := plot\big(Uw21sing, \mathrm{nu} = \nu_c..3, color = yellow\big)$:

> $plots[display](\{plotUrho3, plotUw21sing\});$



Same impossibility at before, except at nu_c where w21=rho.

>

Real roots of P1 for nu >=3

We check when rho and -rho are roots of P1:

```
> factor(resultant(P1, subs(w=-w, P1), w)); solve(%); evalf(%);
```

$$1099511627776\,\nu^{27}\left(4\,\nu^{2}-8\,\nu-23\right)\left(81\,\nu^{4}-324\,\nu^{3}-602\,\nu^{2}+1852\,\nu+2449\right)^{2}\left(\nu-1\right)^{3}$$

$$1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1+\frac{3\sqrt{3}}{2}, 1-\frac{3\sqrt{3}}{2}, 1-\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1+\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1-\frac{2\sqrt{136+10\sqrt{10}}}{9}, 1$$

$$+\frac{2\sqrt{136+10\sqrt{10}}}{9}, 1-\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1$$
$$+\frac{2\sqrt{136-10\sqrt{10}}}{9}, 1-\frac{2\sqrt{136+10\sqrt{10}}}{9}, 1+\frac{2\sqrt{136+10\sqrt{10}}}{9}$$

1., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3.598076212, −1.598076212, −1.270337153, 3.270337153, −1.877093669, 3.877093669, −1.270337153, 3.270337153, −1.877093669, 3.877093669 **(1.4.5.1)**

We have three possible values for nu:

> $nu1 := 1+\frac{3}{2}\sqrt{3}$: *evalf*(%); $nu2 := 1+\frac{2}{9}\sqrt{136-10\sqrt{10}}$: *evalf*(%);
> $nu3 := 1+\frac{2}{9}\sqrt{136+10\sqrt{10}}$: *evalf*(%);

3.598076212
3.270337153
3.877093669 **(1.4.5.2)**

First value is when one of the roots of P1 is 0, which is not singular for U

> *evalf*(*solve*(*simplify*(*subs*(nu = *nu1*, *P1*))));

0., 0.0009428090128, −0.001208340163 **(1.4.5.3)**

nu3 does not work either:

> *evalf*(*solve*(*simplify*(*subs*(nu = *nu3*, *P1*))));

0.0009447149241, −0.0009447149241, 0.000759232603 **(1.4.5.4)**

When do the roots of P1 meet (to know if nu3 is before or after)

> *factor*(*discrim*(*P1*, *w*)); *fsolve*(%);

$$82556485632\,\nu^{18}(\nu-1)^2(\nu^2-2\nu-7)^2(\nu+1)^3(\nu-3)^3$$

−1.828427125, −1.828427125, −1., −1., −1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3.828427125, 3.828427125 **(1.4.5.5)**

nu3 is after, so rho_nu3 is the smallest positive root of P1, .000759... and the negative root is outside the circle of convergence

>
>
>

For nu2 we have to check by hand since a negative root of P1 is on the circle of convergence

> *factor*(*simplify*(*subs*(nu = *nu2*, *P1*))); *fsolve*(%);

$$\Big(\big(340668182080\sqrt{136-10\sqrt{10}}\sqrt{10}-1888752594322\sqrt{136-10\sqrt{10}}$$
$$+4743730980000\sqrt{10}-23288005045449\big)$$
$$\big(672918721623116460\sqrt{136-10\sqrt{10}}\sqrt{10}$$

$$+ 2209187958577267433482\,24\, w^2$$

$$+ 2711575949339267856 \sqrt{136 - 10\sqrt{10}} - 6645649759606551105\sqrt{10}$$

$$- 2876864050949948500\Big) \Big(1378055795\sqrt{136 - 10\sqrt{10}}\sqrt{10}$$

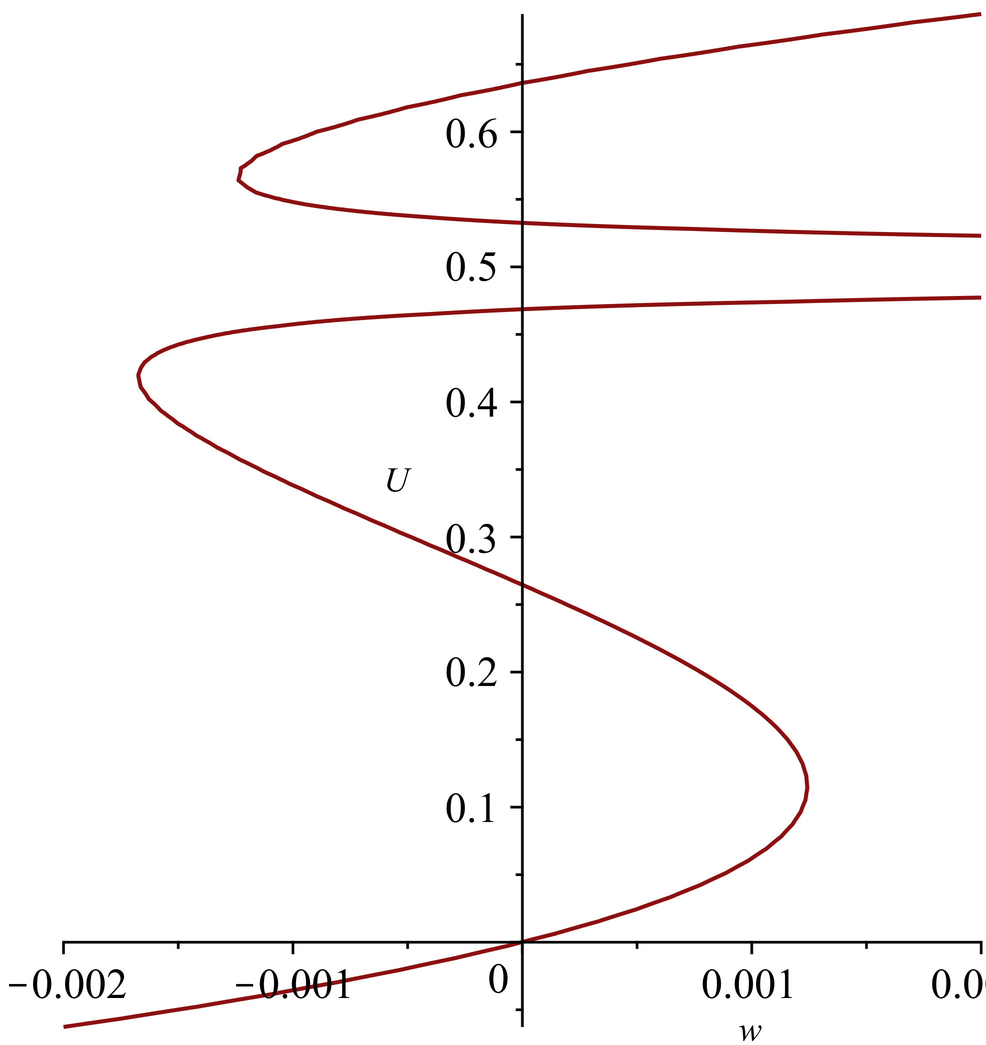
$$+ 3065912636\sqrt{136 - 10\sqrt{10}} - 16806994560\sqrt{10} - 5640239942016\, w$$

$$- 32140229112\Big)\Big) \Big/ 368301006577386634182621525837553305588\,3$$

$$-0.001674427933, -0.001242162516, 0.001242162516 \quad \textbf{(1.4.5.6)}$$

We could try to check the singular behavior at -0.0012... with puiseux or algeqtoseries but Maple does not handle it well ... Instead we can see that our branch of U is not singular directly:

> *implicitplot*(*factor*(*subs*(nu = *nu2*, *algU*)), *w* = -0.002 ..0.002, *U* = -0.5 ..1, *numpoints* = 10000);



At w=-0.0012 there is a double root for U but its modulus is too large to be our branch. The other roots are simple and not singular.

> *factor*(*subs*(nu = *nu2*, *eqUrho*)); *fsolve*(%);

$$- \frac{1}{2179240250625} \Big(64 \Big(1340550 \sqrt{136 - 10\sqrt{10}}\,\sqrt{10} - 13094217 \sqrt{136 - 10\sqrt{10}} + 21225290\sqrt{10} - 157590473\Big) \Big(\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} + 540\,U - 54\sqrt{10} - 35\sqrt{136 - 10\sqrt{10}} + 270\Big) \Big(190\,U\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} - 347\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} + 640\,U\sqrt{136 - 10\sqrt{10}} - 2160\,U\sqrt{10} - 5400\,U^2 - 1220\sqrt{136 - 10\sqrt{10}} + 3618\sqrt{10} - 2160\,U + 11880\Big)(-1 + 2\,U)\Big(2\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} - 270\,U^2 + 11\sqrt{136 - 10\sqrt{10}} - 18\sqrt{10} + 270\,U - 189\Big)\Big)$$

0.1154879305, 0.4185807983, 0.5000000000, 0.5671915934 **(1.4.5.7)**

Puiseux (and algeqtoseries) mishandle approximations :

> $puiseux\Big(subs(\mathrm{nu} = nu2, algU), w = -\frac{3}{470019995168}\Big(3196515612166609500 - 74768746847012940\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136 - 10\sqrt{10}}\Big)^{1/2}, U, 0\Big) : evalf(allvalues(\%));$

$$\{0.3584377386 + 0.0001072443646\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}, \{0.4519346757 - 0.0000869795\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}, \{-0.0428651114 - 0.00002026476455\,\mathrm{I}, 0.5671915904 + 2.21141033\sqrt{0.4522001125\,w + 0.0005617060241}\}$$ **(1.4.5.8)**

> $puiseux\Big(subs(\mathrm{nu} = nu2, algU), w = \frac{3}{470019995168}\Big(3196515612166609500 - 74768746847012940\sqrt{136 - 10\sqrt{10}}\,\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136 - 10\sqrt{10}}\Big)^{1/2}, U, 0\Big) : evalf(allvalues(\%));$

$$\{0.4881285434 - 0.008985456863\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\}, \{0.5085662128 + 0.01009404578\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\}, \{0.6742198609 - 0.001108588910\,I, 0.1154879334 - 12.99194739\sqrt{-0.07697075504\,w + 0.00009561018582}\} \tag{1.4.5.9}$$

> *with*(*gfun*) :

> $algeqtoseries\left(factor\left(simplify\left(subs\left(\mathrm{nu} = nu2, w = -\frac{3}{470019995168}\left(319651561216660 9500 - 747687468470129 40\sqrt{136 - 10\sqrt{10}}\sqrt{10} + 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136 - 10\sqrt{10}}\right)^{1/2} \cdot (1 - x), algU\right)\right)\right), x, U, 2\right) : evalf(allvalues(\%));$

$$[0.3584377386 + 0.0001072443646\,I + (-0.1089652047 + 0.0001248040815\,I)\,x + O(x^2), 0.5671915904 + 0.05241117185\sqrt{x} + O(x)], [0.4519346757 - 0.0000869795\,I + (0.0330416126 - 0.0001629945963\,I)\,x + O(x^2), 0.5671915904 + 0.05241117185\sqrt{x} + O(x)], [-0.0428651114 - 0.00002026476455\,I + (0.03567313526 + 0.00003819055156\,I)\,x + O(x^2), 0.5671915904 + 0.05241117185\sqrt{x} + O(x)], [0.3584377386 + 0.0001072443646\,I + (-0.1089652047 + 0.0001248040815\,I)\,x + O(x^2), 0.5671915904 - 0.05241117185\sqrt{x} + O(x)], [0.4519346757 - 0.0000869795\,I + (0.0330416126 - 0.0001629945963\,I)\,x + O(x^2), 0.5671915904 - 0.05241117185\sqrt{x} + O(x)], [-0.0428651114 - 0.00002026476455\,I + (0.03567313526 + 0.00003819055156\,I)\,x + O(x^2), 0.5671915904 - 0.05241117185\sqrt{x} + O(x)] \tag{1.4.5.10}$$

> $algeqtoseries\left(factor\left(simplify\left(subs\left(\mathrm{nu} = nu2, w = \frac{3}{470019995168}\left(3196515612166609500 - 74768746847012940\sqrt{136 - 10\sqrt{10}}\sqrt{10}\right.\right.\right.\right.\right.$

$$+ 738405528845172345\sqrt{10} - 301286216593251984\sqrt{136 - 10\sqrt{10}}\Big)^{1/2}$$

$$\cdot(1-x), algU\Big)\Big)\Big)\Big], x, U, 2\Big) : evalf(allvalues(\%));$$

$$[0.4881285434 - 0.008985456863\,I + (-0.0007414499 - 0.002082455741\,I)\,x + O(x^2), 0.1154879334 + 0.1270358605\sqrt{x} + O(x)], [0.5085662128 + 0.01009404578\,I + (0.0031474460 + 0.001457362074\,I)\,x + O(x^2), 0.1154879334 + 0.1270358605\sqrt{x} + O(x)], [0.6742198609 - 0.001108588910\,I + (-0.0318124224 + 0.000625093674\,I)\,x + O(x^2), 0.1154879334 + 0.1270358605\sqrt{x} + O(x)], [0.4881285434 - 0.008985456863\,I + (-0.0007414499 - 0.002082455741\,I)\,x + O(x^2), 0.1154879334 - 0.1270358605\sqrt{x} + O(x)], [0.5085662128 + 0.01009404578\,I + (0.0031474460 + 0.001457362074\,I)\,x + O(x^2), 0.1154879334 - 0.1270358605\sqrt{x} + O(x)], [0.6742198609 - 0.001108588910\,I + (-0.0318124224 + 0.000625093674\,I)\,x + O(x^2), 0.1154879334 - 0.1270358605\sqrt{x} + O(x)] \quad \textbf{(1.4.5.11)}$$

Complex roots of P1 for nu_c =< nu =< 3

the product of the roots is rho^3 (since rho is a root of P1 in this domain):

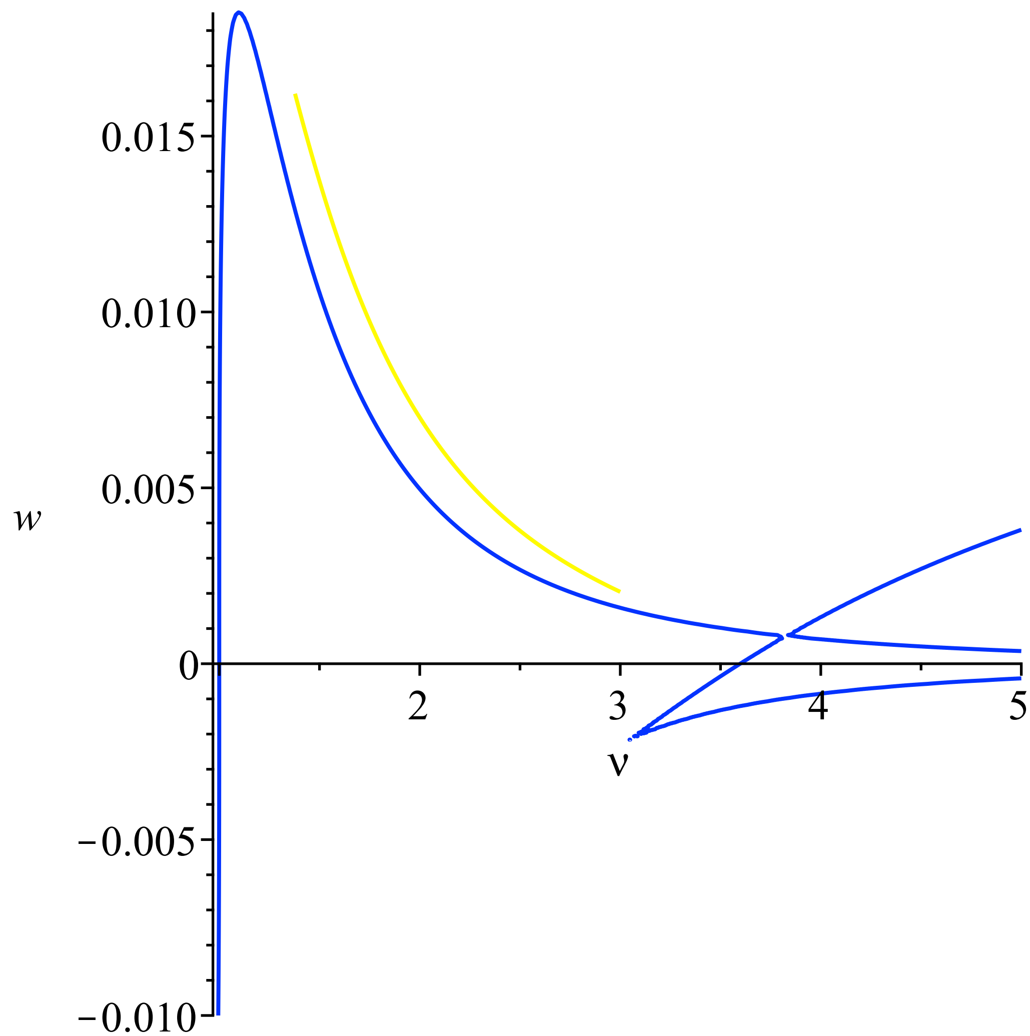
> $w3mod := factor\left(-\dfrac{(subs(w=0, P1))}{coeff(P1, w, 3)}\right);$

$$w3mod := -\frac{(\nu-1)\,(4\nu^2 - 8\nu - 23)}{131072\,\nu^9} \quad \textbf{(1.4.6.1)}$$

> $plotw3mod := plot\left((w3mod)^{\frac{1}{3}}, \mathrm{nu} = \nu_c..3, color = yellow\right):$

In this range of nu P1 does not have 3 roots with the same modulus:

> $plots[display](\{plotw3mod, plotrho1\});$



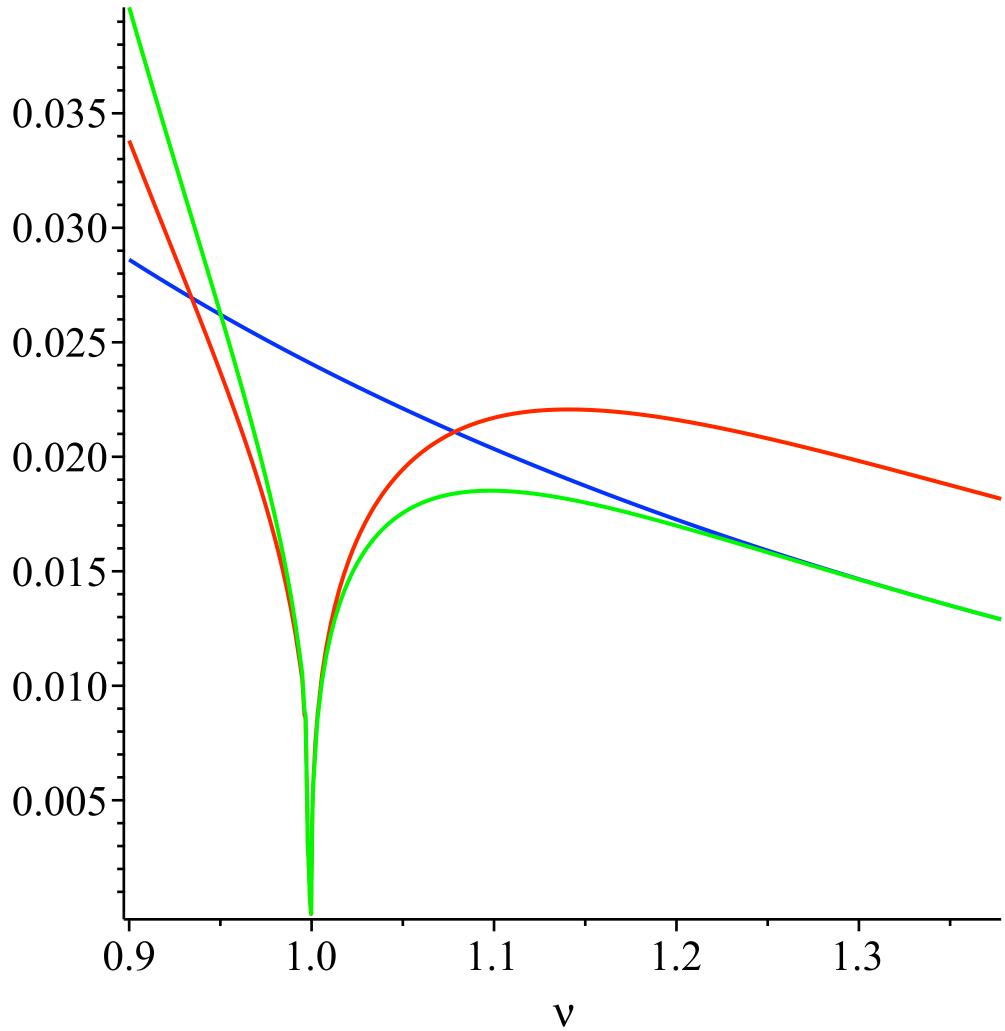
Roots of P1 for nu<nu_c

> $w11, w12, w13 := solve(P1, w)$:

> $Pl1 := plot(|w11|, \mathrm{nu} = 0.9 .. \nu_c, color = green)$: $Pl2 := plot(|w12|, \mathrm{nu} = 0.9 .. \nu_c, color = red)$: $Pl3 := plot(|w13|, \mathrm{nu} = 0.9 .. \nu_c, color = yellow)$:

> $PlT := plot(w21, \mathrm{nu} = 0.9 .. \nu_c, color = blue)$:

> $plots[display](\{PlT, Pl1, Pl2, Pl3\})$;



We have 3 candidates for nu ! (the green curve meets the blue one at nu_c. Maple cannot handle the expressions of w1i correctly so we will check that they are never singular before nu_c with Newton polygon method.

Important : 0 is a triple root at nu=1

```
> subs(nu = 1, P1);
```

$$131072\, w^3 \tag{1.4.7.1}$$

We have to write an algebraic equation for (w-w1i) and (U-U(w1i))

First, an equation for U(w1i)

```
> eqUw1i := factor(resultant(algU, P1, w));
```

$$eqUw1i := -2048\,\nu^9\,\big(2048\,U^9\,\nu^5 + 10240\,U^9\,\nu^4 - 5376\,U^8\,\nu^5 + 20480\,U^9\,\nu^3 - 34560\,U^8\,\nu^4 + 5472\,U^7\,\nu^5 + 20480\,U^9\,\nu^2 - 84480\,U^8\,\nu^3 + 48480\,U^7\,\nu^4 - 2972\,U^6\,\nu^5 + 10240\,U^9\,\nu - 99840\,U^8\,\nu^2 + 142656\,U^7\,\nu^3 - 35332\,U^6\,\nu^4 + 1428\,U^5\,\nu^5 + 2048\,U^9 - 57600\,U^8\,\nu + 191808\,U^7\,\nu^2 - 127176\,U^6\,\nu^3 + 13548\,U^5\,\nu^4 - 843\,U^4\,\nu^5 - 13056\,U^8 + 122208\,U^7\,\nu - 191480\,U^6\,\nu^2 + 61656\,U^5\,\nu^3 - 2925\,U^4\,\nu^4 + 328\,U^3\,\nu^5 + 30048\,U^7 - 127900\,U^6\,\nu \tag{1.4.7.2}$$

$$+ 105000\, U^5 \nu^2 - 11610\, U^4 \nu^3 + 1076\, U^3 \nu^4 - 48\, U^2 \nu^5 - 31236\, U^6$$
$$+ 72084\, U^5 \nu - 28470\, U^4 \nu^2 - 3760\, U^3 \nu^3 - 552\, U^2 \nu^4 + 16620\, U^5$$
$$- 24411\, U^4 \nu + 1976\, U^3 \nu^2 + 2592\, U^2 \nu^3 + 96\, U \nu^4 - 5469\, U^4$$
$$+ 7528\, U^3 \nu + 360\, U^2 \nu^2 - 432\, U \nu^3 + 1044\, U^3 - 2976\, U^2 \nu + 24\, U \nu^2$$
$$+ 16\, \nu^3 + 624\, U^2 + 864\, U \nu - 48\, \nu^2 - 552\, U - 60\, \nu + 92\big)\, \big(4\, U^3 \nu^2$$
$$+ 8\, U^3 \nu - 3\, U^2 \nu^2 + 4\, U^3 - 12\, U^2 \nu - 9\, U^2 + 6\, U \nu + 6\, U - 2\big)^2$$

There are two factors. We will see that the right one is the first before nu_c and the second after nu_c

> $eqUw1i1 := op(3, eqUw1i) : eqUw1i2 := op(1, op(4, eqUw1i)) :$

they do meet at nu_c

> $factor\big(subs\big(\mathrm{nu} = \nu_c, eqUw1i1\big)\big);$

$$-\frac{1}{4921675101}\Big(\big(14966 + 4201\sqrt{7}\big)\,\big(4199040\, U^5 \sqrt{7} + 15116544\, U^6 - 9716112\, U^4 \sqrt{7} - 47449152\, U^5 + 7910136\, U^3 \sqrt{7} + 56270052\, U^4 - 2959524\, U^2 \sqrt{7} - 30304044\, U^3 + 649746\, U \sqrt{7} + 7914645\, U^2 - 108262\sqrt{7} - 1479366\, U + 312872\big)\,\big(54\, U\sqrt{7} - 216\, U^2 - 25\sqrt{7} + 189\, U - 55\big)\,\big(9\, U - 5 + \sqrt{7}\big)\Big) \tag{1.4.7.3}$$

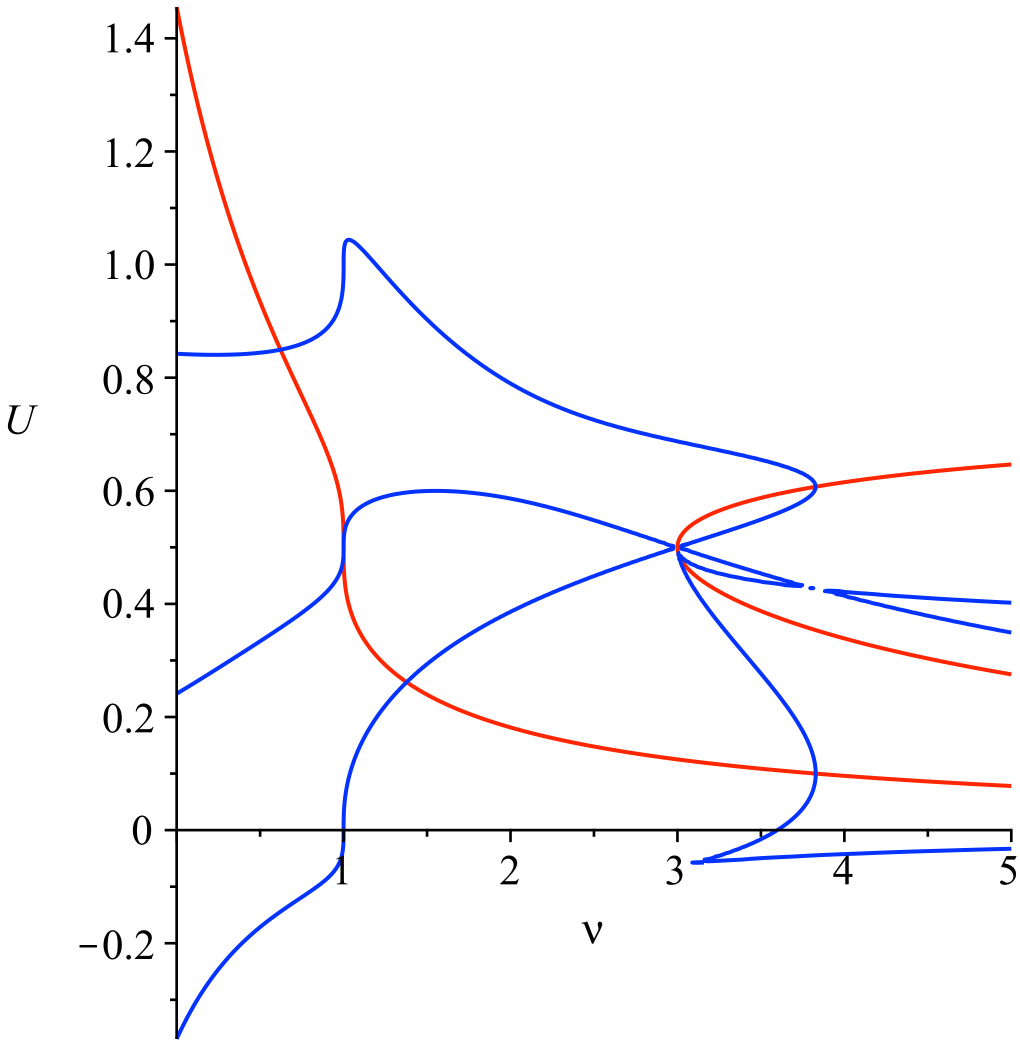
> $factor\big(subs\big(\mathrm{nu} = \nu_c, eqUw1i2\big)\big);$

$$-\frac{1}{5103}\Big(\big(29 + 4\sqrt{7}\big)\,\big(18\, U\sqrt{7} - 324\, U^2 - 7\sqrt{7} + 315\, U - 91\big)\,\big(9\, U - 5 + \sqrt{7}\big)\Big) \tag{1.4.7.4}$$

When else ?

> $aa1 := implicitplot(eqUw1i1, \mathrm{nu} = 0 ..5, U = -0.5 ..2, numpoints = 100000, color = blue) : aa2 := implicitplot(eqUw1i2, \mathrm{nu} = 0 ..5, U = -0.5 ..2, numpoints = 100000, color = red) :$

> $plots[display](\{aa1, aa2\});$



> *factor*(*resultant*(*eqUw1i1*, *eqUw1i2*, *U*)); *solve*(%); *evalf*(%);

$$-47775744\left(7\nu^2-14\nu+6\right)\left(\nu^2-2\nu-7\right)^2(\nu-1)^3(\nu-3)^7(\nu+1)^{13}$$

$1, 1, 1, 3, 3, 3, 3, 3, 3, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+2\sqrt{2}, 1-2\sqrt{2}, 1+\frac{\sqrt{7}}{7}, 1-\frac{\sqrt{7}}{7}$

1., 1., 1., 3., 3., 3., 3., 3., 3., 3., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., 3.828427124, −1.828427124, 3.828427124, −1.828427124, 1.377964473, 0.6220355269 **(1.4.7.5)**

Before nu_c, there is 1 and 0.622,

When nu = 1 the meeting point is 1/2 but w3i=0

> *factor*(*subs*(nu = 1 , *eqUw1i2*)); *solve*(%); *factor*(*subs*(nu = 1, *eqUw1i1*)); *solve*(%);

$$2\,(-1+2U)^3$$

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$$

$$8192\, U^3 \, (-1+2\, U)^3 \, (U-1)^3$$

$$0, 0, 0, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \quad \textbf{(1.4.7.6)}$$

Metting point for nu=0.622

> $factor\left(subs\left(\mathrm{nu} = 1 - \frac{1}{7}\sqrt{7}, eqUw1i2\right)\right); fsolve(\%); factor\left(subs\left(\mathrm{nu} = 1 - \frac{1}{7}\sqrt{7}, eqUw1i1\right)\right); fsolve(\%);$

$$\frac{1}{5103}\left(\left(-29+4\sqrt{7}\right)\left(18\, U\sqrt{7}+324\, U^2-7\sqrt{7}-315\, U+91\right)\left(-9\, U+5+\sqrt{7}\right)\right)$$

$$0.8495279234$$

$$-\frac{1}{4921675101}\left(\left(-14966+4201\sqrt{7}\right)\left(4199040\, U^5\sqrt{7}-15116544\, U^6-9716112\, U^4\sqrt{7}+47449152\, U^5+7910136\, U^3\sqrt{7}-56270052\, U^4-2959524\, U^2\sqrt{7}+30304044\, U^3+649746\, U\sqrt{7}-7914645\, U^2-108262\sqrt{7}+1479366\, U-312872\right)\left(54\, U\sqrt{7}+216\, U^2-25\sqrt{7}-189\, U+55\right)\left(-9\, U+5+\sqrt{7}\right)\right)$$

$$-0.1442045455, 0.3577667178, 0.8495279234 \quad \textbf{(1.4.7.7)}$$

> $resultant\left(\left(18\, U\sqrt{7}+324\, U^2-7\sqrt{7}-315\, U+91\right), \left(4199040\, U^5\sqrt{7}-15116544\, U^6-9716112\, U^4\sqrt{7}+47449152\, U^5+7910136\, U^3\sqrt{7}-56270052\, U^4-2959524\, U^2\sqrt{7}+30304044\, U^3+649746\, U\sqrt{7}-7914645\, U^2-108262\sqrt{7}+1479366\, U-312872\right)\left(54\, U\sqrt{7}+216\, U^2-25\sqrt{7}-189\, U+55\right), U\right);$

$$\left(14161808609399144719872000\sqrt{7}+38236667941785472123507200\right)\left(14883264\sqrt{7}+61136856\right) \quad \textbf{(1.4.7.8)}$$

Value of U(rho_nu)

> $factor\left(subs\left(w = w21, \mathrm{nu} = 1 - \frac{1}{7}\sqrt{7}, algU\right)\right); fsolve(\%);$

$$\frac{1}{964467}\left(\left(-434+85\sqrt{7}\right)\left(594\, U^2\sqrt{7}-1944\, U^3-630\, U\sqrt{7}+3213\, U^2+182\sqrt{7}-2016\, U+469\right)\left(9\, U-4+\sqrt{7}\right)^2\right)$$

$$0.1504720766, 0.1504720766, 1.278680597 \quad \textbf{(1.4.7.9)}$$

The meeting point is larger than U(rho_c) and corresponds to wrong branches or values of w1i outside the circle of convergence. Therefore, for nu<nu_c and values of w1i inside the disk of convergence, U(w1i) satisfies eqUw1i1 and not eqUw1i2

the factor of degree 3 was also in the caracteristic equation of U(rho)

> $eqUrho3;$

$$4\,U^3\nu^2+8\,U^3\nu-3\,U^2\nu^2+4\,U^3-12\,U^2\nu-9\,U^2+6\,U\nu+6\,U-2 \quad \textbf{(1.4.7.10)}$$

> *eqUw1i2*;

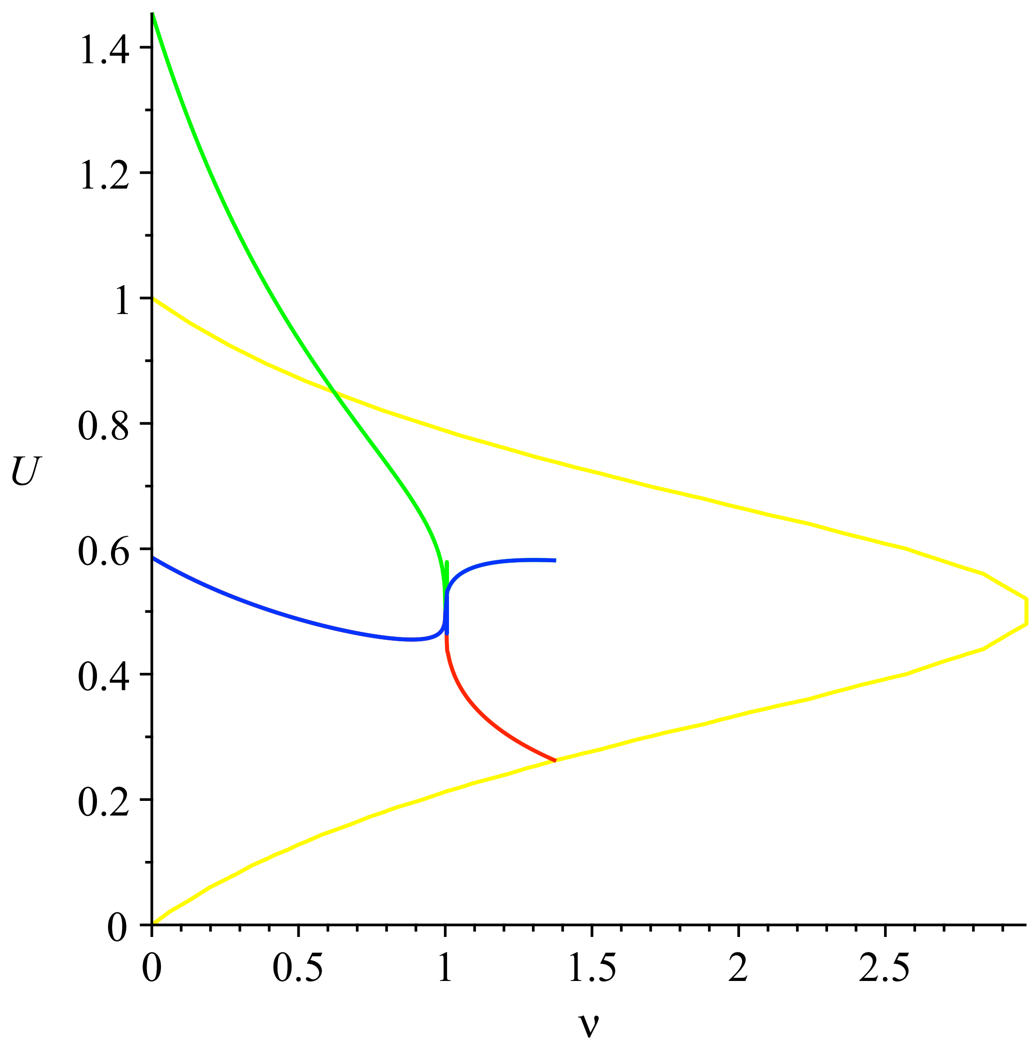
$$4\,U^3\nu^2+8\,U^3\nu-3\,U^2\nu^2+4\,U^3-12\,U^2\nu-9\,U^2+6\,U\nu+6\,U-2 \quad \textbf{(1.4.7.11)}$$

For nu < nu_c, we already know that the real root of this equation is larger than U(rho). We also know that if w1i is on the circle of convergence, |U(w1i)| < U(rho), we will see that is is never the case for this factor il nu < nu_c

> $u31, u32, u33 := solve(eqUrho3, U)$:

> $U1 := plot(|u31|, \mathrm{nu} = 0..\nu_c, color = green)$: $U2 := plot(|u32|, \mathrm{nu} = 0..\nu_c, color = red)$: $U3 := plot(|u33|, \mathrm{nu} = 0..\nu_c, color = blue)$:

> $plots[display](\{plotUrho2, U1, U2, U3\})$;



EqUrho3 has two complex conjuguate roots :

> $factor(discrim(eqUrho3, U))$;

$$108\,(\nu-3)\,(\nu-1)^2\,(\nu+1)^3 \quad \textbf{(1.4.7.12)}$$

Equation for the modulus of these roots:

> $eqmodUw1i := 6\,(\mathrm{nu}+1)\cdot 4\,(\mathrm{nu}+1)^2\cdot U^4-\left(4\,(\mathrm{nu}+1)^2\right)^2 U^6+4-2\cdot 3\,(\mathrm{nu}$

$+ 3) \cdot (\mathrm{nu} + 1)\ U^2;$

$$eqmodUw1i := 24\ (\nu + 1)^3\ U^4 - 16\ (\nu + 1)^4\ U^6 + 4 - 6\ (\nu + 3)\ (\nu + 1)\ U^2 \quad \textbf{(1.4.7.13)}$$

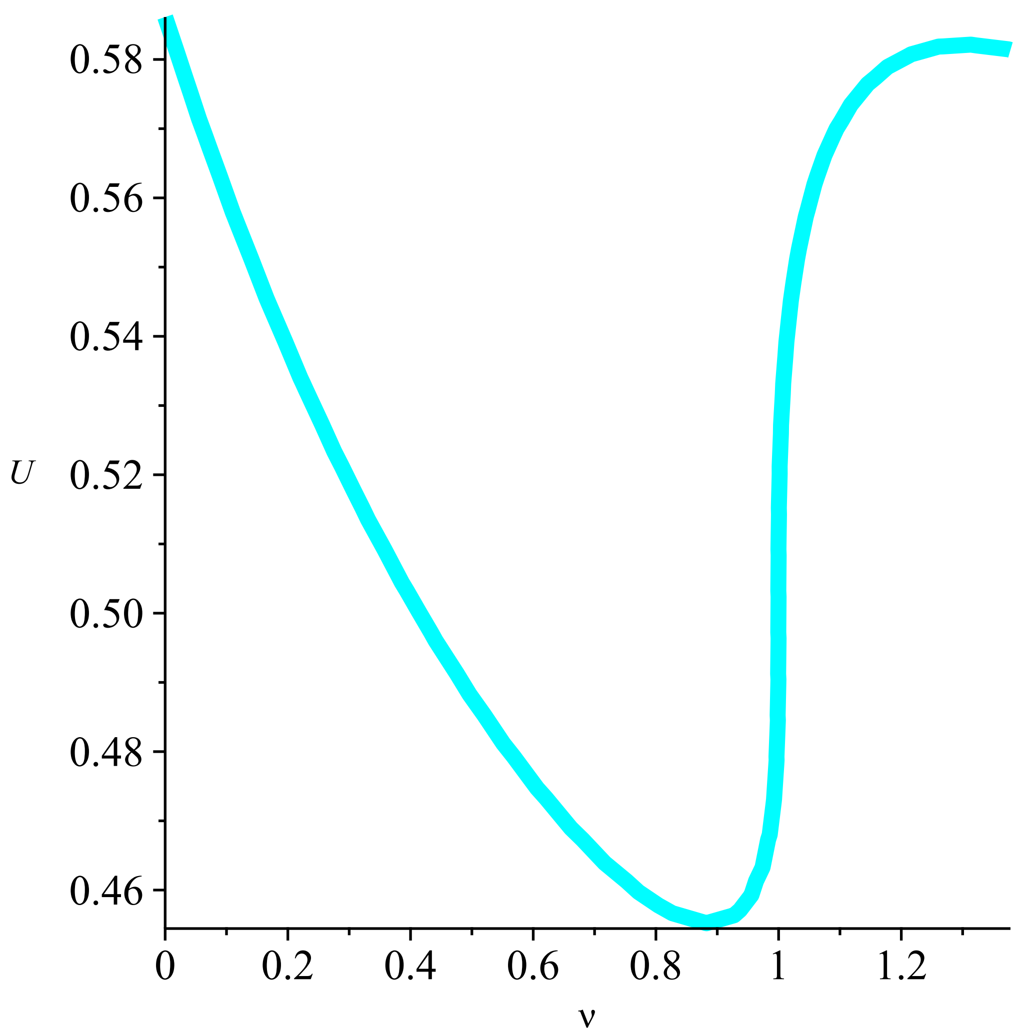
When can the modulus become smaller than U(rho) : never !

> $factor(resultant(eqmodUw1i, eqUrho2, U)); fsolve(\%);$

$$4\ (\nu - 3)\ \left(64\ \nu^6 - 32\ \nu^5 - 168\ \nu^4 - 396\ \nu^3 + 405\ \nu^2 + 1458\right)\ (\nu + 1)^7$$

$$-1., -1., -1., -1., -1., -1., -1., 3. \quad \textbf{(1.4.7.14)}$$

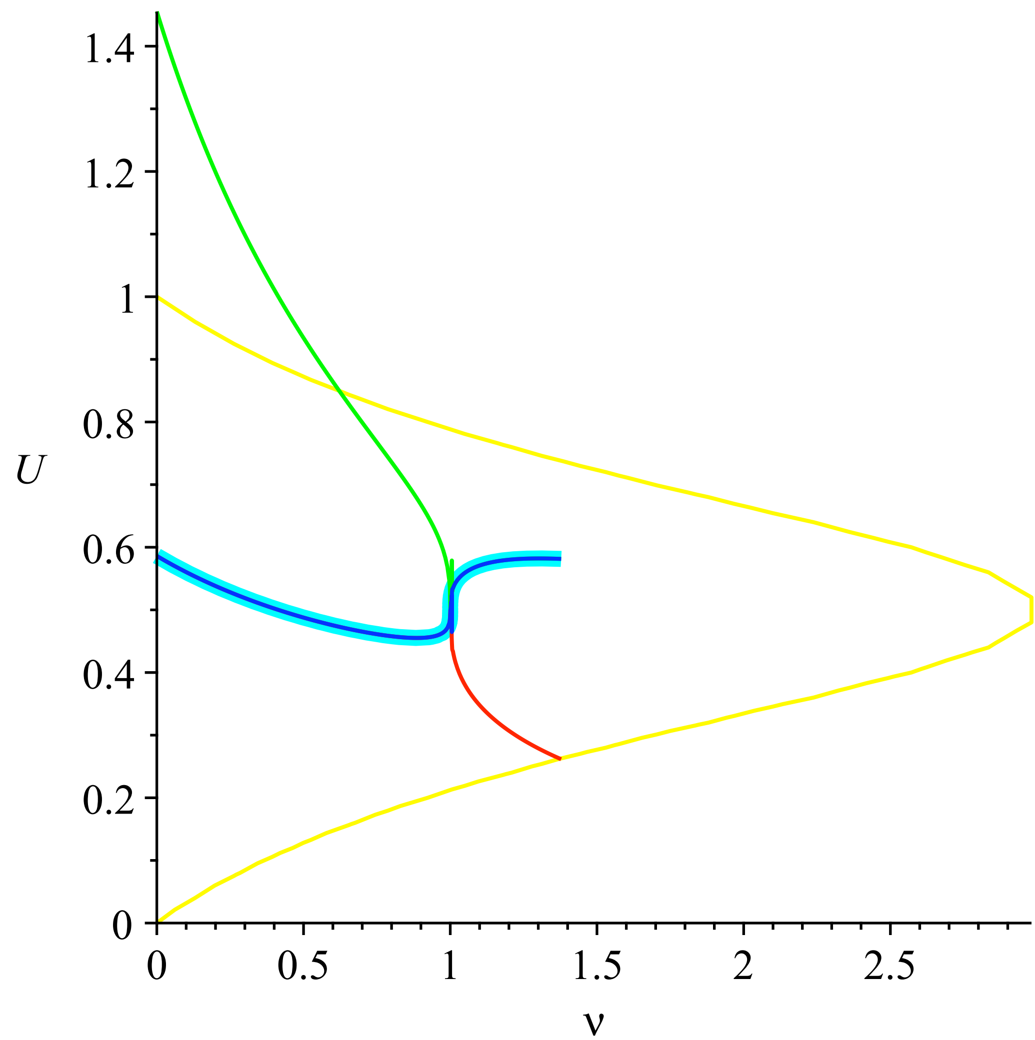
> $MU := implicitplot\left(eqmodUw1i, \mathrm{nu} = 0 .. \nu_c, U = 0 .. 2, color = cyan, thickness = 6\right);$



> $factor(subs(\mathrm{nu} = 1, eqmodUw1i));$

$$-4\ (-1 + 2\ U)^3\ (2\ U + 1)^3 \quad \textbf{(1.4.7.15)}$$

> $plots[display](\{plotUrho2, U1, U2, U3, MU\});$



>

Now we know that U(w1i) is given by the big factor. Starting form *eqUw1i1*, we write an equation satisfied by w and UU with U=U(w1i) - UU (Maple does not factorize it)

> $eqUUw1i1 := resultant(eqUw1i1, subs(U = U - UU, algU), U) : indets(\%);$

$$\{UU, \nu, w\} \qquad \textbf{(1.4.7.16)}$$

Then we write an equation for UU and WW with w=w1i - WW et U= U(w1i) - UU, w1i being a root of P1 (2 min computing time)

> $eqWW1UU1 := resultant(P1, subs(w = w - WW, eqUUw1i1), w) :$

Maple can factorize it by it can take a few minutes !!! (20 on my laptop):

> $eqWW1UU1 := factor(eqWW1UU1) :$

> $nops(\%);$

$$5 \qquad \textbf{(1.4.7.17)}$$

> $op(1, eqWW1UU1); op(2, eqWW1UU1); op(3, eqWW1UU1);$

$$-490398573077084434674671048688098938757996519098752696\,32$$

$$(\nu + 1)^{45}$$

$$\nu^{81} \qquad \textbf{(1.4.7.18)}$$

> $subs(UU = 0, WW = 0, op(4, eqWW1UU1));$

$$0 \quad (1.4.7.19)$$

> $factor(subs(UU = 0, WW = 0, op(5, eqWW1UU1))); fsolve(\%); degree(op(5, eqWW1UU1), WW)$

$$-1310067672419160639406008 (\nu^2 - 2\nu - 7)^6 (\nu + 1)^9 (\nu - 1)^{10} (\nu - 3)^{25}$$

−1.828427125, −1.828427125, −1.828427125, −1.828427125, −1.828427125, −1.828427125, −1., −1., −1., −1., −1., −1., −1., −1., −1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125, 3.828427125, 3.828427125, 3.828427125, 3.828427125

$$18 \quad (1.4.7.20)$$

The first factor is always the right one before nu_c ! We can do Newton's method

> $eqWW1UU1good := collect(op(4, eqWW1UU1), \{WW, UU\}, factor) : degree(eqWW1UU1good, WW);$

$$9 \quad (1.4.7.21)$$

> **for** i **from** 0 **to** 9 **do**
$ldegree(coeff(eqWW1UU1good, WW, i), UU);$
$coeff(coeff(eqWW1UU1good, WW, i), UU, \%); fsolve(\%);$ **od**;

$$9$$

$$191102976 (13573\nu^4 - 54292\nu^3 + 69811\nu^2 - 31038\nu + 67482)(\nu - 1)^2 (\nu^2 - 2\nu - 7)^2 (7\nu^2 - 14\nu + 6)^2 (\nu - 3)^9 (\nu + 1)^{10}$$

−1.828427125, −1.828427125, −1., −1., −1., −1., −1., −1., −1., −1., −1., −1., 0.6220355270, 0.6220355270, 1., 1., 1.377964473, 1.377964473, 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125

$$8$$

$$-1019215872\nu^3 (49\nu^4 - 196\nu^3 + 339\nu^2 - 286\nu + 102)(13573\nu^4 - 54292\nu^3 + 69811\nu^2 - 31038\nu + 67482)(\nu - 1)^2 (\nu^2 - 2\nu - 7)^2 (\nu + 1)^9 (\nu - 3)^9$$

−1.828427125, −1.828427125, −1., −1., −1., −1., −1., −1., −1., −1., −1., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125, 3.828427125

$$7$$

$$56623104\nu^6 (\nu^2 - 2\nu - 7)(15961848\nu^{12} - 191542176\nu^{11} + 957162619\nu^{10} - 2548413070\nu^9 + 3210739098\nu^8 + 1764577920\nu^7 - 10717931194\nu^6 + 3195829884\nu^5 + 25842319124\nu^4 - 35230532064\nu^3 + 23728532391\nu^2 - 19564752366\nu + 14587990002)(\nu - 1)^2 (\nu + 1)^7 (\nu - 3)^8$$

−1.828427125, −1., −1., −1., −1., −1., −1., −1., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3., 3.828427125

6

$$-16777216\,\nu^9\,(\nu^2-2\,\nu-7)\,(808177139\,\nu^{12}-9698125668\,\nu^{11}+46556200397\,\nu^{10}-109964062810\,\nu^9+159720819568\,\nu^8-345499166672\,\nu^7+990817163826\,\nu^6-1670019200108\,\nu^5+1266409702955\,\nu^4+38943843492\,\nu^3-530185174623\,\nu^2+44916452694\,\nu-414979921710)\,(\nu-1)^2\,(\nu+1)^5\,(\nu-3)^7$$

−1.828427125, −1., −1., −1., −1., −1., −0.6940748849, 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.694074885, 3., 3., 3., 3., 3., 3., 3., 3.828427125

5

$$12582912\,\nu^{12}\,(17823292487\,\nu^{14}-249526094818\,\nu^{13}+1407043935773\,\nu^{12}-3909170298740\,\nu^{11}+4957978223199\,\nu^{10}-1396155435454\,\nu^9-4219025596163\,\nu^8+31721054006056\,\nu^7-103825901192355\,\nu^6+96699512390594\,\nu^5+54116108603583\,\nu^4-58253103431028\,\nu^3-15238553021955\,\nu^2-67969070267490\,\nu+2300590998567)\,(\nu-1)^2\,(\nu+1)^3\,(\nu-3)^6$$

−1.800513327, −1., −1., −1., −0.9085016737, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.03356371495, 1., 1., 1.966436285, 2.908501674, 3., 3., 3., 3., 3., 3., 3.800513327

4

$$-12884901888\,\nu^{15}\,(277982901\,\nu^{12}-3335794812\,\nu^{11}+16122301322\,\nu^{10}-38910536780\,\nu^9+36257967195\,\nu^8+56689258248\,\nu^7-220515526388\,\nu^6+263055018888\,\nu^5-116228389445\,\nu^4+53162531700\,\nu^3-81758241270\,\nu^2+12201729732\,\nu+179748005013)\,(\nu-1)^2\,(\nu+1)^2\,(\nu-3)^6$$

−1.738685063, −1., −1., −0.7154269072, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.715426907, 3., 3., 3., 3., 3., 3., 3.738685063

3

$$35184372088832\,\nu^{18}\,(\nu+1)\,(1467508\,\nu^{10}-14675080\,\nu^9+51504055\,\nu^8-59830520\,\nu^7-68361826\,\nu^6+261632860\,\nu^5-219224808\,\nu^4-105745968\,\nu^3+2509110\,\nu^2+415307412\,\nu+116574633)\,(\nu-1)^2\,(\nu-3)^6$$

$-1.643450035, -1., -0.2823650610, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.282365061, 3., 3., 3., 3., 3., 3., 3.643450035$

$$2$$

$$-108086391056891904\, \nu^{21} \left(4077\, \nu^8 - 32616\, \nu^7 + 71231\, \nu^6 + 29238\, \nu^5 - 218739\, \nu^4 + 71204\, \nu^3 + 137493\, \nu^2 + 8478\, \nu - 127710\right) (\nu - 1)^2 (\nu - 3)^6$$

$-1.469961881, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3.469961881$

$$1$$

$$138350580552821637120\, \nu^{24} (\nu + 1) \left(161\, \nu^2 - 322\, \nu - 159\right) (\nu - 1)^2 (\nu - 3)^8$$

$-1., -0.4098147537, 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 2.409814754, 3., 3., 3., 3., 3., 3., 3., 3.$

$$0$$

$$-4722366482869645213696\, \nu^{27} (\nu - 1)^2 (\nu - 3)^8$$

$0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 3., 3., 3., 3., 3., 3., 3., 3.$ **(1.4.7.22)**

The behavior is non singular for a generic nu < nu_c, the only possible change is for nu=0.622... where the coef UU^9WW^0 vanishes

> $evalf\left(1 - \frac{1}{7}\sqrt{7}\right);$

$$0.6220355269 \qquad \textbf{(1.4.7.23)}$$

It is the meeting point we saw earlier and we know that it corresponds to vales w1i outside the disk of convergence of U so it does not concerns us

Singular behavior at the radius of convergence

We apply Newton polygon method before and after nu_c

After nu_c

For nu > nu_c, the radius of convergence is a root w3i of P1. Recall the two factors of the equation for U(w1i)

> *eqUw1i1*; *eqUw1i2*;

$$2048 U^9 \nu^5 + 10240 U^9 \nu^4 - 5376 U^8 \nu^5 + 20480 U^9 \nu^3 - 34560 U^8 \nu^4 + 5472 U^7 \nu^5 + 20480 U^9 \nu^2 - 84480 U^8 \nu^3 + 48480 U^7 \nu^4 - 2972 U^6 \nu^5 + 10240 U^9 \nu - 99840 U^8 \nu^2 + 142656 U^7 \nu^3 - 35332 U^6 \nu^4 + 1428 U^5 \nu^5 + 2048 U^9 - 57600 U^8 \nu + 191808 U^7 \nu^2 - 127176 U^6 \nu^3 + 13548 U^5 \nu^4 - 843 U^4 \nu^5 - 13056 U^8 + 122208 U^7 \nu - 191480 U^6 \nu^2 + 61656 U^5 \nu^3 - 2925 U^4 \nu^4 + 328 U^3 \nu^5 + 30048 U^7 - 127900 U^6 \nu + 105000 U^5 \nu^2 - 11610 U^4 \nu^3 + 1076 U^3 \nu^4 - 48 U^2 \nu^5 - 31236 U^6 + 72084 U^5 \nu - 28470 U^4 \nu^2 - 3760 U^3 \nu^3 - 552 U^2 \nu^4 + 16620 U^5 - 24411 U^4 \nu + 1976 U^3 \nu^2 + 2592 U^2 \nu^3 + 96 U \nu^4 - 5469 U^4 + 7528 U^3 \nu + 360 U^2 \nu^2 - 432 U \nu^3 + 1044 U^3 - 2976 U^2 \nu + 24 U \nu^2 + 16 \nu^3 + 624 U^2 + 864 U \nu - 48 \nu^2 - 552 U - 60 \nu + 92$$

$$4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2 \qquad \textbf{(1.5.1.1)}$$

We also have an equation for U(rho_nu):

> *eqUrho*;

$$128 (-1 + 2 U) \nu^3 \left(4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2\right) \left(3 U^2 \nu + 3 U^2 - 3 U \nu - 3 U + \nu\right) \qquad \textbf{(1.5.1.2)}$$

> *factor*(*resultant*(*eqUw1i1*, (-1 + 2 U)·(3 U^2 ν + 3 U^2 − 3 U ν − 3 U + ν), U));
> *fsolve*(%);

$$-128 (\nu - 1) (\nu - 3)^{10} \left(7 \nu^2 - 14 \nu + 6\right) \left(13573 \nu^4 - 54292 \nu^3 + 69811 \nu^2 - 31038 \nu + 67482\right) (\nu + 1)^7$$

$$-1., -1., -1., -1., -1., -1., -1., 0.6220355270, 1., 1.377964473, 3., 3., 3., 3., 3., 3., 3., 3., 3., 3., 3. \qquad \textbf{(1.5.1.3)}$$

For nu>nu_c, the first branch can be the good one only for nu=3, but in this case U(rho) is also a root of the second factor

> *factor*(*subs*(nu = 3, *eqUw1i1*)); *factor*(*subs*(nu = 3, *eqUw1i2*)); *factor*(*subs*(nu = 3, *eqUrho*));

$$8 (-11 + 16 U) (16 U + 1)^2 (-1 + 2 U)^6$$

$$2 (8 U - 1) (-1 + 2 U)^2$$

$$20736 (-1 + 2 U)^5 (8 U - 1) \qquad \textbf{(1.5.1.4)}$$

Starting from eqUw1i2, we write an equation for w and UU with U=Uw1i-UU (Maple does not factorize it)

> *eqUUw1i2* := *resultant*(*eqUw1i2*, *subs*(U = U − UU, *algU*), U) : *indets*(%);

$$\{UU, \nu, w\} \qquad \textbf{(1.5.1.5)}$$

Then an equation for UU and WW with w=w1i - WW

```
> eqWW1iUU2 := resultant(P1, subs(w = w − WW, eqUUw1i2), w) :
> eqWW1iUU2 := factor(eqWW1iUU2) :
> nops(%);
```

$$5 \tag{1.5.1.6}$$

There are two factors

```
> op(1, eqWW1iUU2); op(2, eqWW1iUU2); op(3, eqWW1iUU2)
```

$$-562949953421312$$
$$(\nu+1)^{18}$$
$$\nu^{27} \tag{1.5.1.7}$$

```
> eqWW1iUU21 := collect(op(4, eqWW1iUU2), {UU, WW}, factor) :
      eqWW1iUU22 := collect(op(5, eqWW1iUU2), {UU, WW}, factor) :
> subs(UU = 0, WW = 0, eqWW1iUU21); subs(UU = 0, WW = 0, eqWW1iUU22);
```

$$0$$
$$-5038848\left(\nu^2-2\nu-7\right)^2(\nu+1)^3(\nu-1)^6(\nu-3)^7 \tag{1.5.1.8}$$

The first factor is the good one

```
> degree(eqWW1iUU21, WW); degree(eqWW1iUU22, WW);
```

$$3$$
$$6 \tag{1.5.1.9}$$

```
> for i from 0 to 3 do
  ldegree(coeff(eqWW1iUU21, WW, i), UU);
  coeff(coeff(eqWW1iUU21, WW, i), UU, %); od;
```

$$6$$
$$432(\nu-1)\left(7\nu^2-14\nu+6\right)(\nu-3)^2(\nu+1)^6$$
$$4$$
$$-20736\nu^3(\nu-1)^2(\nu-3)^2(\nu+1)^4$$
$$2$$
$$-64512\nu^6(\nu+1)^2(\nu-3)^2(\nu-1)^3$$
$$0$$
$$-131072\nu^9(\nu-1)^2(\nu-3)^2 \tag{1.5.1.10}$$

For a generic nu we have a square root singularity, we need to check 3 and nu_c

```
> eqWW1iUU21badnu3 := factor(subs(nu = 3, eqWW1iUU21)) :
> degree(eqWW1iUU21badnu3, WW);
```

$$3 \tag{1.5.1.11}$$

```
> for i from 0 to 3 do
  ldegree(coeff(eqWW1iUU21badnu3, WW, i), UU);
  coeff(coeff(eqWW1iUU21badnu3, WW, i), UU, %); od;
```

$$10$$
$$13759414272$$
$$8$$

$$165112971264$$
$$6$$
$$-1981355655168$$
$$4$$
$$-23776267862016 \qquad \textbf{(1.5.1.12)}$$

Also a square root singularity

>

Finally at nu_c

> $eqWW1iUU21nuc := factor\left(subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7}, eqWW1iUU21\right)\right):$

> $degree(eqWW1iUU21nuc, WW);$

$$3 \qquad \textbf{(1.5.1.13)}$$

> **for** i **from** 0 **to** 3 **do**
$ldegree(coeff(eqWW1iUU21nuc, WW, i), UU);$
$coeff(coeff(eqWW1iUU21nuc, WW, i), UU, \%);$ **od**;

$$7$$
$$-\frac{1}{333745342838270677185398 1}\left(\left(12016033849\sqrt{7} + 32234505926\right)\left(-629856\sqrt{7} + 4566456\right)\left(-82281568762008 + 25407896603040\sqrt{7}\right)\right)$$
$$4$$
$$-\frac{1}{3337453428382706771853981}\left(\left(12016033849\sqrt{7} + 32234505926\right)\left(-160832\sqrt{7} - 1831616\right)\left(-82281568762008 + 25407896603040\sqrt{7}\right)\right)$$
$$2$$
$$-\frac{1}{3337453428382706771853981}\left(\left(12016033849\sqrt{7} + 32234505926\right)\left(-160832\sqrt{7} - 1831616\right)\left(-4587623780544\sqrt{7} + 3944498814144\right)\right)$$
$$0$$
$$-\frac{1}{3337453428382706771853981}\left(\left(12016033849\sqrt{7} + 32234505926\right)\left(-160832\sqrt{7} - 1831616\right)\left(-2356659716096\sqrt{7} - 14143542788096\right)\right) \qquad \textbf{(1.5.1.14)}$$

We have a possible 1/3 singularity to check (see later)

>

Before nu_c

For nu < nu_c, the radius of convergence is w21 the root of P2.

We have to write an algebraic equation for (w-w21) and (U-U(w21))

First, an equation for U(w2i)

> *eqUw2i* := *factor*(*resultant*(*algU*, *P2*, *w*));

$$eqUw2i := 1024\,\nu^4\,\big(192\,U^6\,\nu^4 + 768\,U^6\,\nu^3 - 144\,U^5\,\nu^4 + 1152\,U^6\,\nu^2 - 1440\,U^5\,\nu^3 - 53\,U^4\,\nu^4 + 768\,U^6\,\nu - 3456\,U^5\,\nu^2 + 720\,U^4\,\nu^3 + 94\,U^3\,\nu^4 + 192\,U^6 - 3168\,U^5\,\nu + 3450\,U^4\,\nu^2 + 24\,U^3\,\nu^3 - 21\,U^2\,\nu^4 - 1008\,U^5 + 4528\,U^4\,\nu - 1572\,U^3\,\nu^2 - 198\,U^2\,\nu^3 - 14\,U\,\nu^4 + 1851\,U^4 - 2840\,U^3\,\nu + 504\,U^2\,\nu^2 + 126\,U\,\nu^3 + 7\,\nu^4 - 1338\,U^3 + 870\,U^2\,\nu - 186\,U\,\nu^2 - 42\,\nu^3 + 189\,U^2 - 158\,U\,\nu + 75\,\nu^2 + 168\,U - 20\,\nu - 36\big)\,\big(3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu\big)^2 \qquad \textbf{(1.5.2.1)}$$

> *eqUrho*;

$$128\,(-1 + 2\,U)\,\nu^3\,\big(4\,U^3\,\nu^2 + 8\,U^3\,\nu - 3\,U^2\,\nu^2 + 4\,U^3 - 12\,U^2\,\nu - 9\,U^2 + 6\,U\,\nu + 6\,U - 2\big)\,\big(3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu\big) \qquad \textbf{(1.5.2.2)}$$

> *factor*(*resultant*((192 U^6 ν^4 + 768 U^6 ν^3 − 144 U^5 ν^4 + 1152 U^6 ν^2 − 1440 U^5 ν^3 − 53 U^4 ν^4 + 768 U^6 ν − 3456 U^5 ν^2 + 720 U^4 ν^3 + 94 U^3 ν^4 + 192 U^6 − 3168 U^5 ν + 3450 U^4 ν^2 + 24 U^3 ν^3 − 21 U^2 ν^4 − 1008 U^5 + 4528 U^4 ν − 1572 U^3 ν^2 − 198 U^2 ν^3 − 14 U ν^4 + 1851 U^4 − 2840 U^3 ν + 504 U^2 ν^2 + 126 U ν^3 + 7 ν^4 − 1338 U^3 + 870 U^2 ν − 186 U ν^2 − 42 ν^3 + 189 U^2 − 158 U ν + 75 ν^2 + 168 U − 20 ν − 36), (−1 + 2 U) ν^3 (4 U^3 ν^2 + 8 U^3 ν − 3 U^2 ν^2 + 4 U^3 − 12 U^2 ν − 9 U^2 + 6 U ν + 6 U − 2), U)); *fsolve*(%);

$$1728\,(\nu - 1)^6\,(\nu - 3)^6\,\nu^{18}\,\big(7\,\nu^2 - 14\,\nu + 6\big)\,\big(13573\,\nu^4 - 54292\,\nu^3 + 69811\,\nu^2 - 31038\,\nu + 67482\big)\,(\nu + 1)^{10}$$

−1., −1., −1., −1., −1., −1., −1., −1., −1., −1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.6220355270, 1., 1., 1., 1., 1., 1., 1.377964473, 3., 3., 3., 3., 3., 3. **(1.5.2.3)**

> *factor*(*subs*(nu = 1 − sqrt(7)/7, (192 U^6 ν^4 + 768 U^6 ν^3 − 144 U^5 ν^4 + 1152 U^6 ν^2 − 1440 U^5 ν^3 − 53 U^4 ν^4 + 768 U^6 ν − 3456 U^5 ν^2 + 720 U^4 ν^3 + 94 U^3 ν^4 + 192 U^6 − 3168 U^5 ν + 3450 U^4 ν^2 + 24 U^3 ν^3 − 21 U^2 ν^4 − 1008 U^5 + 4528 U^4 ν − 1572 U^3 ν^2 − 198 U^2 ν^3 − 14 U ν^4 + 1851 U^4 − 2840 U^3 ν + 504 U^2 ν^2 + 126 U ν^3 + 7 ν^4 − 1338 U^3 + 870 U^2 ν − 186 U ν^2 − 42 ν^3 + 189 U^2 − 158 U ν + 75 ν^2 + 168 U − 20 ν − 36)));

$$-\frac{1}{964467}\Big(\big(-953 + 232\sqrt{7}\big)\,\big(594\,U^2\sqrt{7} - 1944\,U^3 - 630\,U\sqrt{7} + 3213\,U^2 + 182\sqrt{7} - 2016\,U + 469\big)\,\big(54\,U\sqrt{7} + 216\,U^2 - 25\sqrt{7} - 189\,U \qquad \textbf{(1.5.2.4)}$$

$+ 55)\,(-9\,U + 5 + \sqrt{7}))$

> $factor\left(subs\left(nu = 1 - \frac{\text{sqrt}(7)}{7}, 3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu\right)\right);$

$$\frac{(-14 + \sqrt{7})\,(9\,U - 4 + \sqrt{7})\,(-9\,U + 5 + \sqrt{7})}{189} \quad (1.5.2.5)$$

The rightfactor is always the second one

> $eqUw2i2 := (3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu);$

$$eqUw2i2 := 3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu \quad (1.5.2.6)$$

Starting from it, we write an equation for w and UU with U=Uw2i-UU

> $eqUUw2i2 := factor(resultant(eqUw2i2, subs(U = U - UU, algU), U)):$
> $indets(\%);$

$$\{UU, \nu, w\} \quad (1.5.2.7)$$

Then an equation for UU and WW with w=w2i - WW

> $eqWW2iUU2 := resultant(P2, subs(w = w - WW, eqUUw2i2), w):$

> $eqWW2iUU2 := factor(eqWW2iUU2):$

> $nops(\%);$

$$5 \quad (1.5.2.8)$$

There are two factors

> $op(1, eqWW2iUU2); op(2, eqWW2iUU2); op(3, eqWW2iUU2)$

$$20639121408$$

$$(\nu + 1)^6$$

$$\nu^8 \quad (1.5.2.9)$$

> $eqWW2iUU21 := collect(op(4, eqWW2iUU2), \{UU, WW\}, factor):$
> $eqWW2iUU22 := collect(op(5, eqWW2iUU2), \{UU, WW\}, factor):$

> $subs(UU = 0, WW = 0, eqWW2iUU21); subs(UU = 0, WW = 0, eqWW2iUU22);$

$$0$$

$$(\nu + 1)^3\,(\nu - 3)^5 \quad (1.5.2.10)$$

The first factor is the good one

> $degree(eqWW2iUU21, WW);$

$$2 \quad (1.5.2.11)$$

> **for** i **from** 0 **to** 2 **do**
> $ldegree(coeff(eqWW2iUU21, WW, i), UU);$
> $coeff(coeff(eqWW2iUU21, WW, i), UU, \%);$ **od**;

$$4$$

$$12\,(7\,\nu^2 - 14\,\nu + 6)\,(\nu - 3)^2\,(\nu + 1)^4$$

$$2$$

$$576\,\nu^3\,(\nu - 1)\,(\nu + 1)^2\,(\nu - 3)^2$$

$$0$$

(1.5.2.12)

$$1024\,\nu^6\,(\nu-3)^2 \qquad \textbf{(1.5.2.12)}$$

Again a generic square root singularity except maybe at 1-sqrt(7)/7 and nu_c

```
> eqWW2iUU21bad := collect(factor(subs(nu = 1 - sqrt(7)/7, eqWW2iUU22)), WW):
> degree(eqWW2iUU21bad, WW);
```

$$2 \qquad \textbf{(1.5.2.13)}$$

```
> for i from 0 to 2 do
  ldegree(coeff(eqWW2iUU21bad, WW, i), UU);
  coeff(coeff(eqWW2iUU21bad, WW, i), UU, %); od;
```

$$0$$

$$\frac{\left(442192\sqrt{7}-1284977\right)\left(-57726\sqrt{7}-113427\right)^2}{25115308040403}$$

$$0$$

$$\frac{\left(442192\sqrt{7}-1284977\right)\left(-57726\sqrt{7}-113427\right)\left(160832\sqrt{7}-1831616\right)}{25115308040403} + \frac{1}{25115308040403}\left(\left(442192\sqrt{7}-1284977\right)\left(-160832\sqrt{7}+1831616\right)\left(-57726\sqrt{7}-113427\right)\right)$$

$$0$$

$$\frac{1}{25115308040403}\left(\left(442192\sqrt{7}-1284977\right)\left(-160832\sqrt{7}+1831616\right)\left(160832\sqrt{7}-1831616\right)\right) \qquad \textbf{(1.5.2.14)}$$

we have to check:

```
> simplify(puiseux(subs(nu = 1 - sqrt(7)/7, algU), w = subs(nu = 1 - sqrt(7)/7, w21), U, 0));
```

$$\left\{\frac{1}{81\left(-7+\sqrt{7}\right)^2\left(101\sqrt{7}-179\right)}\left(179\sqrt{7-\sqrt{7}}\,\sqrt{101\sqrt{7}-179}\,\left(\sqrt{7}-\frac{707}{179}\right)\sqrt{(704\,w+7)\sqrt{7}-2240\,w}+473130\sqrt{7}-1231398\right),\ \mathit{RootOf}\left(1944\,\_Z^3+\left(-594\sqrt{7}-3213\right)\_Z^2+\left(630\sqrt{7}+2016\right)\_Z-182\sqrt{7}-469\right)\right\} \qquad \textbf{(1.5.2.15)}$$

A square root singularity

```
>
```

Finally at nu_c

```
> eqWW2iUU22nuc := collect(factor(subs(nu = 1 + sqrt(7)/7, eqWW2iUU22)),
      WW) :
> degree(eqWW2iUU22nuc, WW);
```

$$2 \tag{1.5.2.16}$$

```
> for i from 0 to 2 do
  ldegree(coeff(eqWW2iUU22nuc, WW, i), UU);
  coeff(coeff(eqWW2iUU22nuc, WW, i), UU, %); od;
```

$$0$$

$$-\frac{\left(1284977+442192\sqrt{7}\right)\left(-57726\sqrt{7}+113427\right)^{2}}{25115308040403}$$

$$0$$

$$-\frac{1}{25115308040403}\left(\left(1284977+442192\sqrt{7}\right)\left(-57726\sqrt{7}+113427\right)\left(-160832\sqrt{7}-1831616\right)\right)-\frac{1}{25115308040403}\left(\left(1284977+442192\sqrt{7}\right)\left(160832\sqrt{7}+1831616\right)\left(-57726\sqrt{7}+113427\right)\right)$$

$$0$$

$$-\frac{1}{25115308040403}\left(\left(1284977+442192\sqrt{7}\right)\left(-160832\sqrt{7}-1831616\right)\left(160832\sqrt{7}+1831616\right)\right) \tag{1.5.2.17}$$

we have to check to be sure that the singularity is 1/3

At nu_c

```
> simplify(puiseux(subs(nu = 1 + sqrt(7)/7, algU), w = subs(nu = 1 + sqrt(7)/7,
      w21), U, 0)); evalf(allvalues(%));
```

$$\left\{\frac{1}{-2282+188\sqrt{7}}\left(-14^{1/3}\left(1235-257\sqrt{7}\right)^{2/3}\left(\left(176\,w-5\right)\sqrt{7}+560\,w\right)^{1/3}+358\sqrt{7}-1414\right), RootOf\left(216\,\_Z^{2}+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)\right\}$$

$$\{0.5970188747, 0.09121213316\,(1025.652231\,w-13.22875656)^{1/3}+0.2615831876\}, \{0.9394189531, 0.09121213316\,(1025.652231\,w \tag{1.5.3.1}$$

$- 13.22875656)^{1/3} + 0.2615831876\}$

A 1/3 singularity !

> $algeqtoseries\left(subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7}, w = subs\left(\mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7}, w21\right) \cdot (1 - x), algU\right), x, U, 2\right);$

$$\left[RootOf\left(216\_Z^2 + \left(-54\sqrt{7} - 189\right)\_Z + 25\sqrt{7} + 55\right) + \left(\frac{1715}{12672} + \frac{515\sqrt{7}}{19008} - \frac{1415\, RootOf\left(216\_Z^2 + \left(-54\sqrt{7} - 189\right)\_Z + 25\sqrt{7} + 55\right)}{4752}\right)x + \mathrm{O}\left(x^2\right), \frac{5}{9} - \frac{\sqrt{7}}{9} + RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right)x^{1/3} + \mathrm{O}\left(x^{2/3}\right)\right] \quad (1.5.3.2)$$

>