```
> restart;
> with(plots) : with(gfun) :
```

Ising on triangulations, spins on vertices, simple boundary, counted by edges

Equation with 2 catalytic variables (Sec 2.4)

Variables: t counts edges an nu count monochromatic edges

Catalytic variables: x counts number of + on the boundary, y counts number of -

Zxy(x,y): one group of +s and one of -s on the boundary (Z^{+,-} in the paper)

Z1x(x) : exactly one - on the boundary and one group of + (= [y]Z^{+,-}=Z_1^{+,-}(x) in the paper)

Z1y(y) : exactly one + on the boundary and one group of - (= [x]Z^{+,-}=Z_1^{+,-}(y) in the paper)

Zx(x): only +s on the boundary (Z^+(x) in the paper)

Zy(y): only +s on the boundary (Z^+(y) in the paper)

Z1: exactly one + on the boundary (= [x]Z^+(x) in the paper=Z_1^+)

Z2: exactly two + on the boundary (= [x^2]Z^+(x)=Z_2^+ in the paper)

Z11 : exactly one + and one - on the boundary (= [xy]Z^{+,-} in the paper)

We open the edge at the interface between +s and -s. It corresponds to Equation (14) of the article.

$$> eqZxy := -Zxy + t\cdot x\cdot y + \frac{t}{x}\cdot Zxy\cdot Zx + \frac{t}{y}Zxy\cdot Zy + \frac{t}{x}\cdot (Zxy - x\cdot Z1y) + \frac{t}{y}\cdot (Zxy - y\cdot Z1x) :$$

$$> eqZx := -Zx + \nu\cdot t\cdot x^2 + \frac{\nu\cdot t}{x}\cdot Zx^2 + \frac{\nu\cdot t}{x}\cdot (Zx - x\cdot Z1) + \nu\cdot t\cdot Z1x :$$

```
> ZxySer := proc(n) option remember :
  if n = 0 then 0 else
  convert(normal(series(subs(Zxy = ZxySer(n-1), Zx = ZxSer(n-1), Zy = subs(x = y,
      ZxSer(n-1)), Z1y = coeff(ZxySer(n-1), x, 1), Z1x = coeff(ZxySer(n-1), y, 1), eqZxy
      + Zxy), t, n + 1)), polynom) :fi: end:

> ZxSer := proc(n) option remember :
  if n = 0 then 0 else
  convert(normal(series(subs(Zx = ZxSer(n-1), Z1 = coeff(ZxSer(n-1), x, 1), Z1x
      = coeff(ZxySer(n-1), y, 1), eqZx + Zx), t, n + 1)), polynom); fi; end:

> map(factor, ZxSer(2));
```

$$\nu\, t\, x^2 + \nu\, x\, (\nu + 1)\, t^2 \qquad \mathbf{(1.1.1)}$$

```
> map(factor, ZxySer(4));
```

$$t\, x\, y + x\, y\, \nu\, (x + y)\, t^3 + 2\, x\, y\, \nu\, (\nu + 2)\, t^4 \qquad \mathbf{(1.1.2)}$$

Kernel Method

> $K := coeff(eqZxy, Zxy);$

$$K := -1 + \frac{t\,Zx}{x} + \frac{t\,Zy}{y} + \frac{t}{x} + \frac{t}{y} \qquad (1.2.1)$$

> $R := coeff(eqZxy, Zxy, 0);$

$$R := t\,x\,y - t\,Z1x - t\,Z1y \qquad (1.2.2)$$

We want two series Y(t) such that K(t,x,Y(t))=0.

> $eqYx := simplify((K+1)\cdot y);$

$$eqYx := \frac{t\,(Zx\,y + Zy\,x + x + y)}{x} \qquad (1.2.3)$$

There is only one such series in t, whose coefficients can be defined recursively. To find another one, we relax the hypothesis and ask for a formal power series Y(x) in t, such that K(x,Y(x)/t)=0. (This is well-defined since subs(y=Y/t,t*Zy) is a formal power series in t.

> $collect\left(numer\left(factor\left(subs\left(Zx = ZxSer(4), Zy = subs(x = y, ZxSer(4)), y = \frac{Y}{t}, K\right)\right)\right), t\right);$

$$\left(3\,Y\,\nu^4\,x^2 + 2\,Y\,\nu^3\,x^2 + Y\,\nu^2\,x^2\right)t^5 + \left(3\,Y^2\,\nu^4\,x + Y\,\nu^3\,x^3 + 2\,Y^2\,\nu^3\,x + Y^2\,\nu^2\,x\right)t^4 + \left(2\,Y\,\nu^2\,x + 2\,Y\,\nu\,x\right)t^3 + \left(Y^3\,\nu^3\,x + Y\,\nu\,x^2 + x\right)t^2 + \left(Y^2\,\nu\,x + Y\right)t - x\,Y \qquad (1.2.4)$$

We have only one Y solution. Substitute x=t+at^2 to get a second one, we obtain Equation (15) of the article.

> $map(factor, collect(numer(factor(subs(x = t + a\cdot t^2, (1.2.4)))), t));$

$$Y\,a^3\,\nu^3\,t^{10} + Y\,a^2\,\nu^2\,(3\,\nu^2 + 5\,\nu + 1)\,t^9 + Y\,a\,\nu^2\,(3\,\nu + 2)\,(2\,\nu + 1)\,t^8 + Y\,\nu^2\,(3\,\nu^2 + 3\,\nu + 1)\,t^7 + Y\,a\,\nu\,(3\,Y\,\nu^3 + 2\,Y\,\nu^2 + Y\,\nu + a)\,t^6 + Y\,\nu\,(3\,Y\,\nu^3 + 2\,Y\,\nu^2 + Y\,\nu + 2\,a\,\nu + 4\,a)\,t^5 + (Y^3\,a\,\nu^3 + 2\,Y\,\nu^2 + 3\,Y\,\nu + a)\,t^4 + (\nu^3\,Y^3 + Y^2\,a\,\nu + 1)\,t^3 + Y\,(Y\,\nu - a)\,t^2 \qquad (1.2.5)$$

We compute the possible constant term b for Y, if it a solution of the latter equation:

> $factor\left(subs\left(Zx = ZxSer(5), Zy = subs(x = y, ZxSer(5)), x = t + a\cdot t^2, y = \frac{b}{t}, K\right)\right)$

$$\frac{1}{(a\,t+1)\,b}\Big(t\,\big(2\,a^4\,b\,\nu^5\,t^{12} + 8\,a^3\,b\,\nu^5\,t^{11} + 12\,a^2\,b\,\nu^5\,t^{10} + 8\,a\,b\,\nu^5\,t^9 + a^3\,b\,\nu^3\,t^8 + 3\,a^2\,b\,\nu^4\,t^7 + 2\,b\,\nu^5\,t^8 + 5\,a^2\,b\,\nu^3\,t^7 + 2\,a\,b^4\,\nu^5\,t^3 + 8\,a\,b\,\nu^5\,t^6 + a^2\,b\,\nu^2\,t^7 + 14\,a\,b\,\nu^4\,t^6 + 3\,a\,b^2\,\nu^4\,t^4 + 15\,a\,b\,\nu^3\,t^6 + 2\,b^4\,\nu^5\,t^2 + 8\,b\,\nu^5\,t^5 + 2\,a\,b^2\,\nu^3\,t^4 + 10\,a\,b\,\nu^2\,t^6 + 11\,b\,\nu^4\,t^5 + a\,b^3\,\nu^3\,t^2 + a\,b^2\,\nu^2\,t^4 + 3\,b^2\,\nu^4\,t^3 + 11\,b\,\nu^3\,t^5 \qquad (1.2.6)$$

$+ a^2 b \nu t^4 + 2 b^2 \nu^3 t^3 + 9 b \nu^2 t^5 + 2 a b \nu^2 t^3 + b^3 \nu^3 t + b^2 \nu^2 t^3 + 4 a b \nu t^3$
$+ a b^2 \nu t + 2 b \nu^2 t^2 + 3 \nu b t^2 + a t^2 + \nu b^2 - a b + t))$

> $subs\left(t = 0, simplify\left((a t + 1) b \cdot \frac{\mathbf{(1.2.6)}}{t}\right)\right);$

$$\nu b^2 - a b \qquad \mathbf{(1.2.7)}$$

> $n := 6 : pol := op\left(2, numer\left(factor\left(subs\left(Zx = ZxSer(n), Zy = subs(x = y, ZxSer(n)), x = t + a \cdot t^2, y = \frac{b}{t}, K\right)\right)\right)\right);$

$pol := 2 a^4 b \nu^5 t^{12} + 10 a^3 b \nu^6 t^{11} + 14 a^3 b \nu^5 t^{11} + 3 a^3 b \nu^4 t^{11} + 30 a^2 b \nu^6 t^{10}$ **(1.2.8)**
$+ a^3 b \nu^3 t^{11} + 30 a^2 b \nu^5 t^{10} + 9 a^2 b \nu^4 t^{10} + 30 a b \nu^6 t^9 + 3 a^2 b \nu^3 t^{10}$
$+ 26 a b \nu^5 t^9 + a^3 b \nu^3 t^8 + 10 a b^3 \nu^6 t^5 + 9 a b \nu^4 t^9 + 10 b \nu^6 t^8 + 3 a^2 b \nu^4 t^7$
$+ 6 a b^3 \nu^5 t^5 + 3 a b \nu^3 t^9 + 8 b \nu^5 t^8 + 5 a^2 b \nu^3 t^7 + 2 a b^4 \nu^5 t^3 + 3 a b^3 \nu^4 t^5$
$+ 8 a b \nu^5 t^6 + 10 b^3 \nu^6 t^4 + 3 b \nu^4 t^8 + a^2 b \nu^2 t^7 + a b^3 \nu^3 t^5 + 14 a b \nu^4 t^6$
$+ 6 b^3 \nu^5 t^4 + b \nu^3 t^8 + 3 a b^2 \nu^4 t^4 + 15 a b \nu^3 t^6 + 2 b^4 \nu^5 t^2 + 3 b^3 \nu^4 t^4$
$+ 8 b \nu^5 t^5 + 2 a b^2 \nu^3 t^4 + 10 a b \nu^2 t^6 + b^3 \nu^3 t^4 + 11 b \nu^4 t^5 + a b^3 \nu^3 t^2$
$+ a b^2 \nu^2 t^4 + 3 b^2 \nu^4 t^3 + 11 b \nu^3 t^5 + a^2 b \nu t^4 + 2 b^2 \nu^3 t^3 + 9 b \nu^2 t^5$
$+ 2 a b \nu^2 t^3 + b^3 \nu^3 t + b^2 \nu^2 t^3 + 4 a b \nu t^3 + a b^2 \nu t + 2 b \nu^2 t^2 + 3 \nu b t^2$
$+ a t^2 + \nu b^2 - a b + t$

> $SerY := algeqtoseries(pol, t, b, n-5, true);$

$$SerY := \left[\frac{a}{\nu} + O(t), \frac{1}{a} t + O(t^2)\right] \qquad \mathbf{(1.2.9)}$$

> $Y1Ser := op(1, SerY); Y2Ser := op(2, SerY);$

$$Y1Ser := \frac{a}{\nu} + O(t)$$
$$Y2Ser := \frac{1}{a} t + O(t^2) \qquad \mathbf{(1.2.10)}$$

We found our two series Y1 and Y2 !

> $series\left(subs\left(x = t + a \cdot t^2, y = \frac{Y}{t}, Zx = x \cdot t^3 \cdot ZZx, Zy = \frac{\nu \cdot Y^2}{t} + \nu^3 \cdot Y^3 + t \cdot Y \cdot \nu \cdot (1 + \nu) + t^2 \cdot Y^2 \cdot ZZZy, Y \cdot K\right), t, 4\right);$

$$Y (Y \nu - a) t + Y \left(Y^2 \nu^3 + a^2 + \frac{1}{Y}\right) t^2 + O(t^3) \qquad \mathbf{(1.2.11)}$$

Invariants

Find I(Y_1) = I(Y_2).

Notation: Zy1 = subs(y=Y_1/t, Zy), same for Zy2 and Z1y1=subs(y=Y_1/t,Z1y), same for Z1y2.

We have 4 equations:

$$> e1 := simplify\left(subs\left(y = \frac{Y[1]}{t}, Zy = Zy1, K\right)\right);$$
$$e2 := simplify\left(subs\left(y = \frac{Y[2]}{t}, Zy = Zy2, K\right)\right);$$
$$e3 := subs\left(y = \frac{Y[1]}{t}, Z1y = Z1y1, R\right);$$
$$e4 := subs\left(y = \frac{Y[2]}{t}, Z1y = Z1y2, R\right);$$

$$e1 := \frac{((Zx+1)\,t - x)\,Y_1 + t^2\,x\,(Zy1+1)}{x\,Y_1}$$
$$e2 := \frac{((Zx+1)\,t - x)\,Y_2 + t^2\,x\,(Zy2+1)}{x\,Y_2}$$
$$e3 := -Z1x\,t - Z1y1\,t + x\,Y_1$$
$$e4 := -Z1x\,t - Z1y2\,t + x\,Y_2 \quad \textbf{(1.3.1)}$$

First Invariant

We can easily eliminate Zx between e1 and e2,   x is also eliminated !

$$> normal(e1 - e2);$$
$$\frac{t^2\,(Zy1\,Y_2 - Zy2\,Y_1 - Y_1 + Y_2)}{Y_1\,Y_2} \quad \textbf{(1.3.1.1)}$$

$$> op(2, numer(normal(e1 - e2)));$$
$$Zy1\,Y_2 - Zy2\,Y_1 - Y_1 + Y_2 \quad \textbf{(1.3.1.2)}$$

We have our first invariant ! Inv1 = (Zy(Y/t)+1)/Y given in Equation (16) of the article :

$$> Inv1 := \frac{Zy\left(\frac{y}{t}\right) + 1}{y};$$
$$Inv1 := \frac{Zy\left(\frac{y}{t}\right) + 1}{y} \quad \textbf{(1.3.1.3)}$$

Series of this invariant (needed later to check the second invariant we are going to find).

> $InvISer := simplify\left(series\left(subs\left(x = \frac{Y1Ser}{t}, \frac{ZxSer(10)+1}{Y1Ser}\right), t, 10\right)\right);$

$$InvISer := a\, t^{-1} + O\left(t^{0}\right) \qquad \textbf{(1.3.1.4)}$$

Second Invariant

We start by solving e3, e4 for x and Zpm1x :

> $solve(\{e3, e4\}, \{x, Z1x\});$

$$\left\{Z1x = \frac{Z1y1\; Y_2 - Z1y2\; Y_1}{Y_1 - Y_2}, x = \frac{t\,(Z1y1 - Z1y2)}{Y_1 - Y_2}\right\} \qquad \textbf{(1.3.2.1)}$$

> $xSol := \frac{t\cdot(-Z1y2 + Z1y1)}{-Y_2 + Y_1}; Z1xSol := \frac{-Y_1\cdot Z1y2 + Y_2\cdot Z1y1}{-Y_2 + Y_1};$

$$xSol := \frac{t\,(Z1y1 - Z1y2)}{Y_1 - Y_2}$$

$$Z1xSol := \frac{Z1y1\; Y_2 - Z1y2\; Y_1}{Y_1 - Y_2} \qquad \textbf{(1.3.2.2)}$$

We want everything expressed in terms of Yi and Inv1:

Expressions of Zyi in terms of Inv1:

> $Zy1I := Y_1\cdot Inv - 1; Zy2I := Y_2\cdot Inv - 1;$

$$Zy1I := Y_1\, Inv - 1$$

$$Zy2I := Y_2\, Inv - 1 \qquad \textbf{(1.3.2.3)}$$

Expressions of Z1yi in terms of Inv1 :

> $solve(eqZx, Z1x)$

$$\frac{-\nu\, t\, x^3 + Z1\, \nu\, t\, x - \nu\, t\, Zx^2 - Zx\, \nu\, t + Zx\, x}{x\, \nu\, t} \qquad \textbf{(1.3.2.4)}$$

> $Z1y1I := collect\left(simplify\left(subs\left(Zx = Zy1I, x = \frac{Y_1}{t}, solve(eqZx, Z1x)\right)\right), Inv\right);$

$Z1y2I := collect\left(simplify\left(subs\left(Zx = Zy2I, x = \frac{Y_2}{t}, solve(eqZx, Z1x)\right)\right), Inv\right);$

$$Z1y1I := -Y_1\, t\, Inv^2 + \frac{\left(\nu\, t^3 + Y_1\, t\right) Inv}{t^2\, \nu} + \frac{Z1\, \nu\, t^2 - \nu\, Y_1^2 - t}{t^2\, \nu}$$

$$Z1y2I := -Y_2\, t\, Inv^2 + \frac{\left(\nu\, t^3 + Y_2\, t\right) Inv}{t^2\, \nu} + \frac{Z1\, \nu\, t^2 - \nu\, Y_2^2 - t}{t^2\, \nu} \qquad \textbf{(1.3.2.5)}$$

Expression of Z1x in terms of Inv: (Equation (18) in the paper)

> $Z1xI := collect(simplify(subs(Z1y1 = Z1y1I, Z1y2 = Z1y2I, Z1xSol)), Inv);$

$$Z1xI := -t\,Inv + \frac{\left(-Z1\,t^2 - Y_1\,Y_2\right)\nu + t}{t^2\,\nu} \qquad \textbf{(1.3.2.6)}$$

Expression of x in terms of Inv (Equation (17) of the paper)

> $xI := collect(simplify(subs(Z1y1 = Z1y1I, Z1y2 = Z1y2I, xSol)), Inv);$

$$xI := -t^2\,Inv^2 + \frac{Inv}{\nu} + \frac{-Y_1 - Y_2}{t} \qquad \textbf{(1.3.2.7)}$$

> $solve(e1, Zx)$

$$-\frac{t^2\,x\,Zy1 + t^2\,x + t\,Y_1 - x\,Y_1}{Y_1\,t} \qquad \textbf{(1.3.2.8)}$$

Expression of Zx in terms of Inv1 (Equation (19) in the paper)

> $ZxI := collect(simplify(subs(x = xI, Zy1 = Zy1I, solve(e1, Zx))), Inv);$

$$ZxI := t^3\,Inv^3 + \frac{\left(-t^3\,\nu - t^3\right)Inv^2}{t^2\,\nu} + \frac{\left(\left(Y_1 + Y_2\right)t^2\,\nu + t\right)Inv}{t^2\,\nu} + \frac{-t^2 - Y_1 - Y_2}{t^2} \qquad \textbf{(1.3.2.9)}$$

Put everything in the equation of Zx to get an equation between t, Z1, Y1, Y2 and Inv (this is Equation (20) of the article).

> $eqI2 := collect\left(numer(simplify(subs(Zx = ZxI, Z1x = Z1xI, x = xI, eqZx))), \{Y_1, Y_2\}, distributed\right);$

$$eqI2 := \nu^2\,t\,Y_1^2 + \nu^2\,t\,Y_1\,Y_2 + \left(Inv^2\,\nu^2\,t^4 + 2\,Inv\,\nu^2\,t^2 - 3\,Inv\,\nu\,t^2 - \nu^2 + \nu\right)Y_1 + \nu^2\,t\,Y_2^2 + \left(Inv^2\,\nu^2\,t^4 + 2\,Inv\,\nu^2\,t^2 - 3\,Inv\,\nu\,t^2 - \nu^2 + \nu\right)Y_2 + 2\,Inv^3\,\nu^2\,t^5 - 2\,Inv^3\,\nu\,t^5 - Inv^2\,\nu^2\,t^3 - Inv^2\,\nu\,t^3 - 2\,Z1\,\nu^2\,t^3 + 2\,Inv^2\,t^3 - t^2\,\nu^2 + Inv\,\nu\,t + 2\,t^2\,\nu - t\,Inv \qquad \textbf{(1.3.2.10)}$$

We want no linear terms in Yi:

> $eqI2X := collect\left(subs\left(Y_1 = X1 + b, Y_2 = X2 + b, eqI2\right), [X1, X2], factor\right);$

$$eqI2X := \nu^2\,t\,X1^2 + \left(\nu^2\,t\,X2 + \nu\left(Inv^2\,\nu\,t^4 + 2\,Inv\,\nu\,t^2 - 3\,Inv\,t^2 + 3\,b\,\nu\,t - \nu + 1\right)\right)X1 + \nu^2\,t\,X2^2 + \nu\left(Inv^2\,\nu\,t^4 + 2\,Inv\,\nu\,t^2 - 3\,Inv\,t^2 + 3\,b\,\nu\,t - \nu + 1\right)X2 + 2\,Inv^3\,\nu^2\,t^5 - 2\,Inv^3\,\nu\,t^5 + 2\,Inv^2\,b\,\nu^2\,t^4 - Inv^2\,\nu^2\,t^3 - Inv^2\,\nu\,t^3 + 4\,Inv\,b\,\nu^2\,t^2 - 2\,Z1\,\nu^2\,t^3 + 2\,Inv^2\,t^3 - 6\,Inv\,b\,\nu\,t^2 + 3\,\nu^2\,t\,b^2 - t^2\,\nu^2 + Inv\,\nu\,t - 2\,b\,\nu^2 + 2\,t^2\,\nu - t\,Inv + 2\,b\,\nu \qquad \textbf{(1.3.2.11)}$$

> $bsol := solve(coeff(subs(X2=0, eqI2X), X1, 1), b);$

$$bsol := -\frac{Inv^2\,\nu\,t^4 + 2\,Inv\,\nu\,t^2 - 3\,Inv\,t^2 - \nu + 1}{3\,\nu\,t} \quad (1.3.2.12)$$

> $eqI2X2 := collect\left(\frac{subs(Y_1 = X1 + bsol, Y_2 = X2 + bsol, eqI2)}{\nu^2 \cdot t}, [X1, X2], factor\right);$

$$eqI2X2 := X1^2 + X2\,X1 + X2^2 - \frac{1}{3\,\nu^2\,t^2}\left(Inv^4\,\nu^2\,t^8 - 2\,Inv^3\,\nu^2\,t^6 + 5\,Inv^2\,\nu^2\,t^4 - 7\,Inv^2\,\nu\,t^4 + 6\,Z1\,\nu^2\,t^4 + 3\,Inv^2\,t^4 - 4\,Inv\,\nu^2\,t^2 + 3\,\nu^2\,t^3 + 7\,Inv\,\nu\,t^2 - 6\,t^3\,\nu - 3\,Inv\,t^2 + \nu^2 - 2\,\nu + 1\right) \quad (1.3.2.13)$$

Change this in an expression of the form u^2 + v^2 +uv - 1:

> $Ju := u^3 - u; Jv := v^3 - v;$

$$Ju := u^3 - u$$
$$Jv := v^3 - v \quad (1.3.2.14)$$

> $simplify\left(\frac{(Ju - Jv)}{u - v}\right);$

$$u^2 + v\,u + v^2 - 1 \quad (1.3.2.15)$$

> $op(4, eqI2X2);$

$$-\frac{1}{3\,\nu^2\,t^2}\left(Inv^4\,\nu^2\,t^8 - 2\,Inv^3\,\nu^2\,t^6 + 5\,Inv^2\,\nu^2\,t^4 - 7\,Inv^2\,\nu\,t^4 + 6\,Z1\,\nu^2\,t^4 + 3\,Inv^2\,t^4 - 4\,Inv\,\nu^2\,t^2 + 3\,\nu^2\,t^3 + 7\,Inv\,\nu\,t^2 - 6\,\nu\,t^3 - 3\,Inv\,t^2 + \nu^2 - 2\,\nu + 1\right) \quad (1.3.2.16)$$

> $CC := collect(-op(4, eqI2X2), [Inv, t], factor);$

$$CC := \frac{t^6\,Inv^4}{3} - \frac{2\,t^4\,Inv^3}{3} + \frac{\left(5\,\nu^2 - 7\,\nu + 3\right)t^2\,Inv^2}{3\,\nu^2} - \frac{(4\,\nu - 3)\,(\nu - 1)\,Inv}{3\,\nu^2} + 2\,Z1\,t^2 + \frac{(\nu - 2)\,t}{\nu} + \frac{(\nu - 1)^2}{3\,\nu^2\,t^2} \quad (1.3.2.17)$$

> $U1 := \frac{X1}{\mathrm{sqrt}(CC)} : U2 := \frac{X2}{\mathrm{sqrt}(CC)} :$

Second invariant : J(U1) = J(U2), that we multiply by CC^{3/2} to get a polynom.

> $Inv2 := \left(CC^{\frac{3}{2}} \cdot \frac{y - bsol}{\mathrm{sqrt}(CC)} \cdot \left(\left(\frac{y - bsol}{\mathrm{sqrt}(CC)}\right)^2 - 1\right)\right) :$

> $Inv2 := map\left(simplify, collect\left(simplify\left((y - bsol) \cdot \left((y - bsol)^2 - CC\right)\right), [Inv, y],\right.\right.$

distributed));

$$Inv2 := -\frac{2\,t^9\,Inv^6}{27} + \frac{2\,t^7\,Inv^5}{9} + \frac{t^5\,(3\,\nu^2 - 11\,\nu + 6)\,Inv^4}{9\,\nu^2} + \frac{2\,t^4\,(\nu - 1)\,Inv^3\,y}{\nu} - \frac{2\,t^3\,(14\,\nu^2 - 33\,\nu + 18)\,Inv^3}{27\,\nu^2} + t^3\,Inv^2\,y^2 - \frac{t^2\,(\nu^2 + \nu - 2)\,Inv^2\,y}{\nu^2} - \frac{t\,(6\,t^4\,\nu^3\,Z1 + 3\,\nu^3\,t^3 - 6\,\nu^2\,t^3 - 9\,\nu^3 + 22\,\nu^2 - 16\,\nu + 3)\,Inv^2}{9\,\nu^3} + \frac{t\,(2\,\nu - 3)\,Inv\,y^2}{\nu} + \frac{(\nu - 1)\,y\,Inv}{\nu^2} - \frac{1}{9\,t\,\nu^3}\Big((12\,t^4\,\nu^3\,Z1 - 18\,Z1\,\nu^2\,t^4 + 6\,\nu^3\,t^3 - 21\,\nu^2\,t^3 + 18\,\nu\,t^3 + 4\,\nu^3 - 11\,\nu^2 + 10\,\nu - 3)\,Inv\Big) + y^3 - \frac{(\nu - 1)\,y^2}{\nu\,t} - \frac{t\,(2\,\nu\,t\,Z1 + \nu - 2)\,y}{\nu} + \frac{(\nu - 1)\,(18\,Z1\,\nu^2\,t^4 + 9\,\nu^2\,t^3 - 18\,\nu\,t^3 + 2\,\nu^2 - 4\,\nu + 2)}{27\,\nu^3\,t^3} \quad \textbf{(1.3.2.18)}$$

This is also an invariant, it will give the same equation. We obtain Equation (22) of the paper.

> *Inv3* := *collect*(*Inv2* − *subs*(*y* = 0, *Inv2*), *y*);

$$Inv3 := y^3 + \left(Inv^2\,t^3 + \frac{t\,(2\,\nu - 3)\,Inv}{\nu} - \frac{\nu - 1}{\nu\,t}\right)y^2 + \left(\frac{2\,t^4\,(\nu - 1)\,Inv^3}{\nu} - \frac{t^2\,(\nu^2 + \nu - 2)\,Inv^2}{\nu^2} + \frac{(\nu - 1)\,Inv}{\nu^2} - \frac{t\,(2\,\nu\,t\,Z1 + \nu - 2)}{\nu}\right)y \quad \textbf{(1.3.2.19)}$$

Just to check:

> *map*(*simplify*, *series*(*subs*(*Inv* = *Inv1Ser*, *Inv3*), *t*, 5));

$$\frac{(\nu - 1)\,y\,(-y\,\nu + a)}{\nu^2}\,t^{-1} + O(t^0) \quad \textbf{(1.3.2.20)}$$

> *map*(*simplify*, *series*(*subs*(*y* = *Y1Ser*, *subs*(*Inv* = *Inv1Ser*, *Inv3*)) − *subs*(*y* = *Y2Ser*, *subs*(*Inv* = *Inv1Ser*, *Inv3*)), *t*, 5));

$$O(t^0) \quad \textbf{(1.3.2.21)}$$

New equation for Z^+ with one catalytic variable

Valuation of the pole of Inv3 at y=0:

> *series*(*subs*(*Inv* = *Inv1*, *Inv3*), *y*, 2);

$$\frac{2\,t^4\,(\nu-1)\,(Zy(0)+1)^3}{\nu}\,y^{-2}+\left(\frac{6\,t^3\,(\nu-1)\,(Zy(0)+1)^2\,\mathrm{D}(Zy)(0)}{\nu}-\frac{t^2\,(\nu^2+\nu-2)\,(Zy(0)+1)^2}{\nu^2}\right)y^{-1}+\mathrm{O}(y^0) \quad \textbf{(1.3.3.1)}$$

The invariant equation is then given by :

```
> eqinv := subs(Inv = Inv1, Inv3) - add(c[i]*(Inv1)^i, i = 0..2);
```

$$eqinv := y^3+\left(\frac{\left(Zy\left(\frac{y}{t}\right)+1\right)^2 t^3}{y^2}+\frac{t\,(2\,\nu-3)\left(Zy\left(\frac{y}{t}\right)+1\right)}{\nu\,y}-\frac{\nu-1}{\nu\,t}\right)y^2+\left(\frac{2\,t^4\,(\nu-1)\left(Zy\left(\frac{y}{t}\right)+1\right)^3}{\nu\,y^3}-\frac{t^2\,(\nu^2+\nu-2)\left(Zy\left(\frac{y}{t}\right)+1\right)^2}{\nu^2\,y^2}+\frac{(\nu-1)\left(Zy\left(\frac{y}{t}\right)+1\right)}{\nu^2\,y}-\frac{t\,(2\,Z1\,\nu\,t+\nu-2)}{\nu}\right)y-c_0-\frac{c_1\left(Zy\left(\frac{y}{t}\right)+1\right)}{y}-\frac{c_2\left(Zy\left(\frac{y}{t}\right)+1\right)^2}{y^2} \quad \textbf{(1.3.3.2)}$$

```
> factor(series(normal(eqinv), y, 1));
```

$$\frac{(Zy(0)+1)^2\,\left(2\,Zy(0)\,\nu\,t^4-2\,Zy(0)\,t^4+2\,t^4\,\nu-2\,t^4-\nu\,c_2\right)}{\nu}\,y^{-2}+\mathrm{O}(y^{-1}) \quad \textbf{(1.3.3.3)}$$

```
> c2sol := subs(Zy(0) = 0, factor(solve(coeff(%, y, -2), c[2])));
```

$$c2sol := \frac{2\,t^4\,(\nu-1)}{\nu} \quad \textbf{(1.3.3.4)}$$

```
> factor(series(subs(c[2] = c2sol, normal(eqinv)), y, 2));
```

$$\frac{2\,t^4\,Zy(0)\,(Zy(0)+1)^2\,(\nu-1)}{\nu}\,y^{-2}+\frac{1}{\nu^2}\Big((Zy(0)+1)\,\big(6\,Zy(0)\,\mathrm{D}(Zy)(0)\,t^3\,\nu^2-6\,Zy(0)\,\mathrm{D}(Zy)(0)\,t^3\,\nu+2\,\mathrm{D}(Zy)(0)\,t^3\,\nu^2-Zy(0)\,\nu^2\,t^2-2\,\mathrm{D}(Zy)(0)\,t^3\,\nu-Zy(0)\,\nu\,t^2-t^2\,\nu^2+2\,Zy(0)\,t^2-c_1\,\nu^2-t^2\,\nu+2\,t^2\big)\Big)\,y^{-1}+\mathrm{O}(y^0) \quad \textbf{(1.3.3.5)}$$

```
> c1sol := subs(Zy(0) = 0, D(Zy)(0) = Z1, factor(solve(coeff(%, y, -1), c[1])));
```

$$c1sol := \frac{t^2 (\nu - 1)(2\, Z1\, \nu\, t - \nu - 2)}{\nu^2} \quad \textbf{(1.3.3.6)}$$

```
> factor(series(subs(c[2] = c2sol, c[1] = c1sol, normal(eqinv)), y, 3));
```

$$\frac{2 t^4 Zy(0) (Zy(0) + 1)^2 (\nu - 1)}{\nu} y^{-2} + \frac{1}{\nu^2}\big(t^2 (\nu - 1)(Zy(0) + 1)(6\, Zy(0)\, D(Zy)(0)\, \nu\, t + 2\, D(Zy)(0)\, t\, \nu - 2\, Z1\, \nu\, t - Zy(0)\, \nu - 2\, Zy(0))\big) y^{-1} + \frac{1}{\nu^2}\Big(-1 - 2\, Zy(0)\, D(Zy)(0)\, \nu^2\, t - 2\, Zy(0)\, D(Zy)(0)\, \nu\, t + \nu + Zy(0)\, \nu - D(Zy)(0)\, t\, \nu^2 - D(Zy)(0)\, t\, \nu - Zy(0) + 2\, t\, D(Zy)(0) + D^{(2)}(Zy)(0)\, t^2 \nu^2 + 3\, Zy(0)^2 D^{(2)}(Zy)(0)\, \nu^2 t^2 + 6\, Zy(0)\, D(Zy)(0)^2 \nu^2 t^2 - 3\, Zy(0)^2 D^{(2)}(Zy)(0)\, \nu\, t^2 - 6\, Zy(0)\, D(Zy)(0)^2 \nu\, t^2 + 4\, Zy(0)\, D^{(2)}(Zy)(0)\, \nu^2 t^2 - 2\, D(Zy)(0)\, Z1\, \nu^2 t^2 - 4\, Zy(0)\, D^{(2)}(Zy)(0)\, \nu\, t^2 + 2\, D(Zy)(0)\, Z1\, \nu\, t^2 + Zy(0)^2 \nu^2 t^3 + 2\, Zy(0)\, \nu^2 t^3 + \nu^2 t^3 - c_0 \nu^2 + 4\, D(Zy)(0)^2 \nu^2 t^2 - 4\, D(Zy)(0)^2 \nu\, t^2 - D^{(2)}(Zy)(0)\, t^2 \nu + 4\, Zy(0)\, D(Zy)(0)\, t\Big) + O(y) \quad \textbf{(1.3.3.7)}$$

```
> c0sol := map(factor, collect(subs(Zy(0) = 0, D(Zy)(0) = Z1, D^(2)(Zy)(0) = 2·Z2,
      (solve(coeff(%, y, 0), c[0]))), {Z2, Z1}));
```

$$c0sol := \frac{2 t^2 (\nu - 1) Z1^2}{\nu} - \frac{t (\nu + 2)(\nu - 1) Z1}{\nu^2} + \frac{2 t^2 (\nu - 1) Z2}{\nu} + \frac{\nu^2 t^3 + \nu - 1}{\nu^2} \quad \textbf{(1.3.3.8)}$$

The Invariant equation:

```
> neweqInv := collect(simplify(Inv3 − c0sol − c1sol·Inv - c2sol·Inv^2), Inv, factor);
```

$$neweqInv := \frac{2 t^4 (\nu - 1) Inv^3 y}{\nu} - \frac{t^2 (-\nu^2 t y^2 + 2 t^2 \nu^2 + y \nu^2 - 2 t^2 \nu + y \nu - 2 y) Inv^2}{\nu^2} - \frac{1}{\nu^2}\big((2\, Z1\, \nu^2 t^3 - 2\, Z1\, \nu\, t^3 - 2 \nu^2 t y^2 - t^2 \nu^2 + 3 y^2 \nu\, t - t^2 \nu - y \nu + 2 t^2 + y) Inv\big) - \frac{1}{\nu^2 t}\big(2 t^3 Z1^2 \nu^2 + 2\, Z1\, \nu^2 t^3 y - 2 t^3 Z1^2 \nu + 2 t^3 Z2\, \nu^2 + \nu^2 t^4 - y^3 \nu^2 t - Z1\, \nu^2 t^2 - 2 t^3 Z2\, \nu + \nu^2 t^2 y - Z1\, \nu\, t^2 + \nu^2 y^2 - 2 \nu\, t^2 y + 2\, Z1\, t^2 - y^2 \nu + \nu\, t - t\big) \quad \textbf{(1.3.3.9)}$$

We can retrieve the original value of J (=Inv3) given in (22) of the paper :

> collect(simplify(subs(Zy(0) = 0, D(Zy)(0) = Z1, D^(2)(Zy)(0) = 2 · Z2, neweqInv − subs(y = 0 , neweqInv))), Inv, factor)

$$\frac{2\, t^4\, (\nu-1)\, Inv^3\, y}{\nu} + \frac{y\, t^2\, (\nu^2\, t\, y - \nu^2 - \nu + 2)\, Inv^2}{\nu^2} + \frac{y\, (2\, \nu^2\, t\, y - 3\, \nu\, t\, y + \nu - 1)\, Inv}{\nu^2} - \frac{y\, (2\, Z1\, \nu\, t^3 - y^2\, \nu\, t + t^2\, \nu + y\, \nu - 2\, t^2 - y)}{\nu\, t} \quad \textbf{(1.3.3.10)}$$

> neweqZy := collect(simplify(subs(Inv = Inv1, neweqInv)), Zy(y));

$$neweqZy := \frac{1}{t\, y^2\, \nu^2} \Bigg(2\, t^5\, \nu\, (\nu-1)\, Zy\left(\frac{y}{t}\right)^3 + 4\, t^3 \left((\nu^2-\nu)\, t^2 + \frac{\nu^2\, t\, y^2}{4} - \frac{y\, (\nu+2)\, (\nu-1)}{4} \right) Zy\left(\frac{y}{t}\right)^2 + 2 \left((\nu^2-\nu)\, t^4 + y\, ((y - Z1)\, \nu + Z1)\, \nu\, t^3 - \frac{y\, (\nu+2)\, (\nu-1)\, t^2}{2} + y^3\, \nu \left(\nu - \frac{3}{2}\right) t + \frac{y^2\, (\nu-1)}{2} \right) t\, Zy\left(\frac{y}{t}\right) - 2 \left(Z1\, \nu\, (\nu-1)\, t^4 + y\, ((Z1^2 + Z1\, y + Z2)\, \nu - Z1^2 - Z2)\, \nu\, t^3 - \frac{y\, ((y + Z1)\, \nu + 2\, Z1)\, (\nu-1)\, t^2}{2} - \frac{t\, y^4\, \nu^2}{2} + \frac{y^3\, \nu\, (\nu-1)}{2} \right) y \Bigg) \quad \textbf{(1.3.3.11)}$$

Checking the new equation:

> neweqZy;

$$\frac{1}{t\, y^2\, \nu^2} \Bigg(2\, t^5\, \nu\, (\nu-1)\, Zy\left(\frac{y}{t}\right)^3 + 4\, t^3 \left((\nu^2-\nu)\, t^2 + \frac{\nu^2\, t\, y^2}{4} - \frac{y\, (\nu+2)\, (\nu-1)}{4} \right) Zy\left(\frac{y}{t}\right)^2 + 2 \left((\nu^2-\nu)\, t^4 + y\, ((y - Z1)\, \nu + Z1)\, \nu\, t^3 - \frac{y\, (\nu+2)\, (\nu-1)\, t^2}{2} + y^3\, \nu \left(\nu - \frac{3}{2}\right) t + \frac{y^2\, (\nu-1)}{2} \right) t\, Zy\left(\frac{y}{t}\right) - 2 \left(Z1\, \nu\, (\nu-1)\, t^4 + y\, ((Z1^2 + Z1\, y + Z2)\, \nu - Z1^2 - Z2)\, \nu\, t^3 - \frac{y\, ((y + Z1)\, \nu + 2\, Z1)\, (\nu-1)\, t^2}{2} - \frac{t\, y^4\, \nu^2}{2} + \frac{y^3\, \nu\, (\nu-1)}{2} \right) y \Bigg) \quad \textbf{(1.3.4.1)}$$

> catZ2 := simplify(subs(y = t·y, neweqZy));

(1.3.4.2)

$$catZ2 := \frac{1}{\nu^2 y^2}\Big(2 t^2 \nu (\nu-1) Zy(y)^3 + \big(t^2 y^2 \nu^2 + (4\nu^2 - 4\nu) t - y (\nu + 2) (\nu-1)\big) t\, Zy(y)^2 + \Big(2\nu^2 t^3 y^2 + 2\Big(\big(y^3 - Z1\, y + 1\big)\nu - \frac{3 y^3}{2} + Z1\, y - 1\Big)\nu t^2 - t y (\nu+2)(\nu-1) + y^2 (\nu-1)\Big) Zy(y) + y t \big(\nu^2 y^2 (y^2 - 2 Z1) t^2 + \big(y^2 + (-2 Z1^2 - 2 Z2) y - 2 Z1\big)\nu (\nu-1) t - \big((y^2 - Z1)\nu - 2 Z1\big) y (\nu-1)\big)\Big) \quad \textbf{(1.3.4.2)}$$

And we finally obtain the equation given in (24) of the paper :

> $catZ3 := \nu^2 \cdot y^2 \cdot catZ2;$

$$catZ3 := 2 t^2 \nu (\nu-1) Zy(y)^3 + \big(t^2 y^2 \nu^2 + (4\nu^2 - 4\nu) t - y (\nu+2)(\nu-1)\big) t\, Zy(y)^2 + \Big(2\nu^2 t^3 y^2 + 2\Big(\big(y^3 - Z1\, y + 1\big)\nu - \frac{3y^3}{2} + Z1\, y - 1\Big)\nu t^2 - t y (\nu+2)(\nu-1) + y^2(\nu-1)\Big) Zy(y) + y t \big(\nu^2 y^2 (y^2 - 2 Z1) t^2 + \big(y^2 + (-2 Z1^2 - 2 Z2) y - 2 Z1\big)\nu(\nu-1) t - \big((y^2 - Z1)\nu - 2 Z1\big) y (\nu-1)\big) \quad \textbf{(1.3.4.3)}$$

> $indets(\%);$

$$\{Z1, Z2, \nu, t, y, Zy(y)\} \quad \textbf{(1.3.4.4)}$$

> $Eq := collect\left(\frac{subs(Zy(y) = y \cdot B, catZ3)}{y}, y\right);$

$$Eq := \nu^2 t^3 y^4 + \left(\nu^2 t^3 B^2 + 2\left(\nu - \frac{3}{2}\right)\nu t^2 B - (\nu-1)\nu t\right) y^3 + \big(2 t^2 \nu (\nu-1) B^3 - (\nu+2)(\nu-1) t B^2 + (2\nu^2 t^3 + \nu - 1) B + t(-2 Z1 \nu^2 t^2 + (\nu-1)\nu t)\big) y^2 + \big((4\nu^2 - 4\nu) t^2 B^2 + (2(-Z1\,\nu + Z1)\nu t^2 - (\nu+2)(\nu-1) t) B + t((-2 Z1^2 - 2 Z2)\nu(\nu-1) t - (-Z1\,\nu - 2 Z1)(\nu-1))\big) y + 2(\nu-1)\nu t^2 B - 2 t^2 Z1\, \nu (\nu-1) \quad \textbf{(1.3.4.5)}$$

> $-factor(coeff(Eq, y, 0));$

$$-2(\nu-1)\nu t^2 (B - Z1) \quad \textbf{(1.3.4.6)}$$

> $Pol := map\left(simplify, collect\left(\frac{Eq - coeff(Eq, y, 0)}{y}, [B, Z1, Z2], distributed\right)\right);$

$$Pol := 2(\nu-1)\nu t^2 y B^3 + t B^2 \big((t^2 y^2 + 4t - y)\nu^2 + (-4t - y)\nu + 2y\big) - 2(\nu-1)\nu t^2 B\, Z1 + 2\left(\left(t^3 y + t^2 y^2 - \frac{1}{2} t\right)\nu^2 + \left(-\frac{3}{2} t^2 y^2 - \frac{1}{2} t + \frac{1}{2} y\right)\nu + t - \frac{y}{2}\right) B - 2(\nu-1)\nu t^2 Z1^2 - t\big(2\nu^2 t^2 y - \nu^2 - \nu + 2\big) Z1 - 2 t^2 Z2\, \nu(\nu-1) + y\big((t^2 y^2 + t - y)\nu - t + y\big)\nu t \quad \textbf{(1.3.4.7)}$$

> $collect(catZ3, [Zy(y), Z1, Z2, y]);$

$$2 t^2 \nu (\nu-1) Zy(y)^3 + \big(\nu^2 t^3 y^2 - t y (\nu+2)(\nu-1) + (4\nu^2 \quad \textbf{(1.3.4.8)}$$

$$- 4\nu)\, t^2)\, Zy(y)^2 + \left(2\,(-\nu+1)\,\nu\, t^2\, y\, Z1 + 2\left(\nu - \frac{3}{2}\right)\nu\, t^2\, y^3 + \left(2\,\nu^2\, t^3 + \nu - 1\right) y^2 - t\, y\,(\nu+2)\,(\nu-1) + 2\,(\nu-1)\,\nu\, t^2\right) Zy(y) - 2\,(\nu - 1)\,\nu\, t^2\, y^2\, Z1^2 + \left(-2\,\nu^2\, t^3\, y^3 - t\,(-\nu-2)\,(\nu-1)\, y^2 - 2\,(\nu - 1)\,\nu\, t^2\, y\right) Z1 - 2\,(\nu-1)\,\nu\, t^2\, y^2\, Z2 + \nu^2\, t^3\, y^5 - (\nu-1)\,\nu\, t\, y^4 + (\nu - 1)\,\nu\, t^2\, y^3$$

> $neweqZy2 := simplify\left(\frac{subs(Zy(y) = Z, catZ3)}{y^2\cdot(1-\nu)} + Z\right);$

$$neweqZy2 := -\frac{1}{y^2\,(\nu-1)}\Bigg(2\,t\Bigg(\Bigg(\left(\frac{t^2\, y^5}{2} - \frac{y^4}{2} + t\left(-t\, Z1 + Z + \frac{1}{2}\right) y^3 + \left(\left(\frac{1}{2}\, Z^2 + Z\right) t^2 + \left(-Z1^2 - Z2\right) t + \frac{Z1}{2}\right) y^2 - \frac{(Z+1)\,(2\, t\, Z1 + Z)\, y}{2} + Z\, t\,(Z+1)^2\right)\nu^2 + \left(\frac{y^4}{2} - \frac{3\, t\left(Z + \frac{1}{3}\right) y^3}{2} + \left(\left(Z1^2 + Z2\right) t + \frac{Z1}{2}\right) y^2 - \frac{(Z+1)\,(-2\, t\, Z1 + Z)\, y}{2} - Z\, t\,(Z+1)^2\right)\nu + y\,(Z^2 - Z1\, y + Z)\Bigg)\Bigg)\Bigg) \quad (1.3.4.9)$$

```
> newZySer := proc(n) option remember :
  if n = 0 then 0 else
  convert(normal(series(subs(Z = newZySer(n − 1), Z1 = coeff(newZySer(n − 1), y,
     1), Z2 = coeff(newZySer(n − 1), y, 2), neweqZy2), t, n + 1)), polynom) : fi : end :
```

> $newZySer(5);$

$$y^2\,\nu\, t + (\nu+1)\, y\,\nu\, t^2 + y^3\,\nu^3\, t^3 + \left(3\,\nu^2 + 2\,\nu + 1\right) y^2\,\nu^2\, t^4 + 2\left(y^3\,\nu^3 + 2\,\nu^3 + 2\,\nu^2 + 2\,\nu + 2\right) y\,\nu^2\, t^5 \quad (1.3.4.10)$$

> $simplify(newZySer(10) - subs(x = y, ZxSer(10)));$

$$0 \quad (1.3.4.11)$$

Solving the equation with one catalytic variable

The series Z1 and Z2

From Bernardi and Bousquet-Melou:

$munu := \mu = \frac{\nu+1}{\nu-1};$

$$munu := \mu = \frac{\nu+1}{\nu-1} \quad (1.4.1.1)$$

> $numu := isolate(munu, \nu);$

$$numu := \nu = \frac{\mu + 1}{-1 + \mu} \quad \mathbf{(1.4.1.2)}$$

This is (99) of BBM :

> wS := 1/64*(-2*μ-2*S+4*S^3-S^2+μ^2)*(-S+μ)*(S-2+μ)/S^2/(1+μ)^3;

$$wS := \frac{\left(4 S^3 - S^2 + \mu^2 - 2 S - 2 \mu\right) \left(-S + \mu\right) \left(S - 2 + \mu\right)}{64 S^2 \left(\mu + 1\right)^3} \quad \mathbf{(1.4.1.3)}$$

> $wU := factor(simplify(subs(munu, S = \mu\cdot(1 - 2\cdot U), wS)));$

$$wU := \frac{1}{512 \nu^3 (-1 + 2 U)^2 \mu^2 (\nu - 1)} \Big((2 U \mu \nu - 2 U \mu - \mu \nu + \mu + \nu - 3) (2 U \mu \nu - 2 U \mu - \mu \nu + \mu + \nu + 1) \big(32 U^3 \mu^3 \nu^2 - 64 U^3 \mu^3 \nu - 48 U^2 \mu^3 \nu^2 + 32 U^3 \mu^3 + 96 U^2 \mu^3 \nu + 4 U^2 \mu^2 \nu^2 + 24 U \mu^3 \nu^2 - 48 U^2 \mu^3 - 8 U^2 \mu^2 \nu - 48 U \mu^3 \nu - 4 U \mu^2 \nu^2 - 4 \mu^3 \nu^2 + 4 U^2 \mu^2 + 24 U \mu^3 + 8 U \mu^2 \nu - 4 U \mu \nu^2 + 8 \mu^3 \nu + \mu^2 \nu^2 - 4 U \mu^2 + 8 U \mu \nu - 4 \mu^3 - 2 \mu^2 \nu + 2 \mu \nu^2 - 4 U \mu + \mu^2 - 4 \mu \nu + \nu^2 + 2 \mu - 2 \nu - 3\big)\Big) \quad \mathbf{(1.4.1.4)}$$

> $wUnu := factor(subs(munu, wU));$

$$wUnu := \frac{1}{32 (-1 + 2 U)^2 \nu^3} \Big((U \nu + U - 2) U \big(8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4 \nu\big)\Big) \quad \mathbf{(1.4.1.5)}$$

Parametrization w=t^3 = wS, then t*Z1 is Q1 of BBM:

> $Q1S := \frac{(S - \mu)\cdot(-2\cdot\mu + \mu^2 - S\cdot\mu - S^2\cdot\mu + 3\cdot S^3)}{2\cdot(1 + \mu)\cdot(2*\mu + 2*S - 4*S\text{^}3 + S\text{^}2 - \mu\text{^}2)};$

$$Q1S := \frac{(S - \mu)\left(3 S^3 - S^2 \mu - S \mu + \mu^2 - 2 \mu\right)}{2 (\mu + 1)\left(-4 S^3 + S^2 - \mu^2 + 2 S + 2 \mu\right)} \quad \mathbf{(1.4.1.6)}$$

> $Q1U := factor(subs(S = \mu\cdot(1 - 2\cdot U), Q1S));$

$$Q1U := \frac{\mu U \left(12 U^3 \mu^2 - 16 U^2 \mu^2 + 7 U \mu^2 - \mu U - \mu^2 + 1\right)}{2 (\mu + 1)\left(8 U^3 \mu^2 - 12 U^2 \mu^2 + U^2 \mu + 6 U \mu^2 - \mu U - \mu^2 - U + 1\right)} \quad \mathbf{(1.4.1.7)}$$

> $Q1Unu := factor(subs(munu, Q1U));$

$$Q1Unu := \left(\left(6 U^3 \nu^2 + 12 U^3 \nu - 8 U^2 \nu^2 + 6 U^3 - 16 U^2 \nu + 3 U \nu^2 - 8 U^2 + 7 U \nu + 4 U - 2 \nu\right) U (\nu + 1)\right) \Big/ \left(2 \left(8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4 \nu\right) \nu\right) \quad \mathbf{(1.4.1.8)}$$

Recall that in BBM, Q_i is the generating series of triangulations with a (non-necessarily simple) boundary of length i.

Proof of the base case of Proposition 14. We can express Z2 as a combination of Q_3 and

Q_1=Z_1.

$$> ZZ2 := \frac{2\,\nu}{1-\nu}\cdot t + \frac{\nu}{1-\nu}\cdot t\cdot Q3 - \frac{1}{(1-\nu)\cdot t}\cdot Z1 - Z1^2;$$

$$ZZ2 := \frac{2\,\nu\,t}{-\nu+1} + \frac{\nu\,t\,Q3}{-\nu+1} - \frac{Z1}{(-\nu+1)\,t} - Z1^2 \qquad \textbf{(1.4.1.9)}$$

From Theorem 23 in BBM:

$$> Q3S := \frac{1}{wS^2}\,\frac{1}{8192\cdot(1+\mu)^6 S^4}\big((S-\mu)^3\cdot(S-2+\mu)\cdot(-64\cdot S^6 + (232-128\cdot\mu)\cdot S^5 - (67+48\cdot\mu-64\cdot\mu^2)\cdot S^4 + (106-102\cdot\mu+40\cdot\mu^2)\cdot S^3 - 2\cdot(\mu-2)\cdot(32\cdot\mu^2-48\cdot\mu-1)\cdot S^2 + 2\cdot(3\cdot\mu-1)\cdot(\mu-2)^2\cdot S + 3\cdot\mu\cdot(\mu-2)^3)\big);$$

$$Q3S := \Big((S-\mu)^3\,\big(-64\,S^6 + (232-128\,\mu)\,S^5 - (-64\,\mu^2+48\,\mu+67)\,S^4 + (40\,\mu^2-102\,\mu+106)\,S^3 - 2\,(\mu-2)\,(32\,\mu^2-48\,\mu-1)\,S^2 + 2\,(3\,\mu-1)\,(\mu-2)^2\,S + 3\,\mu\,(\mu-2)^3\big)\Big) \Big/ \Big(2\,(S-2+\mu)\,(4\,S^3 - S^2 + \mu^2 - 2\,S - 2\,\mu)^2\,(-S+\mu)^2\Big) \qquad \textbf{(1.4.1.10)}$$

$$> Q3U := subs(S = \mu\cdot(1-2\cdot U), Q3S) :$$

$$> Q3Unu := factor(subs(munu, Q3U));$$

$$Q3Unu := -\Big(U\,\big(256\,U^6\,\nu^5 + 1280\,U^6\,\nu^4 - 560\,U^5\,\nu^5 + 2560\,U^6\,\nu^3 - 3728\,U^5\,\nu^4 + 491\,U^4\,\nu^5 + 2560\,U^6\,\nu^2 - 9312\,U^5\,\nu^3 + 4411\,U^4\,\nu^4 - 200\,U^3\,\nu^5 + 1280\,U^6\,\nu - 11168\,U^5\,\nu^2 + 13002\,U^4\,\nu^3 - 2746\,U^3\,\nu^4 + 32\,U^2\,\nu^5 + 256\,U^6 - 6512\,U^5\,\nu + 17450\,U^4\,\nu^2 - 9088\,U^3\,\nu^3 + 904\,U^2\,\nu^4 - 1488\,U^5 + 11083\,U^4\,\nu - 13212\,U^3\,\nu^2 + 3420\,U^2\,\nu^3 - 128\,U\,\nu^4 + 2715\,U^4 - 9144\,U^3\,\nu + 5380\,U^2\,\nu^2 - 672\,U\,\nu^3 - 2474\,U^3 + 3972\,U^2\,\nu - 1176\,U\,\nu^2 + 64\,\nu^3 + 1140\,U^2 - 880\,\nu\,U + 96\,\nu^2 - 216\,U + 96\,\nu\big)\Big) \Big/ \Big(2\,(8\,U^3\,\nu^2 + 16\,U^3\,\nu - 11\,U^2\,\nu^2 + 8\,U^3 - 24\,U^2\,\nu + 4\,U\,\nu^2 - 13\,U^2 + 14\,\nu\,U + 6\,U - 4\,\nu)^2\,(\nu\,U + U - 2)\Big) \qquad \textbf{(1.4.1.11)}$$

$$> t2Z2Unu := normal\left(simplify\left(factor\left(subs\left(Z1 = \frac{Q1Unu}{t}, Q3 = Q3Unu, t = w^{1/3}, w = wUnu, t^2\cdot ZZ2\right)\right)\right)\right);$$

$$t2Z2Unu := -\Big(U\,(\nu\,U + U - 2)\,\big(1152\,U^8\,\nu^5 + 5760\,U^8\,\nu^4 - 4872\,U^7\,\nu^5 + 11520\,U^8\,\nu^3 - 23832\,U^7\,\nu^4 + 8589\,U^6\,\nu^5 + 11520\,U^8\,\nu^2 - 46608\,U^7\,\nu^3 + 42693\,U^6\,\nu^4 - 8084\,U^5\,\nu^5 + 5760\,U^8\,\nu - 45552\,U^7\,\nu^2 + 83158\,U^6\,\nu^3 - 43450\,U^5\,\nu^4 + 4288\,U^4\,\nu^5 + 1152\,U^8 - 22248\,U^7\,\nu + 79206\,U^6\,\nu^2 - 85872\,U^5\,\nu^3 + 27556\,U^4\,\nu^4 - 1216\,U^3\,\nu^5 - 4344\,U^7 + 36765\,U^6\,\nu - 78884\,U^5\,\nu^2 + 56384\,U^4\,\nu^3 - 11072\,U^3\,\nu^4 + 144\,U^2\,\nu^5 + 6613\,U^6 - 33532\,U^5\,\nu + 49088\,U^4\,\nu^2 - 24320\,U^3\,\nu^3 + 2640\,U^2\,\nu^4 - 5154\,U^5 \qquad \textbf{(1.4.1.12)}$$

$$+ 18048 U^4 \nu - 19480 U^3 \nu^2 + 6928 U^2 \nu^3 - 288 U \nu^4 + 2076 U^4 - 5520 U^3 \nu + 4704 U^2 \nu^2 - 1280 U \nu^3 - 344 U^3 + 752 U^2 \nu - 544 U \nu^2 + 128 \nu^3)) \Big/ \left(64 \nu^2 \left(8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 \nu U + 6 U - 4 \nu\right)^2 (-1 + 2 U)^2\right)$$

> $t8Z2 := expand\left(simplify\left(subs\left(Z1 = \frac{tZZ1}{t}, t = w^{\frac{1}{3}}, t^8 \cdot ZZ2\right)\right)\right);$

$$t8Z2 := -\frac{w^3 \nu Q3}{\nu - 1} - \frac{w^2 tZZ1^2 \nu}{\nu - 1} - \frac{2 w^3 \nu}{\nu - 1} + \frac{w^2 tZZ1^2}{\nu - 1} + \frac{w^2 tZZ1}{\nu - 1} \quad \textbf{(1.4.1.13)}$$

> $t8Z2Unu := factor(subs(tZZ1 = Q1Unu, Q3 = Q3Unu, w = wUnu, t8Z2));$

$$t8Z2Unu := -\frac{1}{65536 \nu^8 (-1 + 2 U)^6} \left((\nu U + U - 2)^3 \left(1152 U^8 \nu^5 + 5760 U^8 \nu^4 - 4872 U^7 \nu^5 + 11520 U^8 \nu^3 - 23832 U^7 \nu^4 + 8589 U^6 \nu^5 + 11520 U^8 \nu^2 - 46608 U^7 \nu^3 + 42693 U^6 \nu^4 - 8084 U^5 \nu^5 + 5760 U^8 \nu - 45552 U^7 \nu^2 + 83158 U^6 \nu^3 - 43450 U^5 \nu^4 + 4288 U^4 \nu^5 + 1152 U^8 - 22248 U^7 \nu + 79206 U^6 \nu^2 - 85872 U^5 \nu^3 + 27556 U^4 \nu^4 - 1216 U^3 \nu^5 - 4344 U^7 + 36765 U^6 \nu - 78884 U^5 \nu^2 + 56384 U^4 \nu^3 - 11072 U^3 \nu^4 + 144 U^2 \nu^5 + 6613 U^6 - 33532 U^5 \nu + 49088 U^4 \nu^2 - 24320 U^3 \nu^3 + 2640 U^2 \nu^4 - 5154 U^5 + 18048 U^4 \nu - 19480 U^3 \nu^2 + 6928 U^2 \nu^3 - 288 U \nu^4 + 2076 U^4 - 5520 U^3 \nu + 4704 U^2 \nu^2 - 1280 U \nu^3 - 344 U^3 + 752 U^2 \nu - 544 U \nu^2 + 128 \nu^3\right) U^3 \right) \quad \textbf{(1.4.1.14)}$$

Asymptotics and criticality, values of the series at their radius of convergence

From BBM

> $algU := collect(numer(wUnu - w), U, factor);$

$$algU := 8 (\nu + 1)^3 U^5 - (11 \nu + 29) (\nu + 1)^2 U^4 + 4 (\nu + 8) (\nu + 1)^2 U^3 + (-128 w \nu^3 - 12 \nu^2 - 32 \nu - 12) U^2 + 8 \nu (16 \nu^2 w + 1) U - 32 w \nu^3 \quad \textbf{(1.5.1)}$$

> $factor(discrim(algU, U));$

$$-4096 (\nu - 1) (\nu - 3)^2 (\nu + 1)^6 \left(27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36\right) \left(131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23\right) \nu^2 \quad \textbf{(1.5.2)}$$

The radius of convergence is among the roots of this:

> $algr1 := 131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 - 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23;$

$$algr1 := 131072 \nu^9 w^3 - 1728 \nu^9 w^2 + 5184 \nu^8 w^2 + 7104 \nu^7 w^2 - 10560 \nu^6 w^2 \quad \textbf{(1.5.3)}$$

$$- 48 \nu^5 w + 96 \nu^4 w - 48 w \nu^3 + 4 \nu^3 - 12 \nu^2 - 15 \nu + 23$$

> $algr2 := 27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36;$

$$algr2 := 27648 \nu^4 w^2 + 864 \nu^4 w + 7 \nu^4 - 2592 w \nu^3 - 42 \nu^3 + 864 \nu^2 w + 75 \nu^2 + 864 \nu w - 20 \nu - 36 \quad \textbf{(1.5.4)}$$

At criticality

We computed in EtudeU.mw the critical value for nu and the corresponding radius of convergence for U :

> $\rho_c := solve(-864 w - 55 + 25 \sqrt{7}, w); \nu_c := 1 + \frac{1}{7} \sqrt{7};$

$$\rho_c := -\frac{55}{864} + \frac{25 \sqrt{7}}{864}$$
$$\nu_c := 1 + \frac{\sqrt{7}}{7} \quad \textbf{(1.5.1.1)}$$

We checked that the only singularity of U at nu_c is at rho_c and we obtain the following asymptotic development :

> $Ucsing := op\left(2, algeqtoseries\left(subs\left(\nu = \nu_c, w = \rho_c \cdot (1 - x), algU\right), x, U, 5, true\right)\right);$

$$Ucsing := \frac{5}{9} - \frac{\sqrt{7}}{9} + RootOf\left(39366\, \_Z^3 + 310 \sqrt{7} - 425\right) x^{1/3} + \left(-\frac{5\, RootOf\left(39366\, \_Z^3 + 310 \sqrt{7} - 425\right)^2}{24} - \frac{5 \sqrt{7}\, RootOf\left(39366\, \_Z^3 + 310 \sqrt{7} - 425\right)^2}{12}\right) x^{2/3} + \left(-\frac{35}{10368} + \frac{35 \sqrt{7}}{5184}\right) x - \frac{1645\, RootOf\left(39366\, \_Z^3 + 310 \sqrt{7} - 425\right) x^{4/3}}{82944} + O\left(x^{5/3}\right) \quad \textbf{(1.5.1.2)}$$

Which translates into the folllowing development for tZ1 :

> $allvalues\left(RootOf\left(39366\, \_Z^3 + 310 \sqrt{7} - 425\right)\right)$

$$\frac{\left(1240 \sqrt{7} - 1700\right)^{1/3}}{108} + \frac{I \sqrt{3} \left(1240 \sqrt{7} - 1700\right)^{1/3}}{108}, -\frac{\left(1240 \sqrt{7} - 1700\right)^{1/3}}{54}, \frac{\left(1240 \sqrt{7} - 1700\right)^{1/3}}{108} - \frac{I \sqrt{3} \left(1240 \sqrt{7} - 1700\right)^{1/3}}{108} \quad \textbf{(1.5.1.3)}$$

You may want to change the index in the op (from 1 to 3) to get the only solution without imaginary part :

> $tZ1csing := op\left(2, \left[allvalues\left(convert\left(collect\left(map\left(simplify, collect\left(map\left(expand,\right.\right.\right.\right.\right.\right.\right.$

map(rationalize, series(subs(U = Ucsing, subs(ν = ν_c, Q1Unu)), x, 4))), x)), x), polynom))]);

$$tZ1csing := -\frac{\left(1240\sqrt{7}-1700\right)^{1/3} x^{4/3} \left(\sqrt{7}-13\right)}{864} + \left(-\frac{\sqrt{7}}{10}+\frac{1}{10}\right)x - \frac{4\sqrt{7}}{15} + \frac{23}{30} \quad \textbf{(1.5.1.4)}$$

Value at the radius of convergence

> $Uc := \frac{5}{9} - \frac{1}{9}\sqrt{7}$:

> tZ1c := subs(x = 0, tZ1csing);

$$tZ1c := -\frac{4\sqrt{7}}{15} + \frac{23}{30} \quad \textbf{(1.5.1.5)}$$

Subcritical regime

Again from EtudeU.mw, when nu < nu_c, the radius of convergence is given by algr2:

> rhosubc := simplify(op(1, [solve(algr2, w)]));

$$rhosubc := \frac{-9\nu^3 + 27\nu^2 + \sqrt{3}\sqrt{-\left(\nu^2 - 2\nu - 3\right)^3} - 9\nu - 9}{576\nu^3} \quad \textbf{(1.5.2.1)}$$

And

> Usubcsing := convert(op(2, algeqtoseries(subs(w = rhosubc·(1 − x), algU), x, U, 2, true)), polynom);

$$Usubcsing := \frac{1}{6\left(\nu^3 - \nu^2 - 5\nu - 3\right)}\Big(3\nu^3 + \sqrt{3}\sqrt{-\nu^6 + 6\nu^5 - 3\nu^4 - 28\nu^3 + 9\nu^2 + 54\nu + 27} - 3\nu^2 - 15\nu - 9\Big) + RootOf\Big(\left(252\nu^6 - 504\nu^5 - 1296\nu^4 + 1008\nu^3 + 1980\nu^2 - 216\nu - 648\right)\_Z^2 - 13\nu^6 + 3\sqrt{3}\sqrt{-\nu^6 + 6\nu^5 - 3\nu^4 - 28\nu^3 + 9\nu^2 + 54\nu + 27}\,\nu^3 + 78\nu^5 - 9\sqrt{3}\sqrt{-\nu^6 + 6\nu^5 - 3\nu^4 - 28\nu^3 + 9\nu^2 + 54\nu + 27}\,\nu^2 - 120\nu^4 - 40\nu^3 + 6\sqrt{3}\sqrt{-\nu^6 + 6\nu^5 - 3\nu^4 - 28\nu^3 + 9\nu^2 + 54\nu + 27} + 171\nu^2 - 54\nu - 54\Big)\sqrt{x} \quad \textbf{(1.5.2.2)}$$

> tZ1subcsing := collect(map(simplify, collect(map(rationalize, map(simplify, convert(series(simplify(subs(U = Usubcsing, Q1Unu)), x, 2), polynom))), x, expand)), x, factor);

$$tZ1subcsing := -\Big(RootOf\Big(\left(252\nu^6 - 504\nu^5 - 1296\nu^4 + 1008\nu^3 + 1980\nu^2 - 216\nu - 648\right)\_Z^2 - 13\nu^6 + 3\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu^3 + 78\nu^5 \quad \textbf{(1.5.2.3)}$$

$$- 9\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu^2 - 120\nu^4 - 40\nu^3 + 6\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3} + 171\nu^2 - 54\nu - 54\Big)\Big(35\nu^6 + 21\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu^3 - 210\nu^5 - 63\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu^2 + 216\nu^4 + 72\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu + 536\nu^3 - 30\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3} - 645\nu^2 - 342\nu + 378\Big)x^{3/2}\Big)\Big/\Big(2\nu(\nu-3)(7\nu^2-14\nu-9)(7\nu^2-14\nu+6)\Big) - \Big(\Big(9\nu^5 + \sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu^2 - 45\nu^4 - 2\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3}\,\nu + 18\nu^3 + 9\sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3} + 126\nu^2 - 27\nu - 81\Big)x\Big)\Big/\Big(12\nu(\nu+1)(\nu-3)(7\nu^2-14\nu-9)\Big) + \frac{2\nu^3 + \sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3} + \nu^2 - 10\nu - 9}{2(7\nu^2-14\nu-9)\nu}$$

Value at the radius of convergence

```
> tZ1subc := subs(x = 0, tZ1subcsing);
```

$$tZ1subc := \frac{2\nu^3 + \sqrt{3}\sqrt{-(\nu+1)^3(\nu-3)^3} + \nu^2 - 10\nu - 9}{2(7\nu^2-14\nu-9)\nu} \qquad \textbf{(1.5.2.4)}$$

Supercritical regime

```
> rhosupc := RootOf(algr1, w);
```

$$rhosupc := RootOf\big(131072\nu^9\,\_Z^3 + (-1728\nu^9 + 5184\nu^8 + 7104\nu^7 - 10560\nu^6)\,\_Z^2 + (-48\nu^5 + 96\nu^4 - 48\nu^3)\,\_Z + 4\nu^3 - 12\nu^2 - 15\nu + 23\big) \qquad \textbf{(1.5.3.1)}$$

```
> Usupcsing := map(simplify, convert(op(2, algeqtoseries(subs(w = rhosupc·(1 − x),
      algU), x, U, 3, true)), polynom)) :
> tZ1supcsing := collect(map(simplify, collect(map(rationalize, map(simplify,
      convert(series(simplify(subs(U = Usupcsing, Q1Unu)), x, 2), polynom))), x,
      expand)), x, factor);
```

$$tZ1supcsing := -\frac{1}{3\nu}\Big((\nu+1)\,RootOf\Big((126\nu^8 - 504\nu^7 - 1026\nu^6 + 3564\nu^5 + 4518\nu^4 - 5616\nu^3 - 7038\nu^2 + 1404\nu + 2268)\,\_Z^2 - 667 + 6952\,RootOf\big(131072\nu^9\,\_Z^3 + (-1728\nu^9 + 5184\nu^8 + 7104\nu^7 - 10560\nu^6)\,\_Z^2 + (-48\nu^5 + 96\nu^4 - 48\nu^3)\,\_Z + 4\nu^3 - 12\nu^2 - 15\nu + 23\big)\nu^3 + 780\nu - 772\nu^3 + 606\nu^2 - 87\nu^4 - 28\nu^6 + 168\nu^5 + 49152\,RootOf\big(131072\nu^9\,\_Z^3 + (-1728\nu^9 + 5184\nu^8 + 7104\nu^7$$

$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23)^{2} \nu^{8} - 392 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7}$$
$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23) \nu^{8} - 98304 \nu^{7} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8}$$
$$+ 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2}$$
$$- 15 \nu + 23)^{2} + 1960 \nu^{7} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8}$$
$$+ 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2}$$
$$- 15 \nu + 23) - 16384 \nu^{6} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9}$$
$$+ 5184 \nu^{8} + 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z$$
$$+ 4 \nu^{3} - 12 \nu^{2} - 15 \nu + 23)^{2} + 784 \nu^{6} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + ($$
$$-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4}$$
$$- 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu + 23)$$
$$- 10192 \nu^{5} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7}$$
$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23) + 888 \nu^{4} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7}$$
$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23)) x^{3/2}) + (x (32768 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7} - 1056$$
$$- 15 \nu + 23)^{2} \nu^{10} - 432 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8}$$
$$+ 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2}$$
$$- 15 \nu + 23) \nu^{10} - 131072 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9}$$
$$+ 5184 \nu^{8} + 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z$$
$$+ 4 \nu^{3} - 12 \nu^{2} - 15 \nu + 23)^{2} \nu^{9} + 3024 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + ($$
$$-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4}$$
$$- 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu + 23) \nu^{9}$$
$$- 49152 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7}$$
$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23)^{2} \nu^{8} - 712 \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8} + 7104 \nu^{7}$$
$$- 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2} - 15 \nu$$
$$+ 23) \nu^{8} + 360448 \nu^{7} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8}$$
$$+ 7104 \nu^{7} - 10560 \nu^{6}) \_Z^{2} + (-48 \nu^{5} + 96 \nu^{4} - 48 \nu^{3}) \_Z + 4 \nu^{3} - 12 \nu^{2}$$
$$- 15 \nu + 23)^{2} - 26680 \nu^{7} \mathit{RootOf}(131072 \nu^{9} \_Z^{3} + (-1728 \nu^{9} + 5184 \nu^{8}$$

$$
\begin{aligned}
&+ 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} \\
&- 15\,\nu + 23\big) + 966656\,\nu^{6}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} \\
&+ 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z \\
&+ 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big)^{2} + 16064\,\nu^{6}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(\\
&-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} \\
&- 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big) - 16\,\nu^{6} \\
&+ 82720\,\nu^{5}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} \\
&- 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu \\
&+ 23\big) + 96\,\nu^{5} - 7880\,\nu^{4}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} + 5184\,\nu^{8} \\
&+ 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} \\
&- 15\,\nu + 23\big) - 27\,\nu^{4} - 66104\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} \\
&+ 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z \\
&+ 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big)\,\nu^{3} - 532\,\nu^{3} - 42\,\nu^{2} + 1404\,\nu - 883\big)\big) \Big/ \\
&\big(36\,(\nu - 1)\,(\nu + 1)\,\nu\,(\nu - 3)\,\big(\nu^{2} - 2\,\nu - 7\big)\,\big(4\,\nu^{2} - 8\,\nu - 23\big)\big) \\
&+ \big(16384\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48 \\
&+ 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} \\
&- 15\,\nu + 23\big)^{2} - 216\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} + 5184\,\nu^{8} \\
&+ 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} \\
&- 15\,\nu + 23\big)\,\nu^{8} - 32768\,\nu^{6}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} \\
&+ 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z \\
&+ 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big)^{2} + 1080\,\nu^{7}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(\\
&-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} \\
&- 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big) + 1048\,\nu^{6}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} \\
&+ \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} \\
&- 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big) - 7464\,\nu^{5}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} \\
&+ \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} \\
&- 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big) - 2976\,\nu^{4}\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} \\
&+ \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} - 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} \\
&- 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu + 23\big) + 12\,\nu^{5} \\
&+ 8528\,\mathit{RootOf}\big(131072\,\nu^{9}\,\_Z^{3} + \big(-1728\,\nu^{9} + 5184\,\nu^{8} + 7104\,\nu^{7} \\
&- 10560\,\nu^{6}\big)\,\_Z^{2} + \big(-48\,\nu^{5} + 96\,\nu^{4} - 48\,\nu^{3}\big)\,\_Z + 4\,\nu^{3} - 12\,\nu^{2} - 15\,\nu \\
&+ 23\big)\,\nu^{3} - 62\,\nu^{4} - 49\,\nu^{3} + 390\,\nu^{2} + 203\,\nu - 494\big) \Big/ \big(6\,\big(\nu^{2} - 2\,\nu \\
&- 7\big)\,\big(4\,\nu^{2} - 8\,\nu - 23\big)\,(\nu - 1)\,\nu\big)
\end{aligned}
$$

Value at the radius of convergence

> tZ1supc := subs(x = 0, tZ1supcsing);

$$tZ1supc := \Big(16384\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big)^2\, \nu^8 - 32768\, \nu^7\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big)^2 - 216\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big)\, \nu^8 - 32768\, \nu^6\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big)^2 + 1080\, \nu^7\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big) + 1048\, \nu^6\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big) - 7464\, \nu^5\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big) - 2976\, \nu^4\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big) + 12\, \nu^5 + 8528\, RootOf\big(131072\, \nu^9\, \_Z^3 + (-1728\, \nu^9 + 5184\, \nu^8 + 7104\, \nu^7 - 10560\, \nu^6)\, \_Z^2 + (-48\, \nu^5 + 96\, \nu^4 - 48\, \nu^3)\, \_Z + 4\, \nu^3 - 12\, \nu^2 - 15\, \nu + 23\big)\, \nu^3 - 62\, \nu^4 - 49\, \nu^3 + 390\, \nu^2 + 203\, \nu - 494\Big) \Big/ \big(6\, (\nu^2 - 2\, \nu - 7)\, (4\, \nu^2 - 8\, \nu - 23)\, (\nu - 1)\, \nu\big) \quad \textbf{(1.5.3.3)}$$

The spectral radius of the matrixM of Proposition 28

Series evaluated at the radius of convergence:

> t2Z2subc := simplify(factor(rationalize(simplify(subs(U = subs(x = 0, Usubcsing), t2Z2Unu))))) :

> $t2Z11subc := simplify\left(subs\left(x = 0, \frac{tZ1subc}{\nu} - (tZ1subc)^2 - t2Z2subc\right)\right):$

> t2Z2supc := simplify(factor(rationalize(simplify(subs(U = subs(x = 0, Usupcsing), subs(munu, t2Z2Unu)))))) :

> $t2Z11supc := simplify\left(subs\left(x = 0, \frac{tZ1supc}{\nu} - (tZ1supc)^2 - t2Z2supc\right)\right):$

The Matrix of expectations in the branching process:

> with(LinearAlgebra) :

Subcritical regime

$$> MMsubc := map\left(expand, map\left(rationalize, simplify\left(Matrix\left(\left[\left[2\cdot\nu\cdot tZ1subc, \frac{\nu\cdot t2Z2subc}{tZ1subc}, 0, \frac{\nu\cdot t2Z11subc}{tZ1subc}, 0\right], \left[\nu\cdot tZ1subc, 2\cdot\nu\cdot tZ1subc, 1-2\cdot\nu\cdot tZ1subc-\frac{\nu\cdot rhosubc}{\mathrm{t2Z2subc}}, 0, 0\right], \left[\nu\cdot tZ1subc, \frac{\min(1,\nu)^2\cdot \mathrm{t2Z2subc}}{1-2\cdot\min(1,\nu)\cdot tZ1subc}, 1-\frac{\min(1,\nu)^2\cdot\mathrm{t2Z2subc}}{1-2\cdot\min(1,\nu)\cdot tZ1subc}, 0, 0\right], \left[tZ1subc, 0, 0, 2\cdot tZ1subc, 1-2\cdot tZ1subc-\frac{rhosubc}{t2Z11subc}\right], \left[tZ1subc, 0, 0, \frac{\min(1,\nu)^2\cdot\mathrm{t2Z11subc}}{1-2\cdot\min(1,\nu)\cdot tZ1subc}, 1-\frac{\min(1,\nu)^2\cdot\mathrm{t2Z11subc}}{1-2\cdot\min(1,\nu)\cdot tZ1subc}\right]\right]\right)\right)\right)\right):$$

$$> with(Student[NumericalAnalysis]) : evalf\left(SpectralRadius\left(subs\left(\nu=\nu_c, MMsubc\right)\right)\right);$$

0.9898449789 **(1.5.4.1)**

$$> seq\left(evalf\left(SpectralRadius\left(evalf\left(subs\left(\nu=\frac{i}{10}, MMsubc\right)\right)\right)\right), i=1\,..\,10\right);$$

1.004230458, 1.000257313, 0.9978336718, 0.9955295948, 0.9928754496, 0.9899024883, 0.9867298650, 0.9835178213, 0.9804473896, 0.9776887315 **(1.5.4.2)**

$$> seq\left(evalf\left(SpectralRadius\left(evalf\left(subs\left(\nu=1+\frac{(\nu_c-1)\cdot i}{10}, MMsubc\right)\right)\right)\right), i=1\,..\,10\right);$$

0.9791947696, 0.9806473642, 0.9820392366, 0.9833664102, 0.9846267034, 0.9858185777, 0.9869401874, 0.9879886529, 0.9889593019, 0.9898450024 **(1.5.4.3)**

Supercritical Regime

$$> MMsupc := map\left(expand, map\left(rationalize, simplify\left(Matrix\left(\left[\left[2\cdot\nu\cdot tZ1supc, \frac{\nu\cdot t2Z2supc}{tZ1supc}, 0, \frac{\nu\cdot t2Z11supc}{tZ1supc}, 0\right], \left[\nu\cdot tZ1supc, 2\cdot\nu\cdot tZ1supc, 1-2\cdot\nu\cdot tZ1supc-\frac{\nu\cdot rhosupc}{\mathrm{t2Z2supc}}, 0, 0\right], \left[\nu\cdot tZ1supc, \frac{\mathrm{t2Z2supc}}{1-2\, tZ1supc}, 1-\frac{\mathrm{t2Z2supc}}{1-2\cdot tZ1subc}, 0, 0\right], \left[tZ1supc, 0, 0, 2\cdot tZ1supc, 1-2\cdot tZ1supc-\frac{rhosupc}{t2Z11supc}\right], \left[tZ1supc, 0, 0, \frac{\mathrm{t2Z11supc}}{1-2\cdot tZ1supc}, 1-\frac{\mathrm{t2Z11supc}}{1-2\cdot tZ1supc}\right]\right]\right)\right)\right)\right)$$

:

> $with(Student[NumericalAnalysis]) : evalf\left(SpectralRadius\left(evalf\left(subs\left(\nu = \nu_c, MMsupc\right)\right)\right)\right);$

$$0.9898450244 \quad (1.5.4.4)$$

> $seq\left(evalf\left(SpectralRadius\left(evalf\left(subs\left(\nu = \nu_c + \frac{i}{10}, MMsupc\right)\right)\right)\right), i = 1 .. 10\right);$

$$0.9918042538, 0.9934053277, 0.9947874178, 0.9960338049, 0.9972134495, 0.9983945548, 0.9996404784, 1.001029890, 1.002704270, 1.004912748 \quad (1.5.4.5)$$

> $evalf\left(\nu_c + \frac{7}{10}\right);$

$$2.077964473 \quad (1.5.4.6)$$

Equation for Z_p : simple boundary summed over boundary conditions (Lemmas 23 and 24)

Z: simple boundary, summed over all spins (Lemma 31 in BBM). QQ: non simple boundary

Equation for Z of BBM after Lemma 31:

> $eqQQy := subs\big(D(QQ)(0) = QQ1, D^{(3)}(QQ)(0) = 6 \cdot QQ3, 3 y^3 - 3 y^3 \nu + 3 y^2 t \nu + 12 t^3 y^4 D(QQ)(0) \nu^2 QQ(y) + 6 t^3 \nu^2 - 3 \nu^2 t y^5 QQ(y)^2 + 12 t^3 y^3 \nu^2 + 3 t y^5 QQ(y)^2 - 6 t D(QQ)(0) y^3 + 3 y^3 QQ(y) \nu - 6 t^3 QQ(y) \nu^2 - 6 \nu t^2 y + 3 \nu^2 y^2 t - 6 \nu^2 t^2 y - 3 y^3 QQ(y) + 6 t^2 y QQ(y) \nu^2 - 3 t y^2 QQ(y) \nu^2 - 3 t y^2 QQ(y) \nu + 6 t^2 y QQ(y) \nu - 24 t^3 y^3 QQ(y)^2 \nu^2 - 3 t D(QQ)(0) y^3 \nu + 3 t D(QQ)(0) y^3 \nu^2 - 6 t^2 y^2 D(QQ)(0) \nu^2 + 6 t^3 y D(QQ)(0) \nu^2 - 6 \nu t^2 y^4 QQ(y) - 6 \nu^2 t^2 y^4 QQ(y) + 12 \nu^2 t^3 y^3 QQ(y) + t^3 \nu^2 D^{(3)}(QQ)(0) y^3 + 12 t^2 y^4 QQ(y)^2 \nu^2 + 12 t^2 y^4 QQ(y)^2 \nu - 24 t^3 y^6 QQ(y)^3 \nu^2\big);$

$$eqQQy := -6 \nu t^2 y - 6 \nu t^2 y^4 QQ(y) - 6 \nu^2 t^2 y^4 QQ(y) + 12 \nu^2 t^3 y^3 QQ(y) - 6 \nu^2 t^2 y - 3 y^3 \nu + 3 \nu^2 t y^2 + 3 y^2 \nu t + 12 \nu^2 t^3 y^3 - 24 t^3 y^3 QQ(y)^2 \nu^2 - 3 t y^2 QQ(y) \nu^2 + 6 t^2 y QQ(y) \nu^2 - 3 \nu^2 t y^5 QQ(y)^2 - 6 t^3 QQ(y) \nu^2 + 3 y^3 - 3 y^3 QQ(y) + 6 \nu^2 t^3 + 12 t^2 y^4 QQ(y)^2 \nu^2 + 6 t^2 y QQ(y) \nu - 3 t y^2 QQ(y) \nu + 12 t^2 y^4 QQ(y)^2 \nu + 6 t^3 y QQ1 \nu^2 + 3 t QQ1 y^3 \nu^2 - 6 t^2 y^2 QQ1 \nu^2 - 3 t QQ1 y^3 \nu + 6 t^3 \nu^2 QQ3 y^3 - 24 t^3 y^6 QQ(y)^3 \nu^2 + 3 y^3 QQ(y) \nu - 6 t QQ1 y^3 + 3 t y^5 QQ(y)^2 + 12 t^3 y^4 QQ1 \nu^2 QQ(y) \quad (1.6.1.1)$$

> $eqZY := collect\left(numer\left(simplify\left(subs\left(QQ(y) = 1 + Z(Y), y = \frac{Y}{1 + Z(Y)},\right.\right.\right.\right.$

$eqQQy\Big)\Big)\Big), Z(Y)\Big);$

$$eqZY := -6\,Z(Y)^4\,\nu^2\,t^3 + \left(6\,t^2\,Y\,\nu^2 - 18\,\nu^2\,t^3 + 6\,t^2\,Y\,\nu\right) Z(Y)^3 + \left(-24\,\nu^2\,t^3\,Y^3 + 6\,t^3\,Y\,QQ1\,\nu^2 - 3\,Y^2\,\nu^2\,t + 12\,t^2\,Y\,\nu^2 - 18\,\nu^2\,t^3 - 3\,Y^2\,\nu\,t + 12\,t^2\,Y\,\nu\right) Z(Y)^2 + \left(12\,\nu^2\,t^2\,Y^4 - 36\,\nu^2\,t^3\,Y^3 - 6\,t^2\,Y^2\,QQ1\,\nu^2 + 12\,t^3\,Y\,QQ1\,\nu^2 + 12\,t^2\,Y^4\,\nu - 3\,Y^2\,\nu^2\,t + 6\,t^2\,Y\,\nu^2 - 6\,\nu^2\,t^3 + 3\,\nu\,Y^3 - 3\,Y^2\,\nu\,t + 6\,t^2\,Y\,\nu - 3\,Y^3\right) Z(Y) - 24\,t^3\,Y^6\,\nu^2 + 12\,t^3\,Y^4\,QQ1\,\nu^2 + 6\,t^3\,\nu^2\,QQ3\,Y^3 - 3\,\nu^2\,t\,Y^5 + 6\,\nu^2\,t^2\,Y^4 + 3\,t\,QQ1\,Y^3\,\nu^2 - 6\,t^2\,Y^2\,QQ1\,\nu^2 + 6\,t^3\,Y\,QQ1\,\nu^2 + 6\,t^2\,Y^4\,\nu - 3\,t\,QQ1\,Y^3\,\nu + 3\,t\,Y^5 - 6\,t\,QQ1\,Y^3 \quad \textbf{(1.6.1.2)}$$

$> eqZty := collect\left(simplify\left(\left(\frac{1}{w}\left(subs\left(Z(Y) = Zty, QQ1 = \frac{Q1t}{t}, QQ3 = \frac{Q3t}{t^3}, Y = t\cdot y, t = w^{\frac{1}{3}}, eqZY\right)\right)\right)\right), Zty\right);$

$$eqZty := -6\,Zty^4\,\nu^2 + \left((6\,y - 18)\,\nu^2 + 6\,y\,\nu\right) Zty^3 + \left(\left(-24\,w\,y^3 - 3\,y^2 + 6\,(2 + Q1t)\,y - 18\right)\nu^2 + 12\left(-\frac{y}{4} + 1\right) y\,\nu\right) Zty^2 + \left(\left(12\,w\,y^4 - 36\,w\,y^3 - 6\left(Q1t + \frac{1}{2}\right) y^2 + 6\,(1 + 2\,Q1t)\,y - 6\right)\nu^2 + 12\left(w\,y^3 + \frac{1}{4}\,y^2 - \frac{1}{4}\,y + \frac{1}{2}\right) y\,\nu - 3\,y^3\right) Zty + \left(-24\,w^2\,y^6 - 3\,w\,y^5 + 12\left(Q1t + \frac{1}{2}\right) w\,y^4 + (3\,Q1t + 6\,Q3t)\,y^3 - 6\,y^2\,Q1t + 6\,Q1t\,y\right)\nu^2 + 12\left(\frac{1}{2}\,w\,y^3 - \frac{1}{4}\,y^2\,Q1t\right) y\,\nu + 3\,y^3\left(w\,y^2 - 2\,Q1t\right) \quad \textbf{(1.6.1.3)}$$

$> indets(\%);$

$$\{Q1t, Q3t, Zty, \nu, w, y\} \quad \textbf{(1.6.1.4)}$$

$> algeqtoseries(eqZty, y, Zty, 3, true);$

$$\Big[-1 + RootOf\left(2\,\_Z^3\,\nu^2 + \left(2\,\nu^2\,Q1t - 2\,\nu^2 - 2\,\nu\right)\_Z^2 + \left(-2\,\nu^2\,Q1t + \nu^2 + \nu\right)\_Z + \nu^2\,Q1t + 2\,\nu^2\,Q3t + 4\,\nu^2\,w - Q1t\,\nu - 2\,Q1t - \nu + 1\right) y + O\left(y^{5/3}\right), Q1t\,y - \frac{Q1t\,(Q1t\,\nu - 1)}{\nu}\,y^2 + \frac{2\,Q1t^3\,\nu - 3\,Q1t^2 + Q3t\,\nu}{\nu}\,y^3 + O\left(y^4\right)\Big] \quad \textbf{(1.6.1.5)}$$

$> eqZtyU := factor(numer(simplify(subs(w = wUnu, Q1t = Q1Unu, Q3t = wUnu \cdot Q3Unu, eqZty)))):$

This eqZtyU, evaluated at nu_c is the polynomial P(z,y,u) that appears in the proof of Lemma

27.

From now on we work at critical nu=nu_c. Taking U=U_c we obtain the equation for A(y)

> $eqZtyU\_nuc := collect\left(op\left(3, factor\left(expand\left(subs\left(\nu = \nu_c, eqZtyU\right)\right)\right)\right), Zty, factor\right) :$

> $eqZtyU\_nuc\_Uc := collect\left(expand\left(subs\left(\nu = \nu_c, U = Uc, eqZtyU\right)\right), Zty, factor\right) :$
$eqAy := op(3, factor(subs(Zty = A, eqZtyU\_nuc\_Uc)))$

$$eqAy := 26625\,y^5 + 176040\,y^4 + 466560\,A + 160200\,A\,y^4 - 87480\,A\,y^3 + 863136\,A\,y^2 - 1726272\,A\,y - 357696\,y + 357696\,y^2 + 53082\,y^3 + 124416\sqrt{7}\,y - 124416\sqrt{7}\,y^2 - 17766\sqrt{7}\,y^3 - 118800\,A^2\,y^3 + 505440\,A^2\,y^2 - 2379456\,A^2\,y + 77760\,A^3\sqrt{7}\,y + 54000\,A^2\sqrt{7}\,y^3 - 38880\,A^2\sqrt{7}\,y^2 + 279936\,A^2\sqrt{7}\,y - 68400\,A\sqrt{7}\,y^4 + 29160\,A\sqrt{7}\,y^3 - 163296\,A\sqrt{7}\,y^2 + 326592\,A\sqrt{7}\,y - 9525\,y^5\sqrt{7} - 6875\,y^6\sqrt{7} + 18500\,y^6 - 70740\,y^4\sqrt{7} - 1010880\,A^3\,y + 466560\,A^4 + 1399680\,A^3 + 1399680\,A^2 \quad \textbf{(1.6.1.6)}$$

> $factor(eqAy);$

$$26625\,y^5 + 176040\,y^4 + 466560\,A + 160200\,A\,y^4 - 87480\,A\,y^3 + 863136\,A\,y^2 - 1726272\,A\,y - 357696\,y + 357696\,y^2 + 53082\,y^3 + 124416\sqrt{7}\,y - 124416\sqrt{7}\,y^2 - 17766\sqrt{7}\,y^3 - 118800\,A^2\,y^3 + 505440\,A^2\,y^2 - 2379456\,A^2\,y + 77760\,A^3\sqrt{7}\,y + 54000\,A^2\sqrt{7}\,y^3 - 38880\,A^2\sqrt{7}\,y^2 + 279936\,A^2\sqrt{7}\,y - 68400\,A\sqrt{7}\,y^4 + 29160\,A\sqrt{7}\,y^3 - 163296\,A\sqrt{7}\,y^2 + 326592\,A\sqrt{7}\,y - 9525\,y^5\sqrt{7} - 6875\,y^6\sqrt{7} + 18500\,y^6 - 70740\,y^4\sqrt{7} - 1010880\,A^3\,y + 466560\,A^4 + 1399680\,A^3 + 1399680\,A^2 \quad \textbf{(1.6.1.7)}$$

> $unassign('Z0','Z1','Z2','Z3','Z4','Z5','Z6')$

> $sub\_devZtyu\_nuc := Zty = series\left(Z0 + Z3\cdot u^3 + Z4\cdot u^4 + Z5\cdot u^5, u, 5\right)$

$$sub\_devZtyu\_nuc := Zty = Z0 + Z3\,u^3 + Z4\,u^4 + \mathrm{O}\left(u^5\right) \quad \textbf{(1.6.1.8)}$$

> $sub\_devUu := U = Uc - u$

$$sub\_devUu := U = \frac{5}{9} - \frac{\sqrt{7}}{9} - u \quad \textbf{(1.6.1.9)}$$

> $eqZi := factor(simplify(series(subs(sub\_devZtyu\_nuc, sub\_devUu, eqZtyU\_nuc), u, 5))) :$

> $eqZ0y := op(3, factor(coeff(eqZi, u, 0)));$

$$eqZ0y := 466560\,Z0^4 + 26625\,y^5 + 863136\,Z0\,y^2 - 87480\,Z0\,y^3 + 160200\,Z0\,y^4 - 1726272\,Z0\,y + 176040\,y^4 - 357696\,y + 1399680\,Z0^3 - 1010880\,Z0^3\,y + 357696\,y^2 + 53082\,y^3 + 326592\,Z0\sqrt{7}\,y + 77760\,Z0^3\sqrt{7}\,y + 54000\,Z0^2\sqrt{7}\,y^3 - 68400\,Z0\,y^4\sqrt{7} + 29160\,Z0\sqrt{7}\,y^3 - 38880\,Z0^2\sqrt{7}\,y^2 + 279936\,Z0^2\sqrt{7}\,y - 163296\,Z0\sqrt{7}\,y^2 - 2379456\,Z0^2\,y + 505440\,Z0^2\,y^2 - 118800\,Z0^2\,y^3 + 124416\sqrt{7}\,y - 124416\sqrt{7}\,y^2 - 17766\sqrt{7}\,y^3 + 466560\,Z0 - 9525\,y^5\sqrt{7} - 6875\,y^6\sqrt{7} + 18500\,y^6 - 70740\,y^4\sqrt{7} + 1399680\,Z0^2 \quad \textbf{(1.6.1.10)}$$

$> \ factor\left(\frac{eqZ0y}{subs(A=Z0, eqAy)}\right);$

$$1 \quad \textbf{(1.6.1.11)}$$

$> \ eqZ3yZ0 := factor(isolate(coeff(eqZi, u, 3), Z3));$

$$eqZ3yZ0 := Z3 = -\left(\left(85 + 62\sqrt{7}\right)\left(56733696\,Z0^4 + 3093825\,y^5 + 105209280\,Z0\,y^2 - 9675288\,Z0\,y^3 + 18615240\,Z0\,y^4 - 210418560\,Z0\,y + 20589984\,y^4 - 43747776\,y + 170201088\,Z0^3 - 122923008\,Z0^3\,y + 43747776\,y^2 + 6837642\,y^3 + 40217472\,Z0\sqrt{7}\,y + 9455616\,Z0^3\sqrt{7}\,y + 6274800\,Z0^2\sqrt{7}\,y^3 - 7948080\,Z0\,y^4\sqrt{7} + 3108456\,Z0\sqrt{7}\,y^3 - 4727808\,Z0^2\sqrt{7}\,y^2 + 34292160\,Z0^2\sqrt{7}\,y - 20108736\,Z0\sqrt{7}\,y^2 - 289593792\,Z0^2\,y + 61461504\,Z0^2\,y^2 - 13804560\,Z0^2\,y^3 + 15380928\sqrt{7}\,y - 15380928\sqrt{7}\,y^2 - 2300022\sqrt{7}\,y^3 + 56733696\,Z0 - 1106805\,y^5\sqrt{7} - 761750\,y^6\sqrt{7} + 2049800\,y^6 - 8266644\,y^4\sqrt{7} + 170201088\,Z0^2\right)\right) \Big/ \left(972\left(25920\,Z0^3 + 1500\,Z0\sqrt{7}\,y^3 - 950\,y^4\sqrt{7} + 3240\,Z0^2\sqrt{7}\,y - 1080\,Z0\sqrt{7}\,y^2 + 405\sqrt{7}\,y^3 + 7776\,Z0\sqrt{7}\,y - 2268\sqrt{7}\,y^2 + 4536\sqrt{7}\,y + 11988\,y^2 - 1215\,y^3 + 2225\,y^4 - 23976\,y + 58320\,Z0^2 - 42120\,Z0^2\,y - 66096\,Z0\,y + 14040\,Z0\,y^2 - 3300\,Z0\,y^3 + 6480 + 38880\,Z0\right)\right) \quad \textbf{(1.6.1.12)}$$

$> \ eqZ4yZ0 := factor(isolate(subs(eqZ3yZ0, coeff(eqZi, u, 4)), Z4));$

$$eqZ4yZ0 := Z4 = \left(\left(953 + 232\sqrt{7}\right)\left(1664686080\,Z0^4 + 94279125\,y^5 + 3082188672\,Z0\,y^2 - 307317240\,Z0\,y^3 + 567268200\,Z0\,y^4 - 6164377344\,Z0\,y + 624699000\,y^4 - 1278778752\,y + 4994058240\,Z0^3 - 3606819840\,Z0^3\,y + 1278778752\,y^2 + 191765826\,y^3 + 1170319104\,Z0\sqrt{7}\,y + 277447680\,Z0^3\sqrt{7}\,y + 191214000\,Z0^2\sqrt{7}\,y^3 - 242204400\,Z0\,y^4\sqrt{7} + 101855880\,Z0\sqrt{7}\,y^3 - 138723840\,Z0^2\sqrt{7}\,y^2 + 1001331072\,Z0^2\sqrt{7}\,y - 585159552\,Z0\sqrt{7}\,y^2 - 8492418432\,Z0^2\,y + 1803409920\,Z0^2\,y^2 - 420670800\,Z0^2\,y^3 + 446435712\sqrt{7}\,y - 446435712\sqrt{7}\,y^2 - 64122462\sqrt{7}\,y^3 + 1664686080\,Z0 - 33728025\,y^5\sqrt{7} - 24158750\,y^6\sqrt{7} + 65009000\,y^6 - 250956900\,y^4\sqrt{7} + 4994058240\,Z0^2\right)\right) \Big/ \left(7776\left(25920\,Z0^3 + 1500\,Z0\sqrt{7}\,y^3 - 950\,y^4\sqrt{7} + 3240\,Z0^2\sqrt{7}\,y - 1080\,Z0\sqrt{7}\,y^2 + 405\sqrt{7}\,y^3 + 7776\,Z0\sqrt{7}\,y - 2268\sqrt{7}\,y^2 + 4536\sqrt{7}\,y + 11988\,y^2 - 1215\,y^3 + 2225\,y^4 - 23976\,y + 58320\,Z0^2 - 42120\,Z0^2\,y - 66096\,Z0\,y + 14040\,Z0\,y^2 - 3300\,Z0\,y^3 + 6480 + 38880\,Z0\right)\right) \quad \textbf{(1.6.1.13)}$$

$> \ DenomAB := \frac{diff(eqZ0y, Z0)}{233280} \cdot 3240;$

$$DenomAB := 25920\,Z0^3 + 1500\,Z0\sqrt{7}\,y^3 - 950\,y^4\sqrt{7} + 3240\,Z0^2\sqrt{7}\,y - 1080\,Z0\sqrt{7}\,y^2 + 405\sqrt{7}\,y^3 + 7776\,Z0\sqrt{7}\,y - 2268\sqrt{7}\,y^2 + 4536\sqrt{7}\,y + 11988\,y^2 - 1215\,y^3 + 2225\,y^4 - 23976\,y + 58320\,Z0^2 \quad \textbf{(1.6.1.14)}$$

$- 42120\, Z0^2\, y - 66096\, Z0\, y + 14040\, Z0\, y^2 - 3300\, Z0\, y^3 + 6480 + 38880\, Z0$

Singular behavior

> $discUc := factor\left(discrim\left(subs\left(\nu = \nu_c, U = Uc, eqZtyU\right), Zty\right)\right);$

$$discUc := -\left(360287970189639680 \left(110728373803430310220983893 5\sqrt{7} + 289826599405166865143952459 1\right) \left(-16\, y + 15 + 15\sqrt{7}\right) y^6 \left(-5\, y + 3 + 3\sqrt{7}\right)^8\right) / 1114039821949850228740653460025483981303042376 9 \quad \textbf{(1.6.2.1)}$$

> $yc1 := solve\left(-16\, y + 15 + 15\sqrt{7}, y\right); evalf(yc1); yc2 := solve\left(3 + 3\sqrt{7} - 5\, y, y\right); evalf(yc2);$

$$yc1 := \frac{15}{16} + \frac{15\sqrt{7}}{16}$$

$$3.417891854$$

$$yc2 := \frac{3}{5} + \frac{3\sqrt{7}}{5}$$

$$2.187450787 \quad \textbf{(1.6.2.2)}$$

> $algeqtoseries\left(subs\left(y = yc2 \cdot (1 - a), \nu = \nu_c, U = Uc, eqZtyU\right), a, Zty, 3, true\right);$

$$\left[-\frac{6}{5} + \frac{3\sqrt{7}}{10} - \frac{81}{20}\, \frac{51724 + 18079\sqrt{7}}{85369\sqrt{7} + 201382}\, a + \frac{81}{160}\, \frac{14741450018594731\sqrt{7} + 38987534921839510}{\left(85369\sqrt{7} + 201382\right)^3}\, a^2 + O\left(a^3\right), -\frac{2}{5} + \frac{3\sqrt{7}}{10} + \left(-\frac{3}{20} - \frac{3\sqrt{7}}{10}\right) a + RootOf\left(128000\, \_Z^3 - 2187\right) a^{4/3} + O\left(a^{13/9}\right)\right] \quad \textbf{(1.6.2.3)}$$

> $algeqtoseries\left(subs\left(y = yc1 \cdot (1 - a), \nu = \nu_c, U = Uc, eqZtyU\right), a, Zty, 3, true\right);$

$$\left[RootOf\left(16384\, \_Z^2 + \left(-15360\sqrt{7} + 11008\right) \_Z - 5160\sqrt{7} + 27849\right) + \left(\frac{375\sqrt{7}}{256} - \frac{1599}{1024} - \frac{1}{8}\left(33\, RootOf\left(16384\, \_Z^2 + \left(-15360\sqrt{7} + 11008\right) \_Z - 5160\sqrt{7} + 27849\right)\right)\right) a + \left(\frac{420275}{442368} - \frac{146375\sqrt{7}}{110592} + \frac{1}{10368}\left(29275\, RootOf\left(16384\, \_Z^2 + \left(-15360\sqrt{7} + 11008\right) \_Z - 5160\sqrt{7} + 27849\right)\right)\right) a^2 + O\left(a^3\right), -\frac{89}{128} + \frac{15\sqrt{7}}{32} \quad \textbf{(1.6.2.4)}$$

$$+ RootOf\left(2048\_Z^2 - 2187\right)\sqrt{a} + \left(-\frac{15\sqrt{7}}{32} - \frac{75}{256}\right)a + O\left(a^{3/2}\right)\Big]$$

The radius of convergence is yc2 !

> $Zpp := rationalize(simplify(subs(\nu = \nu_c, t2Z2subc)));$

$$Zpp := \frac{131}{1800} - \frac{131\sqrt{7}}{7200} \qquad \textbf{(1.6.2.5)}$$

> $simplify\left(Zpp \cdot \frac{yc2}{(\rho_c)^{\frac{1}{3}}}\right); evalf(\%);$

$$-\frac{131\left(-4+\sqrt{7}\right)\left(1+\sqrt{7}\right)}{1000\left(-110+50\sqrt{7}\right)^{1/3}}$$

$$0.2298277941 \qquad \textbf{(1.6.2.6)}$$

> $simplify\left(Zpp \cdot \frac{1}{(\rho_c)^{\frac{1}{3}}}\right); evalf(\%);$

$$-\frac{131\left(-4+\sqrt{7}\right)}{600\left(-110+50\sqrt{7}\right)^{1/3}}$$

$$0.1050664982 \qquad \textbf{(1.6.2.7)}$$

Let us now check that the expression for B and C is not subcritical:

> $factor(resultant(DenomAB, eqZ0y, Z0));$

$$-60935974001049600\left(394475\sqrt{7} - 1043669\right)\left(-16y + 15 + 15\sqrt{7}\right)y^6\left(-5y + 3 + 3\sqrt{7}\right)^8 \qquad \textbf{(1.6.2.8)}$$

It is however critical: the expansion of Z0 at yc2 cancels the denominators, so that the singular behavior of B and C is different than A, but the singularity is the same.

Analycity checks in Lemma 28

Analysis of the discriminant of the polynom P(z,y,u) with respect to z

> $discPz := factor(discrim(eqZtyU\_nuc, Zty));$

$$discPz := -70368744177664\left(5651267143366715843277761 + 2135765424527671862388320\sqrt{7}\right)\left(127528541372376\,U^7\sqrt{7} - 131372093318490\,U^6\sqrt{7} + 94633979763528\,U^5\sqrt{7} - 47442580518384\,U^4\sqrt{7} + 16146783219456\,U^3\sqrt{7} - 3541712816736\,U^2\sqrt{7} + 196919376487668\,U^9 y^3 - 121986056250126\,U^{10} y^3 + 213996873925044\,U^5 y\sqrt{7} - 17162026585668\,U^4 y^2\sqrt{7} - 45418487296\,U^3 y^3\sqrt{7} - 1501103501568\,U + 90459586834974\,U^5 y^2\sqrt{7} + 703986553088\,U^4 y^3\sqrt{7} - 55600684256592\,U^3 - 144603922678272\,U^{13} y^2 - 6694626049920\,U^{12} y^3 + 962017763373504\,U^{12} y^2 + 43229927844576\,U^{11} y^3 + 139248221838336\,U^{12} y - 2902715470511424\,U^{11} y^2 \qquad \textbf{(1.7.1)}$$

$$
\begin{aligned}
&- 833902456707072\, U^{11}\, y + 5234087539591920\, U^{10}\, y^2 \\
&+ 2317231852583904\, U^{10}\, y - 3949965605416032\, U^9\, y \\
&+ 1022251898533536\, \sqrt{7}\, U^9\, y^2 + 141766933786128\, U^8 - 20584878209280\, U^9 \\
&- 4143505176576\, U^{10} - 6252718947420384\, U^9\, y^2 - 7185014295552\, \sqrt{7}\, U^{10} \\
&+ 35453151869760\, \sqrt{7}\, U^9 - 84674533738536\, \sqrt{7}\, U^8 \\
&- 301172170942062\, U^6\, y^2\, \sqrt{7} - 4624314834496\, U^5\, y^3\, \sqrt{7} \\
&- 426614992942158\, U^6\, y\, \sqrt{7} - 2003286603456\, U^2\, y\, \sqrt{7} \\
&+ 112619310336\, U\, y\, \sqrt{7} + 575844892001064\, U^7\, y\, \sqrt{7} \\
&- 508549946247072\, y\, U^8\, \sqrt{7} + 668071189460880\, U^7\, y^2\, \sqrt{7} \\
&+ 16962739458336\, U^6\, y^3\, \sqrt{7} - 25026513408\, \sqrt{7} - 320695365063744\, U^7 \\
&+ 12052874137248\, U^2 + 162868186404648\, U^4 - 314636424261660\, U^5 \\
&+ 400313435138388\, U^6 - 366995784960\, U\, y + 5180301970700652\, U^8\, y^2 \\
&+ 449721068544\, \sqrt{7}\, U + 4584619950460704\, y\, U^8 \\
&+ 30336832250982\, U^{10}\, y^3\, \sqrt{7} - 52424475741900\, U^9\, y^3\, \sqrt{7} \\
&- 1006674881456088\, U^8\, y^2\, \sqrt{7} + 56014843505040\, U^8\, y^3\, \sqrt{7} \\
&- 38457150334560\, U^7\, y^3\, \sqrt{7} + 1338925209984\, \sqrt{7}\, U^{12}\, y^3 \\
&- 44184531929472\, \sqrt{7}\, U^{12}\, y^2 - 9806387417568\, \sqrt{7}\, U^{11}\, y^3 \\
&+ 10711401679872\, \sqrt{7}\, U^{12}\, y + 258263796059136\, \sqrt{7}\, U^{11}\, y^2 \\
&+ 2267703785859786\, U^6\, y - 975077983592964\, U^5\, y + 293534205138504\, U^4\, y \\
&- 58568067179856\, U^3\, y + 6932806827264\, U^2\, y - 3024446597145540\, U^7\, y^2 \\
&+ 1237174345897155\, U^6\, y^2 - 344742257331972\, U^5\, y^2 + 61640388287352\, U^4\, y^2 \\
&- 6264718715808\, U^3\, y^2 + 267601093200\, U^2\, y^2 - 3791347450650648\, U^7\, y \\
&- 199990559776176\, U^8\, y^3 + 132239260063008\, U^7\, y^3 - 56775710297376\, U^6\, y^3 \\
&+ 15205395687104\, U^5\, y^3 - 2294101223680\, U^4\, y^3 + 148006530560\, U^3\, y^3 \\
&- 24596552005632\, \sqrt{7}\, U^{11}\, y - 671606446489056\, \sqrt{7}\, U^{10}\, y^2 \\
&- 43326352499616\, \sqrt{7}\, U^{10}\, y + 260957445157152\, \sqrt{7}\, U^9\, y \\
&+ 15760540216656\, U^3\, y\, \sqrt{7} - 82118247120\, U^2\, y^2\, \sqrt{7} \\
&- 72277443842472\, U^4\, y\, \sqrt{7} + 1832709559464\, U^3\, y^2\, \sqrt{7} + 81554618880\Big) \\
&\Big(54\, U^2\, y\, \sqrt{7} + 5832\, U^3\, y + 243\, U^2\, \sqrt{7} - 50\, U\, y\, \sqrt{7} - 8775\, U^2\, y \\
&- 318\, \sqrt{7}\, U - 4\, \sqrt{7}\, y + 3159\, U^2 + 4372\, U\, y + 104\, \sqrt{7} - 3162\, U - 700\, y \\
&+ 704\Big)^2\, \Big(1646848 - 34012224\, U^6\, \sqrt{7} + 165022272\, U^5\, \sqrt{7} \\
&- 293561010\, U^4\, \sqrt{7} + 267305832\, U^3\, \sqrt{7} - 134804376\, U^2\, \sqrt{7} \\
&- 414760176\, U^5\, y\, \sqrt{7} + 135670545\, U^4\, y^2\, \sqrt{7} + 14618688\, U \\
&- 105500880\, U^5\, y^2\, \sqrt{7} + 845012088\, U^3 - 1646848\, y - 2411360\, y^2 \\
&+ 34012224\, U^6\, y^2\, \sqrt{7} + 127545840\, U^6\, y\, \sqrt{7} + 188738400\, U^2\, y\, \sqrt{7} \\
&- 42614528\, U\, y\, \sqrt{7} - 3898496\, \sqrt{7} - 213345432\, U^2 + 30835168\, U\, y^2 \\
&- 1534639770\, U^4 + 1328996160\, U^5 - 442158912\, U^6 - 7325696\, U\, y
\end{aligned}
$$

$+ 35842560\sqrt{7}\,U + 3898496\sqrt{7}\,y + 462112\sqrt{7}\,y^2 + 3150382248\,U^6\,y$
$- 4397024736\,U^5\,y + 3167198820\,U^4\,y - 1214502336\,U^3\,y + 218026416\,U^2\,y$
$- 170061120\,U^6\,y^2 + 516166992\,U^5\,y^2 - 650796525\,U^4\,y^2 + 433777140\,U^3\,y^2$
$- 160167500\,U^2\,y^2 - 918330048\,U^7\,y - 6293120\,U\,y^2\sqrt{7}$
$- 440304336\,U^3\,y\sqrt{7} + 33626860\,U^2\,y^2\sqrt{7} + 579846600\,U^4\,y\sqrt{7}$
$- 91446300\,U^3\,y^2\sqrt{7}\Big)^2\,\big(5832\,U^3 + 54\,U^2\sqrt{7} - 8775\,U^2 - 50\sqrt{7}\,U$
$+ 4372\,U - 4\sqrt{7} - 700\big)\,y^6\,(-1 + 2\,U)^{18}$

> *nops*(*discPz*);

$$8 \tag{1.7.2}$$

> *op*(1 ..2, *discPz*);

$-70368744177664,\ 5651267143366715843277761$
$+ 2135765424527671862388320\sqrt{7}$ **(1.7.3)**

> *factor1discPz* := *op*(3, *discPz*);

factor1discPz := $127528541372376\,U^7\sqrt{7} - 131372093318490\,U^6\sqrt{7}$ **(1.7.4)**
$+ 94633979763528\,U^5\sqrt{7} - 47442580518384\,U^4\sqrt{7}$
$+ 16146783219456\,U^3\sqrt{7} - 3541712816736\,U^2\sqrt{7} + 196919376487668\,U^9\,y^3$
$- 121986056250126\,U^{10}\,y^3 + 213996873925044\,U^5\,y\sqrt{7}$
$- 17162026585668\,U^4\,y^2\sqrt{7} - 45418487296\,U^3\,y^3\sqrt{7} - 1501103501568\,U$
$+ 90459586834974\,U^5\,y^2\sqrt{7} + 703986553088\,U^4\,y^3\sqrt{7} - 55600684256592\,U^3$
$- 144603922678272\,U^{13}\,y^2 - 6694626049920\,U^{12}\,y^3 + 962017763373504\,U^{12}\,y^2$
$+ 43229927844576\,U^{11}\,y^3 + 139248221838336\,U^{12}\,y$
$- 2902715470511424\,U^{11}\,y^2 - 833902456707072\,U^{11}\,y$
$+ 5234087539591920\,U^{10}\,y^2 + 2317231852583904\,U^{10}\,y$
$- 3949965605416032\,U^9\,y + 1022251898533536\sqrt{7}\,U^9\,y^2$
$+ 141766933786128\,U^8 - 20584878209280\,U^9 - 4143505176576\,U^{10}$
$- 6252718947420384\,U^9\,y^2 - 7185014295552\sqrt{7}\,U^{10}$
$+ 35453151869760\sqrt{7}\,U^9 - 84674533738536\sqrt{7}\,U^8$
$- 301172170942062\,U^6\,y^2\sqrt{7} - 4624314834496\,U^5\,y^3\sqrt{7}$
$- 426614992942158\,U^6\,y\sqrt{7} - 2003286603456\,U^2\,y\sqrt{7}$
$+ 112619310336\,U\,y\sqrt{7} + 575844892001064\,U^7\,y\sqrt{7}$
$- 508549946247072\,y\,U^8\sqrt{7} + 668071189460880\,U^7\,y^2\sqrt{7}$
$+ 16962739458336\,U^6\,y^3\sqrt{7} - 25026513408\sqrt{7} - 320695365063744\,U^7$
$+ 12052874137248\,U^2 + 162868186404648\,U^4 - 314636424261660\,U^5$
$+ 400313435138388\,U^6 - 366995784960\,U\,y + 5180301970700652\,U^8\,y^2$
$+ 449721068544\sqrt{7}\,U + 4584619950460704\,y\,U^8$
$+ 30336832250982\,U^{10}\,y^3\sqrt{7} - 52424475741900\,U^9\,y^3\sqrt{7}$
$- 1006674881456088\,U^8\,y^2\sqrt{7} + 56014843505040\,U^8\,y^3\sqrt{7}$
$- 38457150334560\,U^7\,y^3\sqrt{7} + 1338925209984\sqrt{7}\,U^{12}\,y^3$
$- 44184531929472\sqrt{7}\,U^{12}\,y^2 - 9806387417568\sqrt{7}\,U^{11}\,y^3$

$+ 10711401679872 \sqrt{7} U^{12} y + 258263796059136 \sqrt{7} U^{11} y^2$
$+ 2267703785859786 U^6 y - 975077983592964 U^5 y + 293534205138504 U^4 y$
$- 58568067179856 U^3 y + 6932806827264 U^2 y - 3024446597145540 U^7 y^2$
$+ 1237174345897155 U^6 y^2 - 344742257331972 U^5 y^2 + 61640388287352 U^4 y^2$
$- 6264718715808 U^3 y^2 + 267601093200 U^2 y^2 - 3791347450650648 U^7 y$
$- 199990559776176 U^8 y^3 + 132239260063008 U^7 y^3 - 56775710297376 U^6 y^3$
$+ 15205395687104 U^5 y^3 - 2294101223680 U^4 y^3 + 148006530560 U^3 y^3$
$- 24596552005632 \sqrt{7} U^{11} y - 671606446489056 \sqrt{7} U^{10} y^2$
$- 43326352499616 \sqrt{7} U^{10} y + 260957445157152 \sqrt{7} U^9 y$
$+ 15760540216656 U^3 y \sqrt{7} - 82118247120 U^2 y^2 \sqrt{7}$
$- 72277443842472 U^4 y \sqrt{7} + 1832709559464 U^3 y^2 \sqrt{7} + 81554618880$

> *factor2discPz* := *op*(1, *op*(4, *discPz*));

$factor2discPz := 54 U^2 y \sqrt{7} + 5832 U^3 y + 243 U^2 \sqrt{7} - 50 U y \sqrt{7} - 8775 U^2 y - 318 \sqrt{7} U - 4 \sqrt{7} y + 3159 U^2 + 4372 U y + 104 \sqrt{7} - 3162 U - 700 y + 704$ **(1.7.5)**

> *factor3discPz* := *op*(1, *op*(5, *discPz*));

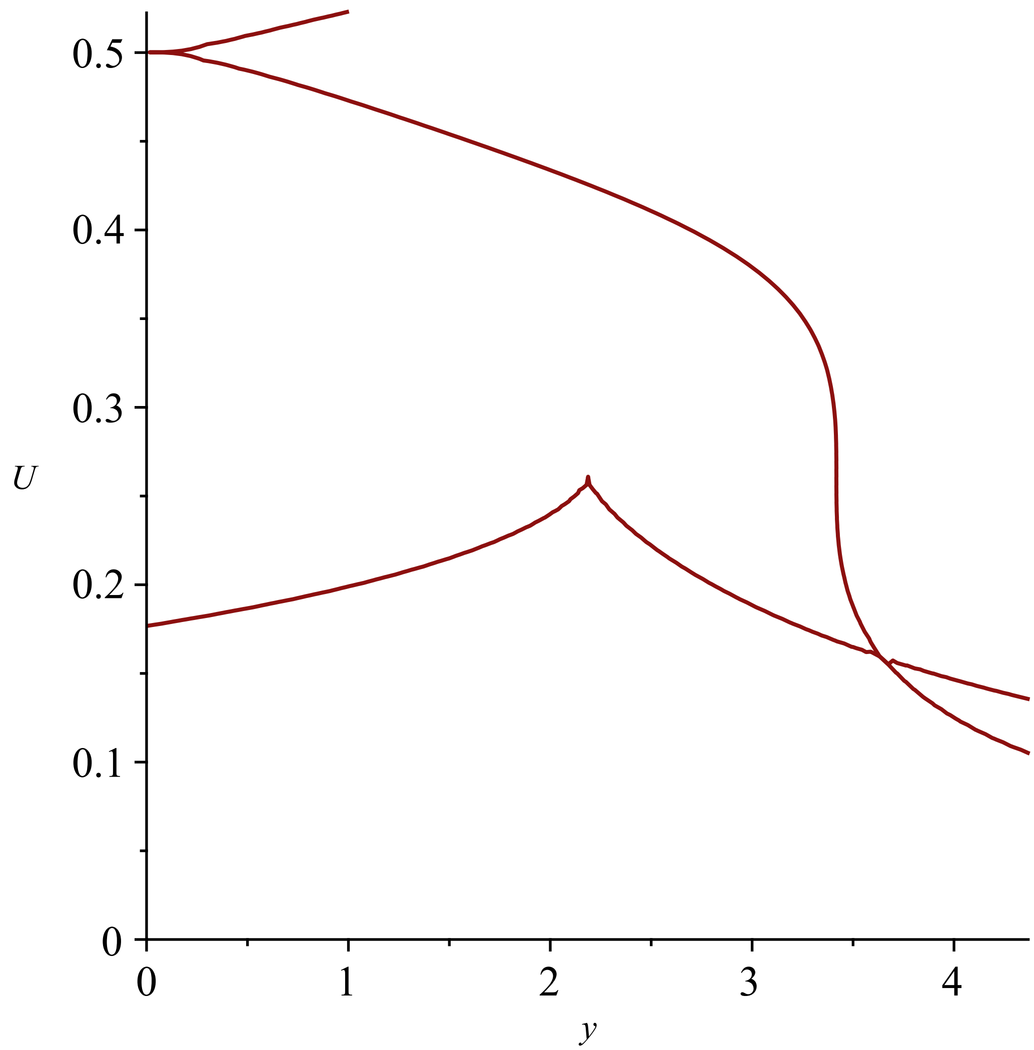
$factor3discPz := 1646848 - 34012224 U^6 \sqrt{7} + 165022272 U^5 \sqrt{7} - 293561010 U^4 \sqrt{7} + 267305832 U^3 \sqrt{7} - 134804376 U^2 \sqrt{7} - 414760176 U^5 y \sqrt{7} + 135670545 U^4 y^2 \sqrt{7} + 14618688 U - 105500880 U^5 y^2 \sqrt{7} + 845012088 U^3 - 1646848 y - 2411360 y^2 + 34012224 U^6 y^2 \sqrt{7} + 127545840 U^6 y \sqrt{7} + 188738400 U^2 y \sqrt{7} - 42614528 U y \sqrt{7} - 3898496 \sqrt{7} - 213345432 U^2 + 30835168 U y^2 - 1534639770 U^4 + 1328996160 U^5 - 442158912 U^6 - 7325696 U y + 35842560 \sqrt{7} U + 3898496 \sqrt{7} y + 462112 \sqrt{7} y^2 + 3150382248 U^6 y - 4397024736 U^5 y + 3167198820 U^4 y - 1214502336 U^3 y + 218026416 U^2 y - 170061120 U^6 y^2 + 516166992 U^5 y^2 - 650796525 U^4 y^2 + 433777140 U^3 y^2 - 160167500 U^2 y^2 - 918330048 U^7 y - 6293120 U y^2 \sqrt{7} - 440304336 U^3 y \sqrt{7} + 33626860 U^2 y^2 \sqrt{7} + 579846600 U^4 y \sqrt{7} - 91446300 U^3 y^2 \sqrt{7}$ **(1.7.6)**

> *op*(6 ..8, *discPz*);

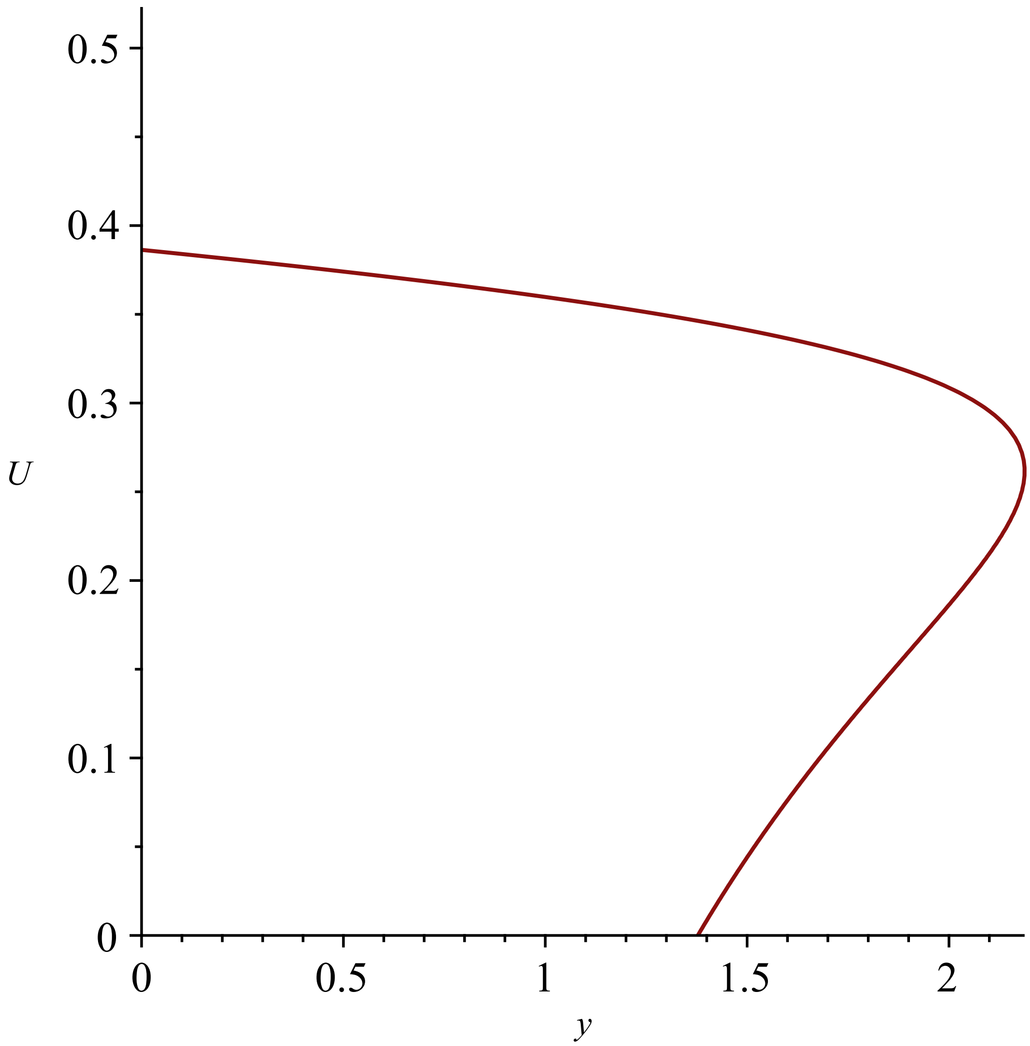
$5832 U^3 + 54 U^2 \sqrt{7} - 8775 U^2 - 50 \sqrt{7} U + 4372 U - 4 \sqrt{7} - 700, y^6, (-1 + 2 U)^{18}$ **(1.7.7)**

There are thus 3 non trivial factors of degree at most 3 in y so a few simple implicit plots allow to understand the landscape

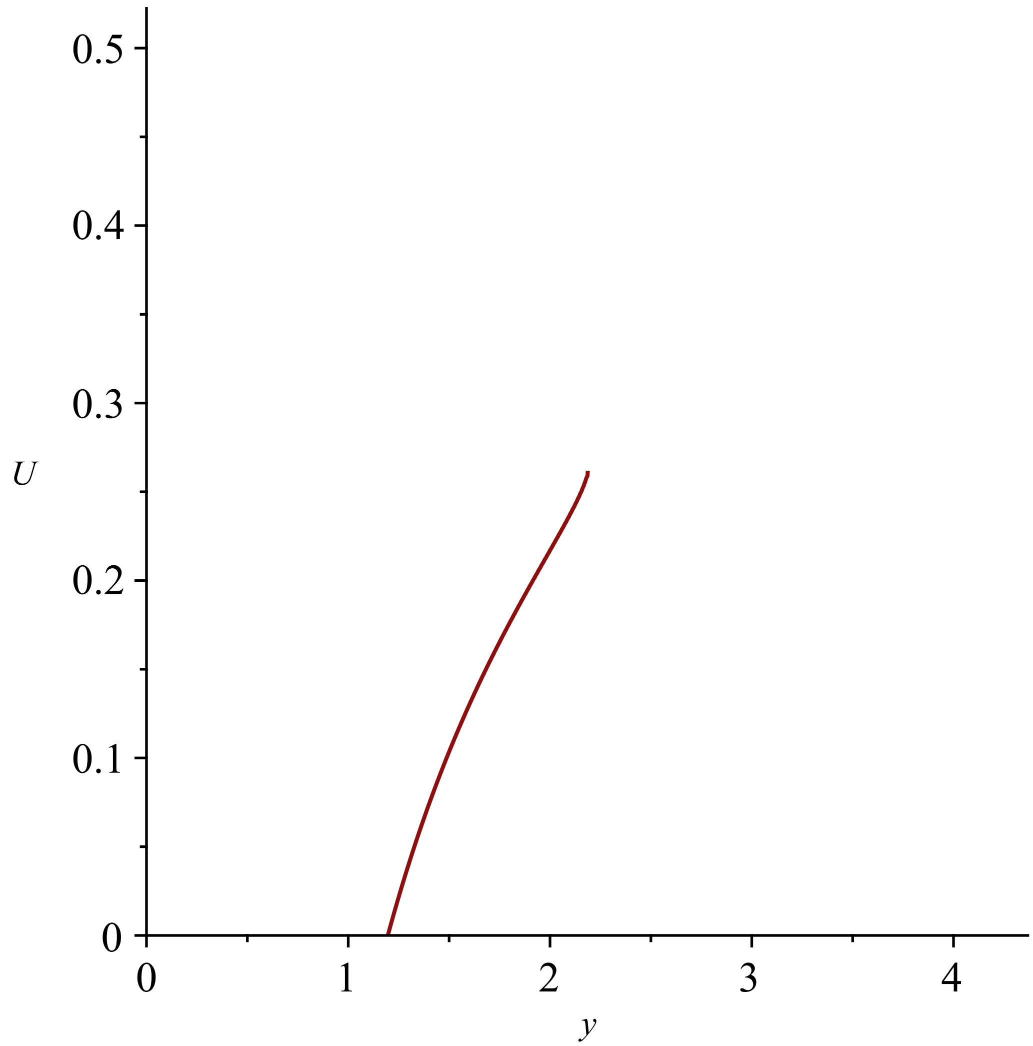
> *implicitplot*(*factor1discPz*, *y* = 0 ..2·*yc2*, *U* = 0 ..2 ·*Uc*, *numpoints* = 10000, *view* = [0 ..2 · *yc2*, 0 ..2·*Uc*]);



> $implicitplot(factor2discPz, y = 0 ..yc2, U = 0 ..2 \cdot Uc, numpoints = 10000, rangeasview = true, view = [0 ..2 \cdot yc2, 0 ..2 \cdot Uc]);$



> *implicitplot*(*factor3discPz*, $y = 0..yc2$, $U = 0..2 \cdot Uc$, *numpoints* = 10000, *view* = $[0..2 \cdot yc2, 0..2 \cdot Uc]$);



0.5
0.4
0.3
U
0.2
0.1
0
0
1
2
3
4
y

>