> *restart*;

> *with*(*algcurves*) : *with*(*gfun*) : *with*(*plots*) : *with*(*CurveFitting*) :

Quantities, equations and parametrizations from the previous paper [AMS 21] (Section 2.2)

Equation for simple monochromatic boundary condition Z(t,y), equation 2.23 of [AMS 21]

$$> eqZ := 2t^2\nu(\nu-1)Z^3 + t\left(\nu^2 t^2 y^2 + (4\nu^2-4\nu)t - y(\nu+2)(\nu-1)\right)Z^2 + \left(2\nu^2 t^3 y^2 + 2\left((y^3 - Z1\,y+1)\nu - \frac{3}{2}y^3 + Z1\,y - 1\right)\nu t^2 - y t(\nu+2)(\nu-1) + y^2(\nu-1)\right)Z + y t\left(y^2\nu^2(y^2-2Z1)t^2 + (\nu-1)\left(y^2 + (-2Z1^2 - 2Z2)y - 2Z1\right)\nu t - y(\nu-1)\left((y^2-Z1)\nu - 2Z1\right)\right) :$$

Rational parametrization

$$> P := collect\left((8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu), U, factor\right);$$

$$P := 8(\nu+1)^2U^3 - (11\nu+13)(\nu+1)U^2 + 2(\nu+3)(2\nu+1)U - 4\nu \quad \textbf{(1.1)}$$

Since all the generating series in t that we consider are actually series in t^3, we set **w := t^3.**
We have the following parametrization of w in terms of U:

$$> wU := \frac{1}{32}\,\frac{(U\cdot(\nu+1)-2)\,U\cdot P}{(-1+2U)^2\nu^3} :$$

$$> tZ1U := \frac{1}{2}\left((6U^3\nu^2 + 12U^3\nu - 8U^2\nu^2 + 6U^3 - 16U^2\nu + 3U\nu^2 - 8U^2 + 7U\nu + 4U - 2\nu)U(\nu+1)\right)\Big/\left((8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu)\nu\right) :$$

$$> t2Z2U := -\frac{1}{64}\left(U(U\nu+U-2)\left(1152U^8\nu^5 + 5760U^8\nu^4 - 4872U^7\nu^5 + 11520U^8\nu^3 - 23832U^7\nu^4 + 8589U^6\nu^5 + 11520U^8\nu^2 - 46608U^7\nu^3 + 42693U^6\nu^4 - 8084U^5\nu^5 + 5760U^8\nu - 45552U^7\nu^2 + 83158U^6\nu^3 - 43450U^5\nu^4 + 4288U^4\nu^5 + 1152U^8 - 22248U^7\nu + 79206U^6\nu^2 - 85872U^5\nu^3 + 27556U^4\nu^4 - 1216U^3\nu^5 - 4344U^7 + 36765U^6\nu - 78884U^5\nu^2 + 56384U^4\nu^3 - 11072U^3\nu^4 + 144U^2\nu^5 + 6613U^6 - 33532U^5\nu + 49088U^4\nu^2 - 24320U^3\nu^3 + 2640U^2\nu^4 - 5154U^5 + 18048U^4\nu - 19480U^3\nu^2 + 6928U^2\nu^3 - 288U\nu^4 + 2076U^4 - 5520U^3\nu + 4704U^2\nu^2 - 1280U\nu^3 - 344U^3 + 752U^2\nu - 544U\nu^2 + 128\nu^3\right)\right)\Big/\left(\nu^2(8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu)^2(-1+2U)^2\right) :$$

$$> algU := collect(numer(wU - w), U, factor) :$$

The two polynoms giving the radius of convergence rho

> $algrhosubc := 27648\,\nu^4 w^2 + 864\,\nu\,(\nu - 1)\,(\nu^2 - 2\,\nu - 1)\,w + (7\,\nu^2 - 14\,\nu - 9)\,(-2 + \nu)^2;$

$$algrhosubc := 27648\,\nu^4 w^2 + 864\,\nu\,(\nu - 1)\,(\nu^2 - 2\,\nu - 1)\,w + (7\,\nu^2 - 14\,\nu - 9)\,(-2 + \nu)^2 \quad \textbf{(1.2)}$$

> $algrhosupc := 131072\,\nu^9 w^3 - 192\,\nu^6\,(3\,\nu + 5)\,(\nu - 1)\,(3\,\nu - 11)\,w^2 - 48\,\nu^3\,(\nu - 1)^2\,w + (\nu - 1)\,(4\,\nu^2 - 8\,\nu - 23);$

$$algrhosupc := 131072\,\nu^9 w^3 - 192\,\nu^6\,(3\,\nu + 5)\,(\nu - 1)\,(3\,\nu - 11)\,w^2 - 48\,\nu^3\,(\nu - 1)^2\,w + (\nu - 1)\,(4\,\nu^2 - 8\,\nu - 23) \quad \textbf{(1.3)}$$

Phase transition at

> $nuc := 1 + \frac{1}{\text{sqrt}(7)};\ Uc := \frac{5}{9} - \frac{\sqrt{7}}{9};$

$$nuc := 1 + \frac{\sqrt{7}}{7}$$

$$Uc := \frac{5}{9} - \frac{\sqrt{7}}{9} \quad \textbf{(1.4)}$$

> $rhoc := simplify(rationalize(subs(\text{nu} = nuc, rhosubc)));$
$simplify(rationalize(\ subs(U = Uc, \text{nu} = nuc, wU)));$

$$rhoc := rhosubc$$

$$-\frac{55}{864} + \frac{25\sqrt{7}}{864} \quad \textbf{(1.5)}$$

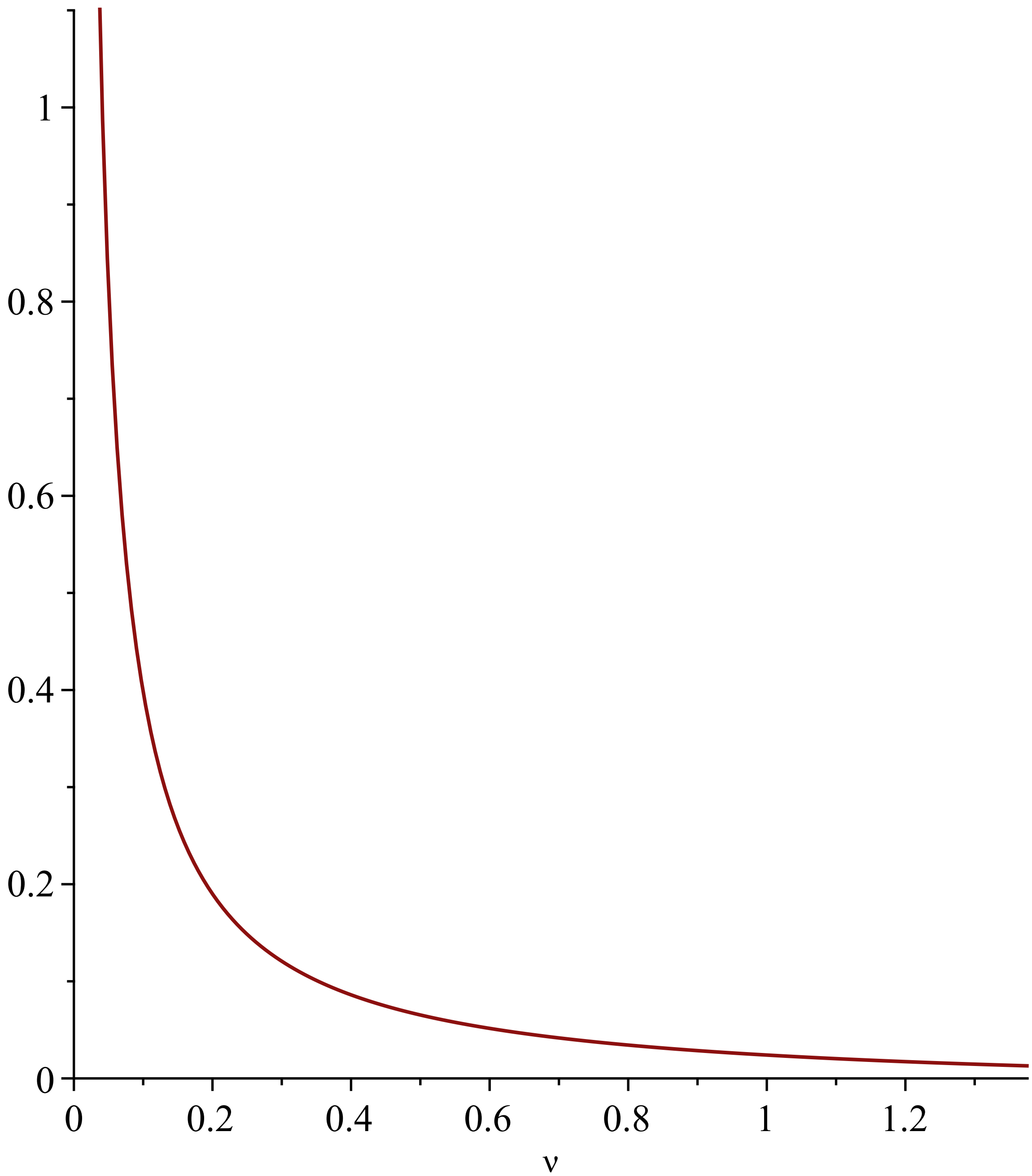
For nu subcritical, we can obtain an explicit expression for the radius of convergence:

> $rhosubc1, rhosubc2 := solve(algrhosubc, w) :$
$simplify(rhosubc1);\ simplify(rhosubc2);$

$$\frac{-9\,\nu^3 + 27\,\nu^2 + \sqrt{3}\,\sqrt{-(\nu^2 - 2\,\nu - 3)^3} - 9\,\nu - 9}{576\,\nu^3}$$

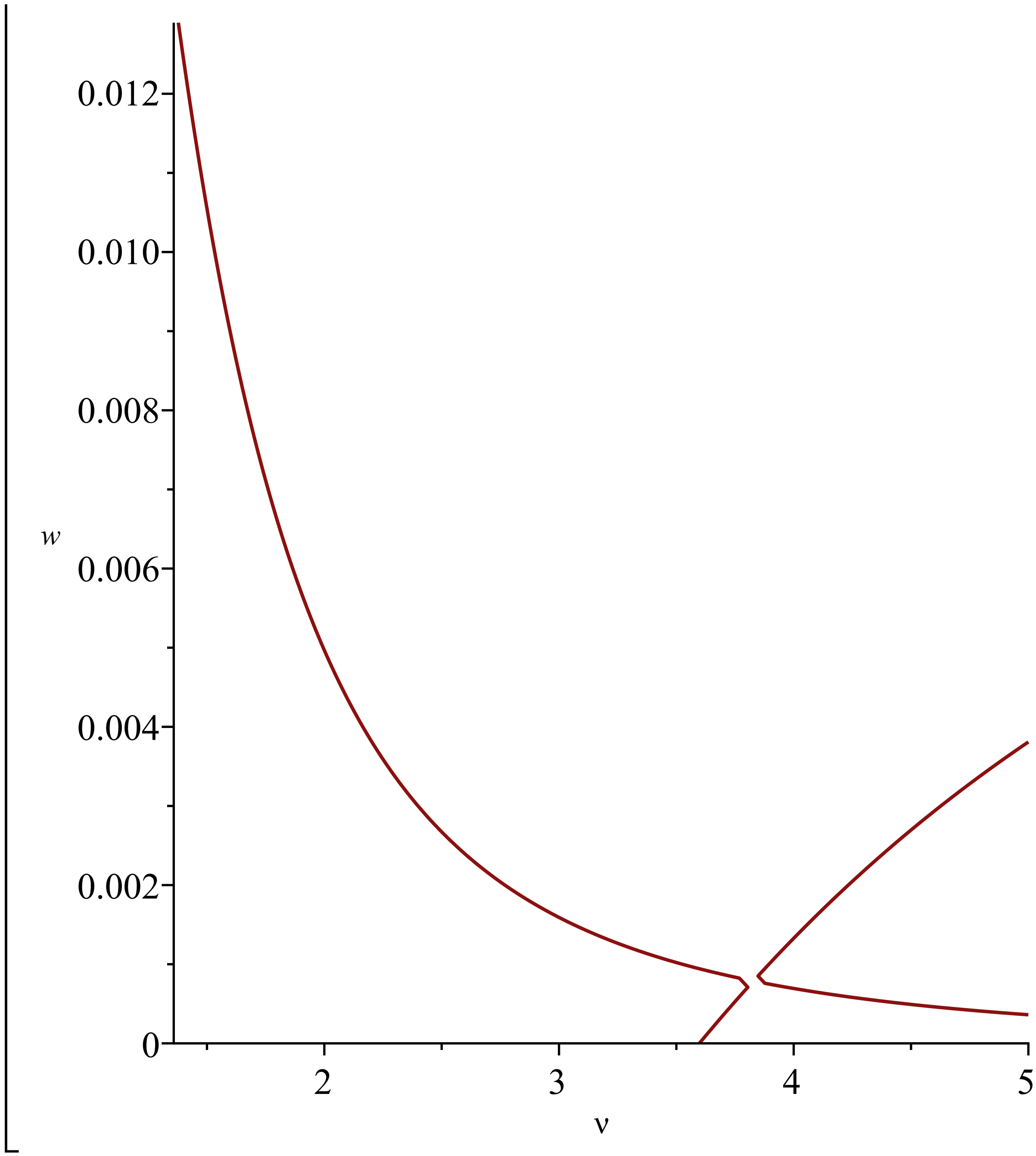
$$\frac{-9\,\nu^3 + 27\,\nu^2 - \sqrt{3}\,\sqrt{-(\nu^2 - 2\,\nu - 3)^3} - 9\,\nu - 9}{576\,\nu^3} \quad \textbf{(1.6)}$$

> $rhosubc := \frac{-9\,\nu^3 + 27\,\nu^2 + \sqrt{3}\,\sqrt{-(\nu^2 - 2\,\nu - 3)^3} - 9\,\nu - 9}{576\,\nu^3} : plot(rhosubc, \text{nu} = 0 ..nuc);$



For nu > nuc, the radius of convergence is the positive decreasing branch of algrosupc:

> *implicitplot*(*algrhosupc*, nu = *nuc* ..5, *w* = 0 ..0.1);



Critical values U(nu,tnu^3) (Proposition 2.2)

Equations for U at criticality and parametrization of the critical line

> *numer*(*factor*(diff(wU, U)));

$$\left(3 U^2 \nu + 3 U^2 - 3 U \nu - 3 U + \nu\right) \left(4 U^3 \nu^2 + 8 U^3 \nu - 3 U^2 \nu^2 + 4 U^3 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U - 2\right) \quad \textbf{(1.1.1)}$$

First factor is for nu < nuc, second for nu>nuc

> $algUsubcrit := \left(3 U^2 \nu - 3 U \nu + \nu - 3 U + 3 U^2\right)$:

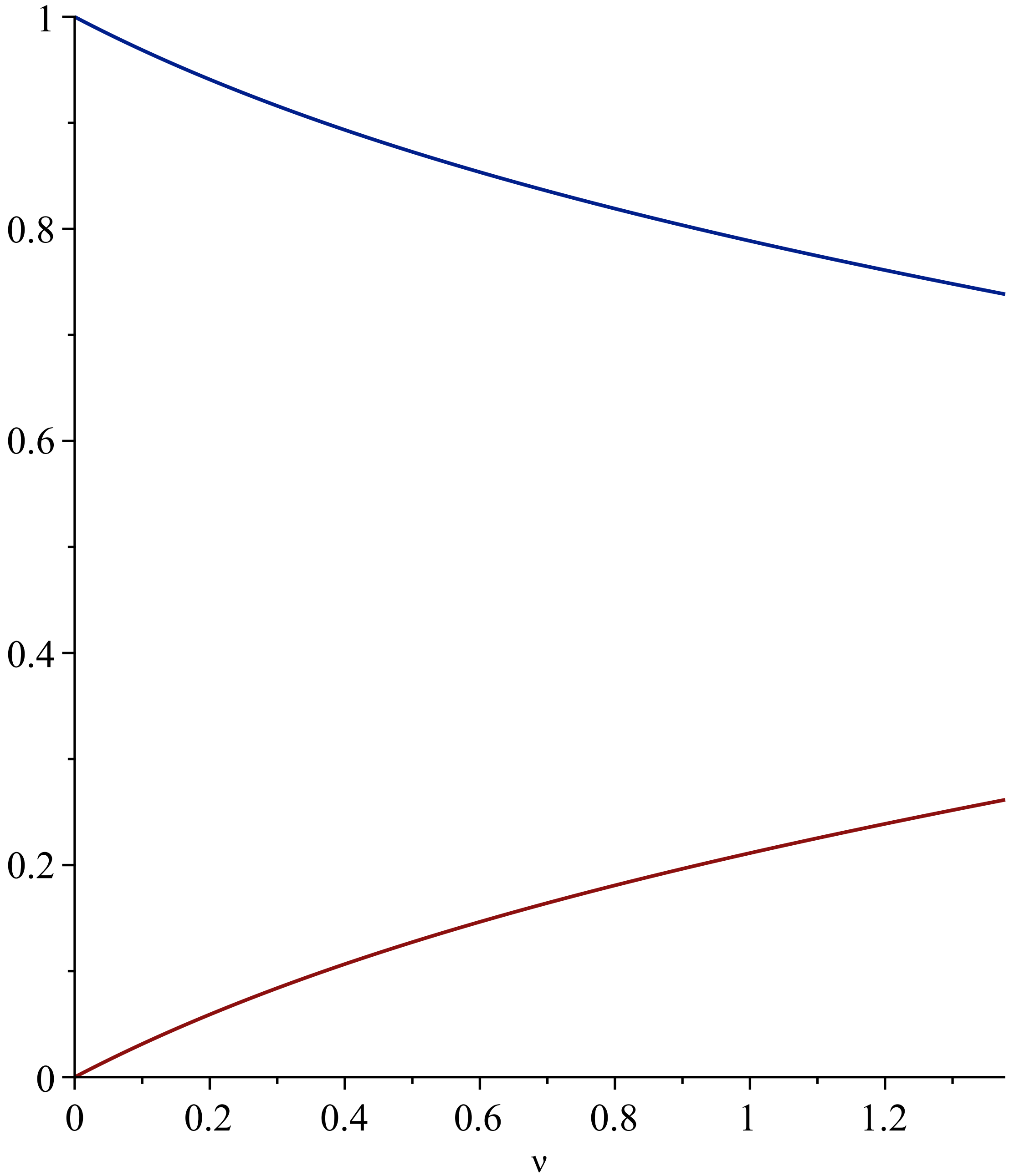
> $algUsupcrit := \left(4 U^3 \nu^2 + 8 U^3 \nu + 4 U^3 - 3 U^2 \nu^2 - 12 U^2 \nu - 9 U^2 + 6 U \nu + 6 U\right.$

$-2)$:

For nu <nuc, the value of Unu is the smallest positive branch of algUsubcrit

> *solve*(*algUsubcrit*, *U*); *plot*({%}, nu = 0 ..*nuc*);

$$\frac{3\nu+3+\sqrt{-3\nu^2+6\nu+9}}{6(\nu+1)}, -\frac{-3\nu-3+\sqrt{-3\nu^2+6\nu+9}}{6(\nu+1)}$$



> $Unusubc := \frac{3\nu+3-\sqrt{-3\nu^2+6\nu+9}}{6(\nu+1)}$; *nuUsub* := *solve*(*algUsubcrit*, nu);

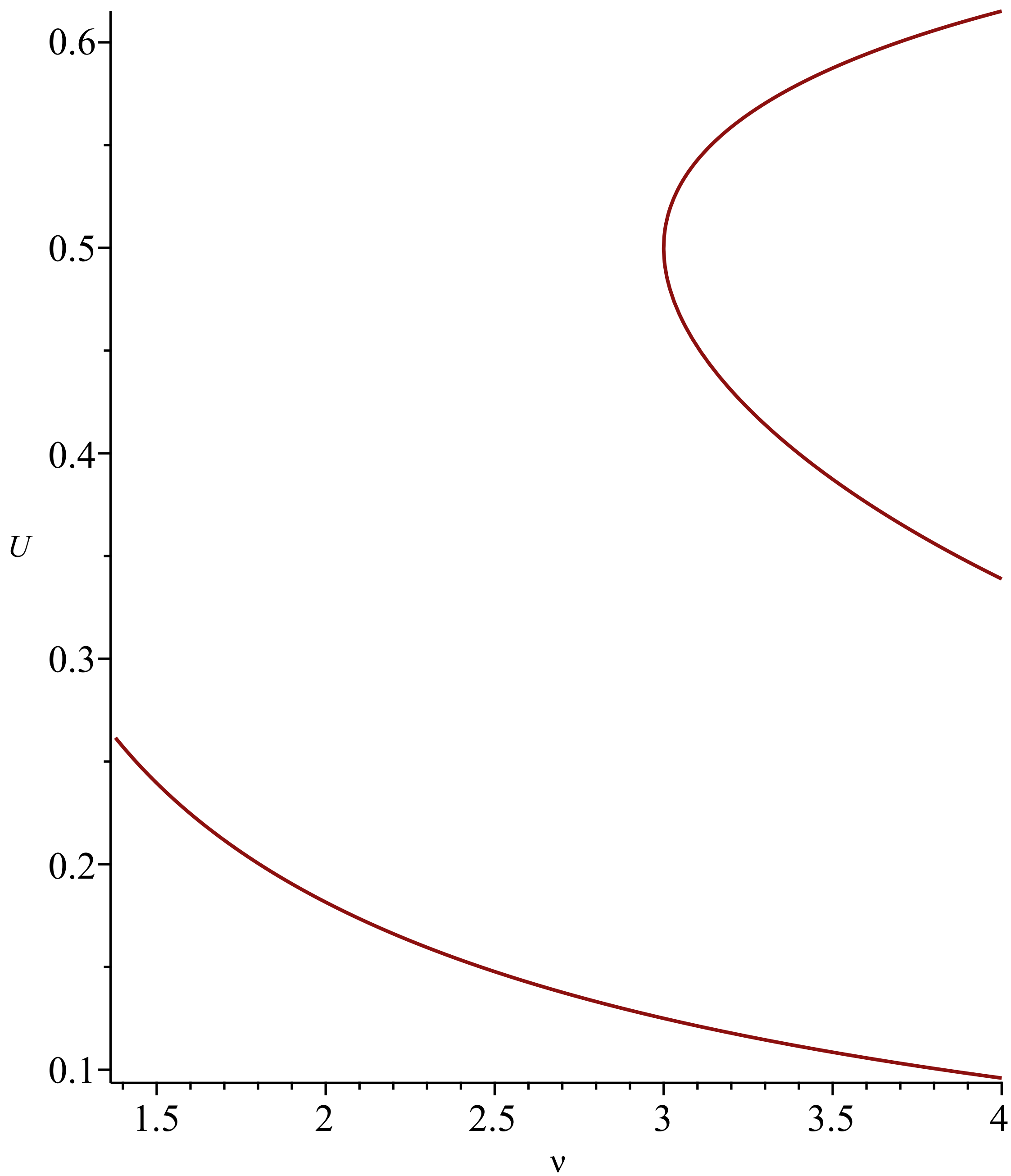
$$Unusubc := \frac{3\nu+3-\sqrt{-3\nu^2+6\nu+9}}{6\nu+6}$$

$$nuUsub := -\frac{3\,U\,(U-1)}{3\,U^2-3\,U+1} \quad \textbf{(1.1.2)}$$

When nu >nuc it is the smallest positive root of algUsupc:

We look at the solutions of algUsubcrit and algUsupcrit, and identify the right branches (in PUsub and PUsur):

> *implicitplot*(*algUsupcrit*, nu = *nuc* ..4, *U* = 0 ..1);

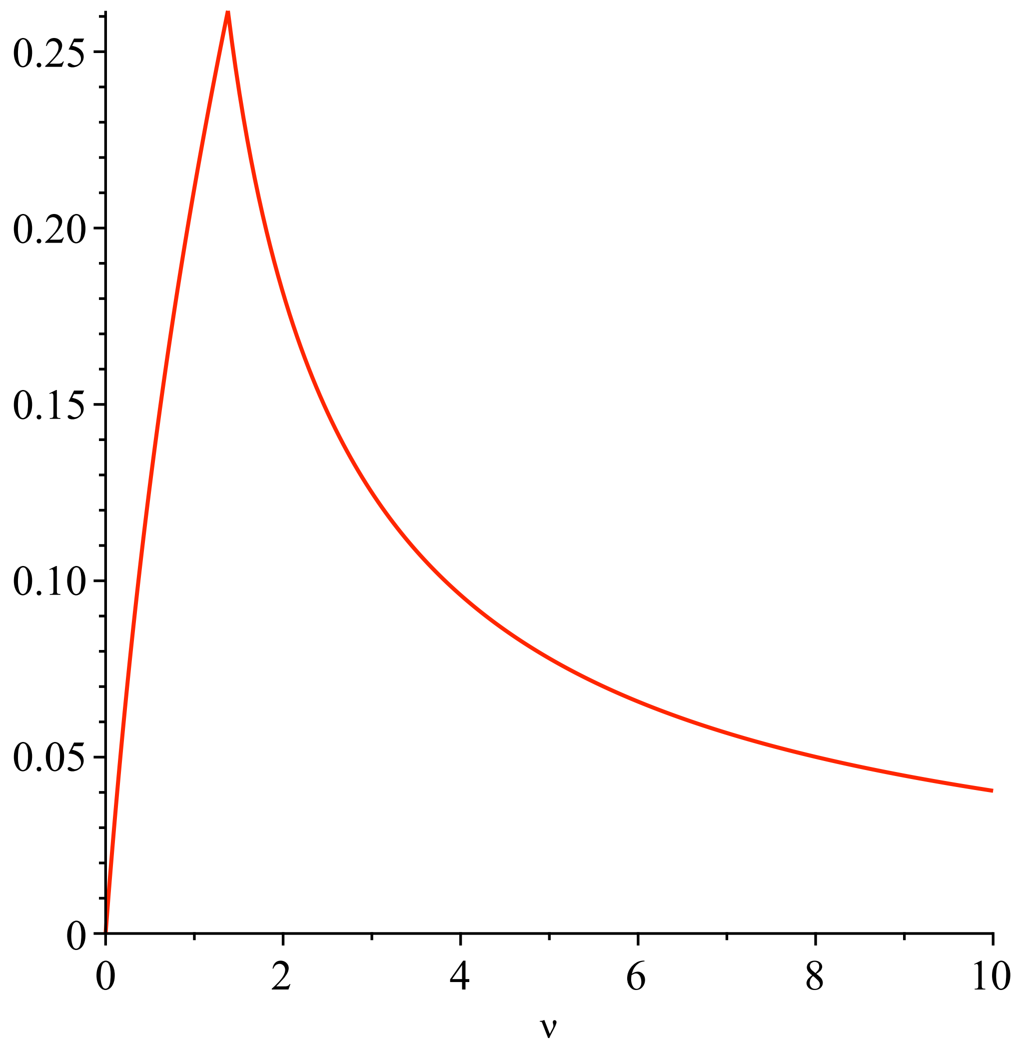


> $PUsub := plot\left(\frac{3\,\nu+3-\sqrt{-3\,\nu^2+6\,\nu+9}}{6\,(\nu+1)}, \mathrm{nu} = 0 \,..nuc, color = red\right):$

> *PUsur* := *plot*(*op*(3, {*solve*(*algUsupcrit*, *U*)}), nu = *nuc* ..10, *color* = *red*) :

The values of Uc

> *display*([*PUsub*, *PUsur*]);



We can parametrize the values of (nu,Unu) with rational functions:

> $UsupK := -\dfrac{K^2-3}{2\,(3\,K+5)}$; $nusupK := factor\left(-\dfrac{K^3+3\,K^2+9\,K+11}{K^3+3\,K^2-3\,K-9}\right)$;
  simplify(*subs*($U = UsupK$, nu $= nusupK$, *algUsupcrit*));

$$UsupK := -\frac{K^2-3}{6\,K+10}$$

$$nusupK := -\frac{K^3+3\,K^2+9\,K+11}{(K+3)\,(K^2-3)}$$

$$0 \qquad \textbf{(1.1.3)}$$

UsupK is clearly a decreasing function of K and to determine if nusupK is decreasing or increasing, we compute its derivative with respect to K:

> *simplify*(*diff*(*nusupK*, *K*));

$$\frac{24\,(K+2)\,(K+1)^2}{(K+3)^2\,(K^2-3)^2} \qquad \textbf{(1.1.4)}$$

So that nusupK is increasing for K>-2 and decreasing otherwise.

We determine the range of values of K which are of interest, Kc=value of K corresponding to Uc and nuc, Kinfini=value of K corresponding to U=0 and nu=infinity.

> $solve(\{UsupK = Uc, nusupK = nuc\}); evalf(\%); Kc := -\frac{2}{3} + \frac{1}{3}\sqrt{7}; Kinfini :=$ sqrt(3);

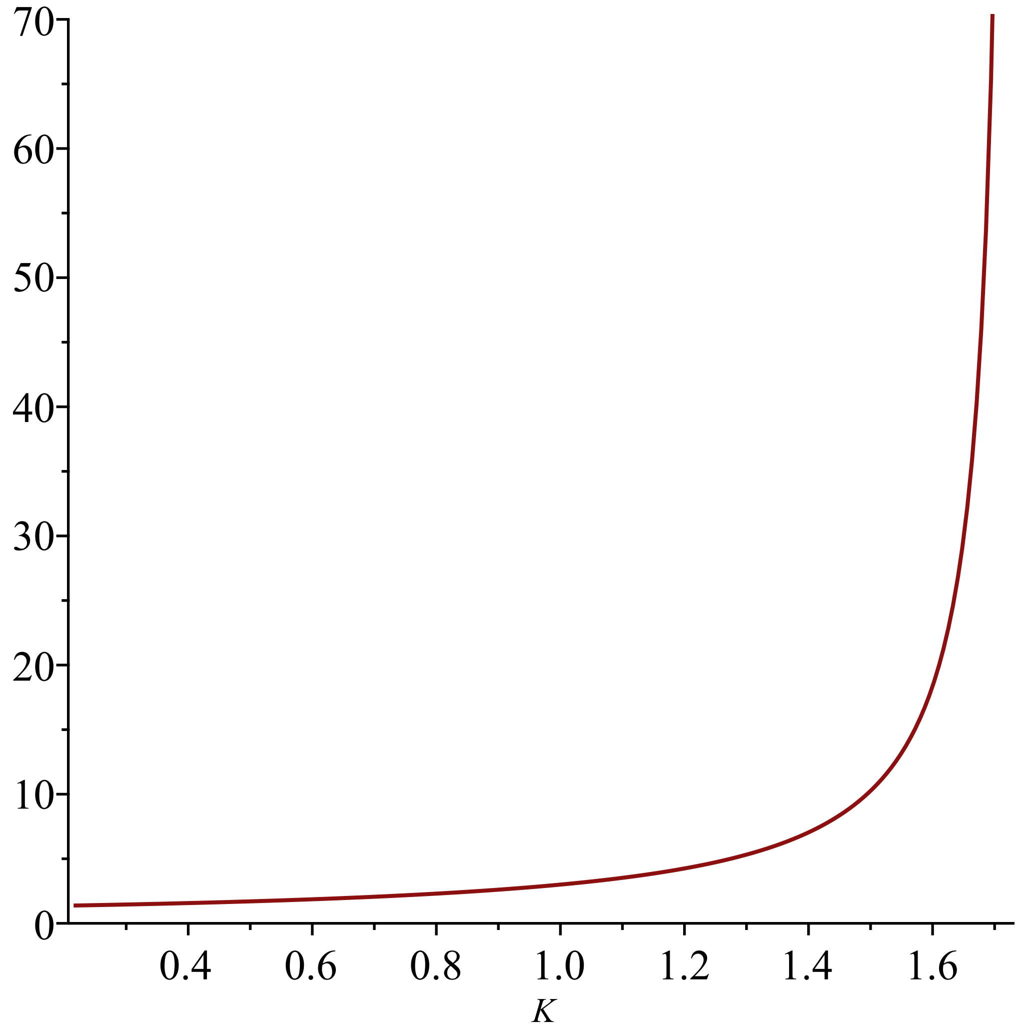
$$\left\{K = -\frac{2}{3} + \frac{\sqrt{7}}{3}\right\}$$

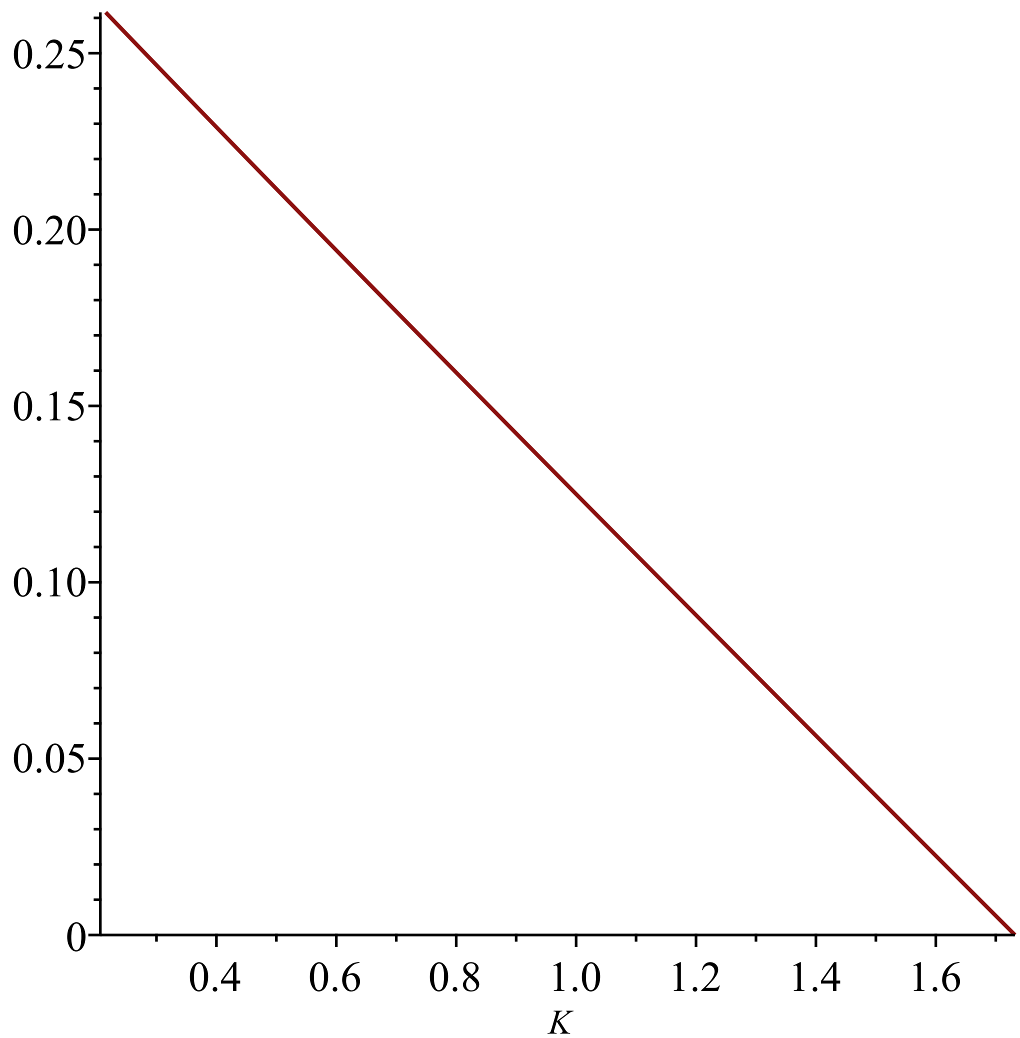
$$\{K = 0.2152504369\}$$

$$Kc := -\frac{2}{3} + \frac{\sqrt{7}}{3}$$

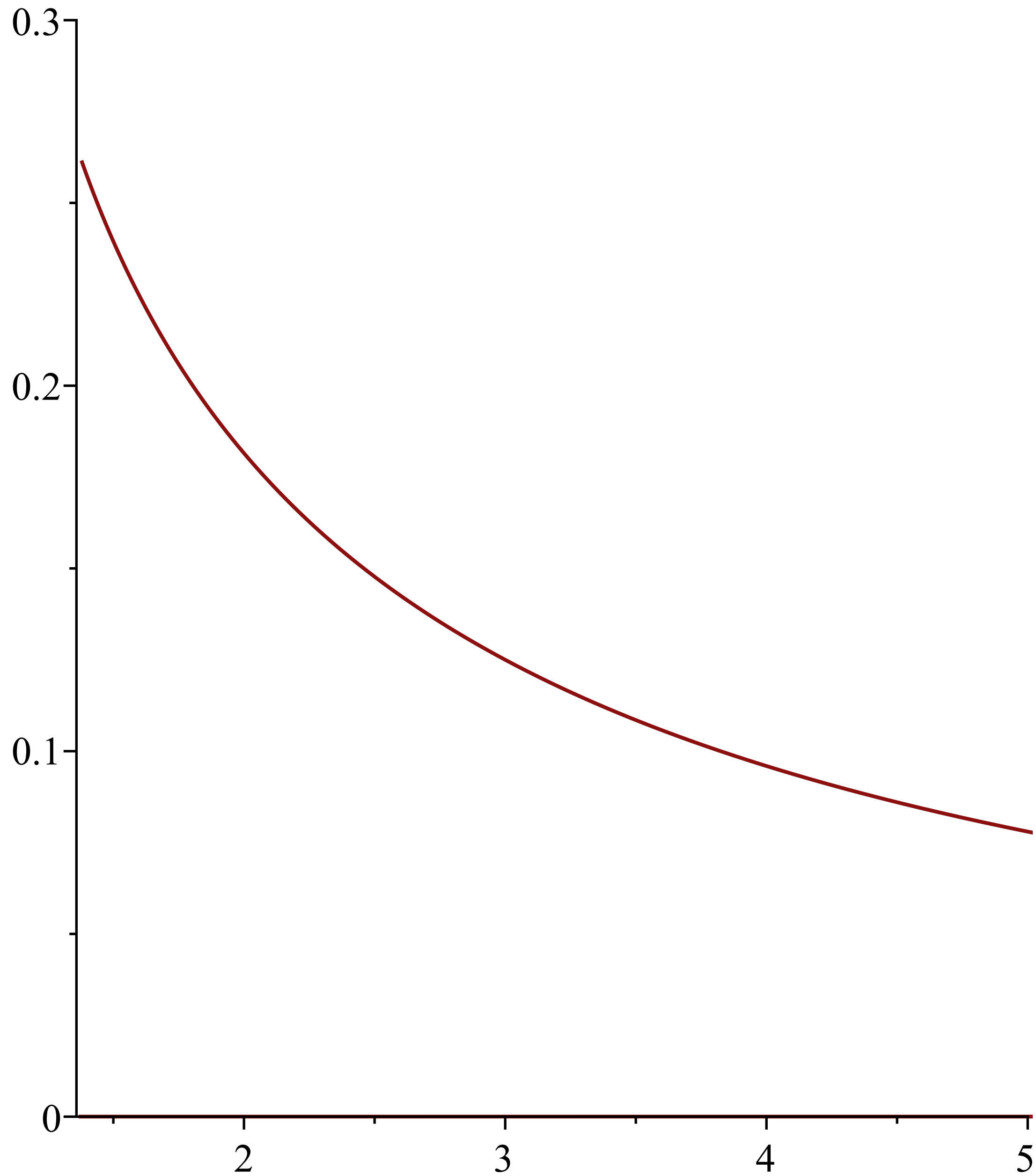
$$Kinfini := \sqrt{3} \quad \textbf{(1.1.5)}$$

> $plot(nusupK, K = Kc..\text{sqrt}(3)); plot(UsupK, K = Kc..\text{sqrt}(3));$





> *plot*([*nusupK*, *UsupK*, *K* = *Kc* ..*Kinfini*], *nuc* ..5, 0 ..0.3);



We always assume Kc<K<Kinfini:

```
> assume(K > Kc and K < Kinfini);
> about(K);
Originally K, renamed K~:
  is assumed to be: RealRange(Open(-2/3+1/3*7^(1/2)),Open(3^
(1/2)))
```

Development in t of U (Lemma 2.3)

Subcritical regime nu < nuc

We start with the equation for U(nu,t^3) here with w=t^3:

> *algU*;

$$8(\nu+1)^3 U^5 - (11\nu+29)(\nu+1)^2 U^4 + 4(\nu+8)(\nu+1)^2 U^3 + (-128 w \nu^3 - 12\nu^2 - 32\nu - 12) U^2 + 8\nu(16\nu^2 w + 1) U - 32 w \nu^3 \quad \textbf{(1.2.1.1)}$$

We replace w by the value of the radius of convergence (=t_\nu^3 in the paper) and compute the corresponding singular behavior of U, (with XX=(1-w/rho)^1/2)

> $op(2, algeqtoseries(subs(w = rhosubc \cdot (1 - XX^2), algU), XX, U, 7));$

$$\frac{1}{6(\nu^3-\nu^2-5\nu-3)}\Big(3\nu^3 + \sqrt{3}\sqrt{-\nu^6+6\nu^5-3\nu^4-28\nu^3+9\nu^2+54\nu+27} - 3\nu^2 - 15\nu - 9\Big) + \mathit{RootOf}\Big((252\nu^6 - 504\nu^5 - 1296\nu^4 + 1008\nu^3 + 1980\nu^2 - 216\nu - 648)\_Z^2 - 13\nu^6 + 3\sqrt{3}\sqrt{-\nu^6+6\nu^5-3\nu^4-28\nu^3+9\nu^2+54\nu+27}\,\nu^3 + 78\nu^5 - 9\sqrt{3}\sqrt{-\nu^6+6\nu^5-3\nu^4-28\nu^3+9\nu^2+54\nu+27}\,\nu^2 - 120\nu^4 - 40\nu^3 + 6\sqrt{3}\sqrt{-\nu^6+6\nu^5-3\nu^4-28\nu^3+9\nu^2+54\nu+27} + 171\nu^2 - 54\nu - 54\Big) XX - \frac{1}{108}\Big(297\nu^9 + 233\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\sqrt{3}\,\nu^6 - 2673\nu^8 - 1398\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^5 + 7578\nu^7 + 3237\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^4 - 3150\nu^6 - 3628\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^3 - 17415\nu^5 + 1440\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^2 + 18783\nu^4 + 648\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\sqrt{3}\,\nu + 10512\nu^3 - 540\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)} - 17820\nu^2 - 972\nu + 4860\Big) \Big/ \Big((7\nu^4 - 28\nu^3 + 13\nu^2 + 30\nu - 18)(7\nu^2 - 14\nu + 6)(\nu^3 - \nu^2 - 5\nu - 3)\Big) XX^2 + \frac{5}{216}\Big(\mathit{RootOf}\Big((252\nu^6 - 504\nu^5 - 1296\nu^4 + 1008\nu^3 + 1980\nu^2 - 216\nu - 648)\_Z^2 - 13\nu^6 \quad \textbf{(1.2.1.2)}$$

$+ 3\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^3 + 78\nu^5$

$- 9\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^2 - 120\nu^4 - 40\nu^3$

$+ 6\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)} + 171\nu^2 - 54\nu - 54\Big)$

$\Big(1334\nu^{10} + 789\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^7 - 13340\nu^9$

$- 5523\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\sqrt{3}\,\nu^6 + 46310\nu^8$

$+ 16158\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^5 - 50320\nu^7$

$- 25560\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^4 - 59060\nu^6$

$+ 20754\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^3 + 162392\nu^5$

$- 4206\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\,\nu^2 - 48486\nu^4$

$- 4896\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)}\sqrt{3}\,\nu - 120456\nu^3$

$+ 2484\sqrt{3}\sqrt{(\nu^2-2\nu-3)^2(-\nu^2+2\nu+3)} + 82242\nu^2 + 21708\nu$

$- 22356\Big)\Big)\Big/\Big(\big(343\nu^9 - 2401\nu^8 + 5341\nu^7 - 2009\nu^6 - 8260\nu^5$

$+ 10570\nu^4 - 876\nu^3 - 5508\nu^2 + 3456\nu - 648\big)(\nu-3)\Big)\,XX^3 + O(XX^4)$

We can now replace nu by its value in terms of Usubc (=value of U corresponding to the radius of convergence rhosubc) in the development:

> *simplify*(*subs*(nu = *nuUsub*, *U* = *Usubc*, **(1.2.1.2)**)) assuming $\left(Usubc < \frac{1}{2}\right)$ **and** $(Usubc > 0)$;

$$Usubc + RootOf\big((54\,Usubc^2 - 60\,Usubc + 12)\,\_Z^2 - 54\,Usubc^6 + 162\,Usubc^5 - 171\,Usubc^4 + 76\,Usubc^3 - 12\,Usubc^2\big)\,XX + \frac{1}{18}\Big(\big(1458\,Usubc^6 - 5778\,Usubc^5 + 9045\,Usubc^4 - 7146\,Usubc^3 + 2984\,Usubc^2 - 616\,Usubc + 48\big)\,Usubc^2\Big)\Big/\Big((2\,Usubc-1)\,(9\,Usubc^2 - 10\,Usubc + 2)^2\Big)\,XX^2 + \frac{5}{216}\Big(Usubc^2\,(135\,Usubc^2 - 134\,Usubc + 22)\,(-2 + 3\,Usubc)^2\,RootOf\big((54\,Usubc^2 - 60\,Usubc + 12)\,\_Z^2 - 54\,Usubc^6 + 162\,Usubc^5 - 171\,Usubc^4 + 76\,Usubc^3 - 12\,Usubc^2\big)\,(6\,Usubc^2 - 10\,Usubc + 3)\Big)\Big/\Big((9\,Usubc^2 - 10\,Usubc + 2)^3\,(2\,Usubc-1)\Big)\,XX^3 + O(XX^4) \quad \textbf{(1.2.1.3)}$$

To get rid of the RootOf in the previous display, we factorize the polynom, and identify as many square terms as possible:

> $factor\left(-\frac{\left(-54\,U^6+162\,U^5-171\,U^4+76\,U^3-12\,U^2\right)}{\left(54\,U^2-60\,U+12\right)}\right);$

$$\frac{U^2\left(6\,U^2-10\,U+3\right)\left(-2+3\,U\right)^2}{6\left(9\,U^2-10\,U+2\right)} \tag{1.2.1.4}$$

> $map\Big(simplify, subs\Big(RootOf\big(\left(54\,Usubc^2-60\,Usubc+12\right)\_Z^2-54\,Usubc^6+162\,Usubc^5-171\,Usubc^4+76\,Usubc^3-12\,Usubc^2\big)=-Usubc\cdot(2-3\cdot Usubc)\cdot\mathrm{sqrt}\left(\frac{\left(6\,Usubc^2-10\,Usubc+3\right)}{6\left(9\,Usubc^2-10\,Usubc+2\right)}\right), \textbf{(1.2.1.3)}\Big)\Big);$

$$Usubc+\frac{1}{6}\,Usubc\left(-2+3\,Usubc\right)\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX+\frac{1}{18}\Big(\left(1458\,Usubc^6-5778\,Usubc^5+9045\,Usubc^4-7146\,Usubc^3+2984\,Usubc^2-616\,Usubc+48\right)Usubc^2\Big)\Big/\Big(\left(2\,Usubc-1\right)\left(9\,Usubc^2-10\,Usubc+2\right)^2\Big)\,XX^2+\frac{5}{1296}\Bigg(Usubc^3\left(135\,Usubc^2-134\,Usubc+22\right)\left(-2+3\,Usubc\right)^3\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\left(6\,Usubc^2-10\,Usubc+3\right)\Bigg)\Bigg/\Big(\left(9\,Usubc^2-10\,Usubc+2\right)^3\left(2\,Usubc-1\right)\Big)\,XX^3+\mathrm{O}\left(XX^4\right) \tag{1.2.1.5}$$

Finally we obtain the following developpement for U around Usubc (recall that XX = (1-w/rhosubc)^{1/2}):

> $Usubcsing3 := Usubc+\frac{1}{6}\,Usubc\left(-2+3\,Usubc\right)\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX+\frac{1}{18}\Big(\left(1458\,Usubc^6-5778\,Usubc^5+9045\,Usubc^4-7146\,Usubc^3+2984\,Usubc^2-616\,Usubc+48\right)Usubc^2\Big)\Big/\Big(\left(9\,Usubc^2-10\,Usubc+2\right)^2\left(-1+2\,Usubc\right)\Big)\,XX^2+\frac{5}{1296}\Big(\left(135\,Usubc^2-134\,Usubc+22\right)\left(6\,Usubc^2-10\,Usubc\right.$

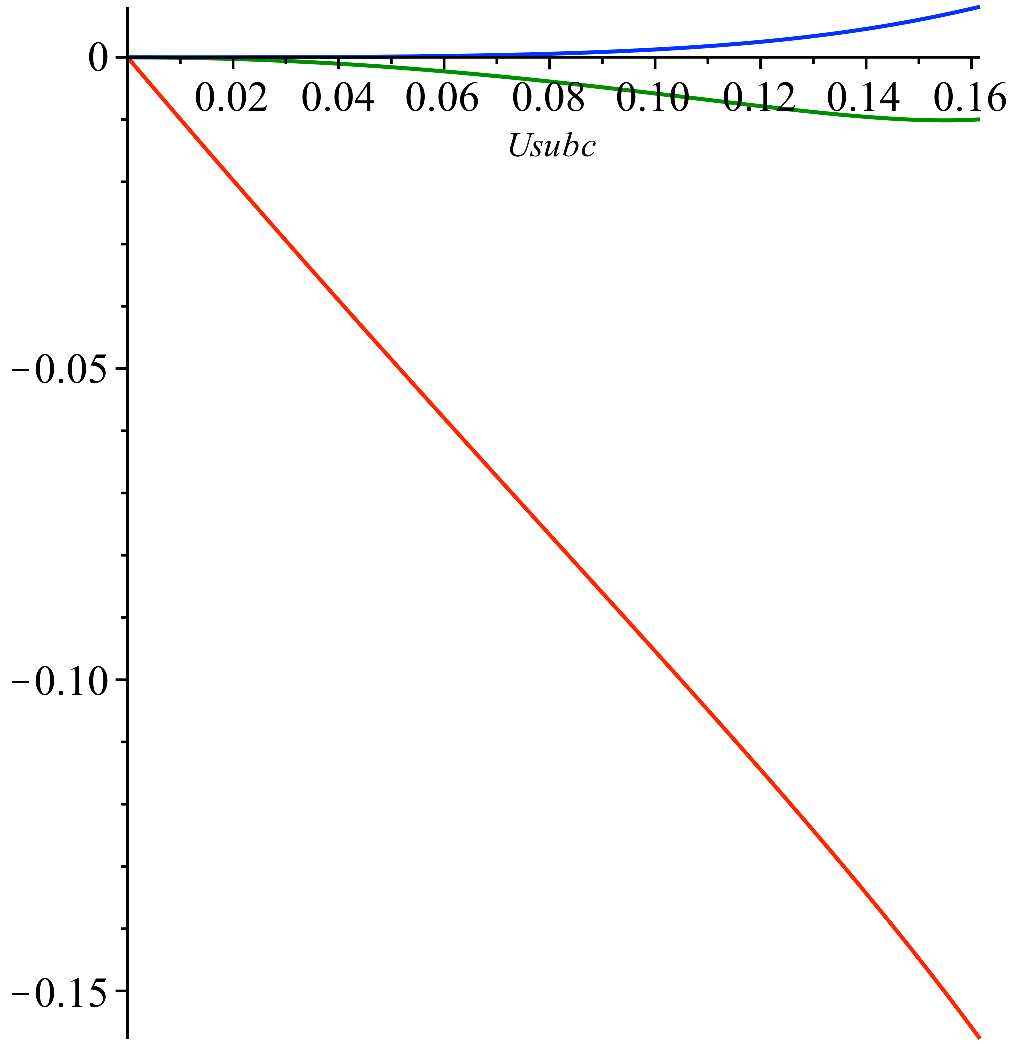
$$+ 3)\, Usubc^3\, (-2 + 3\, Usubc)^3 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\Bigg) \Bigg/ \left((9\, Usubc^2 - 10\, Usubc + 2)^3\, (-1 + 2\, Usubc)\right) XX^3;$$

$$Usubcsing3 := Usubc + \frac{Usubc\, (-2 + 3\, Usubc) \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX}{6}$$

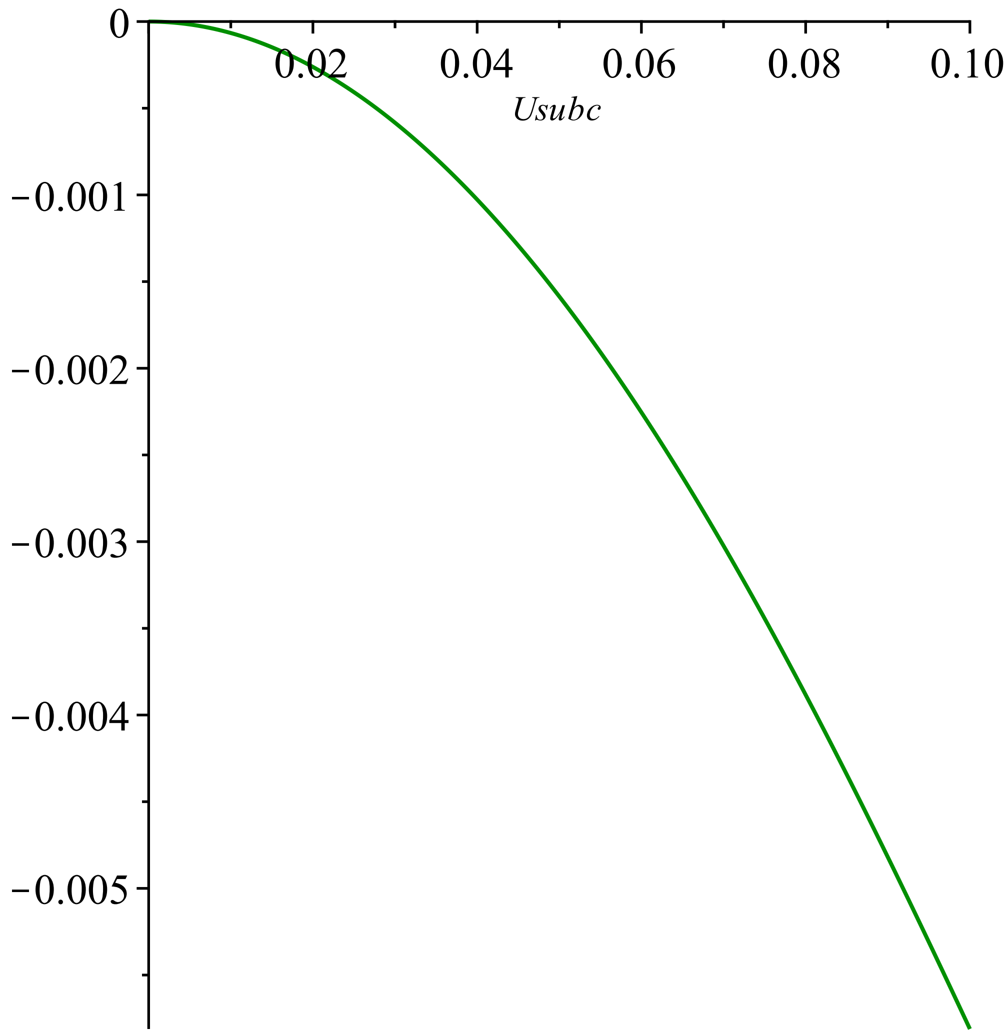
$$+ \Bigg(\left(1458\, Usubc^6 - 5778\, Usubc^5 + 9045\, Usubc^4 - 7146\, Usubc^3 + 2984\, Usubc^2 - 616\, Usubc - 1\right) + \Bigg(5\, (135\, Usubc^2 - 134\, Usubc + 22)\, (6\, Usubc^2 - 10\, Usubc + 3)\, Usubc^3\, (-2 + 3\, Usubc)^3 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX^3\Bigg) \Bigg/ \left(1296\, (9\, Usubc^2 - 10\, Usubc + 2)^3\, (2\, Usubc - 1)\right)$$

We check that the coefficients in the development do not cancel for the considered range of values of U.

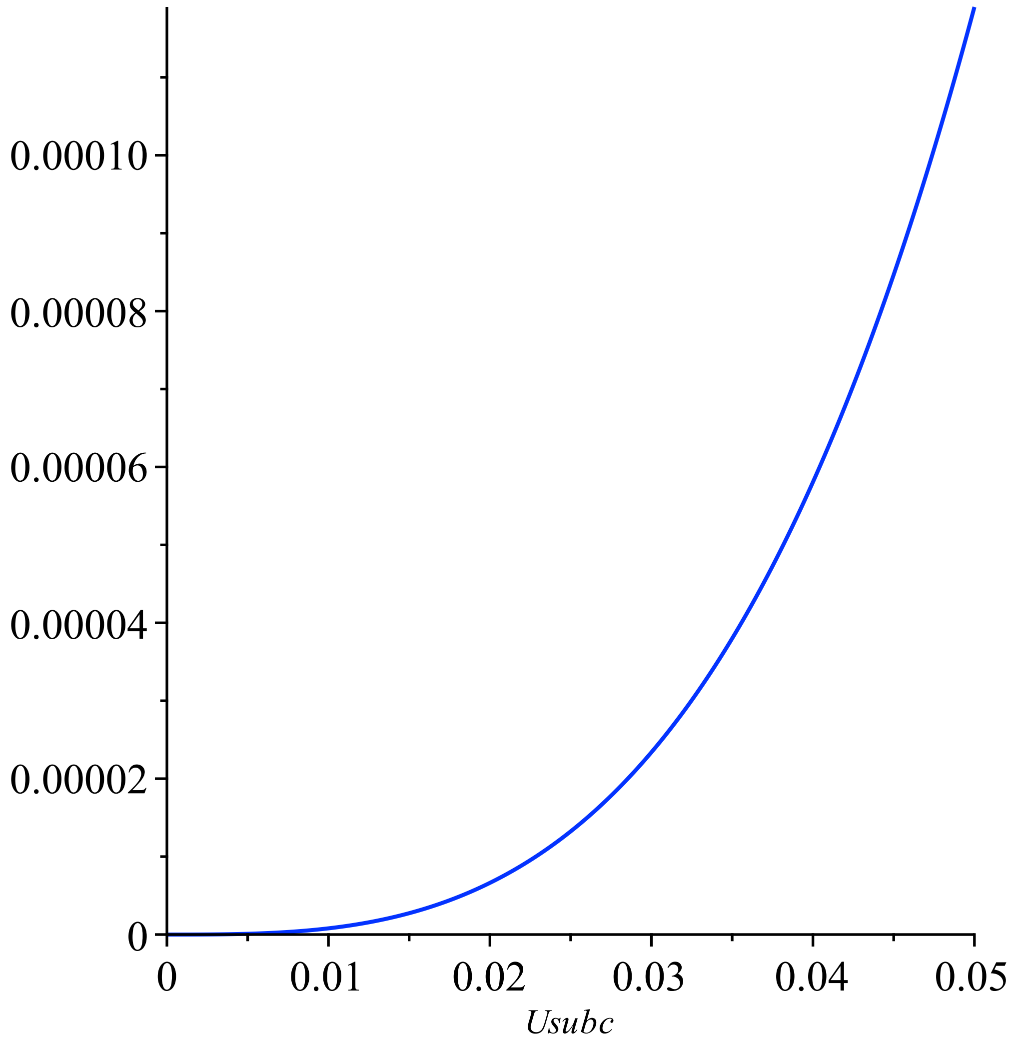
```
> plot([seq(coeff(Usubcsing3, XX, i), i = 1 ..3)], Usubc = 0 ..Uc - 0.1, color = ["Red",
    "Green", "Blue"]);
```



> $plot(coeff(Usubcsing3, XX, 2), Usubc = 0\,..0.1, color = [\text{"Green"}]);$



> *plot*(*coeff*(*Usubcsing3*, *XX*, 3), *Usubc* = 0 ..0.05, *color* = ["Blue"]);



Critical regime nu = nuc

We start again from

> algU;

$$8\,(\nu+1)^3\,U^5-(11\,\nu+29)\,(\nu+1)^2\,U^4+4\,(\nu+8)\,(\nu+1)^2\,U^3+(-128\,w\,\nu^3-12\,\nu^2-32\,\nu-12)\,U^2+8\,\nu\,(16\,\nu^2\,w+1)\,U-32\,w\,\nu^3 \quad \textbf{(1.2.2.1)}$$

We replace w by the value of the radius of convergence rhoc (=t_\nu^3 in the paper) and compute the corresponding singular behavior of U, (with XX=(1-w/rhoc)^1/3)

> map(simplify, algeqtoseries(simplify(subs(w = rhoc·(1 − XX^3), nu = nuc, algU)), XX, U, 3));

$$\left[\mathit{RootOf}\left(216\,\_Z^2+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)+\mathrm{O}\left(XX^3\right),\ \frac{5}{9}-\frac{\sqrt{7}}{9}+\mathit{RootOf}\left(39366\,\_Z^3+310\sqrt{7}-425\right)XX+\mathrm{O}\left(XX^{5/3}\right)\right] \quad \textbf{(1.2.2.2)}$$

There are two possible expansions, but since we know that Uc = 5/9 - sqrt(7)/9, it is necessarily the second one.

> $map\left(simplify, op\left(2, algeqtoseries\left(simplify\left(subs\left(w = rhoc\cdot\left(1 - XX^3\right), \mathrm{nu} = nuc, algU\right)\right), XX, U, 12\right)\right)\right);$

$$\frac{5}{9} - \frac{\sqrt{7}}{9} + RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right) XX - \frac{5\, RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right)^2 \left(2\sqrt{7} + 1\right) XX^2}{24} + \frac{35\left(-1 + 2\sqrt{7}\right) XX^3}{10368} - \frac{1645\, RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right) XX^4}{82944} + \mathrm{O}\left(XX^{14/3}\right) \quad \textbf{(1.2.2.3)}$$

> $allvalues\left(RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right)\right)$

$$\frac{\left(1240\sqrt{7} - 1700\right)^{1/3}}{108} + \frac{\mathrm{I}\sqrt{3}\left(1240\sqrt{7} - 1700\right)^{1/3}}{108}, -\frac{\left(1240\sqrt{7} - 1700\right)^{1/3}}{54}, \frac{\left(1240\sqrt{7} - 1700\right)^{1/3}}{108} - \frac{\mathrm{I}\sqrt{3}\left(1240\sqrt{7} - 1700\right)^{1/3}}{108} \quad \textbf{(1.2.2.4)}$$

There is a unique real root:

> $sort\left(collect\left(simplify\left(subs\left(RootOf\left(39366\_Z^3 + 310\sqrt{7} - 425\right) = -\frac{\left(1240\sqrt{7} - 1700\right)^{1/3}}{54}, \textbf{(1.2.2.3)}\right)\right), XX, simplify\right), XX, ascending\right);$

$$\frac{5}{9} - \frac{\sqrt{7}}{9} + \mathrm{O}\left(XX^{14/3}\right) - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7} - 1700\right)^{2/3}\left(2\sqrt{7} + 1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3 + \frac{1645\left(1240\sqrt{7} - 1700\right)^{1/3} XX^4}{4478976} \quad \textbf{(1.2.2.5)}$$

> $Ucsing4 := \frac{5}{9} - \frac{\sqrt{7}}{9} - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7} - 1700\right)^{2/3}\left(2\sqrt{7} + 1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3$

$$+ \frac{1645 \left(1240 \sqrt{7} - 1700\right)^{1/3} XX^4}{4478976};$$

$$Ucsing4 := \frac{5}{9} - \frac{\sqrt{7}}{9} - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7}-1700\right)^{2/3}\left(2\sqrt{7}+1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3 + \frac{1645\left(1240\sqrt{7}-1700\right)^{1/3} XX^4}{4478976} \qquad \textbf{(1.2.2.6)}$$

Supercritical regime nu >nuc

We consider the rational parametrization of the critical line in this regime given by K:

> *UsupK*; *nusupK*;

$$-\frac{K\sim^2 - 3}{6\,K\sim + 10}$$

$$-\frac{K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11}{(K\sim + 3)\left(K\sim^2 - 3\right)} \qquad \textbf{(1.2.3.1)}$$

We express the value of (t_nu)^3 = rho_c, in terms of K:

> *rhosupcK* := *simplify*(*subs*(nu = *nusupK*, *U* = *UsupK*, *wU*));

$$rhosupcK := -\frac{(K\sim + 1)\left(K\sim^2 + 8\,K\sim + 13\right)\left(K\sim^2 - 3\right)^3}{16\left(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11\right)^3} \qquad \textbf{(1.2.3.2)}$$

We compute the asymptotic behavior of U around rho (with XX=(1-w/rho)^1/2)

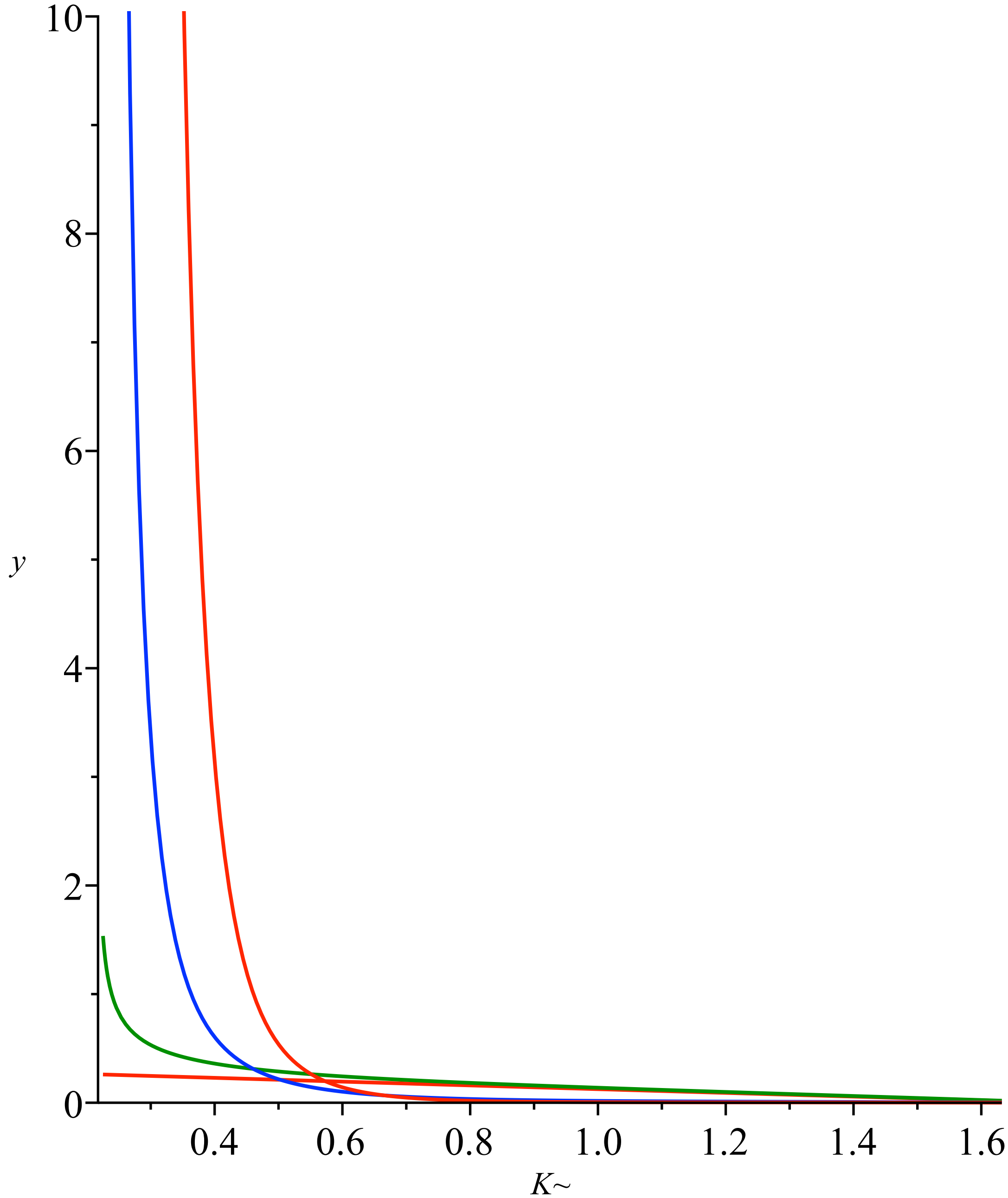
> *Usupcsing* := *collect*(*map*(*factor*, *map*(*simplify*, *convert*(*op*(2, *algeqtoseries*(*subs*(*w* = *rhosupcK*·(1 − *XX*²), *subs*(nu = *nusupK*, *algU*)), *XX*, *U*, 6, *true*)), *polynom*))), *XX*, *factor*);

$$Usupcsing := -\frac{K\sim^2 - 3}{2\,(3\,K\sim + 5)} + RootOf\Big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right)\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\Big)\, XX - \Big(\left(K\sim^2 - 3\right)\left(K\sim^2 + 8\,K\sim + 13\right) XX^2 \left(9\,K\sim^4 + 14\,K\sim^3 - 18\,K\sim^2 - 10\,K\sim + 29\right)(K\sim + 1)\Big) \Big/ \Big(144\,(3\,K\sim + 5)\left(3\,K\sim^2 + 4\,K\sim - 1\right)^2 (2 + K\sim)\Big) + \frac{1}{216\left(3\,K\sim^2 + 4\,K\sim - 1\right)^3 (2 + K\sim)} \Big(5\left(K\sim^2 + 8\,K\sim + 13\right)\big(9\,K\sim^6 + 40\,K\sim^5 + 43\,K\sim^4 - 48\,K\sim^3 - 97\,K\sim^2 + 24\,K\sim + 77\big)\, RootOf\big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right)\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 \qquad \textbf{(1.2.3.3)}$$

$+ 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,XX^3\big)$

The coefficients are non vanishing:

```
> plot([seq(coeff(Usupcsing, XX, i), i = 0 ..3)], K = Kc + 0.01 ..Kinfini − 0.1, y = 0
    ..10, color = ["Red", "Green", "Blue"]);
```



Development in t of the partition function Zplus (Proposition 2.5)

We compute the asymptotic behavior of the generating series of triangulations of the sphere (which corresponds to ((tZ_1)^2+t^2Z_2)/(t^3*nu) by standard manipulations):

Subcritical regime nu < nuc

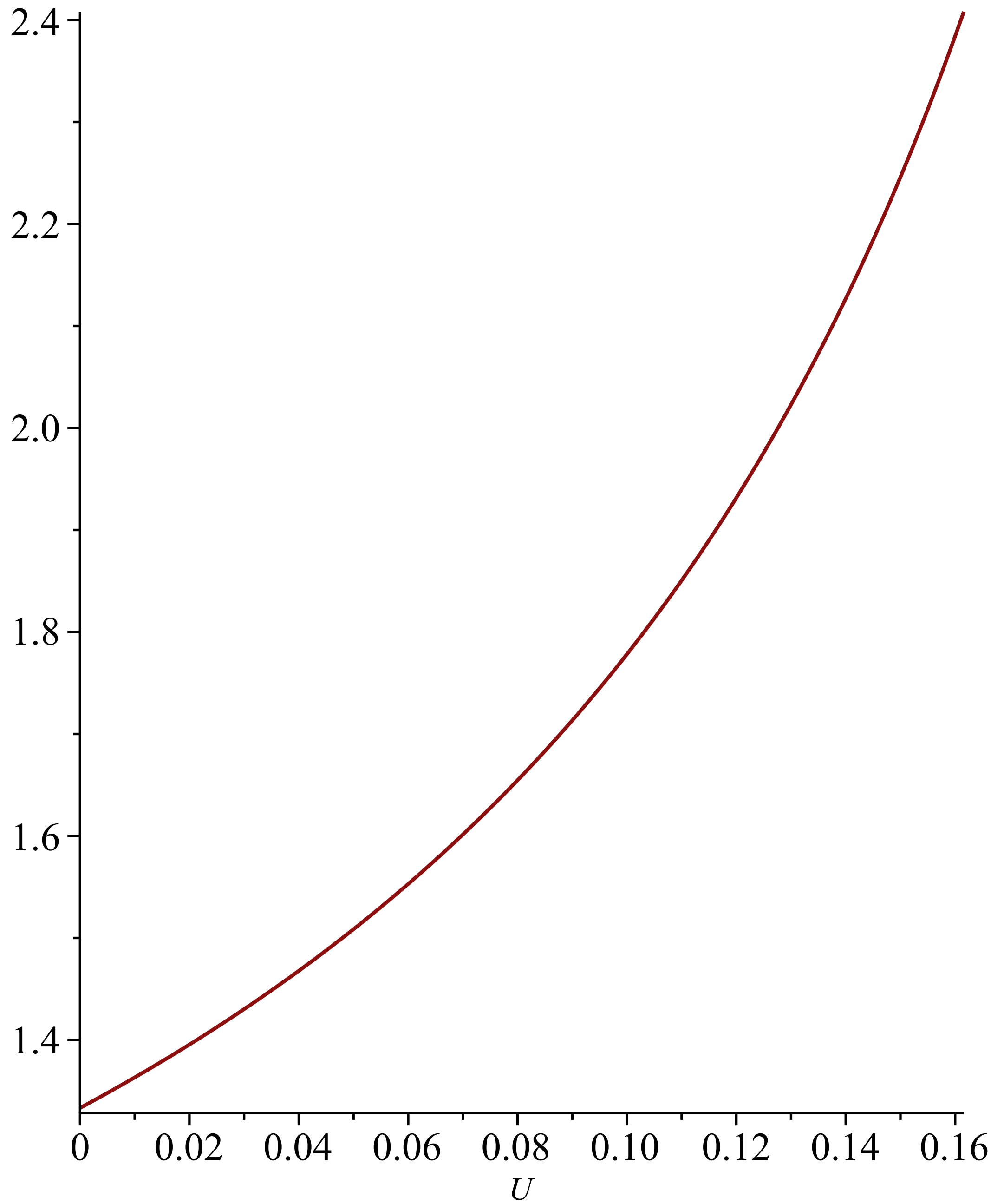
Here U is Unu and XX=(1-w/rho)^1/2

> $Zpsubcdevt := simplify\left(series\left(subs\left(U = Usubcsing3, Usubc = U, \mathrm{nu} = nuUsub, \frac{(tZ1U)^2 + t2Z2U}{wU \cdot \mathrm{nu}}\right), XX, 4\right)\right);$

$$Zpsubcdevt := \frac{810\,U^6 - 3780\,U^5 + 6507\,U^4 - 5805\,U^3 + 2889\,U^2 - 768\,U + 84}{2\left(6\,U^2 - 10\,U + 3\right)^2 (-2 + 3\,U)^2} + \frac{-324\,U^6 + 756\,U^5 - 1008\,U^4 + 900\,U^3 - 516\,U^2 + 168\,U - 24}{(-2 + 3\,U)^2 \left(6\,U^2 - 10\,U + 3\right)^2}\,XX^2 + 12\,\frac{\sqrt{\frac{6\,U^2 - 10\,U + 3}{9\,U^2 - 10\,U + 2}}\,\sqrt{6}\left(U^2 - U + \frac{1}{3}\right)(U - 2)}{54\,U^3 - 126\,U^2 + 87\,U - 18}\,XX^3 + \mathrm{O}\left(XX^4\right) \quad \textbf{(1.3.1.1)}$$

The singular coefficient does not vanish

> $plot(coeff(Zpsubcdevt, XX, 3), U = 0 ..Uc - 0.1);$



Critical regime nu = nuc

with XX=(1-w/rho)^1/3

$$> Zpscritdevt := collect\left(expand\left(rationalize\left(convert\left(series\left(subs\left(U = Ucsing4, \mathrm{nu} = nuc, \frac{(tZ1U)^2 + t2Z2U}{wU \cdot \mathrm{nu}}\right), XX, 5\right), polynom\right)\right)\right), XX, factor\right);$$

$$Zpscritdevt := \frac{3\sqrt{7}\left(1240\sqrt{7} - 1700\right)^{1/3} XX^4}{20} + \left(-\frac{476}{25} + \frac{148\sqrt{7}}{25}\right) XX^3 \quad \textbf{(1.3.2.1)}$$

$$+\frac{263\sqrt{7}}{50}-\frac{308}{25}$$

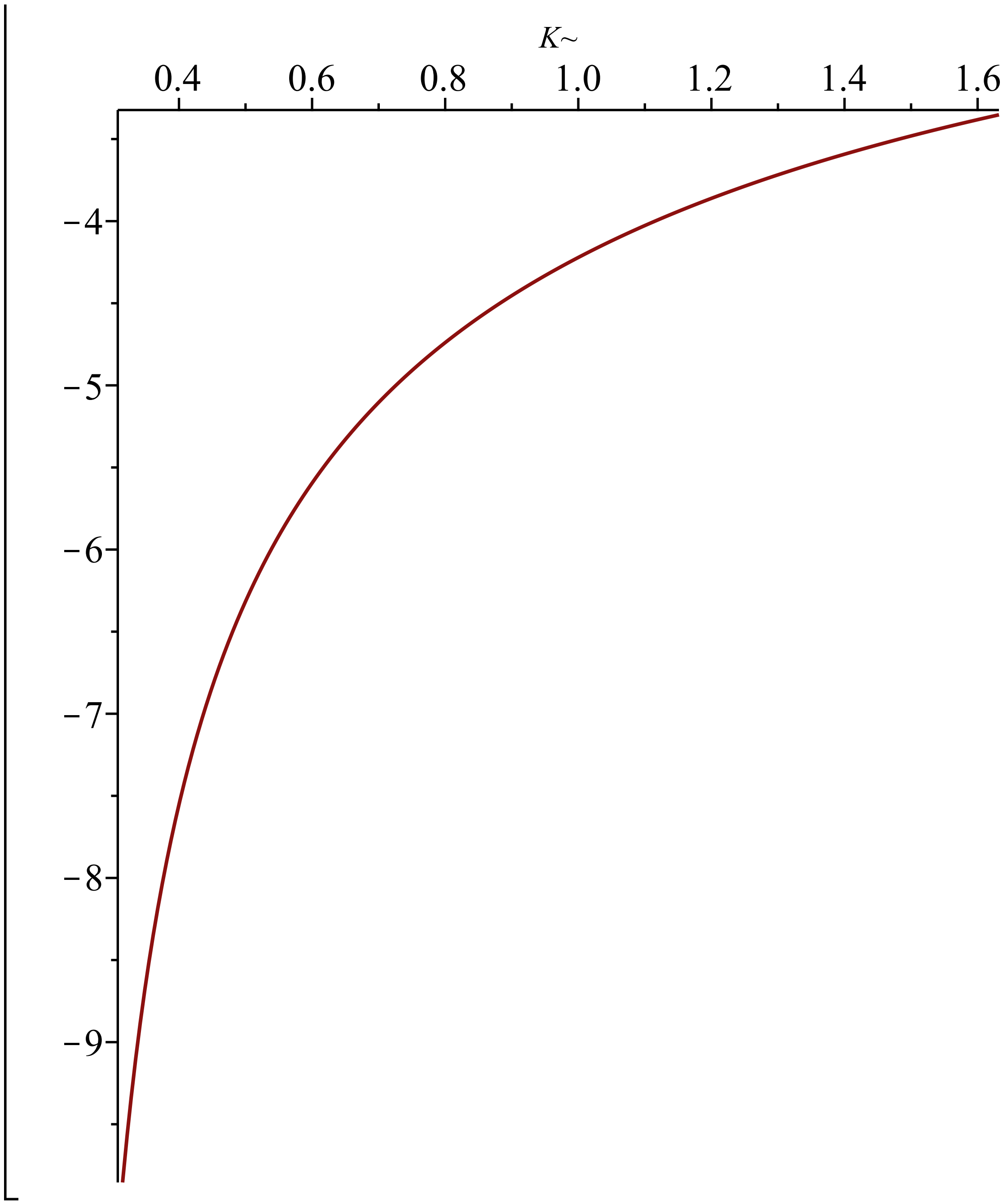
Supercritical regime nu >nuc

With XX=(1-w/rho)^1/2

> $Zpsupcdevt := simplify\left(series\left(subs\left(U = Usupcsing, \mathrm{nu} = nusupK, \frac{(tZ1U)^2 + t2Z2U}{wU\cdot \mathrm{nu}}\right), XX, 4\right)\right);$

$$Zpsupcdevt := \frac{1}{4\,(K\sim+1)^3\,(K\sim^2+8\,K\sim+13)^2}\,(5\,K\sim^7+95\,K\sim^6+675\,K\sim^5+2617\,K\sim^4+6055\,K\sim^3+7845\,K\sim^2+4809\,K\sim+1035) - \frac{1}{(K\sim+1)^3\,(K\sim^2+8\,K\sim+13)^2}\,((K\sim^6+12\,K\sim^5+95\,K\sim^4+344\,K\sim^3+651\,K\sim^2+652\,K\sim+237)\,(K\sim+3))\,XX^2 + \frac{8}{3}\,((21\,K\sim^6+242\,K\sim^5+1083\,K\sim^4+2388\,K\sim^3+2695\,K\sim^2+1410\,K\sim+225)\,RootOf((1296\,K\sim^4+6048\,K\sim^3+8928\,K\sim^2+3360\,K\sim-1200)\,\_Z^2-K\sim^8-10\,K\sim^7-24\,K\sim^6+26\,K\sim^5+158\,K\sim^4+114\,K\sim^3-192\,K\sim^2-306\,K\sim-117))/((K\sim^2+8\,K\sim+13)\,(K\sim+1)^4\,(K\sim^2-3))\,XX^3+\mathrm{O}(XX^4) \quad \textbf{(1.3.3.1)}$$

> $plot(coeff(Zpsupcdevt, XX, 3), K = Kc + 0.1\,..Kinfini - 0.1);$



Theorem 3.1: Rational parametrisation for Q^+(t,ty) (denoted Qt here): g.s of trig with monochromatic non simple boundary

We start with the equation satisfied by Q, in terms of Z1 (=Z_1^+) et Z2 = Z_2^+):

> $eqQ := collect\left(simplify\left(\frac{subs(Z = Q - 1, y = y \cdot Q,\ eqZ)}{Q^2} \right), [Q, y], factor \right);$

$$eqQ := Q^3 \nu^2 t^3 y^5 + \left(-y^4 \nu (\nu-1) t + \nu (2\nu-3) t^2 y^3 + \nu^2 t^3 y^2\right) Q^2 + \left(-t^2 \nu (2 \nu Z1\, t + \nu - 2) y^3 + y^2 (\nu-1) - y t (\nu+2) (\nu-1) + 2\nu(\nu-1) t^2 \right) Q + \left(-2 Z1^2 \nu^2 t^2 + 2 Z1^2 \nu t^2 - 2 Z2 \nu^2 t^2 - \nu^2 t^3 + Z1 \nu^2 t + 2 Z2 \nu t^2 + \nu Z1\, t - 2 Z1\, t - \nu + 1\right) y^2 - (\nu-1) t (2\nu Z1\, t - \nu - 2) y - 2\nu(\nu-1) t^2 \quad (2.1)$$

The equation for Qt= Q(nu,t,ty)

> $eqQt := collect\left(subs\left(t = w^{\frac{1}{3}}, simplify\left(\frac{subs\left(Q = Qt, Z1 = \frac{tZ1}{t}, Z2 = \frac{t2Z2}{t^2}, y = y\cdot t, eqQ\right)}{t^2}\right)\right), [Qt, y, w], recursive\right)$
> ;

$$eqQt := Qt^3 \nu^2 w^2 y^5 + \left(-w \nu (\nu-1) y^4 + 2\nu w \left(\nu - \frac{3}{2}\right) y^3 + \nu^2 w y^2\right) Qt^2 + \left(2\nu w \left(\left(-tZ1 - \frac{1}{2}\right)\nu + 1\right) y^3 + y^2 (\nu-1) - y(\nu+2)(\nu-1) + 2\nu(\nu-1)\right) Qt + \left(-\nu^2 w + \left(-2\, tZ1^2 - 2\, t2Z2 + tZ1\right)\nu^2 + \left(2\, tZ1^2 + 2\, t2Z2 + tZ1 - 1\right)\nu - 2\, tZ1 + 1\right) y^2 - (\nu-1)\left((2\, tZ1 - 1)\nu - 2\right) y - 2\nu(\nu-1) \quad (2.2)$$

> $map(factor, eqQt)$

$$Qt^3 \nu^2 w^2 y^5 - w \nu y^2 \left(\nu y^2 - 2\nu y - y^2 - \nu + 3 y\right) Qt^2 - \left(2\nu^2 w y^3\, tZ1 + \nu^2 w y^3 - 2\nu w y^3 + \nu^2 y - \nu y^2 - 2\nu^2 + \nu y + y^2 + 2\nu - 2y\right) Qt - \left(2\nu^2\, tZ1^2 + 2\nu^2\, t2Z2 - \nu^2\, tZ1 + \nu^2 w - 2\nu\, tZ1^2 - 2\nu\, t2Z2 - \nu\, tZ1 + \nu + 2\, tZ1 - 1\right) y^2 - (\nu-1)(2\nu\, tZ1 - \nu - 2) y - 2\nu(\nu-1) \quad (2.3)$$

We compute the development of the solutions of the equation to identify the right branch (i.e. the one with a formal power series development):

> algeqtoseries(eqQt, y, Qt, 3);

$$\left[-\frac{1}{w} y^{-3} + \frac{1}{\nu w} y^{-2} + O(y^0), -\frac{2(\nu-1)}{\nu w} y^{-2} + \frac{\nu-1}{\nu w} y^{-1} - 1 + O(y), 1 + tZ1\, y + \left(tZ1^2 + t2Z2\right) y^2 + O(y^3)\right] \quad (2.4)$$

> $eqQtU := op(2, factor(numer(subs(w = wU, tZ1 = tZ1U, t2Z2 = t2Z2U, eqQt))))$:
> $indets(eqQtU)$;
> $degree(eqQtU, Qt)$;
> $degree(eqQtU, \{Qt, y\})$;

$$\{Qt, U, \nu, y\}$$

$$3$$

$$8 \qquad (2.5)$$

We have a rational parametrisation y(U,V) and Qt(U,V) such that y(U,0)= 0 and Qt(U,0)=1.

$$> yUV := \frac{8\,\nu\cdot(1-2\,U)}{U\,(U\cdot(\nu+1)-2)}\cdot(V\cdot(V+1)) \Bigg/ \Bigg(V^3 + \frac{9(\nu+1)\cdot U^2 - 2\cdot(3+10\,\nu)\cdot U + 8\,\nu}{U\,(U\cdot(\nu+1)-2)}\cdot V^2 - \frac{9\cdot U\cdot(\nu+1) - 2\cdot(2\,\nu+3)}{(U\cdot(\nu+1)-2)}\cdot V - 1\Bigg):$$

$$> QtUV := \frac{U\,(U\cdot(\nu+1)-2)\cdot(1-\nu)}{P}\cdot\frac{1}{(V+1)^3}\cdot\Bigg(V^3 + \frac{9(\nu+1)\cdot U^2 - 2\cdot(3+10\,\nu)\cdot U + 8\,\nu}{U\,(U\cdot(\nu+1)-2)}\cdot V^2 - \frac{9\cdot U\cdot(\nu+1) - 2\cdot(2\,\nu+3)}{(U\cdot(\nu+1)-2)}\cdot V - 1\Bigg)\cdot\Bigg(V^2 + 2\cdot\frac{(5\cdot(\nu+1)\,U^2 - 2\cdot(3\,\nu+2)\,U + 2\,\nu)}{U\,(U\cdot(\nu+1)-2)}\,V - \frac{P}{U\,(U\cdot(\nu+1)-2)\cdot(1-\nu)}\Bigg):$$

We check that this parametrizes a solution of eqQt:

$$> simplify(subs(y = yUV, Qt = QtUV, eqQtU));$$

$$0 \qquad (2.6)$$

Lastly, we check that this does correspond to the right branch:

$$> collect(eqQtU, Qt, factor);$$

$$U^2\,y^5\,(U\,\nu + U - 2)^2\,(8\,U^3\,\nu^2 + 16\,U^3\,\nu - 11\,U^2\,\nu^2 + 8\,U^3 - 24\,U^2\,\nu + 4\,U\,\nu^2 - 13\,U^2 + 14\,U\,\nu + 6\,U - 4\,\nu)^3\,Qt^3 - 32\,U\,\nu^2\,y^2\,(\nu\,y^2 - 2\,\nu\,y - y^2 - \nu + 3\,y)\,(-1+2\,U)^2\,(U\,\nu + U - 2)\,(8\,U^3\,\nu^2 + 16\,U^3\,\nu - 11\,U^2\,\nu^2 + 8\,U^3 - 24\,U^2\,\nu + 4\,U\,\nu^2 - 13\,U^2 + 14\,U\,\nu + 6\,U - 4\,\nu)^2\,Qt^2 - 64\,\nu^2\,(-1+2\,U)^2\,(8\,U^3\,\nu^2 + 16\,U^3\,\nu - 11\,U^2\,\nu^2 + 8\,U^3 - 24\,U^2\,\nu + 4\,U\,\nu^2 - 13\,U^2 + 14\,U\,\nu + 6\,U - 4\,\nu)\,(3\,U^6\,\nu^4\,y^3 + 12\,U^6\,\nu^3\,y^3 + 18\,U^6\,\nu^2\,y^3 - 18\,U^5\,\nu^3\,y^3 - 4\,U^4\,\nu^4\,y^3 + 12\,U^6\,\nu\,y^3 - 54\,U^5\,\nu^2\,y^3 + 2\,U^3\,\nu^4\,y^3 + 3\,U^6\,y^3 - 54\,U^5\,\nu\,y^3 + 51\,U^4\,\nu^2\,y^3 + 12\,U^3\,\nu^3\,y^3 - 18\,U^5\,y^3 + 86\,U^4\,\nu\,y^3 - 18\,U^3\,\nu^2\,y^3 - 6\,U^2\,\nu^3\,y^3 + 39\,U^4\,y^3 - 64\,U^3\,\nu\,y^3 + 64\,U^2\,\nu^4\,y - 64\,U^2\,\nu^3\,y^2 - 2\,U^2\,\nu^2\,y^3 - 36\,U^3\,y^3 - 128\,U^2\,\nu^4 + 64\,U^2\,\nu^3\,y + 64\,U^2\,\nu^2\,y^2 + 28\,U^2\,\nu\,y^3 - 64\,U\,\nu^4\,y + 64\,U\,\nu^3\,y^2 + 4\,U\,\nu^2\,y^3 + 128\,U^2\,\nu^3 - 128\,U^2\,\nu^2\,y + 12\,U^2\,y^3 + 128\,U\,\nu^4 - 64\,U\,\nu^3\,y - 64\,U\,\nu^2\,y^2 - 8\,U\,\nu\,y^3 + 16\,\nu^4\,y - 16\,\nu^3\,y^2 - 128\,U\,\nu^3 + 128\,U\,\nu^2\,y - 32\,\nu^4 + 16\,\nu^3\,y + 16\,\nu^2\,y^2 + 32\,\nu^3 - 32\,\nu^2\,y)\,Qt - 512\,\nu^3\,(-1+2\,U)^3\,(2\,U^7\,\nu^5\,y^2 \qquad (2.7)$$

$+ 10 U^7 \nu^4 y^2 + 20 U^7 \nu^3 y^2 - 14 U^6 \nu^4 y^2 - 14 U^5 \nu^5 y^2 + 20 U^7 \nu^2 y^2$
$- 56 U^6 \nu^3 y^2 + 24 U^5 \nu^5 y - 26 U^5 \nu^4 y^2 + 23 U^4 \nu^5 y^2 + 10 U^7 \nu y^2 - 84 U^6 \nu^2 y^2$
$+ 48 U^5 \nu^4 y + 44 U^5 \nu^3 y^2 - 76 U^4 \nu^5 y + 85 U^4 \nu^4 y^2 - 14 U^3 \nu^5 y^2 + 2 U^7 y^2$
$- 56 U^6 \nu y^2 + 148 U^5 \nu^2 y^2 + 64 U^4 \nu^5 - 184 U^4 \nu^4 y + 29 U^4 \nu^3 y^2 + 88 U^3 \nu^5 y$
$- 94 U^3 \nu^4 y^2 + 3 U^2 \nu^5 y^2 - 14 U^6 y^2 - 48 U^5 \nu^2 y + 130 U^5 \nu y^2 + 64 U^4 \nu^4$
$- 32 U^4 \nu^3 y - 155 U^4 \nu^2 y^2 - 120 U^3 \nu^5 + 248 U^3 \nu^4 y - 80 U^3 \nu^3 y^2 - 44 U^2 \nu^5 y$
$+ 49 U^2 \nu^4 y^2 - 24 U^5 \nu y + 38 U^5 y^2 - 64 U^4 \nu^3 + 184 U^4 \nu^2 y - 172 U^4 \nu y^2$
$- 136 U^3 \nu^4 + 80 U^3 \nu^3 y + 112 U^3 \nu^2 y^2 + 76 U^2 \nu^5 - 164 U^2 \nu^4 y + 72 U^2 \nu^3 y^2$
$+ 8 U \nu^5 y - 10 U \nu^4 y^2 - 64 U^4 \nu^2 + 108 U^4 \nu y - 50 U^4 y^2 + 120 U^3 \nu^3$
$- 248 U^3 \nu^2 y + 144 U^3 \nu y^2 + 132 U^2 \nu^4 - 80 U^2 \nu^3 y - 66 U^2 \nu^2 y^2 - 16 U \nu^5$
$+ 56 U \nu^4 y - 36 U \nu^3 y^2 + 136 U^3 \nu^2 - 168 U^3 \nu y + 32 U^3 y^2 - 108 U^2 \nu^3$
$+ 180 U^2 \nu^2 y - 72 U^2 \nu y^2 - 72 U \nu^4 + 40 U \nu^3 y + 32 U \nu^2 y^2 - 8 \nu^4 y + 8 \nu^3 y^2$
$- 100 U^2 \nu^2 + 108 U^2 \nu y - 8 U^2 y^2 + 64 U \nu^3 - 80 U \nu^2 y + 16 U \nu y^2 + 16 \nu^4$
$- 8 \nu^3 y - 8 \nu^2 y^2 + 24 U \nu^2 - 24 U \nu y - 16 \nu^3 + 16 \nu^2 y)$

> *factor*(*subs*(y = 0, **(2.7)**));

$$2048 \nu^5 (-1 + 2U)^4 (\nu - 1) (8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4 \nu) (Qt - 1) \quad \textbf{(2.8)}$$

Proposition 3.5

> $yUV := \left(8 \nu (1 - 2U) V (V+1)\right) \Big/ \left(U (U(\nu+1) - 2) \left(V^3 + \frac{(9(\nu+1)U^2 - 2(3+10\nu)U + 8\nu) V^2}{U(U(\nu+1)-2)} - \frac{(9U(\nu+1) - 4\nu - 6) V}{U(\nu+1) - 2} - 1\right)\right);$

$$yUV := \left(8 \nu (1 - 2U) V (V+1)\right) \Big/ \left(U (U(\nu+1) - 2) \left(V^3 + \frac{(9(\nu+1)U^2 - 2(3+10\nu)U + 8\nu) V^2}{U(U(\nu+1)-2)} - \frac{(9U(\nu+1) - 4\nu - 6) V}{U(\nu+1) - 2} - 1\right)\right) \quad \textbf{(3.1)}$$

First we look at the possible poles:

> $factor\left(discrim\left(V^3 + \frac{(9(\nu+1)U^2 - 2(3+10\nu)U + 8\nu)V^2}{U(U(\nu+1)-2)} - \frac{(9U(\nu+1) - 4\nu - 6)V}{U(\nu+1)-2} - 1, V\right)\right);$

$$\frac{1}{U^3(U\nu + U - 2)^4}\left(64(-1+2U)\left(108U^6\nu^4 + 432U^6\nu^3 - 432U^5\nu^4 + 648U^6\nu^2 - 1620U^5\nu^3 + 774U^4\nu^4 + 432U^6\nu - 2268U^5\nu^2 + 2628U^4\nu^3 - 693U^3\nu^4 + 108U^6 - 1404U^5\nu + 3258U^4\nu^2 - 2358U^3\nu^3 + 296U^2\nu^4 - 324U^5 + 1728U^4\nu - 2529U^3\nu^2 + 1308U^2\nu^3 - 48U\nu^4 + 324U^4 - 972U^3\nu + 1044U^2\nu^2 - 432U\nu^3 - 108U^3 + 216U^2\nu - 180U\nu^2 + 64\nu^3\right)\right) \quad \textbf{(3.2)}$$

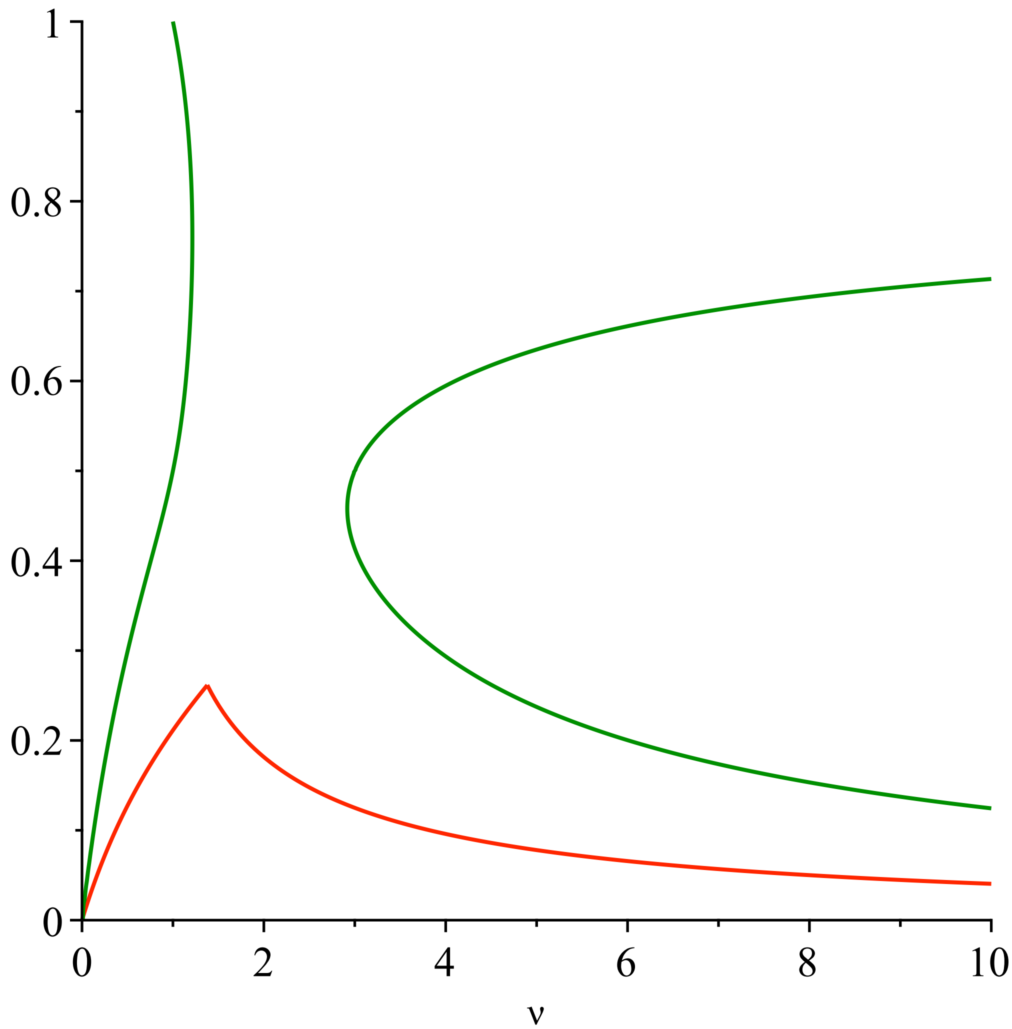
We want to know the sign of this factor:

> $collect(108U^6\nu^4 + 432U^6\nu^3 - 432U^5\nu^4 + 648U^6\nu^2 - 1620U^5\nu^3 + 774U^4\nu^4 + 432U^6\nu - 2268U^5\nu^2 + 2628U^4\nu^3 - 693U^3\nu^4 + 108U^6 - 1404U^5\nu + 3258U^4\nu^2 - 2358U^3\nu^3 + 296U^2\nu^4 - 324U^5 + 1728U^4\nu - 2529U^3\nu^2 + 1308U^2\nu^3 - 48U\nu^4 + 324U^4 - 972U^3\nu + 1044U^2\nu^2 - 432U\nu^3 - 108U^3 + 216U^2\nu - 180U\nu^2 + 64\nu^3, U, factor);$

$$108(\nu+1)^4U^6 - 108(4\nu+3)(\nu+1)^3U^5 + 18(43\nu^2 + 60\nu + 18)(\nu+1)^2U^4 - 9(\nu+1)(77\nu^3 + 185\nu^2 + 96\nu + 12)U^3 + 4\nu(74\nu^3 + 327\nu^2 + 261\nu + 54)U^2 - 12\nu^2(4\nu^2 + 36\nu + 15)U + 64\nu^3 \quad \textbf{(3.3)}$$

Can it be 0 for U \in [0,U_c(nu)]. We plot the points corresponding to its zeroes (in green) and the value of U_\nu (in red).

> pzero := implicitplot(**(3.3)**, nu = 0 ..10 , U = 0 ..1, numpoints = 10000, color = "Green") :
display([PUsub, PUsur, pzero]);



We check when the curves meet. First in the subcritical regime. It is always for U>U_\nu_c

> *factor*(*resultant*(**(3.3)**, $(3U^2\nu + 3U^2 - 3U\nu - 3U + \nu)$, nu)); *fsolve*(%); *evalf*(*Uc*);

$$-81\,U^3\,(23\,U^2 - 28\,U + 8)\,(U-1)^2\,(-1+2\,U)^3$$

0., 0., 0., 0.4580825385, 0.5000000000, 0.5000000000, 0.5000000000, 0.7593087659, 1., 1.

0.2615831877 **(3.4)**

Same in the supercritical regime:

> *factor*(*resultant*(**(3.3)**, $(4U^3\nu^2 + 8U^3\nu - 3U^2\nu^2 + 4U^3 - 12U^2\nu - 9U^2 + 6U\nu + 6U - 2)$, nu)); *fsolve*(%); *evalf*(*Uc*);

$$-4\,U^2\,(200\,U^4 - 432\,U^3 + 447\,U^2 - 232\,U + 48)\,(1633\,U^6 - 5980\,U^5 + 8856\,U^4 - 6856\,U^3 + 2948\,U^2 - 672\,U + 64)\,(-1+2\,U)^4$$

0., 0., 0.5000000000, 0.5000000000, 0.5000000000, 0.5000000000, 0.6745152264, 1.142829079

0.2615831877 **(3.5)**

Hence for U =< U_\nu_c (which is the only values of U of intereset) the discriminant of the denominator does not change sign. To know its sign, we check its value at U=0.

> $subs(U=0, numer((\mathbf{3.2})));$

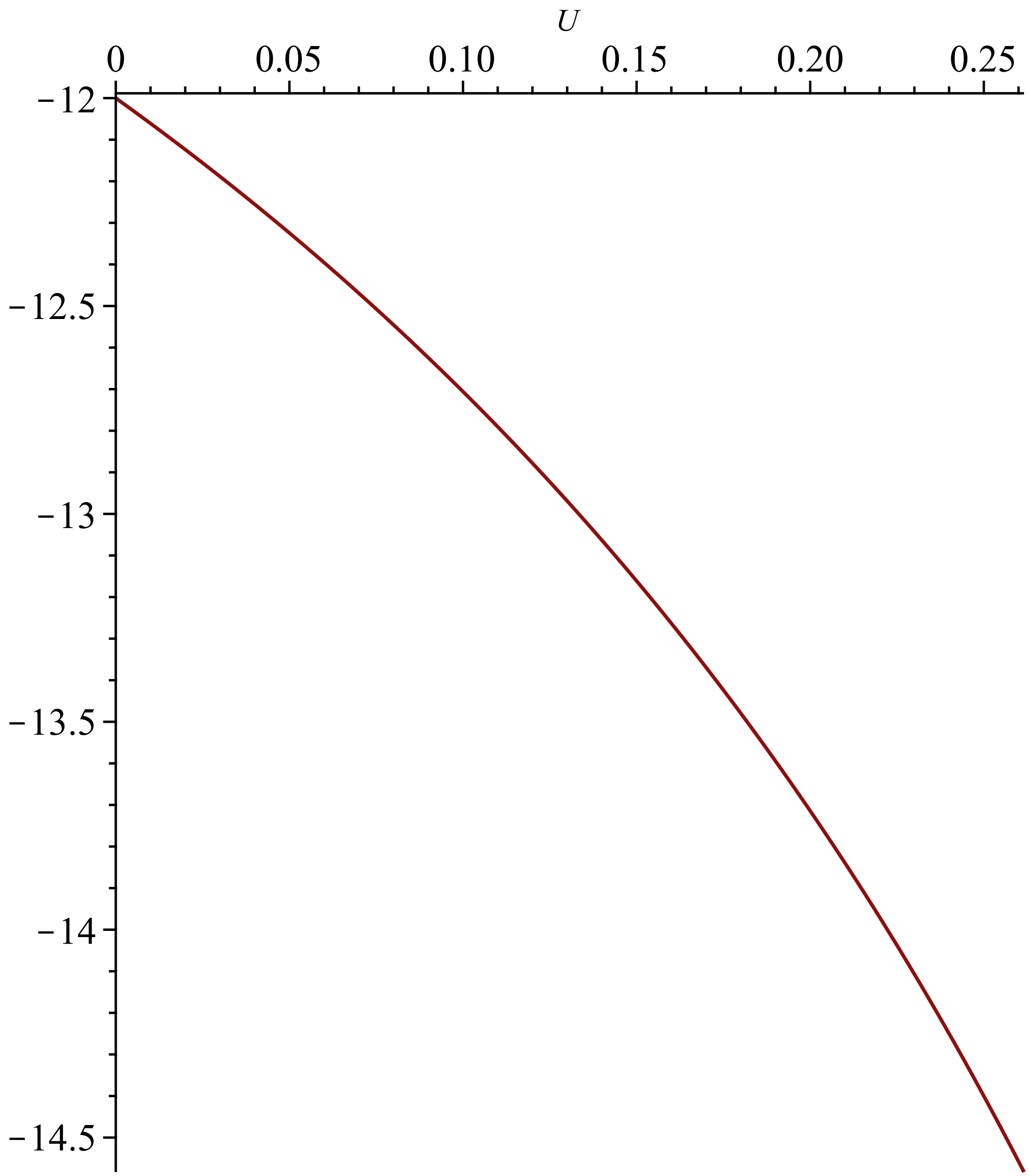
$$-4096\,\nu^3 \tag{3.6}$$

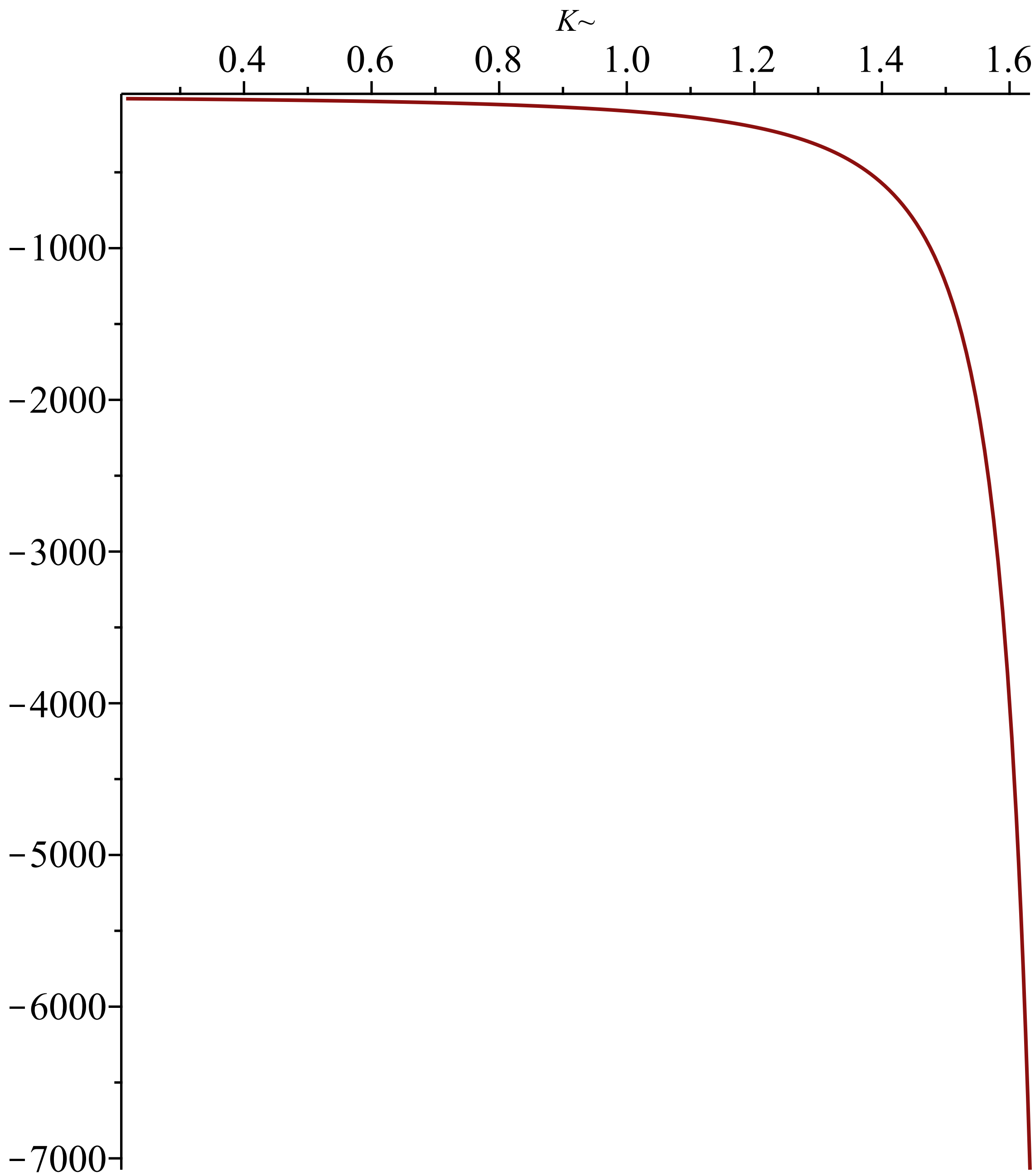
Since it is negative, yUV has a single real pole if U =<U_c:

The leading coefficient of the polynom is positive and the polynom is <0 at V=1 : the pole is after V= 1

> $factor\left(subs\left(V=1, \left(V^3 + \frac{\left(9\,(\nu+1)\,U^2 - 2\,(3+10\,\nu)\,U + 8\,\nu\right)V^2}{U\,(U\,(\nu+1)-2)} - \frac{(9\,U\,(\nu+1) - 4\,\nu - 6)\,V}{U\,(\nu+1)-2} - 1\right)\right)\right); plot(subs(\mathrm{nu} = nuUsub, \%), U = 0..Uc);$
> $plot(subs(\mathrm{nu} = nusupK, U = UsupK, \%\%), K = Kc..Kinfini - 0.1);$

$$-\frac{8\,\nu\,(-1+2\,U)}{U\,(U\,\nu + U - 2)}$$


U
0
0.05
0.10
0.15
0.20
0.25
−12
−12.5
−13
−13.5
−14
−14.5



We now look at the stationary points of y(V):

```
> factor(diff(yUV, V)); factor(subs(V = 0, %));
```

$$\left(8\left(U^2 V^4 \nu + U^2 V^4 + 2 U^2 V^3 \nu + 2 U^2 V^3 + 18 U^2 V^2 \nu - 2 U V^4 + 18 U^2 V^2 + 2 U^2 V \nu - 4 V^3 U - 24 U V^2 \nu + 2 U^2 V + U^2 \nu - 12 V^2 U + 8 V^2 \nu + U^2 - 4 V U - 2 U\right) \nu \left(-1 + 2 U\right)\right) \Big/ \left(U^2 V^3 \nu + U^2 V^3 + 9 U^2 V^2 \nu + 9 U^2 V^2 - 9 U^2 V \nu - 2 V^3 U - 20 U V^2 \nu - 9 U^2 V - U^2 \nu - 6 V^2 U + 4 U V \nu + 8 V^2 \nu - U^2 + 6 V U + 2 U\right)^2$$

$$\frac{8 \nu \left(-1 + 2 U\right)}{U \left(U \nu + U - 2\right)} \tag{3.7}$$

We have to study the roots of the following polynomial of degree 4:

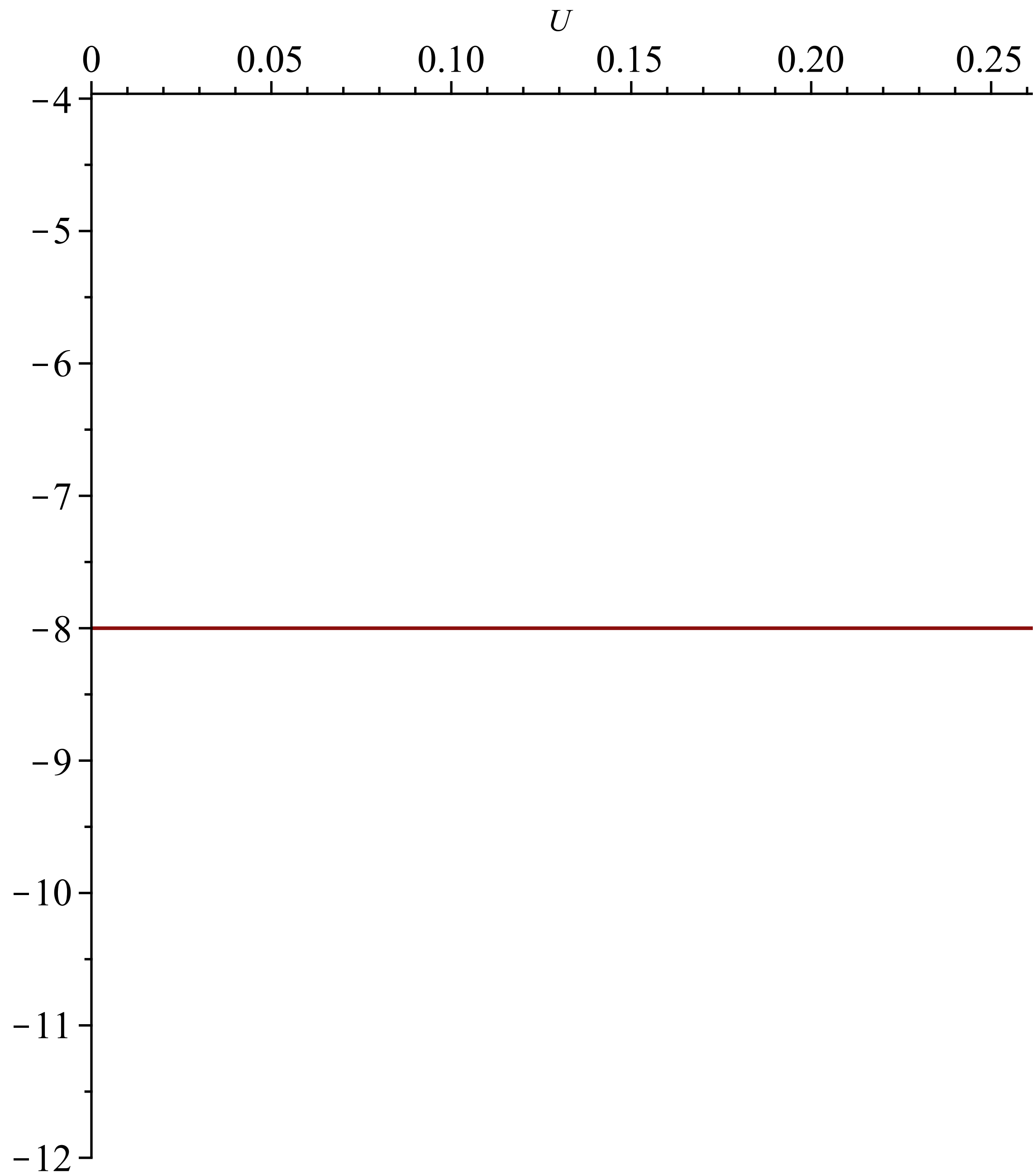
> $eqVcritU := collect\left(\frac{1}{U\,(U\,\nu + U - 2)}\left(\left(U^2\,V^4\,\nu + U^2\,V^4 + 2\,U^2\,V^3\,\nu + 2\,U^2\,V^3 + 18\,U^2\,V^2\,\nu - 2\,U\,V^4 + 18\,U^2\,V^2 + 2\,U^2\,V\,\nu - 4\,V^3\,U - 24\,U\,V^2\,\nu + 2\,U^2\,V + U^2\,\nu - 12\,U\,V^2 + 8\,V^2\,\nu + U^2 - 4\,V\,U - 2\,U\right)\right), V, factor\right);$

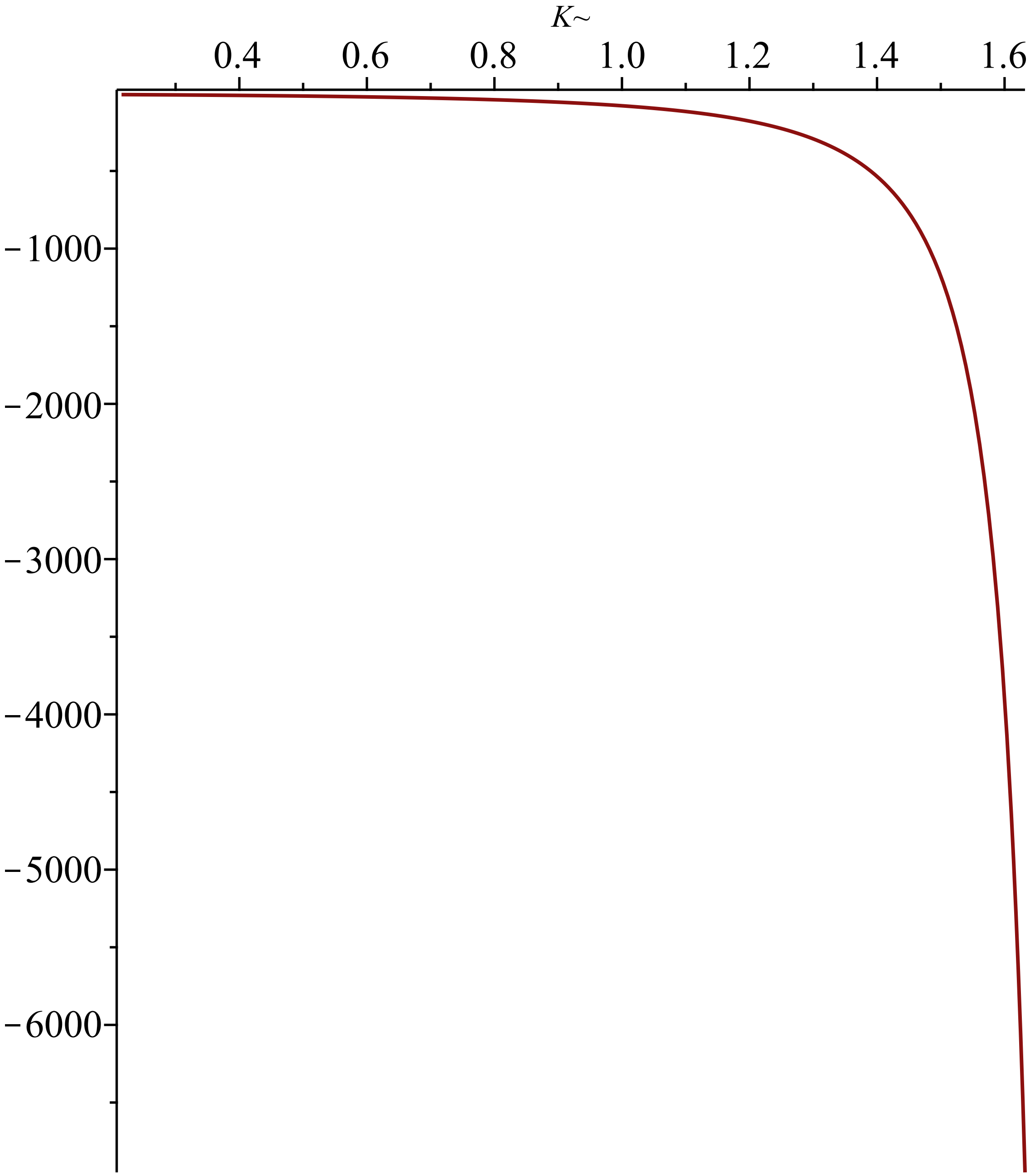
$$eqVcritU := 1 + V^4 + 2\,V^3 + \frac{2\,(-2 + 3\,U)\,(3\,U\,\nu + 3\,U - 2\,\nu)\,V^2}{U\,(U\,\nu + U - 2)} + 2\,V \qquad \textbf{(3.8)}$$

We have four stationary points and we can check that they are all real. Indeed the previous polynom is <0 at V=-1, >0 at V=0 and < at V=1 (with a possible double root here in the subcritical case and if U=U_\nu). It is also positive at +infinity and - infinity.

> $factor(subs(V=-1, eqVcritU)); plot(factor(subs(\text{nu} = nuUsub, \%)), U = 0..Uc);$
> $plot(subs(\text{nu} = nusupK, U = UsupK, \%\%), K = Kc..Kinfini - 0.1);$

$$\frac{8\,(-1 + 2\,U)\,(U\,\nu + U - \nu)}{U\,(U\,\nu + U - 2)}$$


U
0
0.05
0.10
0.15
0.20
0.25
−4
−5
−6
−7
−8
−9
−10
−11
−12



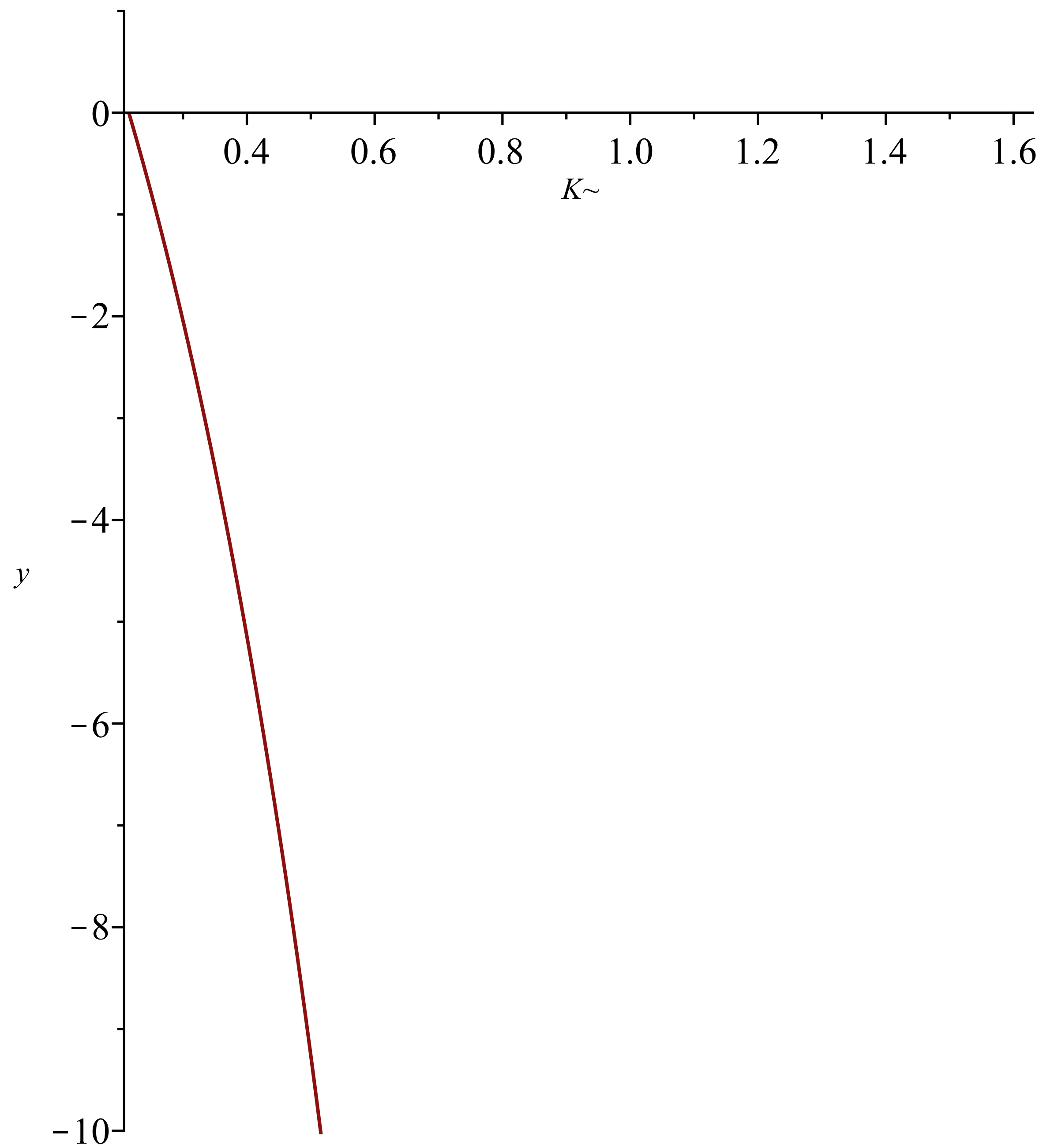
> *factor*(*subs*($V=0$, *eqVcritU*));

$$1 \quad \textbf{(3.9)}$$

> *factor*(*subs*($V=1$, *eqVcritU*)); *factor*(*subs*(nu = *nuUsub*, %)); *plot*(*subs*(nu = *nusupK*, U = *UsupK*, %%), K = *Kc* ..*Kinfini* − 0.1, y = -10 ..1);

$$\frac{8\left(3U^2\nu+3U^2-3U\nu-3U+\nu\right)}{U\left(U\nu+U-2\right)}$$

$$0$$



The polynom has four real roots : one <-1, one between -1 and 0, one between 0 and +1, and one after 1.

Asymptotic expansion of V in y (on the critical line, i.e when t= t_\nu is fixed and equal to the radius of convergence, Lemma 3.8)

For nu<=nu_c

Recall the values of Unu and nuU in this regime:

> *Unusubc*;

$$\frac{3\nu+3-\sqrt{-3\nu^2+6\nu+9}}{6\nu+6} \tag{4.1.1}$$

> *nuUsub* := *solve*(*algUsubcrit*, nu);

$$nuUsub := -\frac{3U(U-1)}{3U^2-3U+1} \tag{4.1.2}$$

In the parametrization of y in terms of V, we replace nu by its expression in terms of Unusub.

> *yUVsubc* := *factor*(*subs*(nu = *nuUsub*, *yUV*));

$$yUVsubc := -\frac{24(U-1)V(V+1)}{3UV^3-21V^2U-2V^3-3UV+18V^2-3U+6V+2} \tag{4.1.3}$$

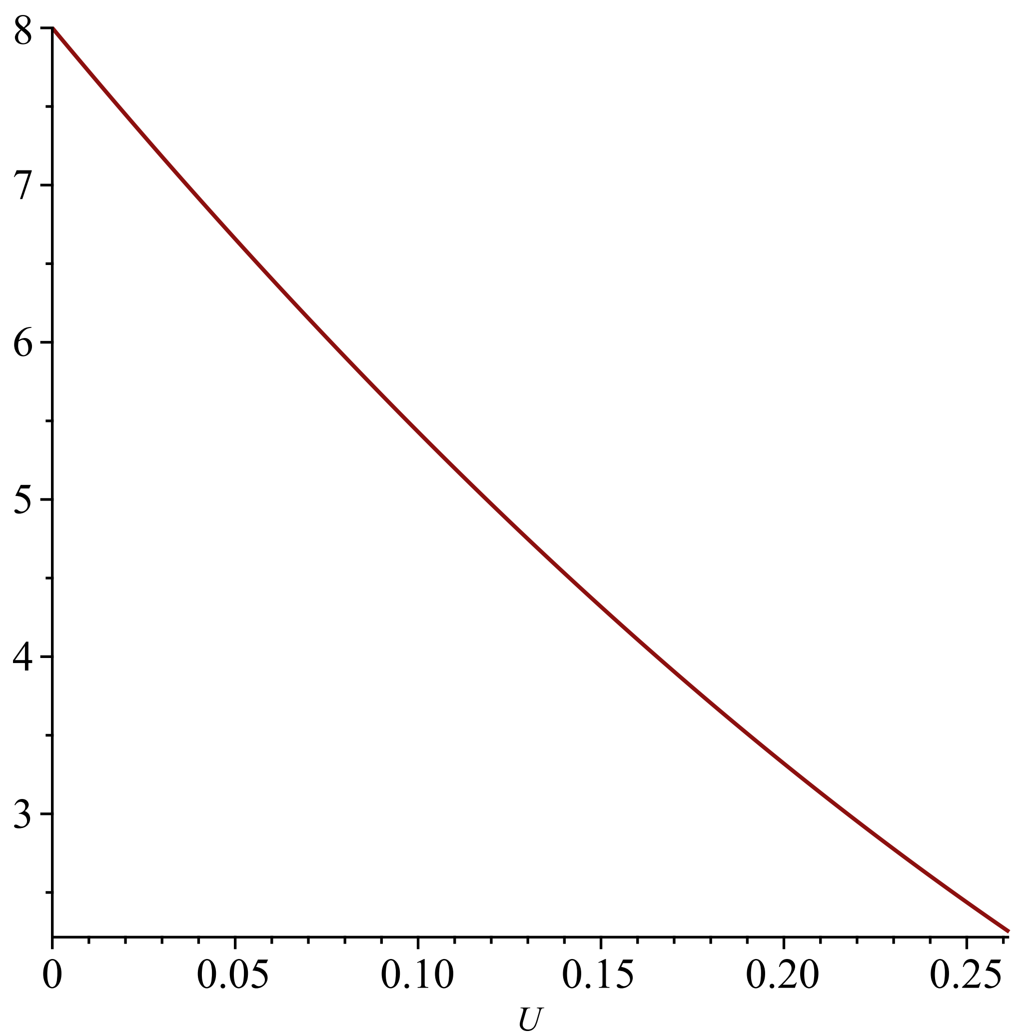
First we look at the poles of y:

> *collect*(*denom*(*yUVsubc*), *V*, *factor*); *factor*(*discrim*(%, *V*));

$$(-2+3U)V^3+(-21U+18)V^2+(-3U+6)V-3U+2$$

$$-5184\left(23U^2-28U+8\right)(U-1)^2 \tag{4.1.4}$$

> *plot*((23 U^2 − 28 U + 8), U = 0 ..*Uc*);



Since the discriminant is negative for U\in [0,Uc], yUVsubc has a unique pole. Moreover the leading coefficient of the denominator of yUVsubc is negative, hence by evaluating it at any value of V, we can determine whether it cancels before or after V.

> $factor(subs(V = 1, denom(yUVsubc)));$

$$-24\,U + 24 \tag{4.1.5}$$

Hence, there is a unique pole, located after V=1.

Now we look at the critical values for (y,V):

> $factor(numer(diff(yUVsubc, V))); solve(\%, V);$

$$24\,(U-1)\,\left(V^2+4\,V+1\right)\,(V-1)^2\,(-2+3\,U)$$

$$1,\,1,\,-2+\sqrt{3},\,-2-\sqrt{3} \tag{4.1.6}$$

> $Vsubl := -2+\sqrt{3} : Vsubc := 1 :$

y(V) is increasing in [Vsubl,Vsubc], y(V) is critical at 1 (corresponding to y=2). We compute the corresponding expansion:

> $simplify(series(yUVsubc, V = 1, 4));$

$$2 + \frac{-2+3\,U}{12\,U-12}\,(V-1)^3 + \mathrm{O}\left((V-1)^4\right) \tag{4.1.7}$$

This gives the development of V around y=2, (YY=(1-y/2)),

> $algeqtoseries(numer(2\cdot(1-YY) - yUVsubc), YY, V, 5);$

$$\left[1 + RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,YY^{1/3} + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)^2\,YY^{2/3}}{2} - \frac{4\,(U-1)\,YY}{-2+3\,U} + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,YY^{4/3}}{3\,(-2+3\,U)} + \mathrm{O}\left(YY^{5/3}\right)\right] \tag{4.1.8}$$

with Y3Y=YY^1/3=(1-y/2)^1/3:

> $allvalues(RootOf((3\,U-2)\,\_Z^3 + 24\,U - 24));$

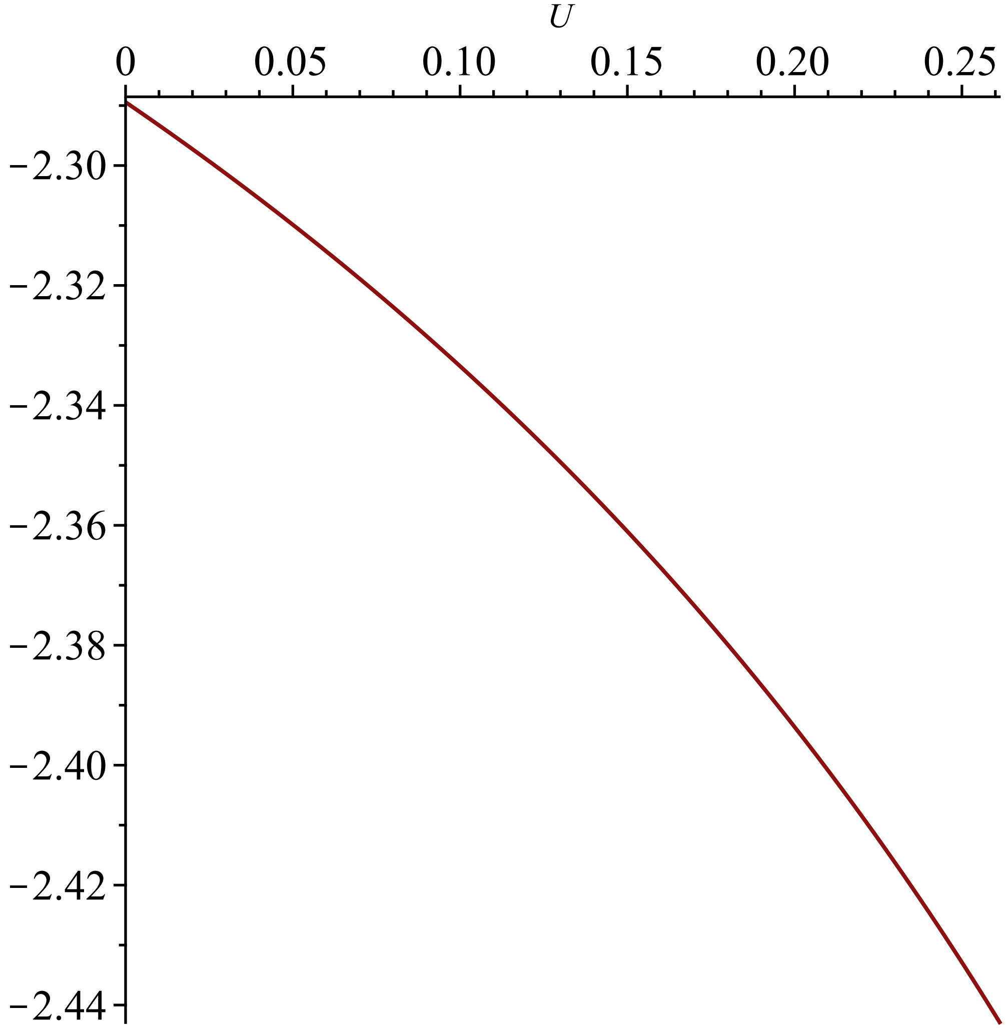
$$\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3},\,\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3},\,-\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{1/3} \tag{4.1.9}$$

> $Vsubsingy := 1 + RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)\,Y3Y + \frac{RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)^2\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{3\,U-2} + \frac{RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)\,Y3Y^4}{3\,(3\,U-2)};$

$$Vsubsingy := 1 + RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,Y3Y + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)^2\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{-2+3\,U} + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,Y3Y^4}{9\,U-6} \tag{4.1.10}$$

We check which solution of the RootOf is real, when U is real:

$$> plot\left(\left(-\frac{24\,U-24}{3\,U-2}\right)^{1/3}(-1)^{2/3}, U=0..Uc\right);$$

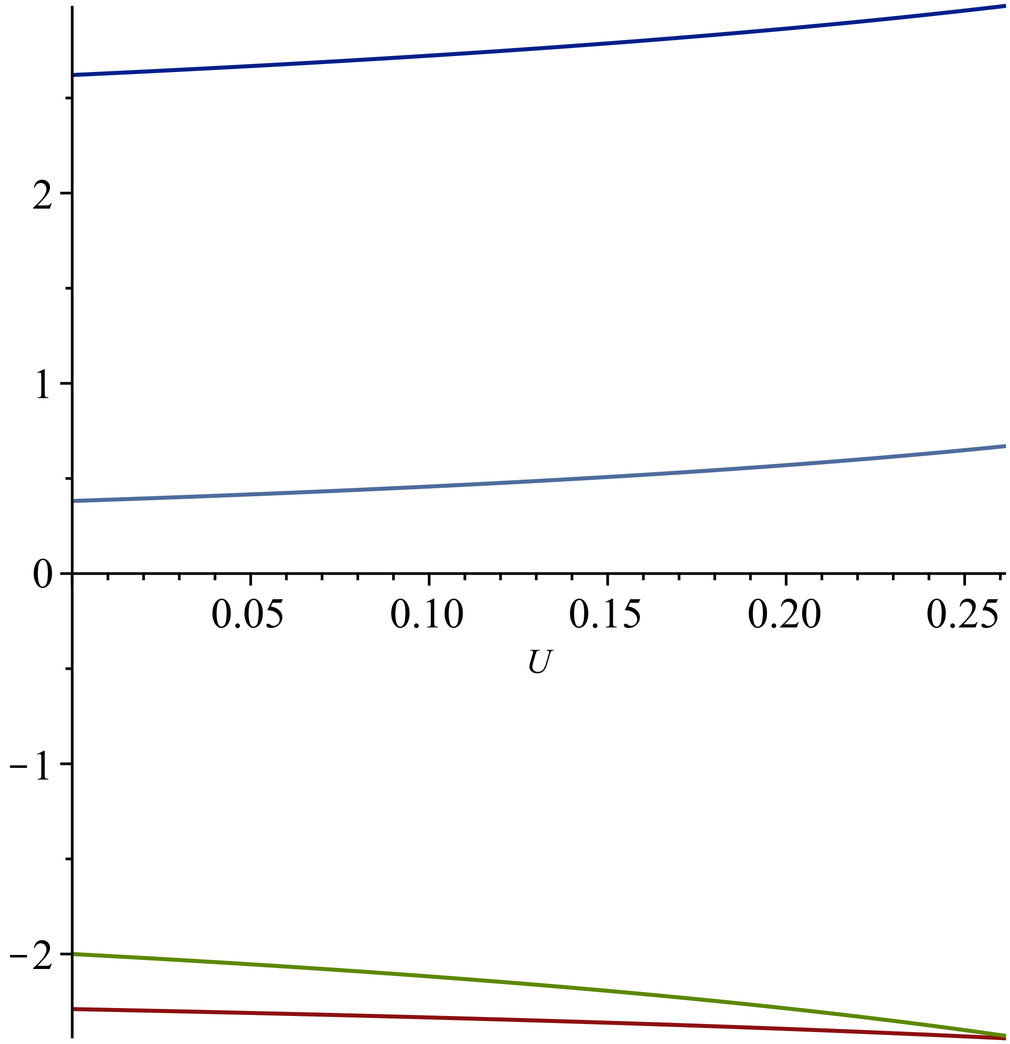


$$> Vsubsingy := subs\left(RootOf\left((3\,U-2)\,\_Z^3+24\,U-24\right)=\left(-\frac{24\,U-24}{3\,U-2}\right)^{1/3}(-1)^{2/3}, Vsubsingy\right);$$

$$Vsubsingy := 1+\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y - \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3}(-1)^{1/3}\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{-2+3\,U} + \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y^4}{9\,U-6} \quad \textbf{(4.1.11)}$$

We check that the coefficients do not vanish :

> $plot([seq(coeff(\mathbf{(4.1.11)}, Y3Y, i), i = 1..4)], U = 0..Uc)$



For nu>nu_c

For nu>nuc, things get slighly more complicated because we cannot express simply U in terms of nu. Indeed the expression is cubic in U, but we have our parametric expression:

> $nusupK; UsupK;$

$$-\frac{K\sim^3 + 3K\sim^2 + 9K\sim + 11}{(K\sim + 3)(K\sim^2 - 3)}$$

$$-\frac{K\sim^2 - 3}{6K\sim + 10} \quad \mathbf{(4.2.1)}$$

Specialisation of y to the critical line for nu > nuc, we first look at the poles of yUVsup

> $yUVsupc := factor(subs(\mathrm{nu} = nusupK, U = UsupK, yUV));$

$$yUVsupc := -\left(8\left(K\sim^3 + 3K\sim^2 + 9K\sim + 11\right)(K\sim + 1)V(V + 1)\right)/\left(K\sim^4 V^3 - 7K\sim^4 V^2 - K\sim^4 V - 40K\sim^3 V^2 - 6K\sim^2 V^3 - K\sim^4 + 8K\sim^3 V - 110K\sim^2 V^2\right. \quad \mathbf{(4.2.2)}$$

$$+ 14 K\sim^2 V - 136 K\sim V^2 + 9 V^3 + 6 K\sim^2 - 24 K\sim V - 55 V^2 - 33 V - 9)$$

We rearrange its denominator :

> *collect*(*denom*(*yUVsupc*), *V*, *factor*);

$$(K\sim^2 - 3)^2 V^3 + (-7 K\sim^4 - 40 K\sim^3 - 110 K\sim^2 - 136 K\sim - 55) V^2 - (K\sim^2 - 8 K\sim - 11) (K\sim^2 - 3) V - (K\sim^2 - 3)^2 \quad \textbf{(4.2.3)}$$

This is a polynomial of degree 3 and V, we compute its discriminant:

> *factor*(*discrim*(**(4.2.3)**, *V*));

$$-64 (K\sim^4 + 6 K\sim^3 + 30 K\sim^2 + 62 K\sim + 45) (23 K\sim^6 + 184 K\sim^5 + 593 K\sim^4 + 1008 K\sim^3 + 989 K\sim^2 + 568 K\sim + 163) (K\sim + 1)^2 (K\sim^2 - 3)^2 \quad \textbf{(4.2.4)}$$

This is clearly nonpositive for K between Kc and Kinfini. Hence the denominator of yUVsupc has only one real root. To determine its position with respect to 1, we compute:

> *factor*(*subs*(*V* = 1, *denom*(*yUVsupc*)));

$$-8 (K\sim + 1) (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11) \quad \textbf{(4.2.5)}$$

Since this is negative, and the leading term of denom(yUVsupc) is also negative, we deduct that its unique real root is bigger than 1. We now turn our attention to the possible values for V critical.

> *factor*(*numer*(*diff*(*yUVsupc*, *V*)));

$$8 (K\sim + 1) (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11) (K\sim^2 V^2 + 4 K\sim^2 V + K\sim^2 + 8 K\sim V - 3 V^2 + 4 V - 3) (K\sim^2 V^2 - 2 K\sim^2 V + K\sim^2 - 8 K\sim V - 3 V^2 - 10 V - 3) \quad \textbf{(4.2.6)}$$

There are 2 polynomials of degree 2 in V with 4 possible real roots. We first check whether the roots are real:

> *P1* := *collect*($K^2 V^2 + 4 K^2 V + K^2 + 8 K V - 3 V^2 + 4 V - 3$, *V*, *factor*);
factor(*discrim*(%, *V*));

$$P1 := (K\sim^2 - 3) V^2 + 4 (K\sim + 1)^2 V + K\sim^2 - 3$$

$$4 (K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1) \quad \textbf{(4.2.7)}$$

> *simplify*(*subs*($K = Kc, 3 K^2 + 4 K - 1$));

$$0 \quad \textbf{(4.2.8)}$$

> *P2* := *collect*($K^2 V^2 - 2 K^2 V + K^2 - 8 K V - 3 V^2 - 10 V - 3$, *V*, *factor*);
factor(*discrim*(%, *V*));

$$P2 := (K\sim^2 - 3) V^2 + (-2 K\sim^2 - 8 K\sim - 10) V + K\sim^2 - 3$$

$$32 (2 + K\sim) (K\sim + 1)^2 \quad \textbf{(4.2.9)}$$

Since the discriminant are nonnegative in the considered range of values for K, there are 4 possible real roots. (The fact that the first term has a double root at Kc is a strong indication that it should give the critical values for V. Let us check it !)

> *V1sol* := *map*(*factor*, *solve*($K^2 V^2 + 4 K^2 V + K^2 + 8 K V - 3 V^2 + 4 V - 3$, [*V*]));

(4.2.10)

$$V1sol := \left[\left[V = -\frac{2\,K\sim^2 + 4\,K\sim - \sqrt{(K\sim^2 + 4\,K\sim + 5)\,(3\,K\sim^2 + 4\,K\sim - 1)} + 2}{K\sim^2 - 3}\right],\left[V = -\frac{2\,K\sim^2 + 4\,K\sim + \sqrt{(K\sim^2 + 4\,K\sim + 5)\,(3\,K\sim^2 + 4\,K\sim - 1)} + 2}{K\sim^2 - 3}\right]\right] \quad \textbf{(4.2.10)}$$

> $V2sol := map(factor, solve(K^2\,V^2 - 2\,K^2\,V + K^2 - 8\,K\,V - 3\,V^2 - 10\,V - 3, [V]))$

$$V2sol := \left[\left[V = \frac{K\sim^2 + 4\,K\sim + 2\sqrt{2}\,\sqrt{(2 + K\sim)\,(K\sim + 1)^2} + 5}{K\sim^2 - 3}\right],\left[V = \frac{K\sim^2 + 4\,K\sim - 2\sqrt{2}\,\sqrt{(2 + K\sim)\,(K\sim + 1)^2} + 5}{K\sim^2 - 3}\right]\right] \quad \textbf{(4.2.11)}$$

> $VK11 := -\frac{2\,K^2 + 4\,K - \sqrt{(K^2 + 4\,K + 5)\,(3\,K^2 + 4\,K - 1)} + 2}{K^2 - 3}:$

$VK12 := -\frac{2\,K^2 + 4\,K + \sqrt{(K^2 + 4\,K + 5)\,(3\,K^2 + 4\,K - 1)} + 2}{K^2 - 3}:$

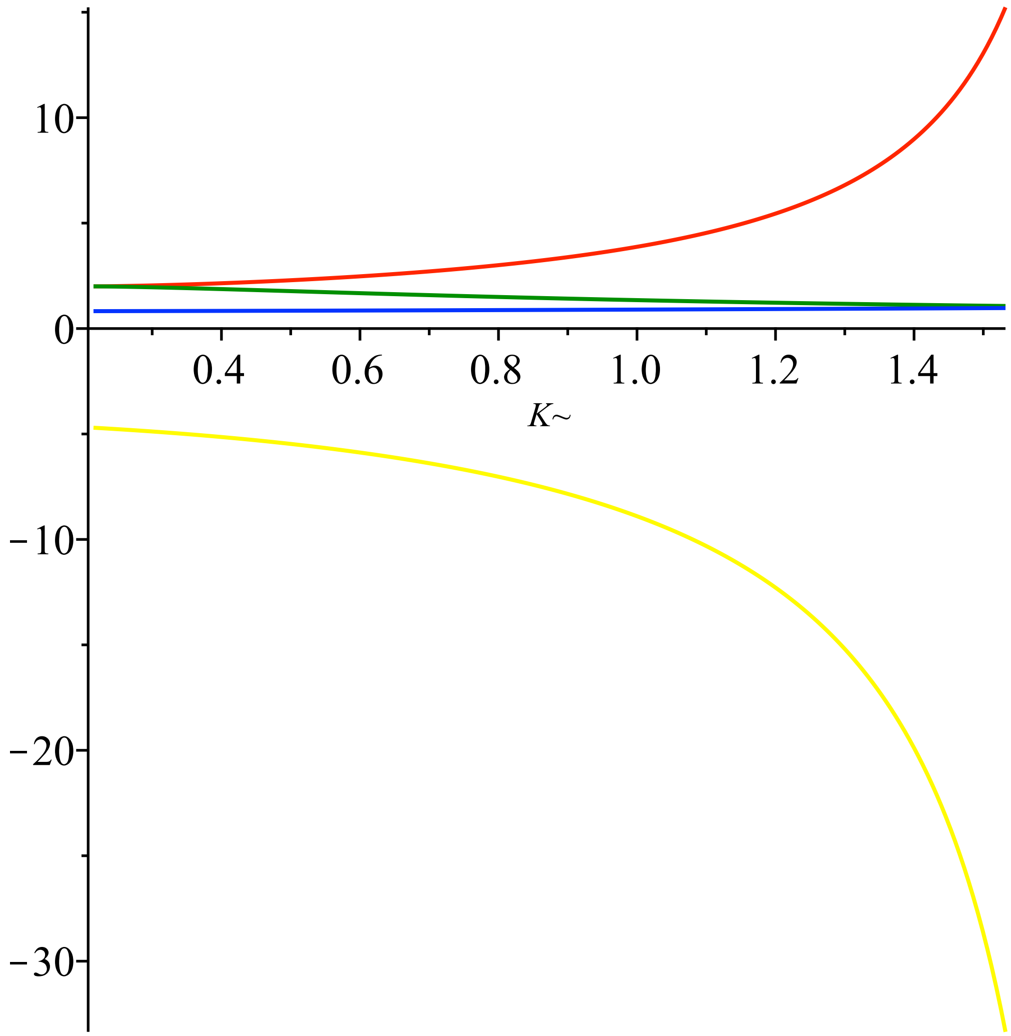
$VK21 := \frac{K^2 + 4\,K + 2\sqrt{2}\,\sqrt{(K + 2)\,(K + 1)^2} + 5}{K^2 - 3}:$

$VK22 := \frac{K^2 + 4\,K - 2\sqrt{2}\,\sqrt{(K + 2)\,(K + 1)^2} + 5}{K^2 - 3}:$

We plot the correponding values of y. Since we know that the coefficients of Qt are nonnegative, its radius of convergence must be singular.

```
> yK11 := simplify(expand(rationalize(subs(V = VK11, yUVsupc)))) :
  yK12 := simplify(expand(rationalize(subs(V = VK12, yUVsupc)))) :
  yK21 := simplify(expand(rationalize(subs(V = VK21, yUVsupc)))) :
  yK22 := simplify(expand(rationalize(subs(V = VK22, yUVsupc)))) :
```

```
> plot([yK11, yK12, yK21, yK22], K = Kc ..Kinfini − 0.2, color = ["Red", "Green", "Blue",
     "Yellow"]);
```



From the analysis for nu<=nuc, we know that the critical value of y for nu=nuc is equal to 2. The plots indicate that only yK11 and yK12 are possible candidates. Let us check:

> *simplify*(*subs* (*K* = *Kc*, *yK11*)); *simplify*(*subs* (*K* = *Kc*, *yK12*));

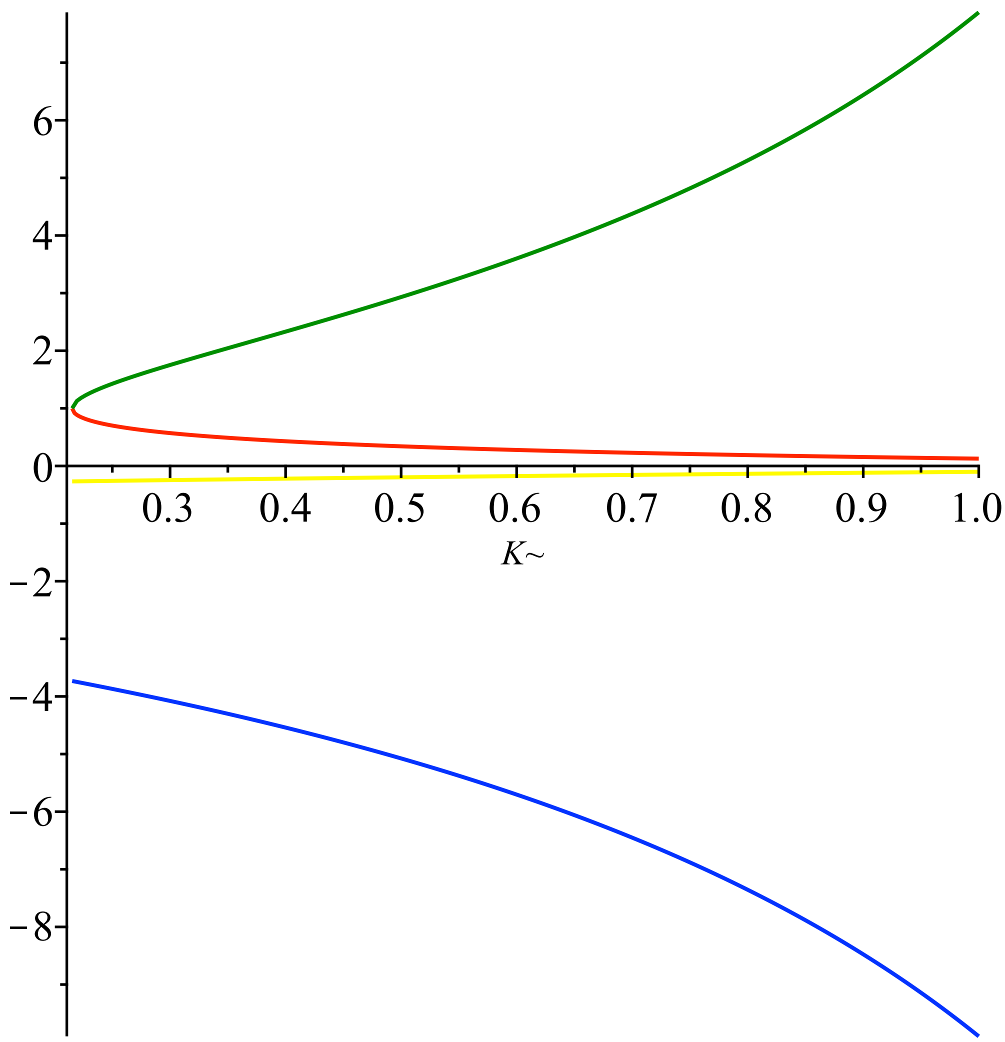
$$2$$

$$2 \tag{4.2.12}$$

We now have to distinguish between these two values. We come back to the corresponding values of V.

For the range of values of K we consider, VK11 is smaller than VK12. Plots of the roots

> *plot*([*VK11*, *VK12*, *VK21*, *VK22*], *K* = *Kc* ..1, *color* = ["Red", "Green", "Blue", "Yellow"]);



So that V+ := VK11 and V-:=VK22

We compute the expansion VK11 around yUVsupc.

```
> map(factor, map(expand, simplify(series(yUVsupc, V = VK11, 3), symbolic)));
```

$$\Big(4\,(K\sim+1)\,(K\sim^3+3\,K\sim^2+9\,K\sim+11)\,\Big(5\,K\sim^4+28\,K\sim^3 - 3\,K\sim^2\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+58\,K\sim^2 - 8\,K\sim\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+44\,K\sim+5 - 7\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}\Big)\Big)\Big/\Big(37\,K\sim^8+348\,K\sim^7 - 21\,K\sim^6\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+1456\,K\sim^6 - 144\,K\sim^5\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+3508\,K\sim^5 - 431\,K\sim^4\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+5314\,K\sim^4 - 704\,K\sim^3\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+5140\,K\sim^3 - 687\,K\sim^2\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+3016\,K\sim^2 \tag{4.2.13}$$

$$- 432 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 956 K\sim$$
$$- 149 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 145\Big) - 4 \Big((K\sim$$
$$+ 1) (3 K\sim^2 + 8 K\sim + 7) (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11) (K\sim^2 - 3)^3 \Big(3 K\sim^4$$
$$- 2 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 16 K\sim^3$$
$$- 4 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 30 K\sim^2$$
$$- 2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 16 K\sim - 5\Big)\Big) \Big/ \Big(37 K\sim^8$$
$$+ 348 K\sim^7 - 21 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1456 K\sim^6$$
$$- 144 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 3508 K\sim^5$$
$$- 431 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 5314 K\sim^4$$
$$- 704 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 5140 K\sim^3$$
$$- 687 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 3016 K\sim^2$$
$$- 432 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 956 K\sim$$
$$- 149 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 145\Big)^2 \Bigg(V$$
$$+ \frac{2 K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} + 2}{K\sim^2 - 3}\Bigg)^2 +$$
$$\mathrm{O}\left(\left(V + \frac{2 K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} + 2}{K\sim^2 - 3}\right)^3\right)$$

The expansion is quadratic, so we introduce the change of variable YY=(1-y/yK11)^(1/2) and get the following development of V around VK11:

```
> devV11ysupc := sort(map(factor, collect(convert(op(2,
      simplify(algeqtoseries(numer(yK11·(1 − YY^2) − yUVsupc), YY, V, 6))),
     polynom), YY)), YY);
```

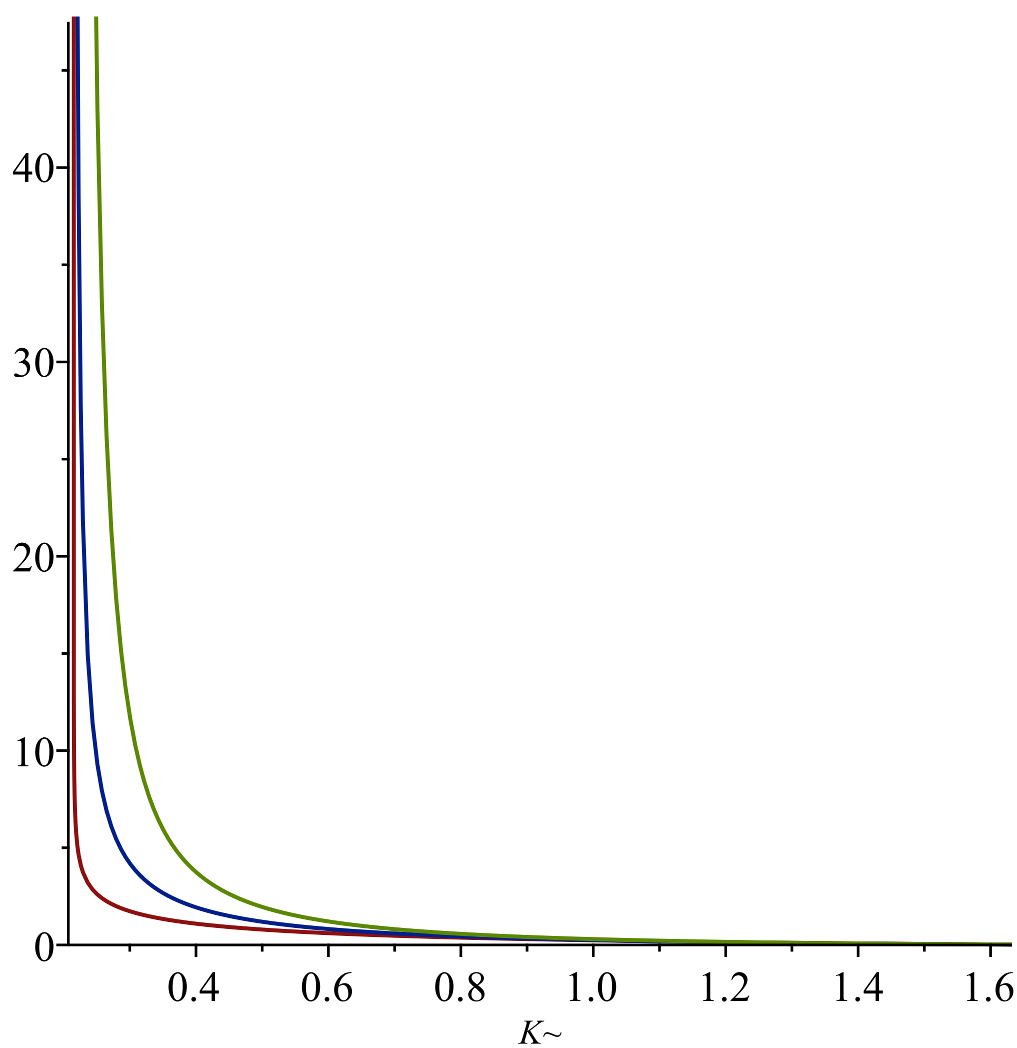
$$devV11ysupc := \Big(RootOf\big(\_Z^2 \,(9 K\sim^{10} + 36 K\sim^9 - 31 K\sim^8 - 304 K\sim^7 - 214 K\sim^6$$
$$+ 792 K\sim^5 + 1170 K\sim^4 - 432 K\sim^3 - 1539 K\sim^2 - 540 K\sim + 189) - 174 K\sim^{10}$$
$$- 1960 K\sim^9 + 100 K\sim^8 \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} - 9950 K\sim^8$$
$$+ 864 K\sim^7 \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} - 29664 K\sim^7$$
$$+ 3304 K\sim^6 \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} - 56972 K\sim^6$$
$$+ 7200 K\sim^5 \sqrt{(K\sim^2 + 4 K\sim + 5) (3 K\sim^2 + 4 K\sim - 1)} - 72752 K\sim^5$$

$$+ 9760 K{\sim}^{4} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} - 61372 K{\sim}^{4}$$
$$+ 8480 K{\sim}^{3} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} - 32608 K{\sim}^{3}$$
$$+ 4504 K{\sim}^{2} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} - 9190 K{\sim}^{2}$$
$$+ 1120 K{\sim} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 152 K{\sim}$$
$$- 4 \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 890\Big) \Big(7445 + 78955 K{\sim}^{12}$$
$$+ 335800 K{\sim}^{11} + 934389 K{\sim}^{10} + 729 K{\sim}^{14} + 11172 K{\sim}^{13} + 1700348 K{\sim}^{9}$$
$$+ 1773231 K{\sim}^{8} + 182160 K{\sim}^{7}$$
$$- 234096 K{\sim}^{6} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 374512 K{\sim}^{5} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 462922 K{\sim}^{4} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 365048 K{\sim}^{3} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 153792 K{\sim}^{2} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 19800 K{\sim} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 81796 K{\sim}$$
$$- 238 K{\sim}^{12} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 2840 K{\sim}^{11} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} - 1264840 K{\sim}^{3}$$
$$- 67049 K{\sim}^{2} - 2565413 K{\sim}^{6} - 4211364 K{\sim}^{5} - 3327599 K{\sim}^{4}$$
$$+ 4118 \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 91230 K{\sim}^{8} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 145520 K{\sim}^{7} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 14928 K{\sim}^{10} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}$$
$$- 45368 K{\sim}^{9} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)}\Big) YY^{3}\Big) \Big/ \Big(4 \big(K{\sim}^{2}$$
$$- 3\big) \left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)^{2} \left(3 K{\sim}^{2} + 8 K{\sim} + 7\right)^{3}\Big)$$
$$- \Big(\Big(98 K{\sim}^{10} + 1048 K{\sim}^{9} - 55 K{\sim}^{8} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 5078 K{\sim}^{8} - 444$$
$$- 3556 K{\sim}^{5} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 35824 K{\sim}^{5}$$
$$- 4866 K{\sim}^{4} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 32332 K{\sim}^{4}$$
$$- 4180 K{\sim}^{3} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 18752 K{\sim}^{3}$$
$$- 2228 K{\sim}^{2} \sqrt{\left(K{\sim}^{2} + 4 K{\sim} + 5\right) \left(3 K{\sim}^{2} + 4 K{\sim} - 1\right)} + 4682 K{\sim}^{2}$$

$$- 812\,K\sim\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 1480\,K\sim$$
$$- 223\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 994\Big)\,YY^{2}\Big)\Big/\Big(\big(3\,K\sim^{2}$$
$$+ 4\,K\sim-1\big)\left(3\,K\sim^{2}+8\,K\sim+7\right)^{2}\left(K\sim^{2}-3\right)^{2}\Big) + RootOf\Big(\_Z^{2}\,\big(9\,K\sim^{10}$$
$$+ 36\,K\sim^{9} - 31\,K\sim^{8} - 304\,K\sim^{7} - 214\,K\sim^{6} + 792\,K\sim^{5} + 1170\,K\sim^{4} - 432\,K\sim^{3}$$
$$- 1539\,K\sim^{2} - 540\,K\sim + 189\big) - 174\,K\sim^{10} - 1960\,K\sim^{9}$$
$$+ 100\,K\sim^{8}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 9950\,K\sim^{8}$$
$$+ 864\,K\sim^{7}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 29664\,K\sim^{7}$$
$$+ 3304\,K\sim^{6}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 56972\,K\sim^{6}$$
$$+ 7200\,K\sim^{5}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 72752\,K\sim^{5}$$
$$+ 9760\,K\sim^{4}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 61372\,K\sim^{4}$$
$$+ 8480\,K\sim^{3}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 32608\,K\sim^{3}$$
$$+ 4504\,K\sim^{2}\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} - 9190\,K\sim^{2}$$
$$+ 1120\,K\sim\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} + 152\,K\sim$$
$$- 4\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)} + 890\Big)\,YY$$
$$- \frac{2\,K\sim^{2}+4\,K\sim-\sqrt{\left(K\sim^{2}+4\,K\sim+5\right)\left(3\,K\sim^{2}+4\,K\sim-1\right)}+2}{K\sim^{2}-3}$$

We check that the coefficients in the development do not cancel for K in [Kc,Kinfini]

```
> plot([seq(coeff(devV11ysupc, YY, i), i = 1..3)], K = Kc..Kinfini - 0.1);
```



Asymptotic behavior (in t) of V(t,ty) (Lemma 3.9)

For nu<nu_c:

Recall the singular expansion of U in this regime (Usubc is U(nu,tnu^3)):

> *Usubcsing3*;

$$Usubc + \frac{Usubc\,(-2 + 3\,Usubc)\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,XX}{6}$$

$$+ \left(\left(1458\,Usubc^6 - 5778\,Usubc^5 + 9045\,Usubc^4 - 7146\,Usubc^3 + 2984\,Usubc^2 - 616\,Usubc + 4\right.\right.$$

$$\left.\left. + 2\right)^2 (2\,Usubc - 1)\right) + \Big(5\,(135\,Usubc^2 - 134\,Usubc + 22)\,(6\,Usubc^2$$

$$- 10\, Usubc + 3\big)\, Usubc^3\, (-2$$

$$+ 3\, Usubc)^3 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX^3\Bigg) \Bigg/ \Big(1296\, \big(9\, Usubc^2$$

$$- 10\, Usubc + 2\big)^3\, (2\, Usubc - 1)\Big)$$

We want to compute the development of V around rhosubc (=t_\nu^3 in the paper), for a fixed y. Recall that y is parametrized by U and V:

> *yUV*;

$$\big(8\, \nu\, (1 - 2\, U)\, V\, (V + 1)\big) \Bigg/ \Bigg(U\, \big(U\, (\nu + 1) - 2\big) \Bigg(V^3 + \frac{\big(9\, (\nu + 1)\, U^2 - 2\, (3 + 10\, \nu)\, U + 8\, \nu\big)\, V^2}{U\, \big(U\, (\nu + 1) - 2\big)} - \frac{\big(9\, U\, (\nu + 1) - 4\, \nu - 6\big)\, V}{U\, (\nu + 1) - 2} - 1\Bigg)\Bigg) \quad \mathbf{(5.1.2)}$$

Since y is fixed, we could write the development of V in terms of this fixed value of y. It turns out that the formulas are simpler when written in terms of V rather than y. Indeed, when U is equal to Usubc (i.e. when w=t^3= rhosubc), nu can be replaced by its value in terms of Usubc, and hence the value of y is fully determined by the value of V in this setting, which we denote by Vsubc:

> *yVsub* := *subs*(*V* = *Vsub*, *U* = *Usubc*, *factor*(*subs*(nu = *nuUsub*, *yUV*)));

$$yVsub := -\big(24\, (Usubc - 1)\, Vsub\, (Vsub + 1)\big) \Big/ \big(3\, Usubc\, Vsub^3 - 21\, Usubc\, Vsub^2 - 2\, Vsub^3 - 3\, Usubc\, Vsub + 18\, Vsub^2 - 3\, Usubc + 6\, Vsub + 2\big) \quad \mathbf{(5.1.3)}$$

When we compute the development of V for w close to rhosubc, we can replace y by the latter value. Here is the new equation we obtain:

> *op*(6, *factor*(*numer*(*yVsub* − *subs*(nu = *subs*(*U* = *Usubc*, *nuUsub*), *yUV*))));
indets(%);

$$-6\, U\, Usubc^2\, V^3\, Vsub^2 + 6\, U\, Usubc^2\, V^2\, Vsub^3 + U^2\, V^3\, Vsub^2 - 6\, U\, Usubc^2\, V^3\, Vsub + 6\, U\, Usubc^2\, V\, Vsub^3 + 6\, U\, Usubc\, V^3\, Vsub^2 - 4\, U\, Usubc\, V^2\, Vsub^3 - 3\, Usubc^2\, V^2\, Vsub^3 + U^2\, V^3\, Vsub + 9\, U^2\, V^2\, Vsub^2 + 36\, U\, Usubc^2\, V^2\, Vsub - 36\, U\, Usubc^2\, V\, Vsub^2 + 6\, U\, Usubc\, V^3\, Vsub - 6\, U\, Usubc\, V^2\, Vsub^2 - 4\, U\, Usubc\, V\, Vsub^3 - 2\, U\, V^3\, Vsub^2 - 3\, Usubc^2\, V^2\, Vsub^2 - 3\, Usubc^2\, V\, Vsub^3 + 2\, Usubc\, V^2\, Vsub^3 + 9\, U^2\, V^2\, Vsub - 9\, U^2\, V\, Vsub^2 - 6\, U\, Usubc^2\, V^2 + 6\, U\, Usubc^2\, Vsub^2 - 30\, U\, Usubc\, V^2\, Vsub + 30\, U\, Usubc\, V\, Vsub^2 - 2\, U\, V^3\, Vsub - 6\, U\, V^2\, Vsub^2 - 21\, Usubc^2\, V^2\, Vsub + 21\, Usubc^2\, V\, Vsub^2 + 6\, Usubc\, V^2\, Vsub^2 + 2\, Usubc\, V\, Vsub^3 - 9\, U^2\, V\, Vsub - U^2\, Vsub^2 - 6\, U\, Usubc^2\, V + 6\, U\, Usubc^2\, Vsub + 4\, U\, Usubc\, V^2 + 6\, U\, Usubc\, V\, Vsub - 6\, U\, Usubc\, Vsub^2 - 6\, U\, V^2\, Vsub + 6\, U\, V\, Vsub^2 + 3\, Usubc^2\, V^2$$

$$+ 3\, Usubc^2\, V\, Vsub + 18\, Usubc\, V^2\, Vsub - 18\, Usubc\, V\, Vsub^2 - U^2\, Vsub + 4\, U\, Usubc\, V - 6\, U\, Usubc\, Vsub + 6\, U\, V\, Vsub + 2\, U\, Vsub^2 + 3\, Usubc^2\, V - 2\, Usubc\, V^2 - 6\, Usubc\, V\, Vsub + 2\, U\, Vsub - 2\, Usubc\, V$$

$$\{U, Usubc, V, Vsub\} \qquad \textbf{(5.1.4)}$$

> $eqyUVsub := -6\, U\, Usubc^2\, V^3\, Vsub^2 + 6\, U\, Usubc^2\, V^2\, Vsub^3 + U^2\, V^3\, Vsub^2 - 6\, U\, Usubc^2\, V^3\, Vsub + 6\, U\, Usubc^2\, V\, Vsub^3 + 6\, U\, Usubc\, V^3\, Vsub^2 - 4\, U\, Usubc\, V^2\, Vsub^3 - 3\, Usubc^2\, V^2\, Vsub^3 + U^2\, V^3\, Vsub + 9\, U^2\, V^2\, Vsub^2 + 36\, U\, Usubc^2\, V^2\, Vsub - 36\, U\, Usubc^2\, V\, Vsub^2 + 6\, U\, Usubc\, V^3\, Vsub - 6\, U\, Usubc\, V^2\, Vsub^2 - 4\, U\, Usubc\, V\, Vsub^3 - 2\, U\, V^3\, Vsub^2 - 3\, Usubc^2\, V^2\, Vsub^2 - 3\, Usubc^2\, V\, Vsub^3 + 2\, Usubc\, V^2\, Vsub^3 + 9\, U^2\, V^2\, Vsub - 9\, U^2\, V\, Vsub^2 - 6\, U\, Usubc^2\, V^2 + 6\, U\, Usubc^2\, Vsub^2 - 30\, U\, Usubc\, V^2\, Vsub + 30\, U\, Usubc\, V\, Vsub^2 - 2\, U\, V^3\, Vsub - 6\, U\, V^2\, Vsub^2 - 21\, Usubc^2\, V^2\, Vsub + 21\, Usubc^2\, V\, Vsub^2 + 6\, Usubc\, V^2\, Vsub^2 + 2\, Usubc\, V\, Vsub^3 - 9\, U^2\, V\, Vsub - U^2\, Vsub^2 - 6\, U\, Usubc^2\, V + 6\, U\, Usubc^2\, Vsub + 4\, U\, Usubc\, V^2 + 6\, U\, Usubc\, V\, Vsub - 6\, U\, Usubc\, Vsub^2 - 6\, U\, V^2\, Vsub + 6\, U\, V\, Vsub^2 + 3\, Usubc^2\, V^2 + 3\, Usubc^2\, V\, Vsub + 18\, Usubc\, V^2\, Vsub - 18\, Usubc\, V\, Vsub^2 - U^2\, Vsub + 4\, U\, Usubc\, V - 6\, U\, Usubc\, Vsub + 6\, U\, V\, Vsub + 2\, U\, Vsub^2 + 3\, Usubc^2\, V - 2\, Usubc\, V^2 - 6\, Usubc\, V\, Vsub + 2\, U\, Vsub - 2\, V\, Usubc :$

We plug the singular behavior of U in the equation, and deduce from it the asympotic behavior of V (we write V=Vsub + XX*VX, so that we obtain the singular behavior of VX)

> $algeqtoseries(simplify(subs(U = Usubcsing3, V = Vsub + XX \cdot VX, eqyUVsub)), XX, VX, 3, true);$

$$\left[\frac{\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \sqrt{6}\, Vsub\, (Vsub + 1)}{3\, (Vsub - 1)} - \frac{1}{18} \Big((Vsub + 1)\, Vsub\, \big(6\, Usubc^2 - 10\, Usubc + 3\big)\, \big(81\, Usubc^4\, Vsub^3 - 243\, Usubc^4\, Vsub^2 - 384\, Usubc^3\, Vsub^3 + 243\, Usubc^4\, Vsub - 288\, Usubc^3\, Vsub^2 + 454\, Vsub^3\, Usubc^2 - 81\, Usubc^4 - 792\, Usubc^3\, Vsub + 894\, Usubc^2\, Vsub^2 \right. \qquad \textbf{(5.1.5)}$$

$$- 188\,Vsub^{3}\,Usubc + 168\,Usubc^{3} + 846\,Usubc^{2}\,Vsub - 492\,Usubc\,Vsub^{2}$$

$$+ 24\,Vsub^{3} - 106\,Usubc^{2} - 348\,Usubc\,Vsub + 72\,Vsub^{2} + 20\,Usubc + 48\,Vsub\Big)$$

$$\Big) \Big/ \Big((2\,Usubc - 1)\,\big(Vsub^{2} + 4\,Vsub + 1\big)\,\big(9\,Usubc^{2} - 10\,Usubc$$

$$+ 2\big)^{2}\,(Vsub - 1)^{2}\Big)\,XX$$

$$- \frac{1}{648}\left(Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^{2} - 10\,Usubc + 3}{9\,Usubc^{2} - 10\,Usubc + 2}}\,\big(103518\,Usubc^{8}\,Vsub^{7}\right.$$

$$- 281394\,Usubc^{8}\,Vsub^{6} - 647622\,Usubc^{7}\,Vsub^{7} + 118098\,Usubc^{8}\,Vsub^{5}$$

$$+ 1074762\,Usubc^{7}\,Vsub^{6} + 1632015\,Usubc^{6}\,Vsub^{7} + 269730\,Usubc^{8}\,Vsub^{4}$$

$$+ 284310\,Usubc^{7}\,Vsub^{5} - 1507545\,Usubc^{6}\,Vsub^{6} - 2172726\,Usubc^{5}\,Vsub^{7}$$

$$- 255150\,Usubc^{8}\,Vsub^{3} - 143802\,Usubc^{7}\,Vsub^{4} - 2474199\,Usubc^{6}\,Vsub^{5}$$

$$+ 840474\,Usubc^{5}\,Vsub^{6} + 1676186\,Usubc^{4}\,Vsub^{7} + 13122\,Usubc^{8}\,Vsub^{2}$$

$$+ 1609686\,Usubc^{7}\,Vsub^{3} - 2504655\,Usubc^{6}\,Vsub^{4} + 4954950\,Usubc^{5}\,Vsub^{5}$$

$$+ 38330\,Usubc^{4}\,Vsub^{6} - 769072\,Usubc^{3}\,Vsub^{7} + 33534\,Usubc^{8}\,Vsub$$

$$+ 187110\,Usubc^{7}\,Vsub^{2} - 4251447\,Usubc^{6}\,Vsub^{3} + 6349878\,Usubc^{5}\,Vsub^{4}$$

$$- 4764858\, Usubc^{4}\, Vsub^{5} - 266800\, Usubc^{3}\, Vsub^{6} + 205560\, Usubc^{2}\, Vsub^{7}$$

$$- 1458\, Usubc^{8} - 126630\, Usubc^{7}\, Vsub - 907551\, Usubc^{6}\, Vsub^{2}$$

$$+ 6094278\, Usubc^{5}\, Vsub^{3} - 6858106\, Usubc^{4}\, Vsub^{4} + 2511600\, Usubc^{3}\, Vsub^{5}$$

$$+ 125208\, Usubc^{2}\, Vsub^{6} - 29376\, Usubc\, Vsub^{7} + 1674\, Usubc^{7}$$

$$+ 179199\, Usubc^{6}\, Vsub + 1586070\, Usubc^{5}\, Vsub^{2} - 5156362\, Usubc^{4}\, Vsub^{3}$$

$$+ 3948272\, Usubc^{3}\, Vsub^{4} - 737016\, Usubc^{2}\, Vsub^{5} - 24192\, Usubc\, Vsub^{6}$$

$$+ 1728\, Vsub^{7} + 5319\, Usubc^{6} - 112086\, Usubc^{5}\, Vsub - 1413642\, Usubc^{4}\, Vsub^{2}$$

$$+ 2630576\, Usubc^{3}\, Vsub^{3} - 1255896\, Vsub^{4}\, Usubc^{2} + 112320\, Usubc\, Vsub^{5}$$

$$+ 1728\, Vsub^{6} - 12006\, Usubc^{5} + 20906\, Usubc^{4}\, Vsub + 703920\, Usubc^{3}\, Vsub^{2}$$

$$- 790968\, Vsub^{3}\, Usubc^{2} + 207360\, Vsub^{4}\, Usubc - 6912\, Vsub^{5} + 9290\, Usubc^{4}$$

$$+ 9104\, Usubc^{3}\, Vsub - 196824\, Usubc^{2}\, Vsub^{2} + 128448\, Vsub^{3}\, Usubc$$

$$- 13824\, Vsub^{4} - 3184\, Usubc^{3} - 4680\, Usubc^{2}\, Vsub + 28800\, Usubc\, Vsub^{2}$$

$$- 8640\, Vsub^{3} + 408\, Usubc^{2} + 576\, Usubc\, Vsub - 1728\, Vsub^{2})\Big) \Big/ \Big(\big(Vsub^{2}$$

$$+ 4\, Vsub + 1\big)\, \big(9\, Usubc^{2} - 10\, Usubc + 2\big)^{3}\, (2\, Usubc - 1)\, (Vsub - 1)^{5}\Big)\, XX^{2}$$

$$+ \mathrm{O}\big(XX^{3}\big)\Big]$$

> $Vsubsing3 := sort\left(collect\left(Vsub + XX\right.\right.$

$$\cdot\left(\frac{Vsub\,(Vsub+1)\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}}{3\,(Vsub-1)} - \frac{1}{18}\,\Big(\big(6\,Usubc^2\right.$$

$$-\,10\,Usubc+3\big)\,(Vsub+1)\,Vsub\,\big(81\,Usubc^4\,Vsub^3-243\,Usubc^4\,Vsub^2$$

$$-\,384\,Usubc^3\,Vsub^3+243\,Usubc^4\,Vsub-288\,Usubc^3\,Vsub^2+454\,Usubc^2\,Vsub^3$$

$$-\,81\,Usubc^4-792\,Usubc^3\,Vsub+894\,Usubc^2\,Vsub^2-188\,Usubc\,Vsub^3$$

$$+\,168\,Usubc^3+846\,Usubc^2\,Vsub-492\,Usubc\,Vsub^2+24\,Vsub^3-106\,Usubc^2$$

$$-\,348\,Usubc\,Vsub+72\,Vsub^2+20\,Usubc+48\,Vsub\big)\Big)\Big/\Big((-1$$

$$+\,2\,Usubc)\,\big(Vsub^2+4\,Vsub+1\big)\,\big(9\,Usubc^2-10\,Usubc+2\big)^2\,(Vsub-1)^2\Big)$$

$$XX - \frac{1}{648}\,\left(Vsub\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,\sqrt{6}\,\big(103518\,Usubc^8\,Vsub^7\right.$$

$$-\,281394\,Usubc^8\,Vsub^6-647622\,Usubc^7\,Vsub^7+118098\,Usubc^8\,Vsub^5$$

$$+\,1074762\,Usubc^7\,Vsub^6+1632015\,Usubc^6\,Vsub^7+269730\,Usubc^8\,Vsub^4$$

$$+\,284310\,Usubc^7\,Vsub^5-1507545\,Usubc^6\,Vsub^6-2172726\,Usubc^5\,Vsub^7$$

$$- 255150\, Usubc^8\, Vsub^3 - 143802\, Usubc^7\, Vsub^4 - 2474199\, Usubc^6\, Vsub^5$$

$$+ 840474\, Usubc^5\, Vsub^6 + 1676186\, Usubc^4\, Vsub^7 + 13122\, Usubc^8\, Vsub^2$$

$$+ 1609686\, Usubc^7\, Vsub^3 - 2504655\, Usubc^6\, Vsub^4 + 4954950\, Usubc^5\, Vsub^5$$

$$+ 38330\, Usubc^4\, Vsub^6 - 769072\, Usubc^3\, Vsub^7 + 33534\, Usubc^8\, Vsub$$

$$+ 187110\, Usubc^7\, Vsub^2 - 4251447\, Usubc^6\, Vsub^3 + 6349878\, Usubc^5\, Vsub^4$$

$$- 4764858\, Usubc^4\, Vsub^5 - 266800\, Usubc^3\, Vsub^6 + 205560\, Usubc^2\, Vsub^7$$

$$- 1458\, Usubc^8 - 126630\, Usubc^7\, Vsub - 907551\, Usubc^6\, Vsub^2$$

$$+ 6094278\, Usubc^5\, Vsub^3 - 6858106\, Usubc^4\, Vsub^4 + 2511600\, Usubc^3\, Vsub^5$$

$$+ 125208\, Usubc^2\, Vsub^6 - 29376\, Usubc\, Vsub^7 + 1674\, Usubc^7$$

$$+ 179199\, Usubc^6\, Vsub + 1586070\, Usubc^5\, Vsub^2 - 5156362\, Usubc^4\, Vsub^3$$

$$+ 3948272\, Usubc^3\, Vsub^4 - 737016\, Usubc^2\, Vsub^5 - 24192\, Usubc\, Vsub^6$$

$$+ 1728\, Vsub^7 + 5319\, Usubc^6 - 112086\, Usubc^5\, Vsub - 1413642\, Usubc^4\, Vsub^2$$

$$+ 2630576\, Usubc^3\, Vsub^3 - 1255896\, Usubc^2\, Vsub^4 + 112320\, Usubc\, Vsub^5$$

$$+ 1728\, Vsub^6 - 12006\, Usubc^5 + 20906\, Usubc^4\, Vsub + 703920\, Usubc^3\, Vsub^2$$

$$- 790968\, Usubc^2\, Vsub^3 + 207360\, Usubc\, Vsub^4 - 6912\, Vsub^5 + 9290\, Usubc^4$$

$$+ 9104\, Usubc^3\, Vsub - 196824\, Usubc^2\, Vsub^2 + 128448\, Usubc\, Vsub^3$$

$$- 13824\, Vsub^4 - 3184\, Usubc^3 - 4680\, Usubc^2\, Vsub + 28800\, Usubc\, Vsub^2$$

$$- 8640\,Vsub^3 + 408\,Usubc^2 + 576\,Usubc\,Vsub - 1728\,Vsub^2)) \Big/ \Big((Vsub^2 + 4\,Vsub + 1) \left(9\,Usubc^2 - 10\,Usubc + 2\right)^3 (-1 + 2\,Usubc)\,(Vsub - 1)^5)\,XX^2 \Big)$$

$$, XX, factor \Big), XX, ascending \Big);$$

$$Vsubsing3 := Vsub + \frac{\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,\sqrt{6}\,Vsub\,(Vsub + 1)\,XX}{3\,(Vsub - 1)}$$

$$- \big((Vsub + 1)\,Vsub\,\left(6\,Usubc^2 - 10\,Usubc + 3\right)\big(81\,Usubc^4\,Vsub^3 - 243\,Usubc^4\,Vsub^2 - 384\,U$$

$$+ 454\,Vsub^3\,Usubc^2 - 81\,Usubc^4 - 792\,Usubc^3\,Vsub + 894\,Usubc^2\,Vsub^2$$

$$- 188\,Vsub^3\,Usubc + 168\,Usubc^3 + 846\,Usubc^2\,Vsub - 492\,Usubc\,Vsub^2$$

$$+ 24\,Vsub^3 - 106\,Usubc^2 - 348\,Usubc\,Vsub + 72\,Vsub^2 + 20\,Usubc + 48\,Vsub\big)$$

$$XX^2 \big) \Big/ \big(18\,(2\,Usubc - 1)\,\left(Vsub^2 + 4\,Vsub + 1\right)\big(9\,Usubc^2 - 10\,Usubc$$

$$+ 2\big)^2\,(Vsub - 1)^2\big) - \left(Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,(Vsub \right.$$

$$+ 1)\,\left(6\,Usubc^2 - 10\,Usubc + 3\right)\big(17253\,Usubc^6\,Vsub^6 - 64152\,Usubc^6\,Vsub^5$$

$$- 79182\,Usubc^5\,Vsub^6 + 83835\,Usubc^6\,Vsub^4 + 180144\,Usubc^5\,Vsub^5$$

$$+ 131406\,Usubc^4\,Vsub^6 - 38880\,Usubc^6\,Vsub^3 - 99954\,Usubc^5\,Vsub^4$$

$$- 190944\,Usubc^4\,Vsub^5 - 103520\,Usubc^3\,Vsub^6 - 3645\,Usubc^6\,Vsub^2$$

$$+ 150912\,Usubc^5\,Vsub^3 - 97614\,Usubc^4\,Vsub^4 + 93888\,Usubc^3\,Vsub^5$$

$$+ 41128\,Usubc^2\,Vsub^6 + 5832\,Usubc^6\,Vsub + 46494\,Usubc^5\,Vsub^2$$

$$- 257376\,Usubc^4\,Vsub^3 + 210912\,Usubc^3\,Vsub^4 - 21024\,Usubc^2\,Vsub^5$$

$$- 7872\,Usubc\,Vsub^6 - 243\,Usubc^6 - 11664\,Usubc^5\,Vsub$$

$$- 100926\,Usubc^4\,Vsub^2 + 230272\,Usubc^3\,Vsub^3 - 120840\,Vsub^4\,Usubc^2$$

$$+ 1728\,Usubc\,Vsub^5 + 576\,Vsub^6 - 126\,Usubc^5 + 6624\,Usubc^4\,Vsub$$

$$+ 89568\,Usubc^3\,Vsub^2 - 109376\,Vsub^3\,Usubc^2 + 28032\,Usubc\,Vsub^4$$

$$+ 798\,Usubc^4 + 192\,Usubc^3\,Vsub - 37800\,Usubc^2\,Vsub^2 + 25728\,Vsub^3\,Usubc$$

$$- 2304\,Vsub^4 - 608\,Usubc^3 - 1056\,Usubc^2\,Vsub + 7488\,Usubc\,Vsub^2$$

$$- 2304\,Vsub^3 + 136\,Usubc^2 + 192\,Usubc\,Vsub - 576\,Vsub^2\big)\,XX^3 \big) \Big/$$

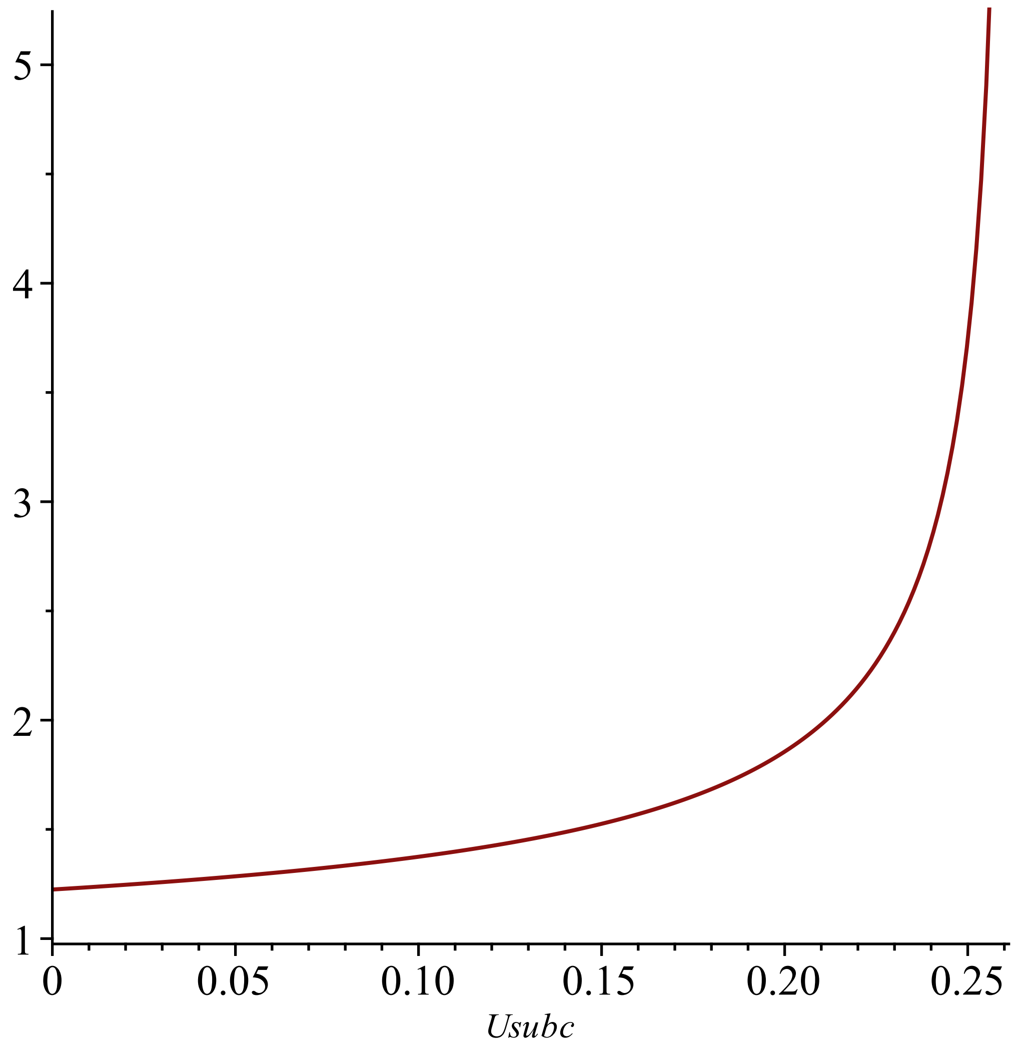
$$\big(648\,\left(Vsub^2 + 4\,Vsub + 1\right)\left(9\,Usubc^2 - 10\,Usubc + 2\right)^3\,(2\,Usubc$$

$- 1)\ (Vsub - 1)^5)$

We check that the coefficients in the development do not cancel. Recall that since y\in(0,2), Vsub\in (0,1) (see the proof of Lemma~\ref{lem:weightsclusters}).

For the coefficient of XX:

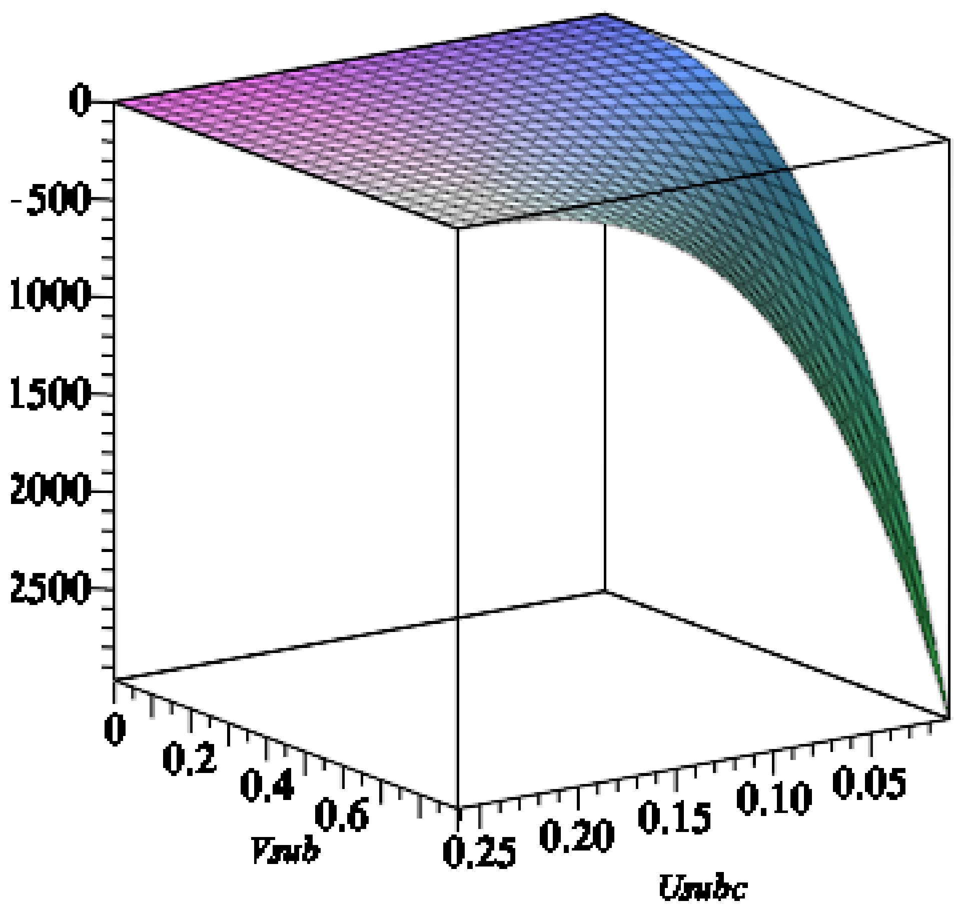
$$> plot\left(\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}, Usubc = 0\,..Uc\right);$$



For the coefficient of XX^2:

$$> plot3d\big(\big(17253\, Usubc^6\, Vsub^6 - 64152\, Usubc^6\, Vsub^5 - 79182\, Usubc^5\, Vsub^6 + 83835\, Usubc^6\, Vsub^4 + 180144\, Usubc^5\, Vsub^5 + 131406\, Usubc^4\, Vsub^6 - 38880\, Vsub^3\, Usubc^6 - 99954\, Usubc^5\, Vsub^4 - 190944\, Usubc^4\, Vsub^5 - 103520\, Usubc^3\, Vsub^6 - 3645\, Vsub^2\, Usubc^6 + 150912\, Vsub^3\, Usubc^5 - 97614\, Usubc^4\, Vsub^4 + 93888\, Usubc^3\, Vsub^5 + 41128\, Usubc^2\, Vsub^6 + 5832\, Usubc^6\, Vsub + 46494\, Vsub^2\, Usubc^5 - 257376\, Vsub^3\, Usubc^4 + 210912\, Usubc^3\, Vsub^4 - 21024\, Usubc^2\, Vsub^5 - 7872\, Usubc\, Vsub^6 - 243\, Usubc^6 - 11664\, Usubc^5\, Vsub - 100926\, Vsub^2\, Usubc^4 + 230272\, Vsub^3\, Usubc^3 - 120840\, Vsub^4\, Usubc^2 + 1728\, Usubc\, Vsub^5 + 576\, Vsub^6 - 126\, Usubc^5$$

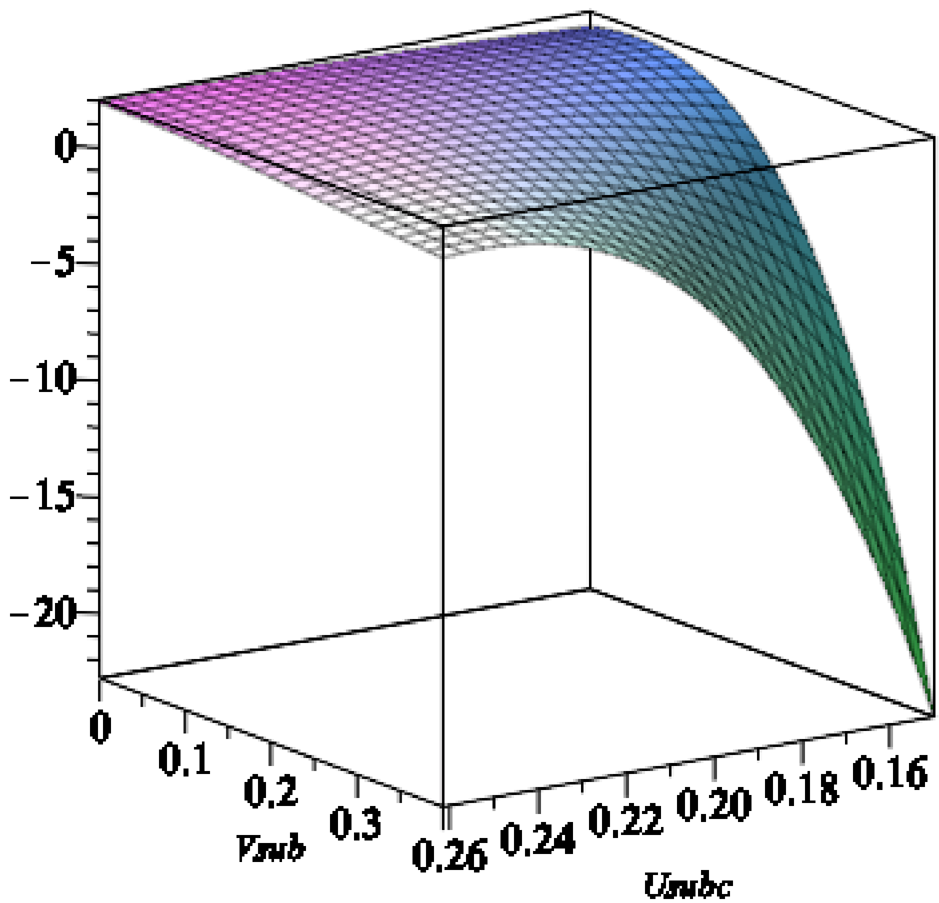
$$+ 6624\, Usubc^4\, Vsub + 89568\, Vsub^2\, Usubc^3 - 109376\, Vsub^3\, Usubc^2 + 28032\, Vsub^4\, Usubc + 798\, Usubc^4 + 192\, Usubc^3\, Vsub - 37800\, Vsub^2\, Usubc^2 + 25728\, Vsub^3\, Usubc - 2304\, Vsub^4 - 608\, Usubc^3 - 1056\, Usubc^2\, Vsub + 7488\, Vsub^2\, Usubc - 2304\, Vsub^3 + 136\, Usubc^2 + 192\, Usubc\, Vsub - 576\, Vsub^2\Big),\ Usubc = 0.01\,..Uc,\ Vsub = 0\,..0.9\Big);$$



For the coefficient of XX^3:

$$> plot3d\Big(17253\, Usubc^6\, Vsub^6 - 64152\, Usubc^6\, Vsub^5 - 79182\, Usubc^5\, Vsub^6 + 83835\, Usubc^6\, Vsub^4 + 180144\, Usubc^5\, Vsub^5 + 131406\, Usubc^4\, Vsub^6 - 38880\, Usubc^6\, Vsub^3 - 99954\, Usubc^5\, Vsub^4 - 190944\, Usubc^4\, Vsub^5 - 103520\, Usubc^3\, Vsub^6 - 3645\, Usubc^6\, Vsub^2 + 150912\, Usubc^5\, Vsub^3 - 97614\, Usubc^4\, Vsub^4 + 93888\, Usubc^3\, Vsub^5 + 41128\, Usubc^2\, Vsub^6 + 5832\, Usubc^6\, Vsub + 46494\, Usubc^5\, Vsub^2 - 257376\, Usubc^4\, Vsub^3 + 210912\, Usubc^3\, Vsub^4 - 21024\, Usubc^2\, Vsub^5 - 7872\, Usubc\, Vsub^6 - 243\, Usubc^6 - 11664\, Usubc^5\, Vsub - 100926\, Usubc^4\, Vsub^2 + 230272\, Usubc^3\, Vsub^3 - 120840\, Usubc^2\, Vsub^4 + 1728\, Usubc\, Vsub^5 + 576\, Vsub^6 - 126\, Usubc^5 + 6624\, Usubc^4\, Vsub + 89568\, Usubc^3\, Vsub^2 - 109376\, Usubc^2\, Vsub^3$$

$+ 28032\, Usubc\, Vsub^4 + 798\, Usubc^4 + 192\, Usubc^3\, Vsub - 37800\, Usubc^2\, Vsub^2$
$+ 25728\, Usubc\, Vsub^3 - 2304\, Vsub^4 - 608\, Usubc^3 - 1056\, Usubc^2\, Vsub$
$+ 7488\, Usubc\, Vsub^2 - 2304\, Vsub^3 + 136\, Usubc^2 + 192\, Usubc\, Vsub - 576\, Vsub^2,$
$Usubc = 0.15\,..Uc,\ Vsub = 0\,..0.4\big);$



For nu = nu_c

Recall the expansion of U on this case:

> *Ucsing4*;

$$\frac{5}{9} - \frac{\sqrt{7}}{9} - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7} - 1700\right)^{2/3}\left(2\sqrt{7} + 1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3 + \frac{1645\left(1240\sqrt{7} - 1700\right)^{1/3} XX^4}{4478976} \quad \textbf{(5.2.1)}$$

We now want to compute the development of V around rhoc (=t_\nu^3 in the paper), for a fixed y. Recall that y is parametrized by U and V:

```
> yUV;
```

$$\left(8\,\nu\,(1-2\,U)\,V\,(V+1)\right)\Big/\left(U\,(U\,(\nu+1)-2)\left(V^3+\frac{(9\,(\nu+1)\,U^2-2\,(3+10\,\nu)\,U+8\,\nu)\,V^2}{U\,(U\,(\nu+1)-2)}-\frac{(9\,U\,(\nu+1)-4\,\nu-6)\,V}{U\,(\nu+1)-2}-1\right)\right) \quad \textbf{(5.2.2)}$$

Since y is fixed, we could write the development of V in terms of this fixed value of y. It turns out that the formulas are simpler when written in terms of V rather than y. Indeed, when U is equal to Uc (i.e. when w = rhoc), the value of y is fully determined by the value of V in this setting, which we denote by Vc:

```
> yVc := factor( rationalize(subs(V = Vc, U = Uc, factor(subs(nu = nuc, yUV) ) ) ) );
```

$$yVc := \frac{4\left(1+\sqrt{7}\right)Vc\,(Vc+1)}{2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1} \quad \textbf{(5.2.3)}$$

When we compute the development of V for w close to rhoc, we can replace y by the latter value. Here is the new equation we obtain:

```
> op(3, factor(numer(yVc - subs(nu = nuc, yUV) ) ) ); indets(%);
```

$$2\,V-4\sqrt{7}\,V-27\,U^2\,Vc^2-27\,U^2\,Vc+28\,U\,Vc^2+28\,Vc\,U-4\,U\,V^2+2\,V^2-2\,V^2\,Vc^3+58\,V^2\,Vc^2-2\,V\,Vc^3+46\,V^2\,Vc-46\,V\,Vc^2-58\,V\,Vc+30\sqrt{7}\,U\,V\,Vc^2-18\sqrt{7}\,U\,V\,Vc+2\sqrt{7}\,U\,V^3\,Vc^2-8\sqrt{7}\,U\,V^2\,Vc^3+2\sqrt{7}\,U\,V^3\,Vc+18\sqrt{7}\,U\,V^2\,Vc^2-8\sqrt{7}\,U\,V\,Vc^3-30\sqrt{7}\,U\,V^2\,Vc-4\sqrt{7}\,V^2-4\,V\,U+8\sqrt{7}\,U\,V^2+8\sqrt{7}\,U\,V-2\sqrt{7}\,U\,Vc^2-2\sqrt{7}\,U\,Vc+27\,U^2\,V^3\,Vc^2+27\,U^2\,V^3\,Vc+243\,U^2\,V^2\,Vc^2-28\,U\,V^3\,Vc^2+4\,U\,V^2\,Vc^3+243\,U^2\,V^2\,Vc-243\,U^2\,V\,Vc^2-28\,U\,V^3\,Vc-252\,U\,V^2\,Vc^2+4\,U\,V\,Vc^3-243\,U^2\,V\,Vc-228\,U\,V^2\,Vc+228\,U\,V\,Vc^2+252\,U\,V\,Vc+4\sqrt{7}\,V^2\,Vc^3-8\sqrt{7}\,V^2\,Vc^2+4\sqrt{7}\,V\,Vc^3+16\sqrt{7}\,V^2\,Vc-16\sqrt{7}\,V\,Vc^2+8\sqrt{7}\,V\,Vc$$

$$\{U, V, Vc\} \quad \textbf{(5.2.4)}$$

```
> eqyUVc := -27 U^2 Vc^2 - 27 U^2 Vc + 28 U Vc^2 + 28 Vc U + 2 V - 4 U V^2 - 2 V^2 Vc^3
    + 58 V^2 Vc^2 - 2 V Vc^3 + 46 V^2 Vc - 46 V Vc^2 - 58 V Vc - 4 sqrt(7) V + 2 V^2
    - 2 sqrt(7) U Vc^2 - 2 sqrt(7) U Vc - 243 U^2 V Vc - 228 U V^2 Vc + 228 U V Vc^2
    + 252 U V Vc + 4 sqrt(7) V^2 Vc^3 - 8 sqrt(7) V^2 Vc^2 + 4 sqrt(7) V Vc^3 + 16 sqrt(7) V^2 Vc
    - 16 sqrt(7) V Vc^2 + 8 sqrt(7) V Vc + 27 U^2 V^3 Vc^2 + 27 U^2 V^3 Vc + 243 U^2 V^2 Vc^2
    - 28 U V^3 Vc^2 + 4 U V^2 Vc^3 + 243 U^2 V^2 Vc - 243 U^2 V Vc^2 - 28 U V^3 Vc
    - 252 U V^2 Vc^2 + 4 U V Vc^3 - 4 sqrt(7) V^2 - 4 V U + 8 sqrt(7) U V^2 + 8 sqrt(7) U V
    + 2 sqrt(7) U V^3 Vc^2 - 8 sqrt(7) U V^2 Vc^3 + 2 sqrt(7) U V^3 Vc + 18 sqrt(7) U V^2 Vc^2
```

$- 8\sqrt{7}\, U\, V\, Vc^3 - 30\sqrt{7}\, U\, V^2\, Vc + 30\sqrt{7}\, U\, V\, Vc^2 - 18\sqrt{7}\, U\, V\, Vc:$

We plug the singular behavior of U in the equation, and deduce from it the asympotic behavior of V (we write V=Vc + XX*VX, so that we obtain the singular behavior of VX, recall that XX = (1-w/rhoc)^{1/3})

```
> simplify(map(simplify, map(expand, map(rationalize, op(1,
    algeqtoseries(simplify(subs(U = Ucsing4, V = Vc + XX·VX, eqyUVc)), XX, VX, 4,
   true))))));
```

$$\frac{(Vc+1)\, Vc\, \left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)}{54\, Vc - 54} + \frac{1}{69984}\,\frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\, Vc\,(17\, Vc+7)}{(Vc-1)^2}\, XX + \frac{5}{384}\,\frac{1}{(Vc-1)^5\left(Vc^2+4\, Vc+1\right)}\Big(Vc\left(97\, Vc^7+241\, Vc^6-521\, Vc^5 - 1673\, Vc^4 - 1113\, Vc^3 - 121\, Vc^2 + Vc + 17\right)\Big) XX^2 + \frac{5}{4478976}\,\frac{1}{(Vc-1)^7\left(Vc^2+4\, Vc+1\right)}\Big((Vc+1)\, Vc\left(15763\, Vc^8 - 12590\, Vc^7 - 121424\, Vc^6 + 6094\, Vc^5 + 392002\, Vc^4 + 315982\, Vc^3 + 66784\, Vc^2 + 1234\, Vc - 293\right)\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)\Big) XX^3 + O\left(XX^4\right) \quad \textbf{(5.2.5)}$$

$$> Vcsing4 := sort\left(collect\left(Vc + XX\cdot\left(\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right) Vc\,(Vc+1)}{54\, Vc-54} + \frac{1}{69984}\,\frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\, Vc\,(17\, Vc+7)}{(Vc-1)^2}\, XX + \frac{5}{384}\,\frac{1}{(Vc-1)^5\left(Vc^2+4\, Vc+1\right)}\Big(Vc\left(97\, Vc^7+241\, Vc^6-521\, Vc^5 - 1673\, Vc^4 - 1113\, Vc^3 - 121\, Vc^2 + Vc + 17\right)\Big) XX^2 + \frac{5}{4478976}\,\frac{1}{\left(Vc^2+4\, Vc+1\right)(Vc-1)^7}\Big((Vc+1)\left(2\sqrt{7}+1\right)\left(15763\, Vc^8 - 12590\, Vc^7 - 121424\, Vc^6 + 6094\, Vc^5 + 392002\, Vc^4 + 315982\, Vc^3 + 66784\, Vc^2 + 1234\, Vc - 293\right)\left(1240\sqrt{7}-1700\right)^{1/3} Vc\Big) XX^3\right), XX, factor\right), XX, ascending\right);$$

$$Vcsing4 := Vc + \frac{\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right) Vc\,(Vc+1)\, XX}{54\,(Vc-1)} + \frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\, Vc\,(17\, Vc+7)\, XX^2}{69984\,(Vc-1)^2} + \frac{1}{384\,(Vc-1)^5\left(Vc^2+4\, Vc+1\right)}\Big(5\, Vc\,(Vc+1)\left(97\, Vc^6 + 144\, Vc^5 \right. \quad \textbf{(5.2.6)}$$

$- 665\, Vc^4 - 1008\, Vc^3 - 105\, Vc^2 - 16\, Vc + 17\big)\, XX^3\big)$

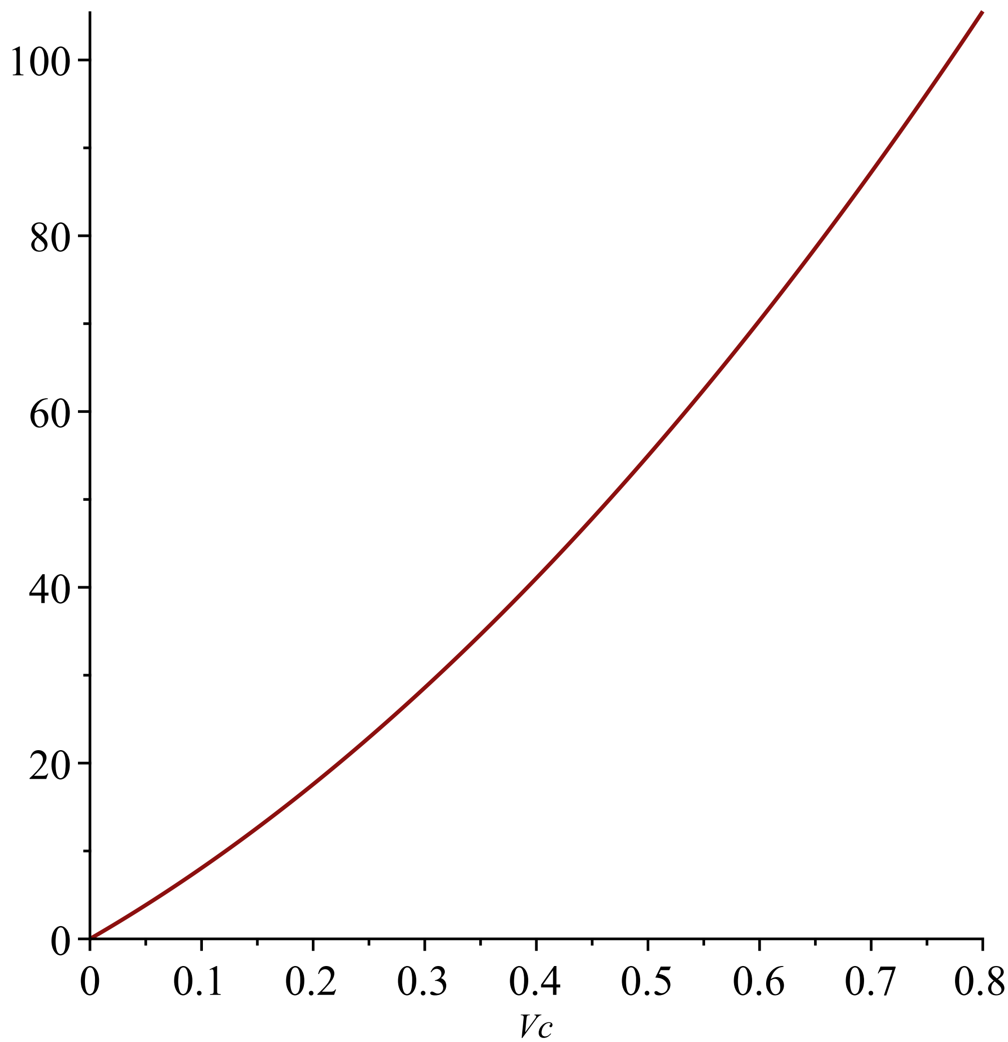
$+ \frac{1}{4478976\, (Vc-1)^7\, (Vc^2 + 4\, Vc + 1)} \big(5\, (Vc+1)\, Vc\, \big(15763\, Vc^8$

$- 12590\, Vc^7 - 121424\, Vc^6 + 6094\, Vc^5 + 392002\, Vc^4 + 315982\, Vc^3 + 66784\, Vc^2$

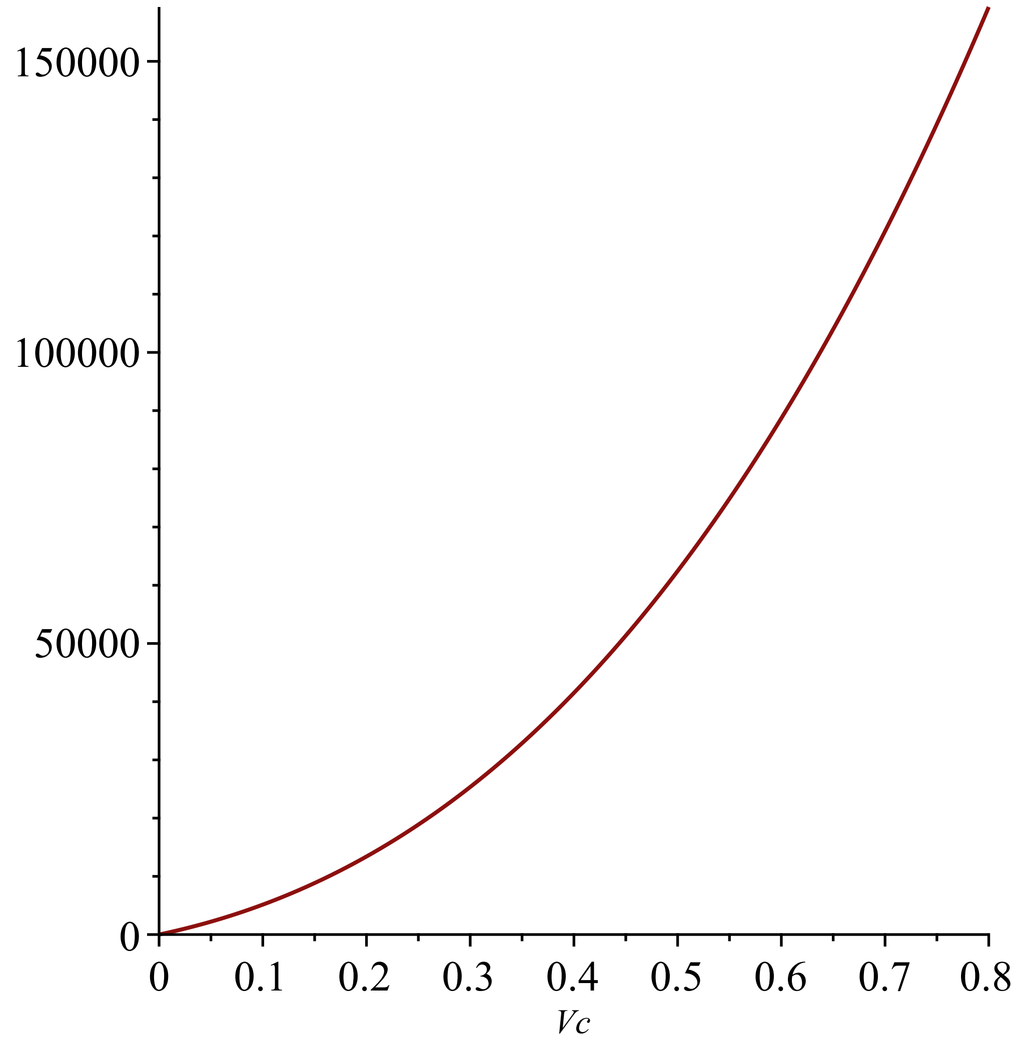
$+ 1234\, Vc - 293\big)\, \big(1240\sqrt{7} - 1700\big)^{1/3}\, \big(2\sqrt{7} + 1\big)\, XX^4\big)$

We check that the coefficient does not vanish for Vc\in (0,1)

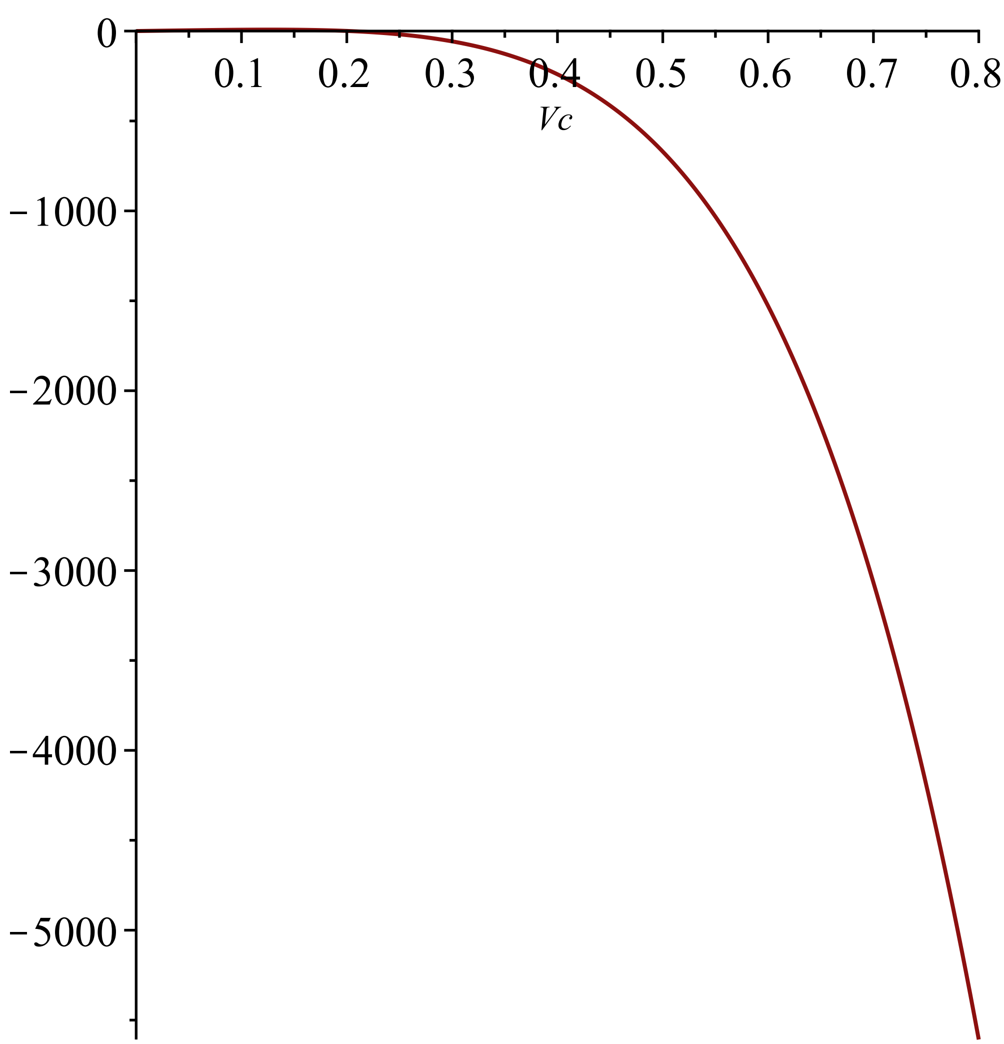
> *plot*(*numer*(*coeff*(*Vcsing4*, *XX*, 1)), *Vc* = 0 ..0.8);



> *plot*(*numer*(*coeff*(*Vcsing4*, *XX*, 2)), *Vc* = 0 ..0.8);



> $plot(numer(coeff(Vcsing4, XX, 3)), Vc = 0 ..0.8);$



For nu > nu_c

We consider again the rational parametrization of the critical line in this regime given by K:

> *UsupK*; *nusupK*;

$$-\frac{K\sim^2-3}{6\,K\sim+10}$$

$$-\frac{K\sim^3+3\,K\sim^2+9\,K\sim+11}{(K\sim+3)\,(K\sim^2-3)} \qquad \textbf{(5.3.1)}$$

And the expansion of U in this regime

> *Usupcsing*;

$$-\frac{K\sim^2-3}{2\,(3\,K\sim+5)}+RootOf\Big(\big(1296\,K\sim^4+6048\,K\sim^3+8928\,K\sim^2+3360\,K\sim \qquad \textbf{(5.3.2)}$$
$$-1200\big)\,\_Z^2-K\sim^8-10\,K\sim^7-24\,K\sim^6+26\,K\sim^5+158\,K\sim^4+114\,K\sim^3$$
$$-192\,K\sim^2-306\,K\sim-117\big)\,XX-\Big(\big(K\sim^2-3\big)\,\big(K\sim^2+8\,K\sim$$

$+ 13) XX^2 \left(9 K\sim^4 + 14 K\sim^3 - 18 K\sim^2 - 10 K\sim + 29\right) (K\sim + 1)\right) \Big/ \left(144 (3 K\sim + 5) \left(3 K\sim^2 + 4 K\sim - 1\right)^2 (2 + K\sim)\right)$

$+ \frac{1}{216 \left(3 K\sim^2 + 4 K\sim - 1\right)^3 (2 + K\sim)} \left(5 \left(K\sim^2 + 8 K\sim + 13\right) \left(9 K\sim^6 + 40 K\sim^5 + 43 K\sim^4 - 48 K\sim^3 - 97 K\sim^2 + 24 K\sim + 77\right) RootOf\left(\left(1296 K\sim^4 + 6048 K\sim^3 + 8928 K\sim^2 + 3360 K\sim - 1200\right) \_Z^2 - K\sim^8 - 10 K\sim^7 - 24 K\sim^6 + 26 K\sim^5 + 158 K\sim^4 + 114 K\sim^3 - 192 K\sim^2 - 306 K\sim - 117\right) XX^3\right)$

As before, we turn our attention to V, and replace U and nu by their expression in terms of K, in the rational parametrization of y. Hence we obtain an expression between y (which is fixed) and K and Vsup

```
> yKsup := simplify(subs(V = Vsup, U = UsupK, nu = nusupK, yUV));
```

$$yKsup := -\left(8 \left(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\right) (K\sim + 1) Vsup (Vsup + 1)\right) \Big/ \left(\left(K\sim^2 - 3\right)^2 Vsup^3 + \left(-7 K\sim^4 - 40 K\sim^3 - 110 K\sim^2 - 136 K\sim - 55\right) Vsup^2 - \left(K\sim^2 - 8 K\sim - 11\right) \left(K\sim^2 - 3\right) Vsup - \left(K\sim^2 - 3\right)^2\right) \quad \textbf{(5.3.3)}$$

```
> op(2, numer(factor(yKsup - subs(nu = nusupK, yUV))));
```

$$2 K\sim^4 U V^3 Vsup^2 - 2 K\sim^4 U V^2 Vsup^3 + 2 K\sim^4 U V^3 Vsup - 2 K\sim^4 U V Vsup^3 + K\sim^4 V^2 Vsup^3 + 8 K\sim^3 U V^3 Vsup^2 + 12 K\sim^2 U^2 V^3 Vsup^2 - 12 K\sim^4 U V^2 Vsup + 12 K\sim^4 U V Vsup^2 + K\sim^4 V^2 Vsup^2 + K\sim^4 V Vsup^3 + 8 K\sim^3 U V^3 Vsup + 24 K\sim^3 U V^2 Vsup^2 + 12 K\sim^2 U^2 V^3 Vsup + 108 K\sim^2 U^2 V^2 Vsup^2 + 12 K\sim^2 U V^2 Vsup^3 + 32 K\sim U^2 V^3 Vsup^2 + 2 K\sim^4 U V^2 - 2 K\sim^4 U Vsup^2 + 7 K\sim^4 V^2 Vsup - 7 K\sim^4 V Vsup^2 - 72 K\sim^3 U V^2 Vsup + 72 K\sim^3 U V Vsup^2 - 8 K\sim^3 V^2 Vsup^2 + 108 K\sim^2 U^2 V^2 Vsup - 108 K\sim^2 U^2 V Vsup^2 - 20 K\sim^2 U V^2 Vsup^2 + 12 K\sim^2 U V Vsup^3 - 6 K\sim^2 V^2 Vsup^3 + 32 K\sim U^2 V^3 Vsup + 288 K\sim U^2 V^2 Vsup^2 - 24 K\sim U V^3 Vsup^2 + 20 U^2 V^3 Vsup^2 + 2 K\sim^4 U V - 2 K\sim^4 U Vsup - K\sim^4 V^2 - K\sim^4 V Vsup - 24 K\sim^3 U V Vsup - 8 K\sim^3 U Vsup^2 + 40 K\sim^3 V^2 Vsup - 40 K\sim^3 V Vsup^2 - 108 K\sim^2 U^2 V Vsup - 12 K\sim^2 U^2 Vsup^2 - 268 K\sim^2 U V^2 Vsup + 268 K\sim^2 U V Vsup^2 - 14 K\sim^2 V^2 Vsup^2 - 6 K\sim^2 V Vsup^3 + 288 K\sim U^2 V^2 Vsup - 288 K\sim U^2 V Vsup^2 - 24 K\sim U V^3 Vsup - 200 K\sim U V^2 Vsup^2 + 20 U^2 V^3 Vsup + 180 U^2 V^2 Vsup^2 - 18 U V^3 Vsup^2 - 18 U V^2 Vsup^3 - K\sim^4 V - 8 K\sim^3 U Vsup + 8 K\sim^3 V Vsup - 12 K\sim^2 U^2 Vsup - 12 K\sim^2 U V^2 + 20 K\sim^2 U V Vsup + 110 K\sim^2 V^2 Vsup - 110 K\sim^2 V Vsup^2 - 288 K\sim U^2 V Vsup - 32 K\sim U^2 Vsup^2 - 424 K\sim U V^2 Vsup + 424 K\sim U V Vsup^2 + 24 K\sim V^2 Vsup^2 + 180 U^2 V^2 Vsup - 180 U^2 V Vsup^2 - 18 U V^3 Vsup \quad \textbf{(5.3.4)}$$

$- 164\, U\, V^2\, Vsup^2 - 18\, U\, V\, Vsup^3 + 9\, V^2\, Vsup^3 - 12\, K\sim^2\, U\, V + 6\, K\sim^2\, V^2$
$+ 14\, K\sim^2\, V\, Vsup - 32\, K\sim\, U^2\, Vsup + 200\, K\sim\, U\, V\, Vsup + 24\, K\sim\, U\, Vsup^2$
$+ 136\, K\sim\, V^2\, Vsup - 136\, K\sim\, V\, Vsup^2 - 180\, U^2\, V\, Vsup - 20\, U^2\, Vsup^2$
$- 208\, U\, V^2\, Vsup + 208\, U\, V\, Vsup^2 + 33\, V^2\, Vsup^2 + 9\, V\, Vsup^3 + 6\, K\sim^2\, V$
$+ 24\, K\sim\, U\, Vsup - 24\, K\sim\, V\, Vsup - 20\, U^2\, Vsup + 18\, U\, V^2 + 164\, U\, V\, Vsup$
$+ 18\, U\, Vsup^2 + 55\, V^2\, Vsup - 55\, V\, Vsup^2 + 18\, V\, U + 18\, U\, Vsup - 9\, V^2$
$- 33\, V\, Vsup - 9\, V$

> $eqyKsup := 2\, K^4\, U\, V^3\, Vsup^2 - 2\, K^4\, U\, V^2\, Vsup^3 + 2\, K^4\, U\, V^3\, Vsup - 2\, K^4\, U\, V\, Vsup^3$
$+ K^4\, V^2\, Vsup^3 + 8\, K^3\, U\, V^3\, Vsup^2 + 12\, K^2\, U^2\, V^3\, Vsup^2 - 12\, K^4\, U\, V^2\, Vsup$
$+ 12\, K^4\, U\, V\, Vsup^2 + K^4\, V^2\, Vsup^2 + K^4\, V\, Vsup^3 + 8\, K^3\, U\, V^3\, Vsup$
$+ 24\, K^3\, U\, V^2\, Vsup^2 + 12\, K^2\, U^2\, V^3\, Vsup + 108\, K^2\, U^2\, V^2\, Vsup^2$
$+ 12\, K^2\, U\, V^2\, Vsup^3 + 32\, K\, U^2\, V^3\, Vsup^2 + 2\, K^4\, U\, V^2 - 2\, K^4\, U\, Vsup^2$
$+ 7\, K^4\, V^2\, Vsup - 7\, K^4\, V\, Vsup^2 - 72\, K^3\, U\, V^2\, Vsup + 72\, K^3\, U\, V\, Vsup^2$
$- 8\, K^3\, V^2\, Vsup^2 + 108\, K^2\, U^2\, V^2\, Vsup - 108\, K^2\, U^2\, V\, Vsup^2 - 20\, K^2\, U\, V^2\, Vsup^2$
$+ 12\, K^2\, U\, V\, Vsup^3 - 6\, K^2\, V^2\, Vsup^3 + 32\, K\, U^2\, V^3\, Vsup + 288\, K\, U^2\, V^2\, Vsup^2$
$- 24\, K\, U\, V^3\, Vsup^2 + 20\, U^2\, V^3\, Vsup^2 + 2\, K^4\, U\, V - 2\, K^4\, U\, Vsup - K^4\, V^2$
$- K^4\, V\, Vsup - 24\, K^3\, U\, V\, Vsup - 8\, K^3\, U\, Vsup^2 + 40\, K^3\, V^2\, Vsup - 40\, K^3\, V\, Vsup^2$
$- 108\, K^2\, U^2\, V\, Vsup - 12\, K^2\, U^2\, Vsup^2 - 268\, K^2\, U\, V^2\, Vsup + 268\, K^2\, U\, V\, Vsup^2$
$- 14\, K^2\, V^2\, Vsup^2 - 6\, K^2\, V\, Vsup^3 + 288\, K\, U^2\, V^2\, Vsup - 288\, K\, U^2\, V\, Vsup^2$
$- 24\, K\, U\, V^3\, Vsup - 200\, K\, U\, V^2\, Vsup^2 + 20\, U^2\, V^3\, Vsup + 180\, U^2\, V^2\, Vsup^2$
$- 18\, U\, V^3\, Vsup^2 - 18\, U\, V^2\, Vsup^3 - K^4\, V - 8\, K^3\, U\, Vsup + 8\, K^3\, V\, Vsup$
$- 12\, K^2\, U^2\, Vsup - 12\, K^2\, U\, V^2 + 20\, K^2\, U\, V\, Vsup + 110\, K^2\, V^2\, Vsup$
$- 110\, K^2\, V\, Vsup^2 - 288\, K\, U^2\, V\, Vsup - 32\, K\, U^2\, Vsup^2 - 424\, K\, U\, V^2\, Vsup$
$+ 424\, K\, U\, V\, Vsup^2 + 24\, K\, V^2\, Vsup^2 + 180\, U^2\, V^2\, Vsup - 180\, U^2\, V\, Vsup^2$
$- 18\, U\, V^3\, Vsup - 164\, U\, V^2\, Vsup^2 - 18\, U\, V\, Vsup^3 + 9\, V^2\, Vsup^3 - 12\, K^2\, U\, V$
$+ 6\, K^2\, V^2 + 14\, K^2\, V\, Vsup - 32\, K\, U^2\, Vsup + 200\, K\, U\, V\, Vsup + 24\, K\, U\, Vsup^2$
$+ 136\, K\, V^2\, Vsup - 136\, K\, V\, Vsup^2 - 180\, U^2\, V\, Vsup - 20\, U^2\, Vsup^2$
$- 208\, U\, V^2\, Vsup + 208\, U\, V\, Vsup^2 + 33\, V^2\, Vsup^2 + 9\, V\, Vsup^3 + 6\, K^2\, V$
$+ 24\, K\, U\, Vsup - 24\, K\, V\, Vsup - 20\, U^2\, Vsup + 18\, U\, V^2 + 164\, U\, V\, Vsup$
$+ 18\, U\, Vsup^2 + 55\, V^2\, Vsup - 55\, V\, Vsup^2 + 18\, V\, U + 18\, U\, Vsup - 9\, V^2$
$- 33\, V\, Vsup - 9\, V :$

We can replace U by its singular behavior in terms of K, and compute the corresponding expansion for V

> $Vsupsing := sort(collect(Vsup + convert(simplify(op(2, algeqtoseries(subs(V = Vsup + VV, subs(U = Usupcsing, eqyKsup)), XX, VV, 3, true))), polynom), XX, factor), XX, ascending);$

$Vsupsing := Vsup + \big(4\, RootOf\big(\big(1296\, K\sim^4 + 6048\, K\sim^3 + 8928\, K\sim^2 + 3360\, K\sim$
$- 1200\big)\, \_Z^2 - K\sim^8 - 10\, K\sim^7 - 24\, K\sim^6 + 26\, K\sim^5 + 158\, K\sim^4 + 114\, K\sim^3$
$- 192\, K\sim^2 - 306\, K\sim - 117\big)\, Vsup\, (Vsup - 1)\, (Vsup + 1)\, (3\, K\sim + 5)\, XX\big) \big/$

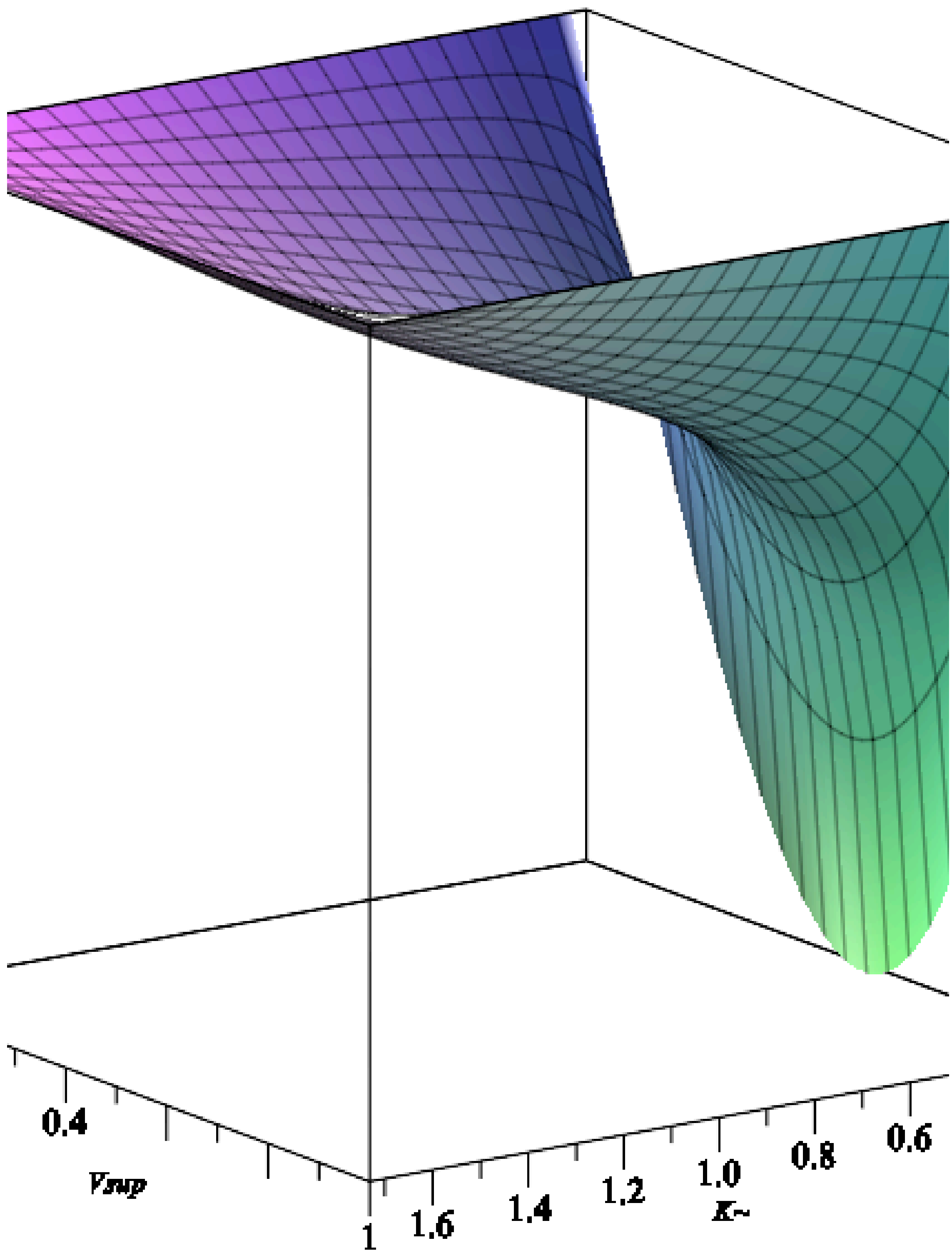
$$\left((K\sim + 1) \left(K\sim^{2}\, Vsup^{2} + 4\, K\sim^{2}\, Vsup + K\sim^{2} + 8\, K\sim\, Vsup - 3\, Vsup^{2} + 4\, Vsup\right.\right.$$
$$\left.\left. - 3\right)\right) + \left(\left(K\sim^{2} - 3\right) (Vsup - 1)\, Vsup \left(K\sim^{2} + 8\, K\sim + 13\right) (Vsup\right.$$
$$+ 1) \left(9\, K\sim^{10}\, Vsup^{6} + 108\, Vsup^{5}\, K\sim^{10} + 38\, K\sim^{9}\, Vsup^{6} + 459\, K\sim^{10}\, Vsup^{4}\right.$$
$$+ 516\, Vsup^{5}\, K\sim^{9} - 27\, K\sim^{8}\, Vsup^{6} + 792\, K\sim^{10}\, Vsup^{3} + 2292\, K\sim^{9}\, Vsup^{4}$$
$$+ 172\, Vsup^{5}\, K\sim^{8} - 328\, K\sim^{7}\, Vsup^{6} + 459\, Vsup^{2}\, K\sim^{10} + 4604\, K\sim^{9}\, Vsup^{3}$$
$$+ 1115\, K\sim^{8}\, Vsup^{4} - 2800\, Vsup^{5}\, K\sim^{7} - 262\, K\sim^{6}\, Vsup^{6} + 108\, Vsup\, K\sim^{10}$$
$$+ 2418\, Vsup^{2}\, K\sim^{9} + 7248\, K\sim^{8}\, Vsup^{3} - 12688\, K\sim^{7}\, Vsup^{4} - 4352\, Vsup^{5}\, K\sim^{6}$$
$$+ 900\, K\sim^{5}\, Vsup^{6} + 9\, K\sim^{10} + 480\, Vsup\, K\sim^{9} + 2399\, Vsup^{2}\, K\sim^{8} - 5136\, K\sim^{7}\, Vsup^{3}$$
$$- 24378\, K\sim^{6}\, Vsup^{4} + 2616\, Vsup^{5}\, K\sim^{5} + 1386\, K\sim^{4}\, Vsup^{6} + 20\, K\sim^{9}$$
$$+ 52\, Vsup\, K\sim^{8} - 7768\, Vsup^{2}\, K\sim^{7} - 20080\, K\sim^{6}\, Vsup^{3} + 3608\, K\sim^{5}\, Vsup^{4}$$
$$+ 9872\, K\sim^{4}\, Vsup^{5} - 648\, K\sim^{3}\, Vsup^{6} - 87\, K\sim^{8} - 2560\, Vsup\, K\sim^{7}$$
$$- 17202\, Vsup^{2}\, K\sim^{6} + 5288\, K\sim^{5}\, Vsup^{3} + 49854\, K\sim^{4}\, Vsup^{4} + 4176\, Vsup^{5}\, K\sim^{3}$$
$$- 1971\, K\sim^{2}\, Vsup^{6} - 208\, K\sim^{7} - 3248\, Vsup\, K\sim^{6} - 628\, Vsup^{2}\, K\sim^{5}$$
$$+ 51520\, K\sim^{4}\, Vsup^{3} + 44720\, K\sim^{3}\, Vsup^{4} - 4044\, K\sim^{2}\, Vsup^{5} - 378\, K\sim\, Vsup^{6}$$
$$+ 290\, K\sim^{6} + 2400\, Vsup\, K\sim^{5} + 21918\, K\sim^{4}\, Vsup^{2} + 47984\, K\sim^{3}\, Vsup^{3}$$
$$+ 3103\, K\sim^{2}\, Vsup^{4} - 2268\, Vsup^{5}\, K\sim + 513\, Vsup^{6} + 792\, K\sim^{5} + 6416\, K\sim^{4}\, Vsup$$
$$+ 9704\, K\sim^{3}\, Vsup^{2} + 12184\, K\sim^{2}\, Vsup^{3} - 9708\, K\sim\, Vsup^{4} + 612\, Vsup^{5} - 342\, K\sim^{4}$$
$$+ 2880\, K\sim^{3}\, Vsup - 12713\, K\sim^{2}\, Vsup^{2} + 2236\, K\sim\, Vsup^{3} - 1641\, Vsup^{4} - 1296\, K\sim^{3}$$
$$- 156\, K\sim^{2}\, Vsup - 9582\, K\sim\, Vsup^{2} + 3760\, Vsup^{3} - 27\, K\sim^{2} - 525\, Vsup^{2} + 756\, K\sim$$
$$\left.\left. - 36\, Vsup + 189\right) XX^{2}\right) \Big/ \left(18\, (2 + K\sim) \left(K\sim^{2}\, Vsup^{2} - 2\, K\sim^{2}\, Vsup + K\sim^{2}\right.\right.$$
$$\left. - 8\, K\sim\, Vsup - 3\, Vsup^{2} - 10\, Vsup - 3\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)^{2} \left(K\sim^{2}\, Vsup^{2}\right.$$
$$\left.\left. + 4\, K\sim^{2}\, Vsup + K\sim^{2} + 8\, K\sim\, Vsup - 3\, Vsup^{2} + 4\, Vsup - 3\right)^{3}\right)$$
$$+ \left(\left(639\, K\sim^{16}\, Vsup^{10} + 10674\, K\sim^{16}\, Vsup^{9} + 2828\, K\sim^{15}\, Vsup^{10} + 71091\, K\sim^{16}\, Vsup^{8} + 63568\, K\sim^{1}\right.\right.$$
$$+ 426006\, Vsup^{6}\, K\sim^{16} + 1800240\, K\sim^{15}\, Vsup^{7} + 679404\, K\sim^{14}\, Vsup^{8}$$
$$- 648048\, K\sim^{13}\, Vsup^{9} - 2396\, K\sim^{12}\, Vsup^{10} + 395820\, K\sim^{16}\, Vsup^{5}$$
$$+ 3218472\, Vsup^{6}\, K\sim^{15} + 3217728\, K\sim^{14}\, Vsup^{7} - 3113492\, K\sim^{13}\, Vsup^{8}$$
$$- 1083928\, K\sim^{12}\, Vsup^{9} + 243660\, K\sim^{11}\, Vsup^{10} + 196614\, K\sim^{16}\, Vsup^{4}$$
$$+ 2711568\, K\sim^{15}\, Vsup^{5} + 5754840\, Vsup^{6}\, K\sim^{14} - 8503920\, K\sim^{13}\, Vsup^{7}$$
$$- 9997948\, K\sim^{12}\, Vsup^{8} + 2001744\, K\sim^{11}\, Vsup^{9} + 204292\, K\sim^{10}\, Vsup^{10}$$
$$+ 52488\, Vsup^{3}\, K\sim^{16} + 1168632\, K\sim^{15}\, Vsup^{4} + 2279280\, K\sim^{14}\, Vsup^{5}$$
$$- 15721032\, Vsup^{6}\, K\sim^{13} - 37331648\, K\sim^{12}\, Vsup^{7} - 197788\, K\sim^{11}\, Vsup^{8}$$
$$+ 6365672\, K\sim^{10}\, Vsup^{9} - 694620\, K\sim^{9}\, Vsup^{10} + 6939\, Vsup^{2}\, K\sim^{16}$$

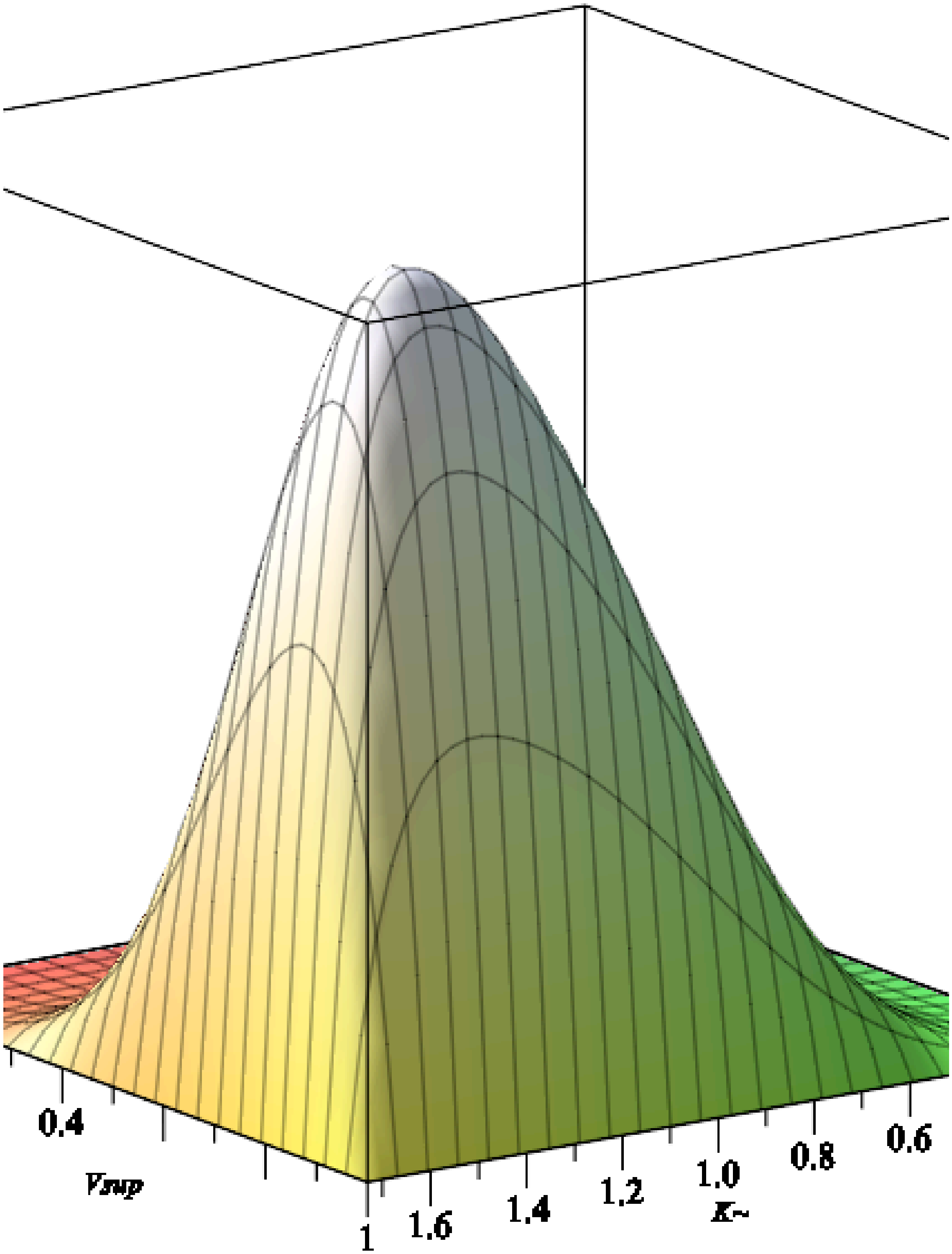
$$
\begin{aligned}
&+ 290832\, Vsup^{3}\, K\sim^{15} - 456456\, K\sim^{14}\, Vsup^{4} - 28678544\, K\sim^{13}\, Vsup^{5} \\
&- 70210936\, K\sim^{12}\, Vsup^{6} - 21406352\, K\sim^{11}\, Vsup^{7} + 33635172\, K\sim^{10}\, Vsup^{8} \\
&- 9456\, K\sim^{9}\, Vsup^{9} - 962430\, K\sim^{8}\, Vsup^{10} + 306\, K\sim^{16}\, Vsup + 40332\, Vsup^{2}\, K\sim^{15} \\
&- 183552\, Vsup^{3}\, K\sim^{14} - 18557016\, K\sim^{13}\, Vsup^{4} - 100072048\, Vsup^{5}\, K\sim^{12} \\
&- 46519576\, K\sim^{11}\, Vsup^{6} + 96135168\, K\sim^{10}\, Vsup^{7} + 34999724\, K\sim^{9}\, Vsup^{8} \\
&- 14324532\, K\sim^{8}\, Vsup^{9} + 919620\, K\sim^{7}\, Vsup^{10} - 9\, K\sim^{16} + 352\, K\sim^{15}\, Vsup \\
&+ 15420\, Vsup^{2}\, K\sim^{14} - 4160976\, K\sim^{13}\, Vsup^{3} - 46512472\, K\sim^{12}\, Vsup^{4} \\
&- 95900080\, Vsup^{5}\, K\sim^{11} + 169920072\, K\sim^{10}\, Vsup^{6} + 166843984\, K\sim^{9}\, Vsup^{7} \\
&- 32436278\, K\sim^{8}\, Vsup^{8} - 10312848\, K\sim^{7}\, Vsup^{9} + 1966068\, K\sim^{6}\, Vsup^{10} \\
&- 124\, K\sim^{15} - 9224\, K\sim^{14}\, Vsup - 300668\, Vsup^{2}\, K\sim^{13} - 7092608\, K\sim^{12}\, Vsup^{3} \\
&- 3437704\, K\sim^{11}\, Vsup^{4} + 95370320\, Vsup^{5}\, K\sim^{10} + 328474808\, K\sim^{9}\, Vsup^{6} \\
&- 5649712\, K\sim^{8}\, Vsup^{7} - 72502036\, K\sim^{7}\, Vsup^{8} + 11587752\, K\sim^{6}\, Vsup^{9} \\
&- 291924\, K\sim^{5}\, Vsup^{10} - 292\, K\sim^{14} - 23232\, Vsup\, K\sim^{13} - 408556\, Vsup^{2}\, K\sim^{12} \\
&+ 10442704\, K\sim^{11}\, Vsup^{3} + 151944168\, K\sim^{10}\, Vsup^{4} + 222863664\, Vsup^{5}\, K\sim^{9} \\
&+ 53109364\, K\sim^{8}\, Vsup^{6} - 232332400\, K\sim^{7}\, Vsup^{7} - 13193356\, K\sim^{6}\, Vsup^{8} \\
&+ 15846192\, K\sim^{5}\, Vsup^{9} - 1845612\, K\sim^{4}\, Vsup^{10} + 1452\, K\sim^{13} + 62792\, Vsup\, K\sim^{12} \\
&+ 805388\, Vsup^{2}\, K\sim^{11} + 41286336\, K\sim^{10}\, Vsup^{3} + 215331560\, K\sim^{9}\, Vsup^{4} \\
&- 30420376\, Vsup^{5}\, K\sim^{8} - 378570056\, K\sim^{7}\, Vsup^{6} - 183172544\, K\sim^{6}\, Vsup^{7} \\
&+ 41919588\, K\sim^{5}\, Vsup^{8} + 160488\, K\sim^{4}\, Vsup^{9} - 339228\, K\sim^{3}\, Vsup^{10} + 5908\, K\sim^{12} \\
&+ 247584\, Vsup\, K\sim^{11} + 1206516\, Vsup^{2}\, K\sim^{10} + 20772976\, K\sim^{9}\, Vsup^{3} \\
&- 2753420\, K\sim^{8}\, Vsup^{4} - 322010064\, Vsup^{5}\, K\sim^{7} - 381755416\, K\sim^{6}\, Vsup^{6} \\
&+ 19549104\, K\sim^{5}\, Vsup^{7} + 18201492\, K\sim^{4}\, Vsup^{8} - 5848848\, K\sim^{3}\, Vsup^{9} \\
&+ 676188\, K\sim^{2}\, Vsup^{10} - 4380\, K\sim^{11} - 73336\, Vsup\, K\sim^{10} - 1953052\, Vsup^{2}\, K\sim^{9} \\
&- 58484464\, K\sim^{8}\, Vsup^{3} - 231714776\, K\sim^{7}\, Vsup^{4} - 174497008\, Vsup^{5}\, K\sim^{6} \\
&- 87082392\, K\sim^{5}\, Vsup^{6} + 76408960\, K\sim^{4}\, Vsup^{7} - 11266452\, K\sim^{3}\, Vsup^{8} \\
&- 703080\, K\sim^{2}\, Vsup^{9} + 102060\, K\sim\, Vsup^{10} - 40460\, K\sim^{10} - 1060608\, Vsup\, K\sim^{9} \\
&- 1505366\, Vsup^{2}\, K\sim^{8} - 78414544\, K\sim^{7}\, Vsup^{3} - 149999800\, K\sim^{6}\, Vsup^{4} \\
&+ 84505936\, Vsup^{5}\, K\sim^{5} + 46216232\, K\sim^{4}\, Vsup^{6} + 18193104\, K\sim^{3}\, Vsup^{7} \\
&- 5913540\, K\sim^{2}\, Vsup^{8} + 1123632\, K\sim\, Vsup^{9} - 132921\, Vsup^{10} - 13140\, K\sim^{9} \\
&- 655956\, Vsup\, K\sim^{8} + 6247748\, Vsup^{2}\, K\sim^{7} - 2589824\, K\sim^{6}\, Vsup^{3} \\
&+ 44865336\, K\sim^{5}\, Vsup^{4} + 31232464\, K\sim^{4}\, Vsup^{5} + 34111032\, K\sim^{3}\, Vsup^{6} \\
&- 11395968\, K\sim^{2}\, Vsup^{7} + 1080324\, K\sim\, Vsup^{8} + 188082\, Vsup^{9} + 136290\, K\sim^{8}
\end{aligned}
$$

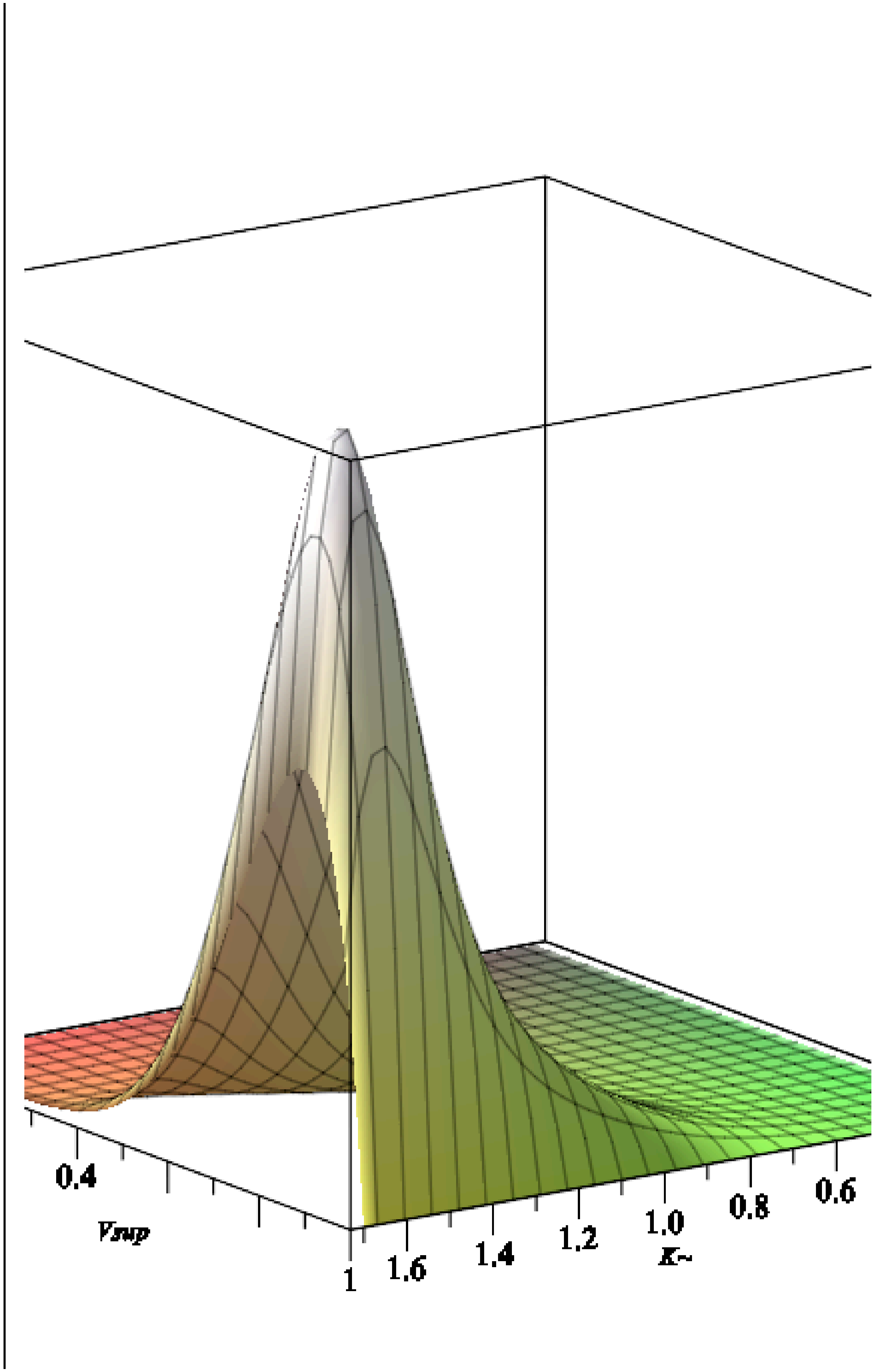
$$+ 1992096\, Vsup\, K\sim^{7} + 4559972\, Vsup^{2}\, K\sim^{6} + 42001296\, K\sim^{5}\, Vsup^{3}$$
$$+ 90548744\, K\sim^{4}\, Vsup^{4} - 97894032\, Vsup^{5}\, K\sim^{3} + 32556024\, K\sim^{2}\, Vsup^{6}$$
$$- 7932816\, K\sim\, Vsup^{7} + 359019\, Vsup^{8} + 119340\, K\sim^{7} + 2550024\, Vsup\, K\sim^{6}$$
$$- 8722260\, Vsup^{2}\, K\sim^{5} + 7858240\, K\sim^{4}\, Vsup^{3} + 65235816\, K\sim^{3}\, Vsup^{4}$$
$$- 77490896\, K\sim^{2}\, Vsup^{5} + 22431528\, K\sim\, Vsup^{6} - 2266776\, Vsup^{7} - 234108\, K\sim^{6}$$
$$- 1073088\, Vsup\, K\sim^{5} - 9440892\, K\sim^{4}\, Vsup^{2} - 25004304\, K\sim^{3}\, Vsup^{3}$$
$$+ 49526232\, K\sim^{2}\, Vsup^{4} - 23701744\, Vsup^{5}\, K\sim + 4257990\, Vsup^{6} - 313308\, K\sim^{5}$$
$$- 3196152\, K\sim^{4}\, Vsup + 951588\, K\sim^{3}\, Vsup^{2} - 22509888\, K\sim^{2}\, Vsup^{3}$$
$$+ 22575672\, K\sim\, Vsup^{4} - 5011444\, Vsup^{5} + 167076\, K\sim^{4} - 997920\, K\sim^{3}\, Vsup$$
$$+ 3048300\, K\sim^{2}\, Vsup^{2} - 8692272\, K\sim\, Vsup^{3} + 2948310\, Vsup^{4} + 366444\, K\sim^{3}$$
$$+ 1036152\, K\sim^{2}\, Vsup + 425196\, K\sim\, Vsup^{2} - 1041624\, Vsup^{3} + 19116\, K\sim^{2}$$
$$+ 777600\, K\sim\, Vsup + 42147\, Vsup^{2} - 160380\, K\sim + 149202\, Vsup - 57105\Big)$$
$$RootOf\Big(\big(1296\, K\sim^{4} + 6048\, K\sim^{3} + 8928\, K\sim^{2} + 3360\, K\sim - 1200\big)\, \_Z^{2} - K\sim^{8}$$
$$- 10\, K\sim^{7} - 24\, K\sim^{6} + 26\, K\sim^{5} + 158\, K\sim^{4} + 114\, K\sim^{3} - 192\, K\sim^{2} - 306\, K\sim - 117\Big)$$
$$(Vsup - 1)\, Vsup\, \big(K\sim^{2} + 8\, K\sim + 13\big)\, (Vsup + 1)\, (3\, K\sim + 5)\, XX^{3}\Big) \Big/ \Big(54\, (2$$
$$+ K\sim)\, (K\sim + 1)\, \big(K\sim^{2}\, Vsup^{2} - 2\, K\sim^{2}\, Vsup + K\sim^{2} - 8\, K\sim\, Vsup - 3\, Vsup^{2}$$
$$- 10\, Vsup - 3\big)\, \big(3\, K\sim^{2} + 4\, K\sim - 1\big)^{3}\, \big(K\sim^{2}\, Vsup^{2} + 4\, K\sim^{2}\, Vsup + K\sim^{2}$$
$$+ 8\, K\sim\, Vsup - 3\, Vsup^{2} + 4\, Vsup - 3\big)^{5}\Big)$$

plots of the coefficients

```
> plot3d(numer(coeff(Vsupsing, XX, 1)), K = Kc..Kinfini, Vsup = 0..1);
    plot3d(numer(coeff(Vsupsing, XX, 2)), K = Kc..Kinfini, Vsup = 0..1);
    plot3d(numer(coeff(Vsupsing, XX, 3)), K = Kc..Kinfini, Vsup = 0..1);
```


0.4
Vsup
1
1.6
1.4
1.2
1.0
0.8
0.6
K~


0.4
Vsup
1
1.6
1.4
1.2
1.0
K~
0.8
0.6



0.4
Vsup
1
1.6
1.4
1.2
1.0
K~
0.8
0.6

Asymptotic behavior (in y) of Q(t,ty) (Proposition 3.10) (on the critical line, i.e when t=t_\nu is fixed and equal to the radius of convergence).

For nu ≤ nu_c:

We plug the development of V obtained above in **(4.1.10)** in the equation for Qt given in the rational parametrization (recall that Y3Y=YY^1/3=(1-y/2)^1/3):

> *Vsubsingy*;

$$1+\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y-\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3}(-1)^{1/3}\,Y3Y^2}{2}-\frac{4\,(U-1)\,Y3Y^3}{-2+3\,U}+\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y^4}{9\,U-6} \quad \textbf{(6.1.1)}$$

In the rational parametrization for Qt (given in QtUV), we replace nu by its value in terms of U:

> *QtUVsubc* := *factor*(*subs*(nu = *nuUsub*, *QtUV*));

$$QtUVsubc := \frac{1}{(-2+3\,U)\,(V+1)^3\,(6\,U^2-10\,U+3)}\big((3\,U\,V^3-21\,U\,V^2-2\,V^3-3\,V\,U+18\,V^2-3\,U+6\,V+2)\,(6\,U^2\,V^2-12\,U^2\,V-6\,U\,V^2-6\,U^2+12\,V\,U+V^2+10\,U-2\,V-3)\big) \quad \textbf{(6.1.2)}$$

> *map*(*factor*, *collect*(*simplify*(*series*(*subs*(*V* = *Vsubsingy*, *QtUVsubc*), *Y3Y*, 4)), *Y3Y*));

$$\frac{12\,(3\,U-1)\,(U-1)^2}{(-2+3\,U)\,(6\,U^2-10\,U+3)}+\frac{-\frac{3\,\mathrm{I}}{2}\,(-\sqrt{3}+\mathrm{I})\,(9\,U^2-10\,U+2)\,(U-1)\left(-\frac{24\,(U-1)}{-2+3\,U}\right)^{2/3}}{(-2+3\,U)\,(6\,U^2-10\,U+3)}\,Y3Y^2+12\,\frac{(45\,U^2-45\,U+8)\,(U-1)^2}{(-2+3\,U)^2\,(6\,U^2-10\,U+3)}\,Y3Y^3+\mathrm{O}(Y3Y^4) \quad \textbf{(6.1.3)}$$

We check if/when the leading term in the development of Qt cancels out. There are two roots either U=1 (which is not possible in this range of nu) or

> *solve*$(9\,U^2-10\,U+2)$; *Uc*

$$\frac{5}{9}+\frac{\sqrt{7}}{9},\ \frac{5}{9}-\frac{\sqrt{7}}{9}$$

$$\frac{5}{9}-\frac{\sqrt{7}}{9} \quad \textbf{(6.1.4)}$$

The leading term cancels for U=Uc, we compute the corresponding development:

> map(factor, collect(simplify(series(subs(V = Vsubsingy, U = Uc, QtUVsubc), Y3Y, 5)), Y3Y));

$$\frac{2}{5} + \frac{2\sqrt{7}}{5} + \left(-\frac{14}{5} - \frac{2\sqrt{7}}{5}\right) Y3Y^3 + \left(\frac{2\left(46+16\sqrt{7}\right)^{1/3}\sqrt{7}}{5} + \frac{2\left(46+16\sqrt{7}\right)^{1/3}}{5}\right) Y3Y^4 + \mathrm{O}\left(Y3Y^5\right) \quad \textbf{(6.1.5)}$$

We obtain a singularity in (1-y/2)^(4/3).

For nu>nu_c

We use the same rational parametrization of U and nu in terms of K, and replace in \hat Q, their expression in terms of K. Then we use the development of V obtained in **(4.2.14)** (with YY=(1-y/yK11)^(1/2)) and substitute it in the expression of Qt:

> QtUVsupc := factor(subs(nu = nusupK, U = UsupK, QtUV)) :

> devQtsur := map(factor, collect(simplify(series(subs(V = devV11ysupc, QtUVsupc), YY, 4)), YY));

$$devQtsur := \Big(4\,\big(37 K\sim^8 + 348 K\sim^7 - 21 K\sim^6 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 1456 K\sim^6 - 144 K\sim^5 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 3508 K\sim^5 - 431 K\sim^4 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 5314 K\sim^4 - 704 K\sim^3 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 5140 K\sim^3 - 687 K\sim^2 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 3016 K\sim^2 - 432 K\sim \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 956 K\sim - 149 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 145\big)\left(K\sim^2+4K\sim+5\right)\big(5K\sim^4 - \left(3K\sim^2+4K\sim-1\right)^{3/2}\sqrt{K\sim^2+4K\sim+5} + 20K\sim^3 + 26K\sim^2 + 4K\sim - 11\big)\Big) \Big/ \Big(\left(K\sim^2+8K\sim+13\right)\left(K\sim^2+4K\sim - \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 5\right)^3 \left(K\sim^2-3\right)^3\Big) + 8\,\Big(\big(-106035 + 34322652 K\sim^{12} + 94811152 K\sim^{11} + 198675880 K\sim^{10} - 52888765 K\sim^6 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 9232696 K\sim^3 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 2484785 K\sim^2 \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 380604 K\sim \sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 10557 K\sim^{16} \quad \textbf{(6.2.1)}$$

$$+ 194672 K\sim^{15} + 1689784 K\sim^{14} + 9134528 K\sim^{13} + 321307488 K\sim^{9}$$
$$+ 404013790 K\sim^{8} + 394520720 K\sim^{7}$$
$$- 6095 K\sim^{14} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 40391073 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 52872400 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 23482148 K\sim^{9} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 3247736 K\sim^{11} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 10235523 K\sim^{10} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 96140 K\sim^{13} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 710421 K\sim^{12} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 326688 K\sim$$
$$- 40181524 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 22752079 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 24891 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 16980208 K\sim^{3}$$
$$+ 1771480 K\sim^{2} + 295981128 K\sim^{6} + 166429952 K\sim^{5} + 66693052 K\sim^{4}\Big) \big(K\sim^{2}$$
$$+ 4 K\sim + 5\big)^{2}\Big) \Big/ \Big(\big(K\sim^{2} - 3\big)^{3} \Big(K\sim^{2} + 4 K\sim$$
$$- \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 5\Big)^{5} \big(3 K\sim^{2} + 8 K\sim + 7\big) \big(K\sim^{2}$$
$$+ 8 K\sim + 13\big)\Big) YY^{2} + 32 \Big(\big(K\sim^{2} + 4 K\sim + 5\big)^{4} \Big(361 K\sim^{12}$$
$$- 208 K\sim^{10} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 4552 K\sim^{11}$$
$$- 2072 K\sim^{9} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 26718 K\sim^{10}$$
$$- 9608 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 96200 K\sim^{9}$$
$$- 27200 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 236399 K\sim^{8}$$
$$- 52192 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 417744 K\sim^{7}$$
$$- 71600 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 544964 K\sim^{6}$$
$$- 71696 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 530576 K\sim^{5}$$
$$- 51968 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 383799 K\sim^{4}$$
$$- 26768 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 198632 K\sim^{3}$$
$$- 9592 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 65662 K\sim^{2}$$

$$- 1960 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 10536 K\sim + 401\Big)$$

$$RootOf\Big(\_Z^2 \left(9 K\sim^{10} + 36 K\sim^9 - 31 K\sim^8 - 304 K\sim^7 - 214 K\sim^6 + 792 K\sim^5 + 1170 K\sim^4 - 432 K\sim^3 - 1539 K\sim^2 - 540 K\sim + 189\right) - 174 K\sim^{10}$$

$$+ 100 K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 1960 K\sim^9$$

$$+ 864 K\sim^7 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 9950 K\sim^8$$

$$+ 3304 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 29664 K\sim^7$$

$$+ 7200 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 56972 K\sim^6$$

$$+ 9760 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 72752 K\sim^5$$

$$+ 8480 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 61372 K\sim^4$$

$$+ 4504 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 32608 K\sim^3$$

$$+ 1120 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 9190 K\sim^2$$

$$- 4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 152 K\sim + 890\Big)\Big) \Big/ \Big(\big(K\sim^2 - 3\big) \left(K\sim^2 + 4 K\sim - \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 5\right)^6 \left(3 K\sim^2 + 8 K\sim + 7\right)^2 \left(K\sim^2 + 8 K\sim + 13\right)\Big) YY^3 + O\left(YY^4\right)$$

We get an expansion in (1-y/y+)^3/2

Asymptotic behavior (in t) of Q(t,ty) (Proposition 3.12)

For nu<nu_c:

We plug the developments of U and V obtained above (in **(5.1.1)** and **(5.1.6)**, with (1-w/rhosubc) ^{1/2}) in the rational parametrization of Q, and check that the coefficient of XX cancels out. Hence it gives the expected singular behavior.

```
> Qtsubcsing3 := convert(simplify(series(subs(U = Usubcsing3, V = Vsubsing3, subs(nu
      = subs(U = Usubc, nuUsub), QtUV)), XX, 4)), polynom);
```

$$Qtsubcsing3 := \left(18 \left(\left(Vsub^3 - 7 Vsub^2 - Vsub - 1\right) Usubc - \frac{2 Vsub^3}{3} + 6 Vsub^2 + 2 Vsub + \frac{2}{3}\right) \left(\left(Vsub^2 - 2 Vsub - 1\right) Usubc^2 + \left(-Vsub^2 + 2 Vsub + \frac{5}{3}\right) Usubc + \frac{Vsub^2}{6} - \frac{Vsub}{3} - \frac{1}{2}\right)\right) \Big/ \Big(\left(-2 + 3 Usubc\right) \left(Vsub\right.$$ **(7.1.1)**

$$+ 1)^3 \left(6\, Usubc^2 - 10\, Usubc + 3\right)\Big) + \Bigg(18\, Vsub \Bigg(\Bigg(\Big(Usubc^2 - Usubc$$

$$+ \frac{1}{6}\Big)\, Vsub^5 + \Big(-10\, Usubc^2 + 10\, Usubc - \frac{5}{3}\Big)\, Vsub^3 + \Big(16\, Usubc^2$$

$$- \frac{52}{3}\, Usubc + \frac{10}{3}\Big)\, Vsub^2 + \Big(9\, Usubc^2 - 17\, Usubc + \frac{11}{2}\Big)\, Vsub - \frac{4\, Usubc}{3}$$

$$+ \frac{2}{3}\Big)\Big(\Big(-\frac{2}{3} + Usubc\Big)\, Vsub^3 + (-7\, Usubc + 6)\, Vsub^2 + (-Usubc + 2)\, Vsub$$

$$- Usubc + \frac{2}{3}\Big)\, XX^2\Bigg) \Bigg/ \Big(\big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^2\, (-2$$

$$+ 3\, Usubc)\, (Vsub + 1)^3\, \big(6\, Usubc^2 - 10\, Usubc + 3\big)\Big) - \Bigg(4\, (Vsub$$

$$+ 1)\, Vsub\, \sqrt{6}\, \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \Big(\big(Vsub^3 - 7\, Vsub^2 - Vsub$$

$$- 1\big)\, Usubc - \frac{2\, Vsub^3}{3} + 6\, Vsub^2 + 2\, Vsub + \frac{2}{3}\Big)\, XX^3\Bigg) \Bigg/ \Bigg(9\, \Big(-\frac{2}{3}$$

$$+ Usubc\Big)\, \big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^4\Bigg)$$

> *coeff*(*Qtsubcsing3*, *XX*, 1);

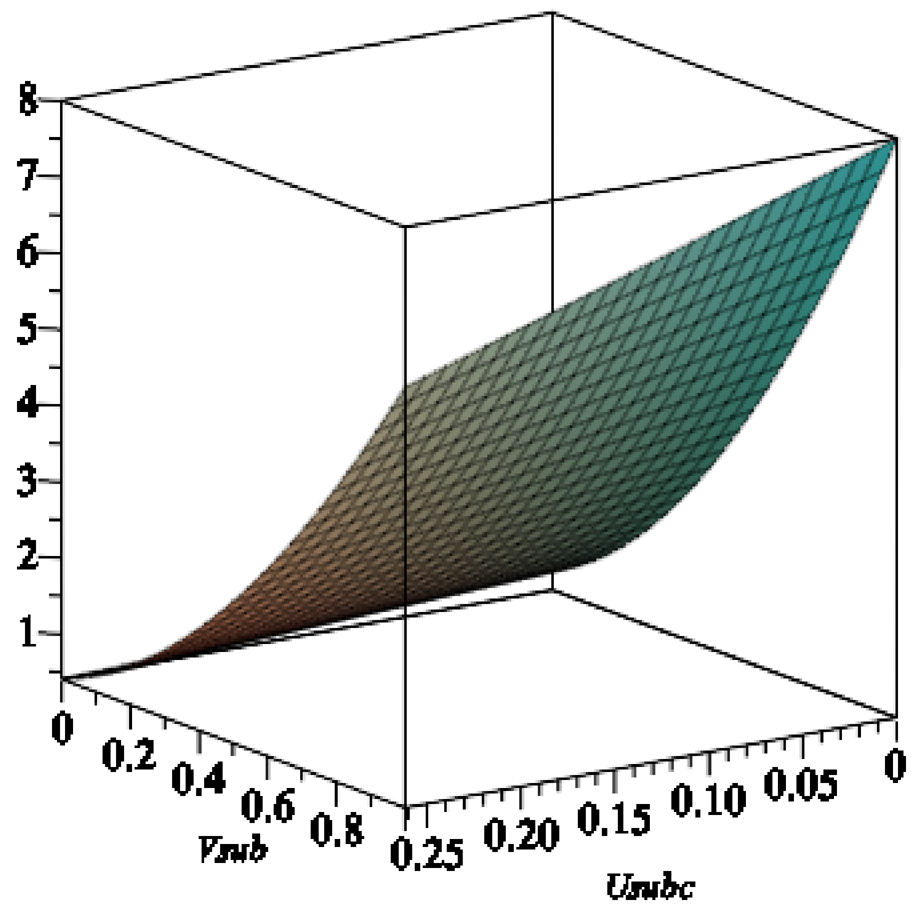
$$0 \tag{7.1.2}$$

> *coeff*(*Qtsubcsing3*, *XX*, 3);

$$-\Bigg(4\, (Vsub + 1)\, Vsub\, \sqrt{6}\, \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \Big(\big(Vsub^3 - 7\, Vsub^2 - Vsub - 1\big)\, Usubc - \frac{2\, Vsub^3}{3} + 6\, Vsub^2 + 2\, Vsub + \frac{2}{3}\Big)\Bigg) \Bigg/ \Bigg(9\, \Big(-\frac{2}{3} + Usubc\Big)\, \big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^4\Bigg) \tag{7.1.3}$$

We check that the coefficient does not vanish (the denominator is clearly not zero in the range of values of interest, V in (0,1) and U in (0,1/2)

> $plot3d\Big(\big(Vsub^3 - 7\, Vsub^2 - Vsub - 1\big)\, Usubc - \frac{2\, Vsub^3}{3} + 6\, Vsub^2 + 2\, Vsub + \frac{2}{3},\ Usubc = 0..Uc,\ Vsub = 0..1\Big)$



For nu = nu_c

We plug again the developments of U and V (obtained in **(5.2.1)** and in **(5.2.6)**, with XX=(1-w/rhoc)^1/3) in the rational parametrization of Q, and check that the coefficient of XX cancels out. Hence it gives the expected singular behavior.

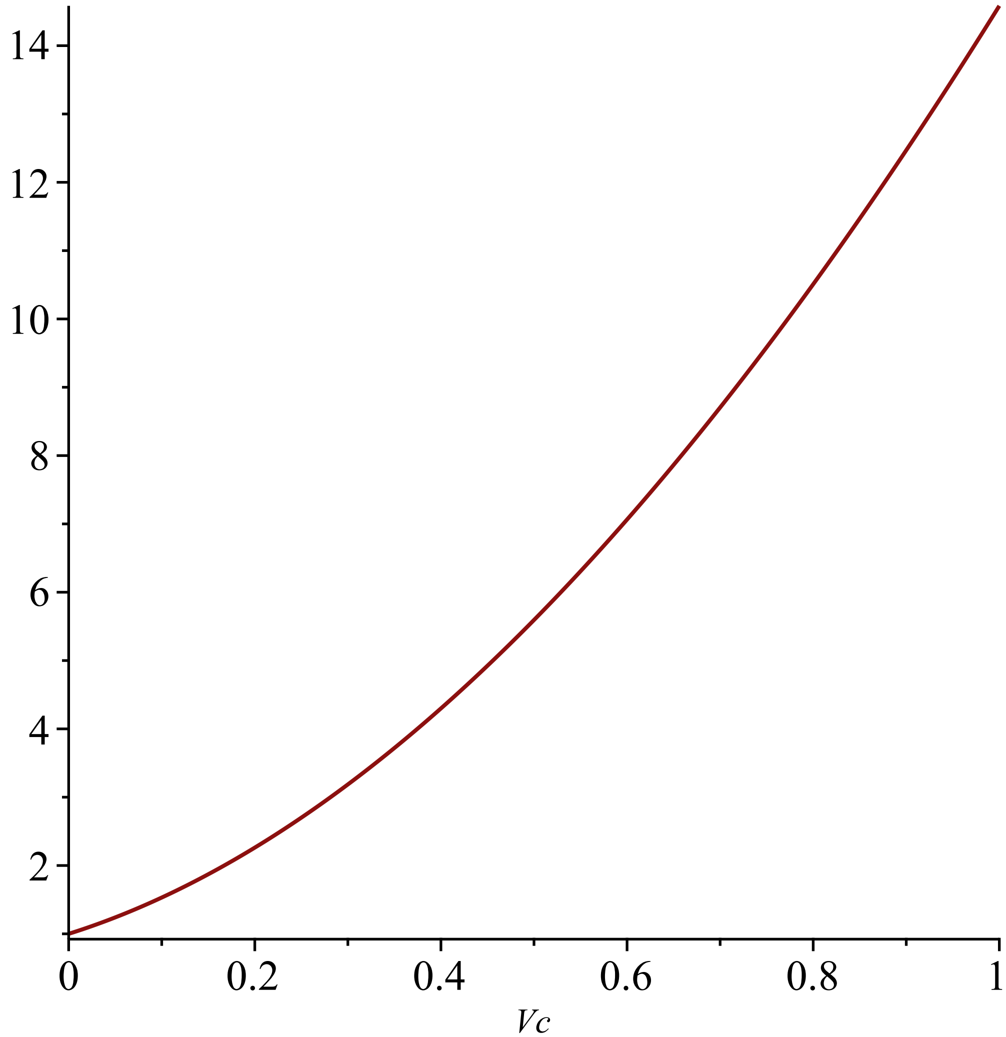
```
> Qtcsing4 := collect(convert(simplify(series(subs(nu = nuc, U = Ucsing4, V = Vcsing4,
    QtUV), XX, 5)), polynom), XX, factor);
```

$$Qtcsing4 := \frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\Big(\big(1240\sqrt{7}-1700\big)^{1/3}\big(2\sqrt{7}+1\big)\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,Vc\,(Vc+1)\,XX^4\Big) + \frac{1}{5\,(Vc-1)^2\,(Vc^2+4\,Vc+1)\,(Vc+1)^3}\Big(\big(Vc^5-10\,Vc^3+4\,Vc^2-63\,Vc-12\big)\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,Vc\,XX^3\Big) \quad \textbf{(7.2.1)}$$

$$+\frac{\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)\left(Vc^2 - 2\,Vc + 5\right)}{5\,(Vc+1)^3}$$

We check that the coefficient does not vanish for Vc\in (0,1)

> $plot\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1,\ Vc = 0\,..1\right);$



For nu > nu_c

We replace U and V by their singular expansion (obtained in **(5.3.2)** and **(5.3.5)**, with XX=(1-w/rho)^1/2) in the expression of Q given by the rational parametrizations :

> $Qtsupsing := simplify(series(subs(U = Usupcsing, V = Vsupsing, subs(\mathrm{nu} = nusupK, QtUV)), XX, 4));$

$$Qtsupsing := \Big(\Big(\big(Vsup^2 - 2\,Vsup - 1\big)\,K\sim^4 + (-24\,Vsup - 8)\,K\sim^3 + \big(-6\,Vsup^2 - 68\,Vsup - 10\big)\,K\sim^2 + (-56\,Vsup + 24)\,K\sim + 9\,Vsup^2 - 2\,Vsup + 39\Big)$$ **(7.3.1)**

$$\Big(\big(Vsup^3 - 7\,Vsup^2 - Vsup - 1\big)\,K\sim^4 + \big(-40\,Vsup^2 + 8\,Vsup\big)\,K\sim^3 + \big(-6\,Vsup^3 - 110\,Vsup^2 + 14\,Vsup + 6\big)\,K\sim^2 + \big(-136\,Vsup^2 - 24\,Vsup\big)\,K\sim + 9\,Vsup^3$$

$$- 55\, Vsup^2 - 33\, Vsup - 9)\big) \Big/ \Big((Vsup + 1)^3 \left(K\sim^2 + 8\,K\sim + 13\right) \left(K\sim^2 - 3\right)^3 \Big)$$
$$+ \Bigg(\Bigg(\left(K\sim^2 - 3\right)^4 Vsup^5 - 24\,(2 + K\sim) \left(K\sim^2 - 3\right)^2 \left(K\sim^2 + \frac{4}{3}\,K\sim - \frac{1}{3}\right) Vsup^4 - 10 \left(K\sim^2 - 3\right)^2 \left(K\sim^2 + \frac{8}{5}\,K\sim + \frac{1}{5}\right) \left(K\sim^2 + 8\,K\sim + 13\right) Vsup^3$$
$$+ 16 \left(K\sim^6 + 14\,K\sim^5 + 83\,K\sim^4 + 236\,K\sim^3 + 307\,K\sim^2 + 126\,K\sim - 31\right) (K\sim + 1)^2\, Vsup^2 + \left(9\,K\sim^8 + 96\,K\sim^7 + 468\,K\sim^6 + 1184\,K\sim^5 + 1062\,K\sim^4 - 2016\,K\sim^3 - 6668\,K\sim^2 - 7200\,K\sim - 2919\right) Vsup - 8\,(2 + K\sim) \left(K\sim^2 + 4\,K\sim + 5\right) \left(K\sim^2 - 3\right)^2 \Big) Vsup \Big(\left(K\sim^2 - 3\right)^2 Vsup^3 + \left(-7\,K\sim^4 - 40\,K\sim^3 - 110\,K\sim^2 - 136\,K\sim - 55\right) Vsup^2 - \left(K\sim^2 - 8\,K\sim - 11\right)\left(K\sim^2 - 3\right) Vsup - \left(K\sim^2 - 3\right)^2 \Big)\Bigg) \Big/ \Big(\left(\left(K\sim^2 - 3\right) Vsup^2 + \left(-2\,K\sim^2 - 8\,K\sim - 10\right) Vsup + K\sim^2 - 3\right) \left(K\sim^2 - 3\right)^3 \left(\left(K\sim^2 - 3\right) Vsup^2 + 4\,(K\sim + 1)^2\, Vsup + K\sim^2 - 3\right) \left(K\sim^2 + 8\,K\sim + 13\right) (Vsup + 1)^3 \Big)$$
$$XX^2 + 32 \Bigg(RootOf\Big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right) \_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\Big) Vsup \Big(\left(K\sim^2 - 3\right)^2 Vsup^3 + \left(-7\,K\sim^4 - 40\,K\sim^3 - 110\,K\sim^2 - 136\,K\sim - 55\right) Vsup^2 - \left(K\sim^2 - 8\,K\sim - 11\right)\left(K\sim^2 - 3\right) Vsup - \left(K\sim^2 - 3\right)^2 \Big) \left((K\sim + 1)^2\, Vsup^2 + \left(K\sim^2 - 3\right) Vsup + (K\sim + 1)^2 \right) \left(K\sim + \frac{5}{3}\right) (Vsup + 1) \Bigg) \Big/ \Big((K\sim + 1) \left(\left(K\sim^2 - 3\right) Vsup^2 + \left(-2\,K\sim^2 - 8\,K\sim - 10\right) Vsup + K\sim^2 - 3\right) \left(\left(K\sim^2 - 3\right) Vsup^2 + 4\,(K\sim + 1)^2\, Vsup + K\sim^2 - 3\right)^3 \Big)$$
$$XX^3 + \mathrm{O}\left(XX^4\right)$$

> *coeff*(*Qtsupsing*, *XX*, 1);

$$0 \qquad \textbf{(7.3.2)}$$

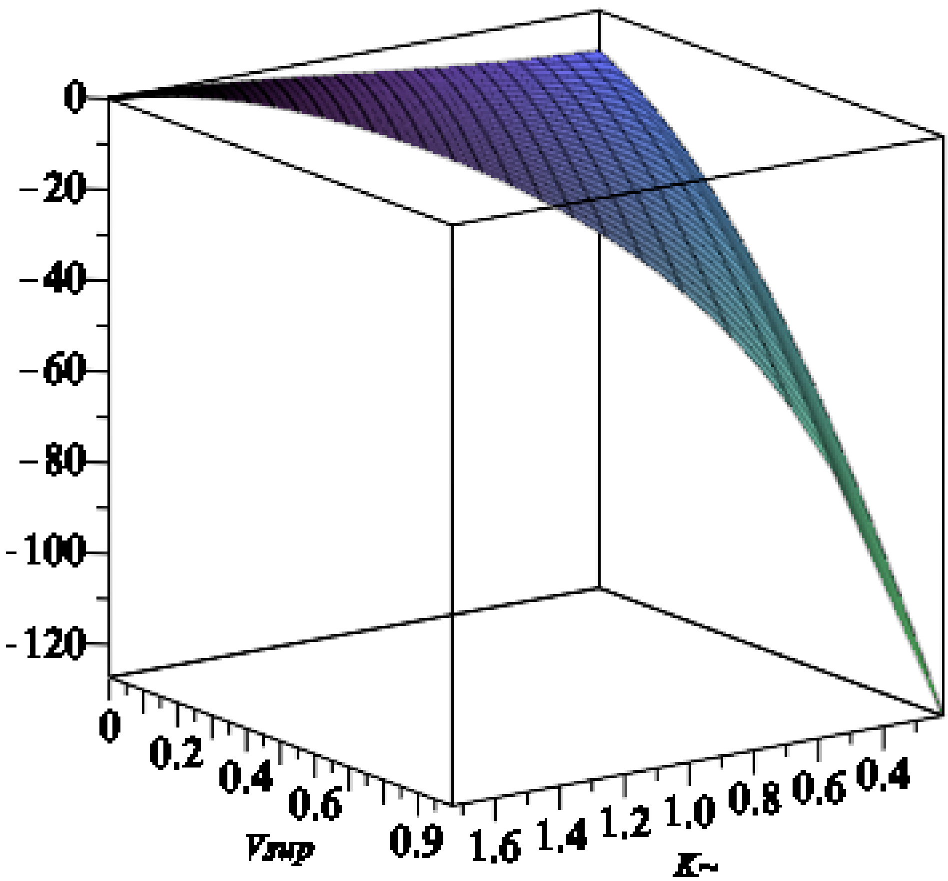
We check that the coefficient of XX^3 does not cancel :

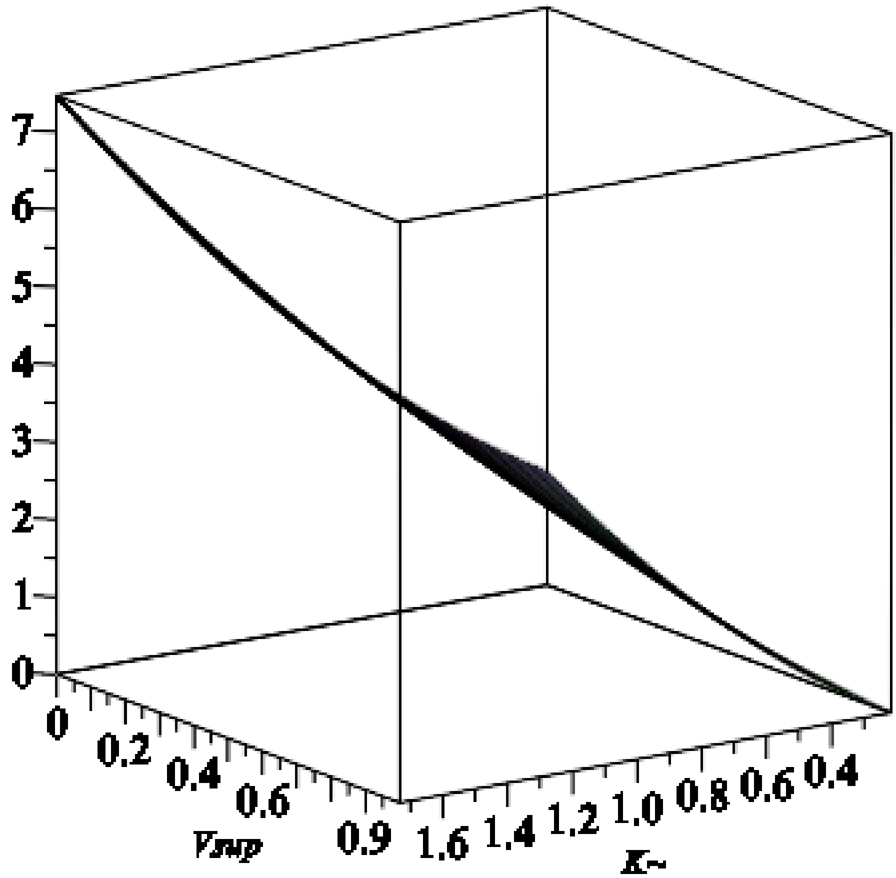
> *factor*(*numer*(*coeff*(*Qtsupsing*, *XX*, 3)));

$$32\, RootOf\Big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right) \_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\Big) Vsup \big(K\sim^4\, Vsup^3 - 7\,K\sim^4\, Vsup^2 - K\sim^4\, Vsup - 40\,K\sim^3\, Vsup^2 - 6\,K\sim^2\, Vsup^3 - K\sim^4 + 8\,K\sim^3\, Vsup - 110\,K\sim^2\, Vsup^2 + 14\,K\sim^2\, Vsup - 136\,K\sim\, Vsup^2 + 9\,Vsup^3 \qquad \textbf{(7.3.3)}$$

$+ 6\,K\sim^2 - 24\,K\sim\,Vsup - 55\,Vsup^2 - 33\,Vsup - 9\big)\,\big(K\sim^2\,Vsup^2 + K\sim^2\,Vsup$
$+ 2\,K\sim\,Vsup^2 + K\sim^2 + Vsup^2 + 2\,K\sim - 3\,Vsup + 1\big)\,(3\,K\sim + 5)\,(Vsup + 1)$

> $plot3d\big(\big(K^4\,Vsup^3 - 7\,K^4\,Vsup^2 - K^4\,Vsup - 40\,K^3\,Vsup^2 - 6\,K^2\,Vsup^3 - K^4$
> $+ 8\,K^3\,Vsup - 110\,K^2\,Vsup^2 + 14\,K^2\,Vsup - 136\,K\,Vsup^2 + 9\,Vsup^3 + 6\,K^2$
> $- 24\,K\,Vsup - 55\,Vsup^2 - 33\,Vsup - 9\big), K = Kc\,..Kinfini, Vsup = 0\,..VK11\,\big);$
> $plot3d\big(K^2\,Vsup^2 + K^2\,Vsup + 2\,K\,Vsup^2 + K^2 + Vsup^2 + 2\,K - 3\,Vsup + 1, K = Kc$
> $..Kinfini, Vsup = 0\,..VK11\,\big);$





\>

We hence get the desired asymptotic behavior for Qt, with a singularity in (1-w/rho)^{3/2}

Asymptotic behavior (in y) of AlephQ+ (Proposition 3.14)

nu <nuc

The function alephQplus is the coefficient of the dominant singular term in the expansion of Q+

```
> Qtsubcsing3;
  alephQplussubc := coeff(Qtsubcsing3, XX, 3);
```

$$\Bigg(18\Bigg(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\Bigg)\Bigg(\left(Vsub^2-2\,Vsub-1\right)Usubc^2+\left(-Vsub^2+2\,Vsub+\frac{5}{3}\right)Usubc+\frac{Vsub^2}{6}-\frac{Vsub}{3}-\frac{1}{2}\Bigg)\Bigg)\Bigg/\Big(\left(-2+3\,Usubc\right)\left(Vsub+1\right)^3\big(6\,Usubc^2$$

$$- 10\, Usubc + 3)\big) + \Bigg(18\, Vsub\, \bigg(\Big(Usubc^2 - Usubc + \frac{1}{6}\Big) Vsub^5 + \Big(-10\, Usubc^2 + 10\, Usubc - \frac{5}{3}\Big) Vsub^3 + \Big(16\, Usubc^2 - \frac{52}{3}\, Usubc + \frac{10}{3}\Big) Vsub^2 + \Big(9\, Usubc^2 - 17\, Usubc + \frac{11}{2}\Big) Vsub - \frac{4\, Usubc}{3} + \frac{2}{3}\bigg) \bigg(\Big(-\frac{2}{3} + Usubc\Big) Vsub^3 + (-7\, Usubc + 6)\, Vsub^2 + (-Usubc + 2)\, Vsub - Usubc + \frac{2}{3}\bigg) XX^2\Bigg) \Big/ \Big(\big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^2\, (-2 + 3\, Usubc)\, (Vsub + 1)^3\, \big(6\, Usubc^2 - 10\, Usubc + 3\big)\Big) - \Bigg(4\, (Vsub + 1)\, Vsub\, \sqrt{6}\, \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \bigg(\big(Vsub^3 - 7\, Vsub^2 - Vsub - 1\big)\, Usubc - \frac{2\, Vsub^3}{3} + 6\, Vsub^2 + 2\, Vsub + \frac{2}{3}\bigg) XX^3\Bigg) \Big/ \bigg(9\, \Big(-\frac{2}{3} + Usubc\Big)\, \big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^4\bigg)$$

$$alephQplussubc := -\Bigg(4\, (Vsub + 1)\, Vsub\, \sqrt{6}\, \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \bigg(\big(Vsub^3 - 7\, Vsub^2 - Vsub - 1\big)\, Usubc - \frac{2\, Vsub^3}{3} + 6\, Vsub^2 + 2\, Vsub + \frac{2}{3}\bigg)\Bigg) \Big/ \bigg(9\, \Big(-\frac{2}{3} + Usubc\Big)\, \big(Vsub^2 + 4\, Vsub + 1\big)\, (Vsub - 1)^4\bigg) \qquad \textbf{(8.1.1)}$$

We use the development of V that we already computed above

```
> Vsubsingy;
```

$$1 + \left(-\frac{24\, U - 24}{-2 + 3\, U}\right)^{1/3} (-1)^{2/3}\, Y3Y - \frac{\left(-\frac{24\, U - 24}{-2 + 3\, U}\right)^{2/3} (-1)^{1/3}\, Y3Y^2}{2} - \frac{4\, (U - 1)\, Y3Y^3}{-2 + 3\, U} + \frac{\left(-\frac{24\, U - 24}{-2 + 3\, U}\right)^{1/3} (-1)^{2/3}\, Y3Y^4}{9\, U - 6} \qquad \textbf{(8.1.2)}$$

And plug it into the expression of alephQplussubc

```
> simplify(series(subs(Vsub = Vsubsingy, Usubc = U, alephQplussubc), Y3Y, 8));
```

(8.1.3)

$$\frac{2\sqrt{3}\sqrt{2}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{27\left(\frac{-24U+24}{-2+3U}\right)^{1/3}} Y3Y^{-4} \quad \textbf{(8.1.3)}$$

$$-2\frac{\sqrt{3}\sqrt{2}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\,(U-1)\left(\mathrm{I}\sqrt{3}+1\right)^2}{\left(\frac{-24U+24}{-2+3U}\right)^{2/3}(-54+81U)} Y3Y^{-2}$$

$$-\frac{2}{81}\frac{\sqrt{3}\sqrt{2}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{\left(\frac{-24U+24}{-2+3U}\right)^{1/3}} Y3Y^{-1}$$

$$+\frac{(-40U+24)\sqrt{6}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}}{-162+243U}$$

$$-\frac{2}{81}\frac{\left(51U^2-77U+26\right)\left(\mathrm{I}\sqrt{3}+1\right)^2\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\sqrt{3}\sqrt{2}}{\left(\frac{-24U+24}{-2+3U}\right)^{2/3}(-2+3U)^2} Y3Y$$

$$+\frac{2}{243}\frac{\left(18U^2-30U+19\right)\sqrt{2}\sqrt{3}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{\left(\frac{-24U+24}{-2+3U}\right)^{1/3}(-2+3U)^2}$$

$$Y3Y^2+\frac{2}{81}\frac{\sqrt{6}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\left(65U^2-81U+26\right)}{(-2+3U)^2} Y3Y^3+$$

$$\mathrm{O}\left(Y3Y^4\right)$$

> *algeqtoseries*(*numer*(2·(1 − *YY*) − *yUVsubc*), *YY*, *V*, 10);

$$\left[1+\mathit{RootOf}\left((-2+3U)\_Z^3+24U-24\right)YY^{1/3} \quad \textbf{(8.1.4)}\right.$$

$$+\frac{\mathit{RootOf}\left((-2+3U)\_Z^3+24U-24\right)^2 YY^{2/3}}{2}-\frac{4(U-1)YY}{-2+3U}$$

$$+\frac{\mathit{RootOf}\left((-2+3U)\_Z^3+24U-24\right)YY^{4/3}}{3(-2+3U)}$$

$$+\frac{(5U-3)\,RootOf\left((-2+3U)\_Z^3+24U-24\right)^2\,YY^{5/3}}{6(-2+3U)}$$

$$-\frac{4(U-1)\,YY^2}{-2+3U}$$

$$-\frac{2\left(10U^2-20U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)YY^{7/3}}{9(-2+3U)^2}$$

$$+\frac{\left(29U^2-33U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 YY^{8/3}}{18(-2+3U)^2}$$

$$-\frac{4(U-1)\,YY^3}{-2+3U}+\mathrm{O}\left(YY^{10/3}\right)\Big]$$

with Y3Y=YY^1/3=(1-y/2)^1/3:

> $allvalues\left(RootOf\left((3U-2)\_Z^3+24U-24\right)\right);$

$$\left(-\frac{24U-24}{-2+3U}\right)^{1/3},\ \left(-\frac{24U-24}{-2+3U}\right)^{1/3}(-1)^{2/3},\ -\left(-\frac{24U-24}{-2+3U}\right)^{1/3}(-1)^{1/3} \quad \textbf{(8.1.5)}$$

> $VsubsingyPrecis := 1+RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^2}{2}-\frac{4(U-1)\,Y3Y^3}{-2+3U}+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y^4}{3(-2+3U)}+\frac{(5U-3)\,RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^5}{6(-2+3U)}-\frac{4(U-1)\,Y3Y^6}{-2+3U}-\frac{2\left(10U^2-20U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y^7}{9(-2+3U)^2}+\frac{\left(29U^2-33U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^8}{18(-2+3U)^2}-\frac{4(U-1)\,Y3Y^9}{-2+3U};$

$$VsubsingyPrecis := 1+RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y \quad \textbf{(8.1.6)}$$

$$+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^2}{2}-\frac{4(U-1)\,Y3Y^3}{-2+3U}$$

$$+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y^4}{9U-6}$$

$$+\frac{(5U-3)\,RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^5}{-12+18U}$$

$$- \frac{4 (U-1) Y3Y^6}{-2+3U}$$

$$- \frac{2 \left(10 U^2 - 20 U + 9\right) RootOf\left((-2+3U) \_Z^3 + 24 U - 24\right) Y3Y^7}{9 (-2+3U)^2}$$

$$+ \frac{\left(29 U^2 - 33 U + 9\right) RootOf\left((-2+3U) \_Z^3 + 24 U - 24\right)^2 Y3Y^8}{18 (-2+3U)^2}$$

$$- \frac{4 (U-1) Y3Y^9}{-2+3U}$$

> $VsubsingyPrecis := subs\left(RootOf\left((3U-2)\_Z^3 + 24U - 24\right) = \left(-\frac{24U-24}{3U-2}\right)^{1/3} (-1)^{2/3}, VsubsingyPrecis\right);$

$$VsubsingyPrecis := 1 + \left(-\frac{24U-24}{-2+3U}\right)^{1/3} (-1)^{2/3} Y3Y - \frac{\left(-\frac{24U-24}{-2+3U}\right)^{2/3} (-1)^{1/3} Y3Y^2}{2} - \frac{4(U-1) Y3Y^3}{-2+3U} + \frac{\left(-\frac{24U-24}{-2+3U}\right)^{1/3} (-1)^{2/3} Y3Y^4}{9U-6} - \frac{(5U-3)\left(-\frac{24U-24}{-2+3U}\right)^{2/3} (-1)^{1/3} Y3Y^5}{-12+18U} - \frac{4(U-1) Y3Y^6}{-2+3U} - \frac{2\left(10U^2-20U+9\right)\left(-\frac{24U-24}{-2+3U}\right)^{1/3} (-1)^{2/3} Y3Y^7}{9(-2+3U)^2} - \frac{\left(29U^2-33U+9\right)\left(-\frac{24U-24}{-2+3U}\right)^{2/3} (-1)^{1/3} Y3Y^8}{18(-2+3U)^2} - \frac{4(U-1) Y3Y^9}{-2+3U} \quad \textbf{(8.1.7)}$$

> $simplify(series(subs(Vsub = VsubsingyPrecis, Usubc = U, alephQplussubc), Y3Y, 5));$

$$\frac{2\sqrt{3}\sqrt{2}\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{27\left(\frac{-24U+24}{-2+3U}\right)^{1/3}} Y3Y^{-4} \quad \textbf{(8.1.8)}$$

$$- 2 \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{6\,U^2 - 10\,U + 3}{9\,U^2 - 10\,U + 2}}\,(U-1)\,\left(\mathrm{I}\sqrt{3}+1\right)^2}{\left(\frac{-24\,U + 24}{-2 + 3\,U}\right)^{2/3}\,(-54 + 81\,U)}\,Y3Y^{-2}$$

$$- \frac{2}{81}\,\frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{6\,U^2 - 10\,U + 3}{9\,U^2 - 10\,U + 2}}\,\left(\mathrm{I}\sqrt{3}+1\right)}{\left(\frac{-24\,U + 24}{-2 + 3\,U}\right)^{1/3}}\,Y3Y^{-1} + \mathrm{O}(Y3Y)$$

\>

\>

\>

nu=nuc

The function alephQplus is the coefficient of the dominant singular term in the expansion of Q+

\> *Qtcsing4*; *alephQplusc* := *coeff*(*Qtcsing4*, *XX*, 4);

$$\frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\Big(\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)Vc\,(Vc+1)\,XX^4\Big) + \frac{1}{5\,(Vc-1)^2\,(Vc^2+4\,Vc+1)\,(Vc+1)^3}\Big(\left(Vc^5 - 10\,Vc^3 + 4\,Vc^2 - 63\,Vc - 12\right)\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)Vc\,XX^3\Big) + \frac{\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)\left(Vc^2 - 2\,Vc + 5\right)}{5\,(Vc+1)^3}$$

$$alephQplusc := \frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\Big(\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)Vc\,(Vc+1)\Big) \qquad \textbf{(8.2.1)}$$

\> *simplify*(*subs*(*U* = *Uc*, *Vsubsingy*));

$$\frac{1}{3\left(1+\sqrt{7}\right)^2}\Big(\left(6\,Y3Y\left(4+\sqrt{7}\right)\right)^{1/3}\left(Y3Y^3 - \sqrt{7} - 1\right)\left(1+\sqrt{7}\right)^{2/3} + 6\,Y3Y^2\left(4+\sqrt{7}\right)^{2/3}\left(1+\sqrt{7}\right)^{4/3} - 20\sqrt{7}\,Y3Y^3 - 44\,Y3Y^3 + 6\sqrt{7} + 24\Big) \qquad \textbf{(8.2.2)}$$

\> *collect*(*expand*(*rationalize*(*simplify*(*series*(*subs*(*Vc* = **(8.2.2)**, *alephQplusc*), *Y3Y*, 4)))), *Y3Y*, *factor*);

(8.2.3)

$$\left(-\frac{5\,20^{1/3}\left(6739+2263\sqrt{7}\right)^{1/3}}{972}+\frac{7\,20^{1/3}\left(6739+2263\sqrt{7}\right)^{1/3}\sqrt{7}}{1944}\right)Y3Y^{-4}+\left(-\frac{13\left(7585+3730\sqrt{7}\right)^{1/3}}{1944}+\frac{\left(7585+3730\sqrt{7}\right)^{1/3}\sqrt{7}}{1944}\right)Y3Y^{-2}+\left(\frac{5\left(134780+45260\sqrt{7}\right)^{1/3}}{2916}-\frac{7\left(134780+45260\sqrt{7}\right)^{1/3}\sqrt{7}}{5832}\right)Y3Y^{-1}+\mathrm{O}\left(Y3Y^{0}\right) \quad \textbf{(8.2.3)}$$

>

Variante avec developpement poussé un cran plus loin :

> *map*(*simplify*, *series*(*simplify*(*subs*(*U* = *Uc*, *VsubsingyPrecis*)), *Y3Y*, 10));

$$1+\frac{\left(-20\sqrt{7}-44\right)\left(4+\sqrt{7}\right)^{1/3}}{\left(1+\sqrt{7}\right)^{10/3}}Y3Y+\frac{\left(20\sqrt{7}+44\right)\left(4+\sqrt{7}\right)^{2/3}}{\left(1+\sqrt{7}\right)^{11/3}}Y3Y^{2}-\frac{8}{3}\frac{79+31\sqrt{7}}{\left(1+\sqrt{7}\right)^{4}}Y3Y^{3}+4\frac{\left(4+\sqrt{7}\right)^{4/3}}{\left(1+\sqrt{7}\right)^{10/3}}Y3Y^{4}+\frac{4}{9}\frac{\left(4+\sqrt{7}\right)^{2/3}\left(22\sqrt{7}+43\right)}{\left(1+\sqrt{7}\right)^{11/3}}Y3Y^{5}-\frac{8}{3}\frac{79+31\sqrt{7}}{\left(1+\sqrt{7}\right)^{4}}Y3Y^{6}+\frac{4}{81}\frac{\left(4+\sqrt{7}\right)^{1/3}\left(229\sqrt{7}+709\right)}{\left(1+\sqrt{7}\right)^{10/3}}Y3Y^{7}+\frac{2}{81}\frac{\left(4+\sqrt{7}\right)^{2/3}\left(179\sqrt{7}+221\right)}{\left(1+\sqrt{7}\right)^{11/3}}Y3Y^{8}-\frac{8}{3}\frac{79+31\sqrt{7}}{\left(1+\sqrt{7}\right)^{4}}Y3Y^{9} \quad \textbf{(8.2.4)}$$

> *map*(*simplify*, *map*(*expand*, *map*(*rationalize*, *series*(*subs*(*Vc* = **(8.2.4)**, *alephQplusc*), *Y3Y*, 5))));

$$\frac{\left(4+\sqrt{7}\right)^{2/3}\left(1240\sqrt{7}-1700\right)^{1/3}\left(1+\sqrt{7}\right)^{1/3}\left(-10+7\sqrt{7}\right)}{1944}Y3Y^{-4}+\frac{\left(-130120450-49180909\sqrt{7}\right)\left(1240\sqrt{7}-1700\right)^{1/3}}{\left(4+\sqrt{7}\right)^{5/3}\left(1+\sqrt{7}\right)^{1/3}\left(303443808+114690840\sqrt{7}\right)}Y3Y^{-2}-\frac{1}{5832}\left(4+\sqrt{7}\right)^{2/3}\left(1240\sqrt{7}-1700\right)^{1/3}\left(1+\sqrt{7}\right)^{1/3}\left(-10+7\sqrt{7}\right)Y3Y^{-1}+\mathrm{O}\left(Y3Y\right) \quad \textbf{(8.2.5)}$$

>

>
>

Proof of proposition 3.15: expansion of the radius of convergence (in y) of Q(t,ty).

To identify the singularities (in y) of the series Q(t,ty) for a fixed t\in (0,t_nu), we start from the parametrization of y by U and V given by:

> *yUV*;

$$\left(8\,\nu\,(1-2\,U)\,V\,(V+1)\right)\Big/\left(U\,(U\,(\nu+1)-2)\left(V^3+\frac{\left(9\,(\nu+1)\,U^2-2\,(3+10\,\nu)\,U+8\,\nu\right)V^2}{U\,(U\,(\nu+1)-2)}-\frac{(9\,U\,(\nu+1)-4\,\nu-6)\,V}{U\,(\nu+1)-2}-1\right)\right) \quad \textbf{(9.1)}$$

Since t is fixed, U is fixed and the possible values for a singularity in y corresponds to the roots of the quantity:

> *eqVcritU*;

$$1+V^4+2\,V^3+\frac{2\,(-2+3\,U)\,(3\,U\,\nu+3\,U-2\,\nu)\,V^2}{U\,(U\,\nu+U-2)}+2\,V \quad \textbf{(9.2)}$$

To see the singularities in U of the roots, we look at the discriminant

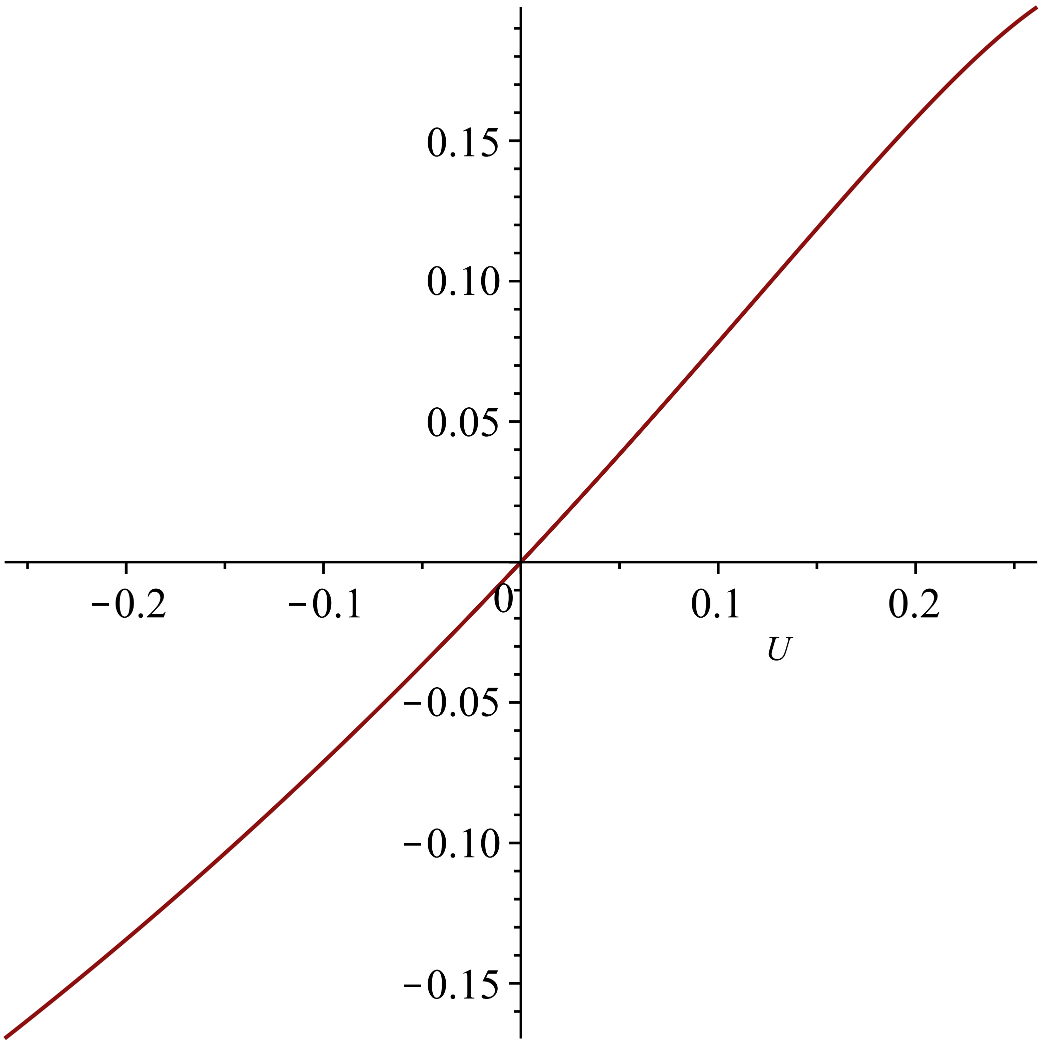
> *factor*(*discrim*(*eqVcritU*, *V*));

$$\frac{1}{U^4\,(U\,\nu+U-2)^4}\left(1024\,(-1+2\,U)\,(U\,\nu+U-\nu)\,(3\,U^2\,\nu+3\,U^2-3\,U\,\nu-3\,U+\nu)\,(15\,U^2\,\nu+15\,U^2-24\,U\,\nu-6\,U+8\,\nu)^2\right) \quad \textbf{(9.3)}$$

Only the last term may pose problems (the one before is the equation for Usubc). We look at wich values of nu may cancel it:

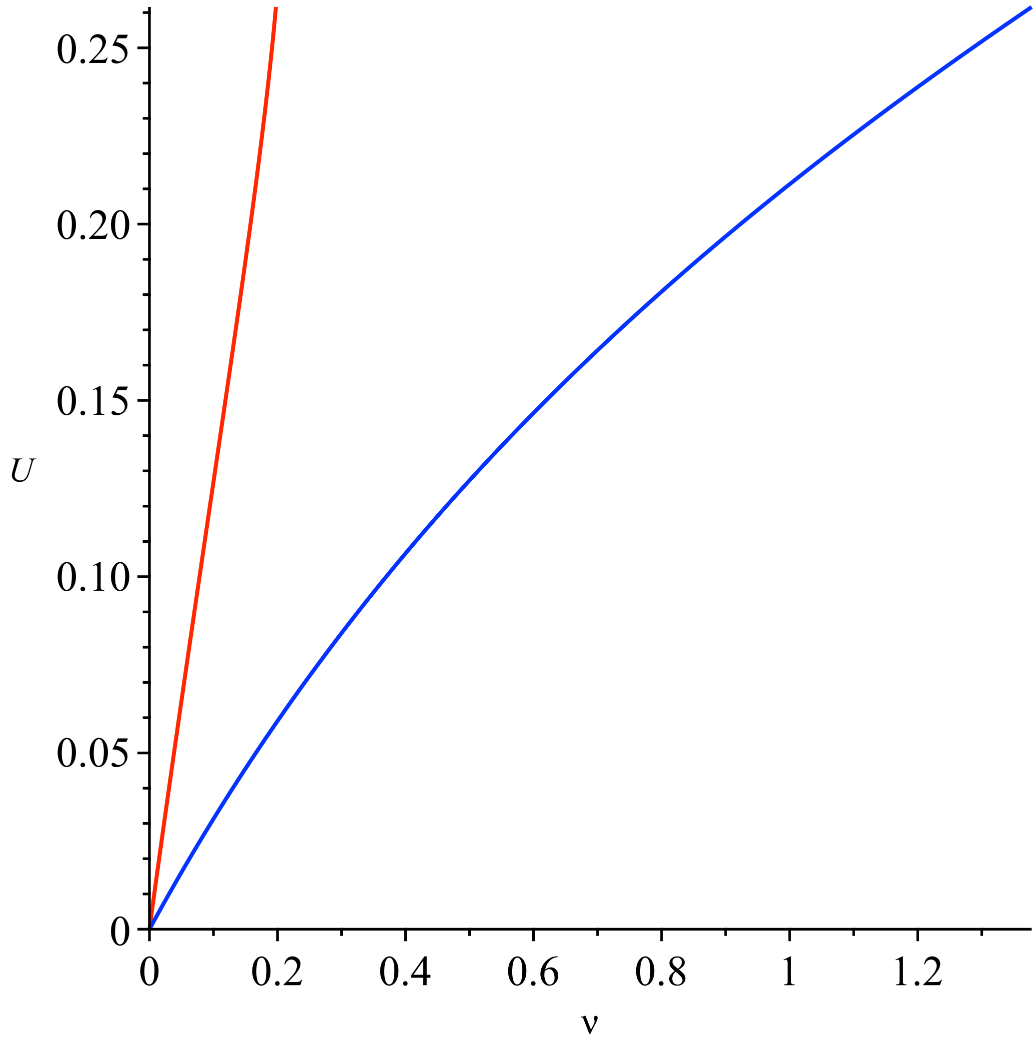
> *solve*((15 U^2 ν + 15 U^2 − 24 U ν − 6 U + 8 ν), nu); *plot*(%, U=-Uc..Uc);

$$-\frac{3\,U\,(5\,U-2)}{15\,U^2-24\,U+8}$$



The discriminant can only be 0 in the subcritical regime. When the roots are real there is no problem:

> $implicitplot\left(\left[\left(15\,U^2\,\nu + 15\,U^2 - 24\,U\,\nu - 6\,U + 8\,\nu\right), algUsubcrit\right], \mathrm{nu} = 0\,..nuc, U = -Uc\,..Uc, color = [\text{"Red"}, \text{"Blue"}]\right)$

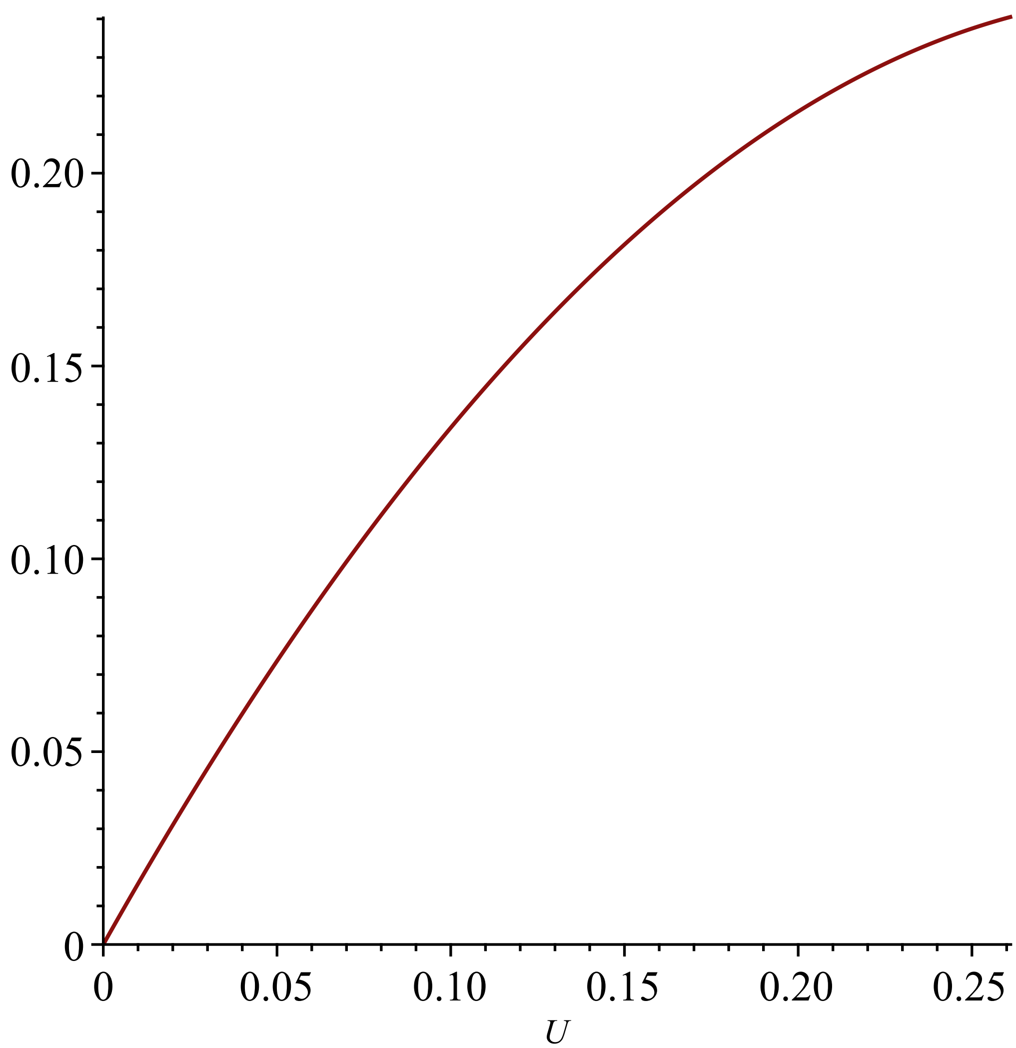


If the roots are complex no problem either as they have modulus larger than Usubc:

> $factor\left(subs\left(\mathrm{nu} = nuUsub, \frac{8\ \mathrm{nu}}{15\,(\mathrm{nu}+1)}\right) - U^2\right);$

$$-\frac{(13\,U-8)\,U}{5} \tag{9.4}$$

> $plot\left(subs\left(\mathrm{nu} = nuUsub, \frac{8\ \mathrm{nu}}{15\,(\mathrm{nu}+1)}\right) - U^2, U = 0\,..Uc\right);$



For nu < nu_c

The strategy of the proof consists in replacing U by its singular behavior around rho obtained above. Recall indeed that, we have the following development for U (with XX=(1-w/rho)^1/2 and Usubc = value of U for t=t_\nu):

> *Usubcsing3*;

$$Usubc + \frac{Usubc\,(-2 + 3\,Usubc)\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,XX}{6}$$

$$+ \big((1458\,Usubc^6 - 5778\,Usubc^5 + 9045\,Usubc^4 - 7146\,Usubc^3 + 2984\,Usubc^2 - 616\,Usubc + 4$$

$$+ 2)^2\,(2\,Usubc - 1)\big) + \Big(5\,(135\,Usubc^2 - 134\,Usubc + 22)\,(6\,Usubc^2$$

$$- 10\,Usubc + 3)\,Usubc^3\,(-2$$

$$+ 3\, Usubc)^3 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX^3 \Bigg) \Bigg/ \Big(1296\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^3\, (2\, Usubc - 1)\Big)$$

In eqUVcritc, we replace U by its development and nu by its expression in termes of Usubc. We also know that for U=Usubc, the radius of convergence in y is 2, corresponding to V=1, so that we set V=1+VV.

```
> eqVVpsubsing := convert(map(factor, series(numer(simplify(subs(U = Usubcsing3, nu
      = subs(U = Usubc, nuUsub), V = 1 + VV, eqVcritU))), XX, 4)), polynom);
```

$$eqVVpsubsing := -1679616\, VV^2\, \big(VV^2 + 6\, VV + 6\big)\, (2\, Usubc - 1)^3\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^7 - 559872 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, (2\, Usubc - 1)^2\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^7\, \big(3\, Usubc^2\, VV^4 + 18\, Usubc^2\, VV^3 - 4\, Usubc\, VV^4 + 18\, Usubc^2\, VV^2 - 24\, Usubc\, VV^3 + VV^4 - 48\, Usubc\, VV^2 + 6\, VV^3 - 48\, Usubc\, VV + 18\, VV^2 - 24\, Usubc + 24\, VV + 12\big)\, XX - 93312\, Usubc\, (2\, Usubc - 1)\, (-2 + 3\, Usubc)\, \big(6\, Usubc^2 - 10\, Usubc + 3\big)\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^5\, \big(162\, Usubc^4\, VV^4 + 972\, Usubc^4\, VV^3 - 426\, Usubc^3\, VV^4 + 972\, Usubc^4\, VV^2 - 2556\, Usubc^3\, VV^3 + 373\, Usubc^2\, VV^4 - 5148\, Usubc^3\, VV^2 + 2238\, Usubc^2\, VV^3 - 126\, Usubc\, VV^4 - 5184\, Usubc^3\, VV + 6222\, Usubc^2\, VV^2 - 756\, Usubc\, VV^3 + 14\, VV^4 - 2592\, Usubc^3 + 7968\, Usubc^2\, VV - 2580\, Usubc\, VV^2 + 84\, VV^3 + 3984\, Usubc^2 - 3648\, Usubc\, VV + 324\, VV^2 - 1824\, Usubc + 480\, VV + 240\big)\, XX^2 + 2592 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, Usubc^2\, (2\, Usubc - 1)\, \big(6\, Usubc^2 - 10\, Usubc + 3\big)\, (-2 + 3\, Usubc)^2\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^4\, \big(891\, Usubc^4\, VV^4 + 5346\, Usubc^4\, VV^3 - 1338\, Usubc^3\, VV^4 + 75330\, Usubc^4\, VV^2 - 8028\, Usubc^3\, VV^3 + 515\, Usubc^2\, VV^4 + 139968\, Usubc^4\, VV - 136980\, Usubc^3\, VV^2 + 3090\, Usubc^2\, VV^3 + 6\, Usubc\, VV^4 + 69984\, Usubc^4 - 257904\, Usubc^3\, VV + 79710\, Usubc^2\, VV^2 + 36\, Usubc\, VV^3 - 14\, VV^4 - 128952\, Usubc^3 + 153240\, Usubc^2\, VV - 15780\, Usubc\, VV^2 - 84\, VV^3 + 76620\, Usubc^2 - 31632\, Usubc\, VV + 900\, VV^2 - 15816\, Usubc + 1968\, VV + 984\big)\, XX^3 \qquad \textbf{(9.1.2)}$$

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

> $algeqtoseries(eqVVpsubsing, XX, VV, 1);$

$$\left[RootOf\left(\_Z^2+6\,\_Z+6\right)+O(XX),\ RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{XX}+O(XX)\right] \quad (9.1.3)$$

The right branch is the second one, and we can compute a full expansion

> $op(2, algeqtoseries(eqVVpsubsing, XX, VV, 3));$

$$RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{XX}+\frac{\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX}{3}+\frac{1}{36\left(9\,Usubc^2-10\,Usubc+2\right)}\left(\left(69\,Usubc^2-74\,Usubc+14\right)\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)XX^{3/2}\right)+O\left(XX^2\right) \quad (9.1.4)$$

> $Vpsubsing := 1+RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{XX}+\frac{\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX}{3}+\frac{1}{36\left(9\,Usubc^2-10\,Usubc+2\right)}\left(RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\sqrt{6}\left(69\,Usubc^2-74\,Usubc+14\right)XX^{3/2}\right):$

We can now plug this expansion in the expression of y in terms of U and V, we get:

> $ypsubsing := map(simplify, series(subs(V = Vpsubsing, U = Usubcsing3, \mathrm{nu} = subs(U = Usubc, nuUsub), yUV), XX, 2));$

$$ypsubsing := 2-\frac{1}{9\,Usubc-9}\left(3\left(-\frac{2}{3}\right.\right. \quad (9.1.5)$$

$$+ Usubc\Big)\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\Bigg(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\Bigg)XX^{3/2}\Bigg)+O(XX^2)$$

We now compute the expansion for the negative value of y which is singular. We first compute the corresponding value of V at t_\nu. We already know from **(4.1.6)** that it is -2+sqrt(3).

```
> eqVVmsubsing := convert(map(factor, series(numer(simplify(subs(U = Usubcsing3, nu
      = subs(U = Usubc, nuUsub), V=-2 + sqrt(3) + VV, eqVcritU))), XX, 4)),
      polynom) :
```

```
> algeqtoseries(eqVVmsubsing, XX, VV, 1);
```

$$\left[-2\sqrt{3}+O(XX),\,3-\sqrt{3}+O\left(\sqrt{XX}\right),\,\frac{\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\sqrt{2}\left(7\sqrt{3}-12\right)}{3\left(2\sqrt{3}-3\right)}XX+O(XX^2)\right] \quad \textbf{(9.1.6)}$$

The right branch is the third one without the constant term and we can compute a full expansion :

```
> map(simplify, op(3, algeqtoseries(eqVVmsubsing, XX, VV, 3)));
```

$$\frac{\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\sqrt{2}\left(7\sqrt{3}-12\right)}{6\sqrt{3}-9}XX-\frac{1}{3}\Big(\big(2217\sqrt{3}\,Usubc^2 - 2314\sqrt{3}\,Usubc-3840\,Usubc^2+418\sqrt{3}+4008\,Usubc-724\big)\big(6\,Usubc^2 - 10\,Usubc+3\big)\Big)\Big/\Big(\big(2\sqrt{3}-3\big)^3\big(9\,Usubc^2-10\,Usubc+2\big)^2\Big)XX^2 - \frac{1}{72}\Bigg(\big(8863911\sqrt{3}\,Usubc^6-15352740\,Usubc^6-48819690\sqrt{3}\,Usubc^5 + 84558168\,Usubc^5+87820434\sqrt{3}\,Usubc^4-152109432\,Usubc^4 - 72112960\sqrt{3}\,Usubc^3+124903296\,Usubc^3+29235912\sqrt{3}\,Usubc^2 - 50638080\,Usubc^2-5621600\sqrt{3}\,Usubc+9736896\,Usubc+409152\sqrt{3} - 708672\big)\big(6\,Usubc^2-10\,Usubc+3\big)\sqrt{2}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\Bigg) \quad \textbf{(9.1.7)}$$

$$\left/ \left(\left(9\, Usubc^2 - 10\, Usubc + 2\right)^3 (2\, Usubc - 1) \left(2\sqrt{3} - 3\right)^5 \right) XX^3 + \mathrm{O}\left(XX^4\right) \right.$$

> $Vmsubsing := -2 + \mathrm{sqrt}(3) + \dfrac{\sqrt{2}\sqrt{\dfrac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}} \left(7\sqrt{3} - 12\right)}{6\sqrt{3} - 9}\, XX$
$- \frac{1}{3} \left(\left(6\, Usubc^2 - 10\, Usubc + 3\right) \left(2217\sqrt{3}\, Usubc^2 - 2314\sqrt{3}\, Usubc - 3840\, Usubc^2 + 418\sqrt{3} + 4008\, Usubc - 724\right) \right) \Big/ \left(\left(2\sqrt{3} - 3\right)^3 \left(9\, Usubc^2 - 10\, Usubc + 2\right)^2 \right) XX^2 - \frac{1}{72} \Big(\big(8863911\sqrt{3}\, Usubc^6 - 15352740\, Usubc^6 - 48819690\sqrt{3}\, Usubc^5 + 84558168\, Usubc^5 + 87820434\sqrt{3}\, Usubc^4 - 152109432\, Usubc^4 - 72112960\sqrt{3}\, Usubc^3 + 124903296\, Usubc^3 + 29235912\sqrt{3}\, Usubc^2 - 50638080\, Usubc^2 - 5621600\sqrt{3}\, Usubc + 9736896\, Usubc + 409152\sqrt{3} - 708672\big) \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}} \left(6\, Usubc^2 - 10\, Usubc + 3\right)\sqrt{2} \Big) \Big/ \left(\left(2\sqrt{3} - 3\right)^5 (2\, Usubc - 1) \left(9\, Usubc^2 - 10\, Usubc + 2\right)^3 \right) XX^3 :$

We can now plug this expansion in the expression of y in terms of U and V, we get:

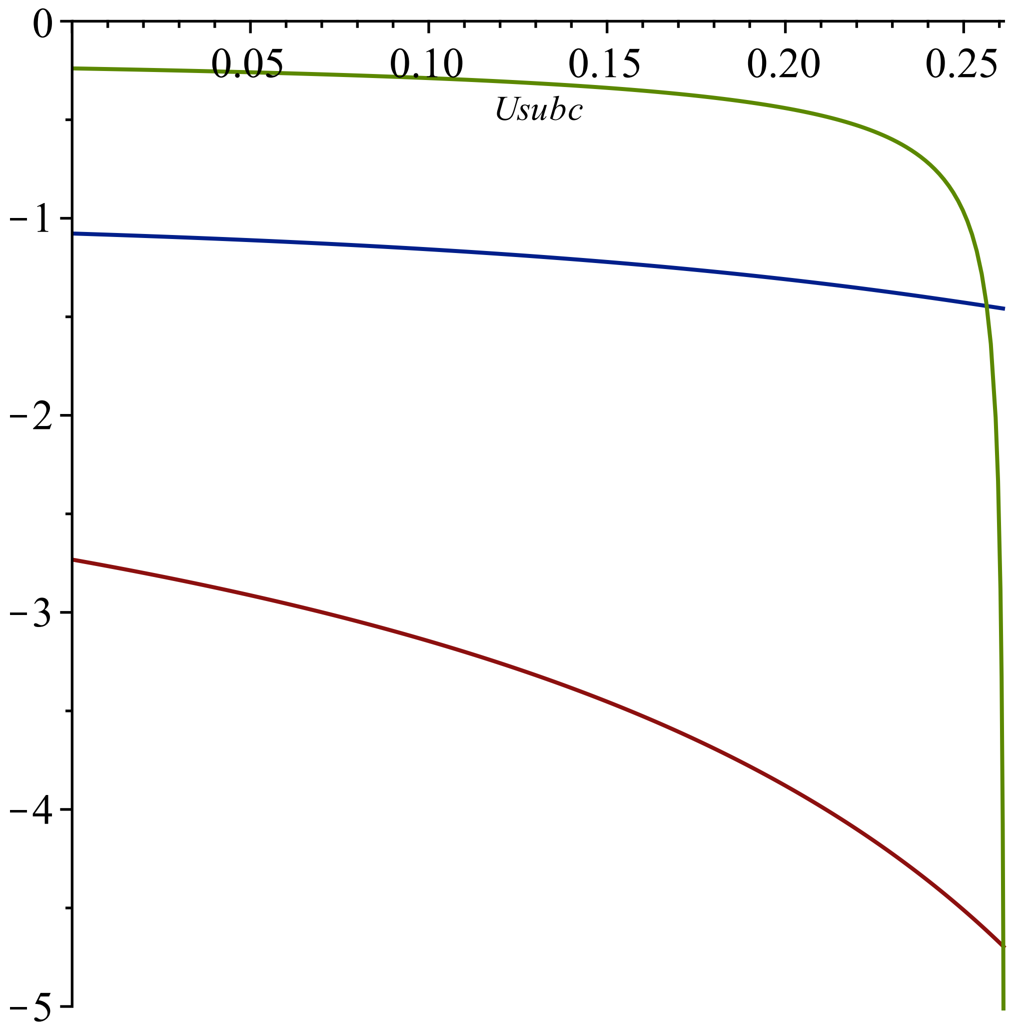
> $ymsubsing := map(simplify, series(subs(V = Vmsubsing, U = Usubcsing3, \mathrm{nu} = subs(U = Usubc, nuUsub), yUV), XX, 4));$

$$ymsubsing := -\frac{4\,(Usubc - 1)\left(-2 + \sqrt{3}\right)\left(\sqrt{3} - 1\right)}{(21\, Usubc - 16)\sqrt{3} - 37\, Usubc + 28} - 4\left(\left(6\, Usubc^2 - 10\, Usubc + 3\right)(-2 + 3\, Usubc)\left(780\sqrt{3} - 1351\right)(Usubc - 1) \right) \Big/ \left(\left(21\sqrt{3}\, Usubc - 16\sqrt{3} - 37\, Usubc + 28\right)^2 (2\, Usubc - 1)\left(2\sqrt{3} - 3\right)^3 \right) XX^2 - \frac{8}{9} \Big(\left(6\, Usubc^2 - 10\, Usubc + 3\right)\sqrt{2}\,(-2 + 3\, Usubc)\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\,(Usubc - 1)\left(1380661\sqrt{3}\, Usubc - 2391375\, Usubc - 1048348\sqrt{3} + 1815792\right) \Big) \Big/ \tag{9.1.8}$$

$$\left((2\,Usubc-1)\left(21\sqrt{3}\,Usubc-16\sqrt{3}-37\,Usubc+28\right)^{3}\left(2\sqrt{3}-3\right)^{5}\right)XX^{3}$$
$$+\,O\left(XX^{4}\right)$$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by plotting the values of the coefficients:

```
> plot([coeff(ymsubsing, XX, 0), coeff(ymsubsing, XX, 2), coeff(ymsubsing, XX, 3)],
      Usubc = 0..Uc);
```



For nu = nu_c

The proof is similar as the subcritical case, except that the singular expansion of U around rho is different. With XX=(1-w/rhoc)^1/3, we have:

```
> Ucsing4;
```

$$\frac{5}{9}-\frac{\sqrt{7}}{9}-\frac{\left(1240\sqrt{7}-1700\right)^{1/3}XX}{54} \tag{9.2.1}$$
$$-\frac{5\left(1240\sqrt{7}-1700\right)^{2/3}\left(2\sqrt{7}+1\right)XX^{2}}{69984}+\left(-\frac{35}{10368}+\frac{35\sqrt{7}}{5184}\right)XX^{3}$$

$$+\frac{1645\left(1240\sqrt{7}-1700\right)^{1/3}XX^4}{4478976}$$

In eqVcritU, we replace U by its development and nu by its expression in termes of Usubc. We also know that for U=Uc, the radius of convergence in y is 2, corresponding to V=1, so that we set V=1+VV.

> $eqVVpcsing := convert(map(factor, series(numer(simplify(subs(U = Ucsing4, \mathrm{nu} = nuc, V = 1 + VV, eqVcritU))), XX, 4)), polynom);$

$$eqVVpcsing := 4458050224128\left(-14+\sqrt{7}\right)VV^2\left(VV^2+6VV+6\right) + 743008370688\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}\left(VV^4+6VV^3+42VV^2+72VV+36\right)XX - 30958682112\,50^{1/3}\left(-14+\sqrt{7}\right)\left(17VV^4+102VV^3+570VV^2+936VV+468\right)XX^2 - 58047528960\left(-14+\sqrt{7}\right)\left(11VV^4+66VV^3+302VV^2+472VV+236\right)XX^3 \quad \textbf{(9.2.2)}$$

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

> $algeqtoseries(eqVVpcsing, XX, VV, 1);$

$$\left[RootOf\left(\_Z^2+6\_Z+6\right)+\mathrm{O}(XX), RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\sqrt{XX}+\mathrm{O}(XX)\right] \quad \textbf{(9.2.3)}$$

The right branch is the second one, and we can compute a full expansion

> $op(2, algeqtoseries(eqVVpcsing, XX, VV, 3));$

$$RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\sqrt{XX} + \left(\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}}{27}+\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{54}\right)XX + \frac{1}{96}\left(5\,50^{2/3}RootOf\left(-2\left(\frac{2\,50^{2/3}\sqrt{7}}{5}-\frac{50^{2/3}}{5}\right)\sqrt{7}+27\_Z^2 - \frac{2\,50^{2/3}\sqrt{7}}{5}+\frac{50^{2/3}}{5}\right)XX^{3/2}\right)+\mathrm{O}\left(XX^2\right) \quad \textbf{(9.2.4)}$$

> $Vpcsing := 1 + RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\sqrt{XX} + \left(\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{54}+\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}}{27}\right)XX + \left(\frac{1}{2592}\left(25\,RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\right.\right.$

$\left(1240\sqrt{7}-1700\right)^{1/3}\Big)+\frac{1}{1296}\Big(25\,RootOf\Big(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\Big)\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}\Big)\Big)$

$XX^{3/2}$:

We can now plug this expansion in the expression of y in terms of U and V, we get the desired asymptotics (recall that XX=(1-w/rhoc)^1/3)

> *ypcsing* := *map*(*simplify*, *series*(*subs*(*V* = *Vpcsing*, *U* = *Ucsing4*, nu = *nuc*, *yUV*), *XX*, 2));

$$ypcsing := 2+\frac{1}{\left(-5+\sqrt{7}\right)^2\left(7+\sqrt{7}\right)^2\left(7+13\sqrt{7}\right)}\Big(\left(-812\sqrt{7}+784\right)XX^{3/2}\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\Big(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\Big)\Big)+O\left(XX^2\right) \quad \textbf{(9.2.5)}$$

We only need to check that the leading coefficient is not zero :

> *evalf*(*allvalues*($\frac{1}{\left(-5+\sqrt{7}\right)^2\left(7+\sqrt{7}\right)^2\left(7+13\sqrt{7}\right)}\Big(\left(-812\sqrt{7}+784\right)\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\Big(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\Big)\Big)$))

$$-1.226670602,\ 1.226670602 \quad \textbf{(9.2.6)}$$

We now compute the expansion for the negative singular value of y. We already know from **??** that y(V) is increasing in V for V between -2+sqrt(3) and 1 so that the corresponding value of V is -2+sqrt(3). We start from the same expression as above, only replacing V by -2+sqrt(3) + VV.

> *eqVVmcsing* := *convert*(*map*(*factor*, *series*(*numer*(*simplify*(*subs*(*U* = *Ucsing4*, nu = *nuc*, *V* = -2 + sqrt(3) + *VV*, *eqVcritU*))), *XX*, 4)), *polynom*);

$$eqVVmcsing := 4458050224128\left(-14+\sqrt{7}\right)\left(2\sqrt{3}+VV\right)VV\left(VV-3+\sqrt{3}\right)^2+743008370688\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}\left(4\sqrt{3}\,VV^3+VV^4-18\sqrt{3}\,VV^2-6\,VV^3+96\sqrt{3}\,VV+60\,VV^2-144\sqrt{3}-180\,VV+252\right)XX-30958682112\,50^{1/3}\left(-14+\sqrt{7}\right)\left(68\sqrt{3}\,VV^3+17\,VV^4-306\sqrt{3}\,VV^2-102\,VV^3+1344\sqrt{3}\,VV+876\,VV^2-1872\sqrt{3}-2484\,VV+3276\right)XX^2-58047528960\left(-14+\sqrt{7}\right)\left(44\sqrt{3}\,VV^3+11\,VV^4-198\sqrt{3}\,VV^2-66\,VV^3+736\sqrt{3}\,VV+500\,VV^2-944\sqrt{3}-1340\,VV+1652\right)XX^3 \quad \textbf{(9.2.7)}$$

> *algeqtoseries*(*eqVVmcsing*, *XX*, *VV*, 1);

$$\Big[-2\sqrt{3}+O(XX),\ 3-\sqrt{3}+O\left(\sqrt{XX}\right),\ \Big(-\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{54}+\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{3}}{81}-\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}}{27} \quad \textbf{(9.2.8)}$$

$$+\frac{2\sqrt{7}\,(1240\sqrt{7}-1700)^{1/3}\sqrt{3}}{81}\Bigg)XX+O(XX^2)\Bigg]$$

The right branch is the third one without the constant term and we can compute a full expansion :

```
> map(simplify, op(3, algeqtoseries(eqVVmcsing, XX, VV, 4)));
```

$$\frac{(1240\sqrt{7}-1700)^{1/3}\,(2\sqrt{3}-3)\,(2\sqrt{7}+1)}{162}XX + \frac{1}{1296}\,\frac{1}{(2\sqrt{3}-3)\,(-14+\sqrt{7})}\Bigg(-2\,(14+\sqrt{7})\,(-2+\sqrt{3})\,(1240\sqrt{7}-1700)^{2/3}-1404\left(\sqrt{3}-\frac{7}{4}\right)(-14+\sqrt{7})\,50^{1/3}\Bigg)XX^2 + \frac{1}{1152}\,\frac{1}{(2\sqrt{3}-3)^2\,(-14+\sqrt{7})^2}\Bigg(728\left(-\frac{1}{2}+\sqrt{7}\right)50^{1/3}\left(\sqrt{3}-\frac{24}{13}\right)(1240\sqrt{7}-1700)^{1/3}-510440\left(\sqrt{7}-\frac{29}{4}\right)\left(\sqrt{3}-\frac{6303}{3646}\right)\Bigg)XX^3 + \frac{1}{4608}\,\frac{1}{(2\sqrt{3}-3)^3\,(-14+\sqrt{7})^3}\Bigg(-10614870\left(\sqrt{3}-\frac{9062}{5229}\right)\left(\sqrt{7}-\frac{28}{29}\right)(1240\sqrt{7}-1700)^{1/3}-151704\left(\sqrt{3}-\frac{4693}{2709}\right)(-14+\sqrt{7})\,50^{1/3}\,(1240\sqrt{7}-1700)^{2/3}-1956955\,50^{2/3}\left(\sqrt{3}-\frac{438}{253}\right)\left(\sqrt{7}-\frac{434}{85}\right)\Bigg)XX^4+O(XX^5) \quad \textbf{(9.2.9)}$$

$$> Vmcsing := -2+\mathrm{sqrt}(3)+\frac{(2\sqrt{7}+1)\,(1240\sqrt{7}-1700)^{1/3}\,(2\sqrt{3}-3)}{162}XX + \frac{1}{2592}\,\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)(-14+\sqrt{7})}\Bigg(-2\,(14+\sqrt{7})\,(-2+\sqrt{3})\,(1240\sqrt{7}-1700)^{2/3}-1404\,50^{1/3}\left(\sqrt{3}-\frac{7}{4}\right)(-14+\sqrt{7})\Bigg)XX^2 + \frac{1}{4608}\,\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)^2(-14+\sqrt{7})^2}\Bigg(728\left(\sqrt{7}-\frac{1}{2}\right)50^{1/3}\left(\sqrt{3}-\frac{24}{13}\right)(1240\sqrt{7}-1700)^{1/3}-510440\left(\sqrt{3}-\frac{6303}{3646}\right)\left(\sqrt{7}-\frac{29}{4}\right)\Bigg)XX^3 + \frac{1}{36864}\,\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)^3(-14+\sqrt{7})^3}\Bigg(-10614870\left(\sqrt{7}-\frac{28}{29}\right)\left(\sqrt{3}-\frac{9062}{5229}\right)(1240\sqrt{7}-1700)^{1/3}-151704\,50^{1/3}\left(\sqrt{3}-\frac{4693}{2709}\right)(-14$$

$$+\sqrt{7}\big)\left(1240\sqrt{7}-1700\right)^{2/3}-1956955\left(\sqrt{7}-\frac{434}{85}\right)\left(\sqrt{3}-\frac{438}{253}\right)50^{2/3}\Big)XX^4:$$

We can now plug this expansion in the expression of y in terms of U and V, we get:

> *ymcsing* := *map*(*simplify*, *series*(*subs*(*V* = *Vmcsing*, *U* = *Ucsing4*, nu = *nuc*, *yUV*), *XX*, 5));

$$ymcsing := -\frac{4\left(-\frac{1}{2}+\sqrt{7}\right)\left(7+\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)}{\left(24\sqrt{7}+231\right)\sqrt{3}-38\sqrt{7}-413} \quad \textbf{(9.2.10)}$$

$$+\frac{27440}{3}\left(\left(362\sqrt{3}-627\right)\left(78806\sqrt{7}-181693\right)\right)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^2\left(7+\sqrt{7}\right)^2\left(-5+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^2\left(2\sqrt{3}-3\right)^2\right)$$

$$XX^3+\frac{1}{8}\Bigg(\Big(\left(330082753200510240\sqrt{3}-571720105508325550\right)\sqrt{7}-1024391999457256185\sqrt{3}+1774299006738717515\Big)\left(1240\sqrt{7}-1700\right)^{1/3}+15704812680490206\left(\left(\sqrt{3}-\frac{33518496652}{19351912887}\right)\sqrt{7}-\frac{8999600785\sqrt{3}}{4300425086}+\frac{140289893863}{38703825774}\right)50^{1/3}\left(1240\sqrt{7}-1700\right)^{2/3}+\left(-586902892127647914\,50^{2/3}\sqrt{3}+1016545638114735092\,50^{2/3}\right)\sqrt{7}+1392843856906107333\,50^{2/3}\sqrt{3}-2412476352951155491\,50^{2/3}\Bigg)\Bigg/$$

$$\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^3\left(7+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^3\left(2\sqrt{3}-3\right)^3\left(-5+\sqrt{7}\right)^4\right)XX^4+\mathrm{O}\left(XX^5\right)$$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by evaluating the values of the coefficients:

> *evalf*(*coeff*(*ymcsing*, *XX*, 0));
evalf(*coeff*(*ymcsing*, *XX*, 3));
evalf(*coeff*(*ymcsing*, *XX*, 4));

$$-4.702978452$$
$$-1.459540327$$
$$22.57932079 \quad \textbf{(9.2.11)}$$

For nu>nu_c

For nu>nu_c, we cannot replace nu by its value in terms of Usupc and have to rely on the rational parametrization by K. In this case the parametrization of y by K and V *

> *yUVsupc*;

$$-\left(8\left(K\sim^3+3K\sim^2+9K\sim+11\right)(K\sim+1)V(V+1)\right)\Big/\left(K\sim^4V^3-7K\sim^4V^2-K\sim^4V-40K\sim^3V^2-6K\sim^2V^3-K\sim^4+8K\sim^3V-110K\sim^2V^2+14K\sim^2V-136K\sim V^2+9V^3+6K\sim^2-24K\sim V-55V^2-33V-9\right) \quad (9.3.1)$$

> *Usupcsing*;

$$-\frac{K\sim^2-3}{2(3K\sim+5)}+RootOf\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\right)XX-\left(\left(K\sim^2-3\right)\left(K\sim^2+8K\sim+13\right)XX^2\left(9K\sim^4+14K\sim^3-18K\sim^2-10K\sim+29\right)(K\sim+1)\right)\Big/\left(144(3K\sim+5)\left(3K\sim^2+4K\sim-1\right)^2(2+K\sim)\right)+\frac{1}{216\left(3K\sim^2+4K\sim-1\right)^3(2+K\sim)}\left(5\left(K\sim^2+8K\sim+13\right)\left(9K\sim^6+40K\sim^5+43K\sim^4-48K\sim^3-97K\sim^2+24K\sim+77\right)RootOf\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\right)XX^3\right) \quad (9.3.2)$$

In eqVcritU, we replace U by its development and nu by its expression in termes of K. We also know that for U=Uc, the radius of convergence coresponds to V=VK11, so that we set V=VK11+VV.

> *eqVVpsupsing* := *convert*(*map*(*factor*, *series*(*numer*(*simplify*(*subs*(*U* = *Usupcsing*, nu = *nusupK*, *V* = *VK11* + *VV*, *eqVcritU*))), *XX*, 4)), *polynom*) :

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

> *simplify*(*algeqtoseries*(*eqVVpsupsing*, *XX*, *VV*, 1));

$$\Big[RootOf\Big(\left(K\sim^6-9K\sim^4+27K\sim^2-27\right)\_Z^3+\Big(-6K\sim^6 \quad (9.3.3)$$

$$+4K\sim^4\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1}-16K\sim^5+22K\sim^4$$

$$- 24 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 96 K\sim^3 + 30 K\sim^2$$

$$+ 36 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 144 K\sim - 126\Big) \_Z^2 + \Big(24 K\sim^6$$

$$- 18 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 120 K\sim^5$$

$$- 48 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 152 K\sim^4$$

$$+ 12 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 208 K\sim^3$$

$$+ 144 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 664 K\sim^2$$

$$+ 126 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 456 K\sim - 24\Big) \_Z - 36 K\sim^6$$

$$+ 24 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 288 K\sim^5$$

$$+ 112 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 956 K\sim^4$$

$$+ 208\, K\sim^2 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} - 1600\, K\sim^3$$

$$+ 176\, K\sim \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} - 1292\, K\sim^2$$

$$+ 56 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} - 288\, K\sim + 140\Big) + \mathrm{O}(XX),$$

$$-\Bigg(64 \left(K\sim + \frac{5}{3}\right) \Bigg(\sqrt{K\sim^2 + 4\, K\sim + 5}\, (K\sim + 1)^2 \sqrt{3\, K\sim^2 + 4\, K\sim - 1} - \frac{7\, K\sim^4}{4}$$
$$- 8\, K\sim^3 - \frac{27\, K\sim^2}{2} - 8\, K\sim + \frac{1}{4}\Bigg) (K\sim + 1)\, (2 + K\sim)\, \mathit{RootOf}\big(\big(1296\, K\sim^4$$
$$+ 6048\, K\sim^3 + 8928\, K\sim^2 + 3360\, K\sim - 1200\big)\, \_Z^2 - K\sim^8 - 10\, K\sim^7 - 24\, K\sim^6$$
$$+ 26\, K\sim^5 + 158\, K\sim^4 + 114\, K\sim^3 - 192\, K\sim^2 - 306\, K\sim - 117\big)\Bigg) \Bigg/$$
$$\Bigg(\Bigg(\sqrt{K\sim^2 + 4\, K\sim + 5}\, (K\sim + 1)^2 \sqrt{3\, K\sim^2 + 4\, K\sim - 1} - \frac{3\, K\sim^4}{2} - 8\, K\sim^3$$
$$- 15\, K\sim^2 - 8\, K\sim + \frac{5}{2}\Bigg) \left(K\sim^2 + \frac{8}{3}\, K\sim + \frac{7}{3}\right) \left(K\sim^2 - 3\right)^2\Bigg) XX + \mathrm{O}\left(XX^2\right)\Bigg]$$

The right branch is the second one, and we can compute a full expansion

> *Vpsupsing* := *VK11* + *collect*(*convert*(*op*(2, *algeqtoseries*(*eqVVpsupsing*, *XX*, *VV*, 3)), *polynom*), *XX*, *factor*) :

We can now plug this expansion in the expression of y in terms of U and V, we get:

> *ypsupsing* := *collect*(*convert*(*map*(*expand*, *map*(*rationalize*, *map*(*simplify*, *series*(*subs*(*V* = *Vpsupsing*, *U* = *Usupcsing*, nu = *nusupK*, *yUV*), *XX*, 2)))), *polynom*), *XX*, *factor*);

$$\mathit{ypsupsing} := -\Big(16\, (3\, K\sim + 5)\, \left(3\, K\sim^2 + 8\, K\sim + 7\right) \left(K\sim^3 + 3\, K\sim^2 + 9\, K\sim\right. \quad \textbf{(9.3.4)}$$
$$\left. + 11\right) \Big(36\, K\sim^{10} + 31\, K\sim^8 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 384\, K\sim^9$$
$$+ 248\, K\sim^7 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 1996\, K\sim^8$$
$$+ 844\, K\sim^6 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 6624\, K\sim^7$$
$$+ 1544\, K\sim^5 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 14952\, K\sim^6$$
$$+ 1818\, K\sim^4 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 22176\, K\sim^5$$

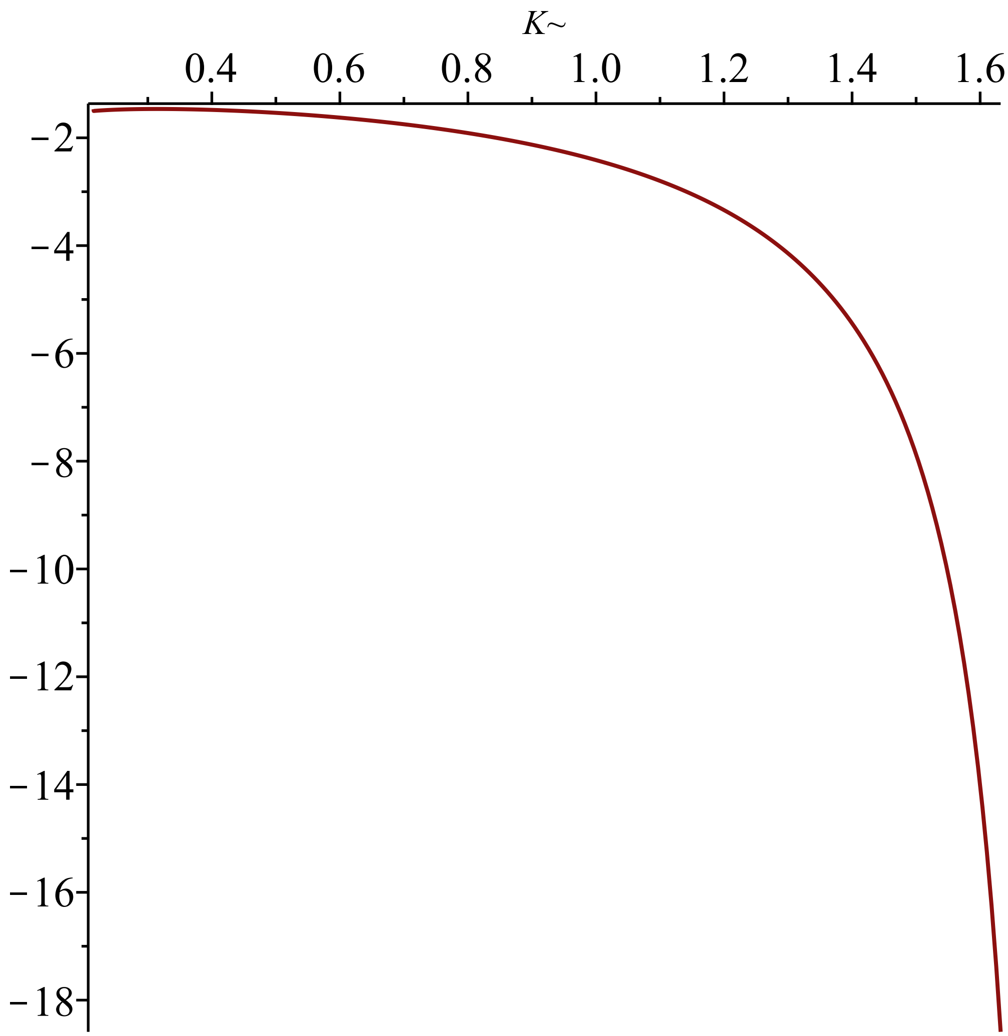
$$+ 2088\,K\sim^3 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 19160\,K\sim^4$$
$$+ 2508\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 6560\,K\sim^3$$
$$+ 1816\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1804\,K\sim^2$$
$$+ 479 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1184\,K\sim + 220\Big)$$
$$RootOf\big((1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200)\,\_Z^2 - K\sim^8$$
$$- 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)$$
$$XX\Big) \Big/ \Big((K\sim^2 + 4\,K\sim + 5)\,(23\,K\sim^6 + 184\,K\sim^5 + 593\,K\sim^4 + 1008\,K\sim^3$$
$$+ 989\,K\sim^2 + 568\,K\sim + 163)^2\,(K\sim^2 - 3)^2\Big) - \Big(4\,(K\sim + 1)\,(K\sim^3 + 3\,K\sim^2$$
$$+ 9\,K\sim + 11)\,\Big(2\,K\sim^4 + 3\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 8\,K\sim^3$$
$$+ 4\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 24\,K\sim^2$$
$$- \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 40\,K\sim + 22\Big)\Big) \Big/ \Big((23\,K\sim^6$$
$$+ 184\,K\sim^5 + 593\,K\sim^4 + 1008\,K\sim^3 + 989\,K\sim^2 + 568\,K\sim + 163)\,(K\sim^2 - 3)\Big)$$

The coefficient in XX does not vanish:

$$> fsolve\Big(36\,K\sim^{10} + 31\,K\sim^8 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 384\,K\sim^9$$
$$+ 248\,K\sim^7 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 1996\,K\sim^8$$
$$+ 844\,K\sim^6 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 6624\,K\sim^7$$
$$+ 1544\,K\sim^5 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 14952\,K\sim^6$$
$$+ 1818\,K\sim^4 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 22176\,K\sim^5$$
$$+ 2088\,K\sim^3 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 19160\,K\sim^4$$
$$+ 2508\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 6560\,K\sim^3$$
$$+ 1816\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1804\,K\sim^2$$
$$+ 479 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1184\,K\sim + 220\Big)$$

$$-1.548583770 \qquad \textbf{(9.3.5)}$$

$$> plot(coeff(ypsupsing, XX, 1), K = Kc\,..Kinfini - 0.1);$$



>

>

We now compute the expansion for the negative singular value of y. At t_\nu, the negative singularity is given by VK22:

> *eqVVmsupsing* := *convert*(*map*(*factor*, *series*(*numer*(*simplify*(*subs*(*U* = *Usupcsing*, nu = *nusupK*, *V* = *VK22* + *VV*, *eqVcritU*))), *XX*, 4)), *polynom*) : *degree*(%, *XX*);

$$3 \tag{9.3.6}$$

> *algeqtoseries*(*eqVVmsupsing*, *XX*, *VV*, 1);

$$\Big[\mathit{RootOf}\Big(\big(K\sim^6 - 9\,K\sim^4 + 27\,K\sim^2 - 27\big)\,\_Z^3 + \big(-8\,K\sim^5\,\sqrt{2}\,\sqrt{2+K\sim} + 6\,K\sim^6 \tag{9.3.7}$$

$$- 8\,K\sim^4\,\sqrt{2}\,\sqrt{2+K\sim} + 16\,K\sim^5 + 48\,K\sim^3\,\sqrt{2}\,\sqrt{2+K\sim} - 22\,K\sim^4$$

$$+ 48\,K\sim^2\,\sqrt{2}\,\sqrt{2+K\sim} - 96\,K\sim^3 - 72\,\sqrt{2}\,\sqrt{2+K\sim}\,K\sim - 30\,K\sim^2$$

$- 72\sqrt{2}\,\sqrt{2+K\sim} + 144\,K\sim + 126\big)\,\_Z^2 + \big(-36\,K\sim^5\sqrt{2}\,\sqrt{2+K\sim} + 6\,K\sim^6$

$- 132\,K\sim^4\sqrt{2}\,\sqrt{2+K\sim} + 72\,K\sim^5 - 72\,K\sim^3\sqrt{2}\,\sqrt{2+K\sim} + 218\,K\sim^4$

$+ 312\,K\sim^2\sqrt{2}\,\sqrt{2+K\sim} + 80\,K\sim^3 + 540\sqrt{2}\,\sqrt{2+K\sim}\,K\sim - 574\,K\sim^2$

$+ 252\sqrt{2}\,\sqrt{2+K\sim} - 888\,K\sim - 402\big)\,\_Z - 24\,K\sim^5\sqrt{2}\,\sqrt{2+K\sim}$

$- 184\,K\sim^4\sqrt{2}\,\sqrt{2+K\sim} + 96\,K\sim^5 - 592\,K\sim^3\sqrt{2}\,\sqrt{2+K\sim} + 640\,K\sim^4$

$- 976\,K\sim^2\sqrt{2}\,\sqrt{2+K\sim} + 1728\,K\sim^3 - 824\sqrt{2}\,\sqrt{2+K\sim}\,K\sim + 2368\,K\sim^2$

$- 280\sqrt{2}\,\sqrt{2+K\sim} + 1632\,K\sim + 448\big) + \mathrm{O}(XX),\ -\big(16\,\mathit{RootOf}\big(\big(1296\,K\sim^4$
$+ 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\big)\,\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6$
$+ 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,\big($
$-12\,K\sim^5\sqrt{2}\,\sqrt{2+K\sim} + 3\,K\sim^6 - 104\,K\sim^4\sqrt{2}\,\sqrt{2+K\sim} + 59\,K\sim^5$
$- 368\,K\sim^3\sqrt{2}\,\sqrt{2+K\sim} + 360\,K\sim^4 - 656\,K\sim^2\sqrt{2}\,\sqrt{2+K\sim} + 1038\,K\sim^3$
$- 580\sqrt{2}\,\sqrt{2+K\sim}\,K\sim + 1583\,K\sim^2 - 200\sqrt{2}\,\sqrt{2+K\sim} + 1251\,K\sim + 410\big)\big)\Big/$
$\big(3\,K\sim^8\sqrt{2}\,\sqrt{2+K\sim} + 20\,K\sim^7\sqrt{2}\,\sqrt{2+K\sim} - 12\,K\sim^8 + 36\,K\sim^6\sqrt{2}\,\sqrt{2+K\sim}$
$- 68\,K\sim^7 - 52\,K\sim^5\sqrt{2}\,\sqrt{2+K\sim} - 76\,K\sim^6 - 262\,K\sim^4\sqrt{2}\,\sqrt{2+K\sim} + 260\,K\sim^5$
$- 228\,K\sim^3\sqrt{2}\,\sqrt{2+K\sim} + 724\,K\sim^4 + 276\,K\sim^2\sqrt{2}\,\sqrt{2+K\sim} + 276\,K\sim^3$
$+ 612\sqrt{2}\,\sqrt{2+K\sim}\,K\sim - 996\,K\sim^2 + 315\sqrt{2}\,\sqrt{2+K\sim} - 1332\,K\sim - 504\big)\,XX$
$+ \mathrm{O}(XX^2)\big]$

The right branch is the second one without the constant term and we can compute a full expansion :

```
> Vmsupsing := VK22 + collect(convert(map(expand, map(rationalize, map(simplify,
      op(2, algeqtoseries(eqVVmsupsing, XX, VV, 4))))), polynom), XX, factor) :
```

We can now plug this expansion in the expression of y in terms of U and V, we get:

```
> ymsupsing := collect(convert(map(expand, map(rationalize, map(simplify, series(subs(V
      = Vmsupsing, U = Usupcsing, nu = nusupK, yUV), XX, 4)))), polynom), XX, factor)
      ;
```

$ymsupsing := \big(4\,\mathit{RootOf}\big(\big(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\big)\,\_Z^2$ **(9.3.8)**
$- K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim$
$- 117\big)\,(3\,K\sim + 5)\,(K\sim + 1)\,\big(K\sim^2 + 8\,K\sim + 13\big)\,\big(K\sim^3 + 3\,K\sim^2 + 9\,K\sim$

$$+ 11)\left(3K{\sim}^4 + 12K{\sim}^3 + 22K{\sim}^2 + 28K{\sim} + 19\right)\left(K{\sim}^6\sqrt{2} + 6K{\sim}^5\sqrt{2} + 27K{\sim}^4\sqrt{2} + 16K{\sim}^4\sqrt{2+K{\sim}} + 108K{\sim}^3\sqrt{2} + 80K{\sim}^3\sqrt{2+K{\sim}} + 339K{\sim}^2\sqrt{2} + 240K{\sim}^2\sqrt{2+K{\sim}} + 582K{\sim}\sqrt{2} + 464\sqrt{2+K{\sim}}\,K{\sim} + 377\sqrt{2} + 352\sqrt{2+K{\sim}}\right)XX^3\Big)\Big/\Big(3\sqrt{2+K{\sim}}\left(3K{\sim}^2 + 8K{\sim} + 7\right)^3\left(K{\sim}^2 - 3\right)^2\left(K{\sim}^4 + 6K{\sim}^3 + 30K{\sim}^2 + 62K{\sim} + 45\right)^2\Big) + \Big(\big(K{\sim}^2 + 8K{\sim} + 13\big)\left(K{\sim}^3 + 3K{\sim}^2 + 9K{\sim} + 11\right)(K{\sim}+1)^2\left(K{\sim}^6\sqrt{2} + 6K{\sim}^5\sqrt{2} + 27K{\sim}^4\sqrt{2} + 16K{\sim}^4\sqrt{2+K{\sim}} + 108K{\sim}^3\sqrt{2} + 80K{\sim}^3\sqrt{2+K{\sim}} + 339K{\sim}^2\sqrt{2} + 240K{\sim}^2\sqrt{2+K{\sim}} + 582K{\sim}\sqrt{2} + 464\sqrt{2+K{\sim}}\,K{\sim} + 377\sqrt{2} + 352\sqrt{2+K{\sim}}\right)XX^2\Big)\Big/\Big(2\sqrt{2+K{\sim}}\left(K{\sim}^2 - 3\right)\left(K{\sim}^4 + 6K{\sim}^3 + 30K{\sim}^2 + 62K{\sim} + 45\right)^2\left(3K{\sim}^2 + 8K{\sim} + 7\right)\Big) + \Big(2\left(K{\sim}^3 + 3K{\sim}^2 + 9K{\sim} + 11\right)\left(K{\sim}^3 + 4\sqrt{2}\sqrt{2+K{\sim}}\,K{\sim} + 3K{\sim}^2 + 8\sqrt{2}\sqrt{2+K{\sim}} + 9K{\sim} + 11\right)\Big)\Big/\Big(\left(K{\sim}^2 - 3\right)\left(K{\sim}^4 + 6K{\sim}^3 + 30K{\sim}^2 + 62K{\sim} + 45\right)\Big)$$

> *coeff*(*ymsupsing*, *XX*, 1);

$$0 \qquad \textbf{(9.3.9)}$$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by plotting the values of the coefficients:

> *coeff*(*ymsupsing*, *XX*, 3) ;

$$\Big(4\,RootOf\Big(\left(1296K{\sim}^4 + 6048K{\sim}^3 + 8928K{\sim}^2 + 3360K{\sim} - 1200\right)\_Z^2 - K{\sim}^8 - 10K{\sim}^7 - 24K{\sim}^6 + 26K{\sim}^5 + 158K{\sim}^4 + 114K{\sim}^3 - 192K{\sim}^2 - 306K{\sim} - 117\Big)(3K{\sim}+5)(K{\sim}+1)\left(K{\sim}^2 + 8K{\sim} + 13\right)\left(K{\sim}^3 + 3K{\sim}^2 + 9K{\sim} + 11\right)\left(3K{\sim}^4 + 12K{\sim}^3 + 22K{\sim}^2 + 28K{\sim} + 19\right)\left(K{\sim}^6\sqrt{2} + 6K{\sim}^5\sqrt{2} + 27K{\sim}^4\sqrt{2} + 16K{\sim}^4\sqrt{2+K{\sim}} + 108K{\sim}^3\sqrt{2} + 80K{\sim}^3\sqrt{2+K{\sim}} + 339K{\sim}^2\sqrt{2} + 240K{\sim}^2\sqrt{2+K{\sim}} + 582K{\sim}\sqrt{2} + 464\sqrt{2+K{\sim}}\,K{\sim} + 377\sqrt{2} + 352\sqrt{2+K{\sim}}\right)\Big)\Big/\Big(3\sqrt{2+K{\sim}}\left(3K{\sim}^2 + 8K{\sim} + 7\right)^3\left(K{\sim}^2 - 3\right)^2\left(K{\sim}^4 + 6K{\sim}^3 + 30K{\sim}^2 + 62K{\sim} + 45\right)^2\Big) \qquad \textbf{(9.3.10)}$$

> *fsolve*$\big(\big(6561K{\sim}^{12} + 78732K{\sim}^{11} + 409698K{\sim}^{10} + 1193292K{\sim}^9 - 81K{\sim}^8\sqrt{3} + 2091447K{\sim}^8 - 558K{\sim}^7\sqrt{3} + 2242008K{\sim}^7 - 1464K{\sim}^6\sqrt{3} + 1680540K{\sim}^6 - 1538K{\sim}^5\sqrt{3} + 1774776K{\sim}^5 + 118K{\sim}^4\sqrt{3} + 2700207K{\sim}^4 + 998K{\sim}^3\sqrt{3} + 2637660K{\sim}^3 - 1072K{\sim}^2\sqrt{3} + 1168962K{\sim}^2 - 2758K{\sim}\sqrt{3} + 36540K{\sim} - 1421\sqrt{3} - 95175\big)\big)$

$$-1.843693340,\ 0.2186477174 \qquad \textbf{(9.3.11)}$$

> *evalf*(*Kc*);

$$0.2152504369 \qquad \textbf{(9.3.12)}$$

Monochromatic simple boundary (Section 3.5)

Rational parametrization (Theorem 3.2)

The algebraic equation for Z(nu,t,t x) in terms of U

$$> eqZtUx := numer\left(factor\left(subs\left(w = wU, numer\left(factor\left(\frac{1}{w^{\frac{2}{3}}}\left(simplify\left(subs\left(y = t\cdot x, Z1 = \frac{tZ1U}{t}, Z2 = \frac{t2Z2U}{t^2}, t = w^{\frac{1}{3}}, Z = Zt, eqZ\right)\right)\right)\right)\right)\right)\right)\right) : indets(\%);$$

$$\{U, Zt, \nu, x\} \quad \textbf{(10.1.1)}$$

The parametrization for x and Z:

$$> XUW := subs(V = W, yUV\cdot QtUV);$$

$$XUW := \left(8\,\nu\,(1-2\,U)\,W\,(-\nu+1)\,\left(W^2 + \frac{2\,(5\,(\nu+1)\,U^2 - 2\,(3\,\nu+2)\,U + 2\,\nu)\,W}{U\,(U\,(\nu+1)-2)} - \frac{1}{U\,(U\,(\nu+1)-2)\,(-\nu+1)}\left(8\,(\nu+1)^2\,U^3 - (11\,\nu+13)\,(\nu+1)\,U^2 + 2\,(\nu+3)\,(2\,\nu+1)\,U - 4\,\nu\right)\right)\right) \Big/ \left((W+1)^2\,\left(8\,(\nu+1)^2\,U^3 - (11\,\nu+13)\,(\nu+1)\,U^2 + 2\,(\nu+3)\,(2\,\nu+1)\,U - 4\,\nu\right)\right) \quad \textbf{(10.1.2)}$$

$$> ZtUW := factor(subs(V = W, QtUV - 1));$$

$$ZtUW := -\left(W\left(U^4 W^4 \nu^3 + 16\,U^5 W^2 \nu^3 + U^4 W^4 \nu^2 + 19\,U^4 W^3 \nu^3 + 48\,U^5 W^2 \nu^2 + 96\,U^5 W \nu^3 - U^4 W^4 \nu + 19\,U^4 W^3 \nu^2 + 59\,U^4 W^2 \nu^3 - 4\,U^3 W^4 \nu^2 - 32\,U^3 W^3 \nu^3 + 48\,U^5 W^2 \nu + 288\,U^5 W \nu^2 - 48\,U^5 \nu^3 - U^4 W^4 - 19\,U^4 W^3 \nu - 21\,U^4 W^2 \nu^2 - 383\,U^4 W \nu^3 - 52\,U^3 W^3 \nu^2 - 296\,U^3 W^2 \nu^3 + 12\,U^2 W^3 \nu^3 + 16\,U^5 W^2 + 288\,U^5 W \nu - 144\,U^5 \nu^2 - 19\,U^4 W^3 - 219\,U^4 W^2 \nu - 927\,U^4 W \nu^2 + 88\,U^4 \nu^3 + 4\,U^3 W^4 + 32\,U^3 W^3 \nu \right.\right. \quad \textbf{(10.1.3)}$$

$$- 28 U^3 W^2 \nu^2 + 480 U^3 W \nu^3 + 4 U^2 W^4 \nu + 64 U^2 W^3 \nu^2 + 356 U^2 W^2 \nu^3 + 96 U^5 W - 144 U^5 \nu - 139 U^4 W^2 - 705 U^4 W \nu + 264 U^4 \nu^2 + 52 U^3 W^3 + 440 U^3 W^2 \nu + 1092 U^3 W \nu^2 - 56 U^3 \nu^3 - 4 U^2 W^4 - 48 U^2 W^3 \nu - 40 U^2 W^2 \nu^2 - 252 U^2 W \nu^3 - 24 U W^3 \nu^2 - 176 U W^2 \nu^3 - 48 U^5 - 161 U^4 W + 264 U^4 \nu + 172 U^3 W^2 + 704 U^3 W \nu - 176 U^3 \nu^2 - 28 U^2 W^3 - 368 U^2 W^2 \nu - 624 U^2 W \nu^2 + 12 U^2 \nu^3 + 24 U W^3 \nu + 88 U W^2 \nu^2 + 48 U W \nu^3 + 32 W^2 \nu^3 + 88 U^4 + 92 U^3 W - 184 U^3 \nu - 60 U^2 W^2 - 352 U^2 W \nu + 56 U^2 \nu^2 + 104 U W^2 \nu + 200 U W \nu^2 - 32 W^2 \nu^2 - 64 U^3 - 20 U^2 W + 60 U^2 \nu + 72 U W \nu - 8 U \nu^2 - 32 W \nu^2 + 16 U^2 - 8 U \nu)) \Big/ \left(U (U \nu + U - 2) (W + 1)^3 \left(8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4 \nu\right)\right)$$

> *simplify*(*subs*(*Zt* = *ZtUW*, *x* = *XUW*, *eqZtUx*));

$$0 \qquad \textbf{(10.1.4)}$$

Asymptotic expansion in x of Z(nu ,tnu , tnu x) (Lemma 3.16)

nu < nuc

> *XUWsub* := *factor*(*subs*(nu = *nuUsub*, *XUW*));

$$XUWsub := -\left(24 (U - 1) W \left(6 U^2 W^2 - 12 U^2 W - 6 U W^2 - 6 U^2 + 12 U W + W^2 + 10 U - 2 W - 3\right)\right) \Big/ \left((-2 + 3 U) \left(6 U^2 - 10 U + 3\right) (W + 1)^2\right) \qquad \textbf{(10.2.1.1)}$$

> *ZtUWsub* := *factor*(*subs*(nu = *nuUsub*, *ZtUW*));

$$ZtUWsub := \left(\left(18 U^3 W^4 - 162 U^3 W^3 - 30 U^2 W^4 + 198 U^3 W^2 + 294 U^2 W^3 + 15 U W^4 + 90 U^3 W - 330 U^2 W^2 - 159 U W^3 - 2 W^4 - 270 U^2 W + 161 U W^2 + 22 W^3 + 219 U W - 18 W^2 + 12 U - 46 W - 4\right) W\right) \Big/ \left((-2 + 3 U) \left(6 U^2 - 10 U + 3\right) (W + 1)^3\right) \qquad \textbf{(10.2.1.2)}$$

Critical points in W of XUWsub:

> *factor*(*diff*(*XUWsub*, *W*));

$$-\left(24 (U - 1) (W - 1) \left(6 U^2 W^2 + 24 U^2 W - 6 U W^2 + 6 U^2 - 24 U W + W^2 - 10 U + 4 W + 3\right)\right) \Big/ \left((-2 + 3 U) \left(6 U^2 - 10 U + 3\right) (W + 1)^3\right) \qquad \textbf{(10.2.1.3)}$$

The critical point corresponding to the radius of convergence will be W=1. We want to rule out the roots of the polynomial of degree 2:

> *BadPol* := *collect*$\left(6 U^2 W^2 + 24 U^2 W - 6 U W^2 + 6 U^2 - 24 U W + W^2 - 10 U + 4 W + 3, W, factor\right)$; *Wsub1*, *Wsub2* := *solve*(*BadPol*, *W*);

$$BadPol := (6U^2-6U+1)W^2+(24U^2-24U+4)W+6U^2-10U+3$$

$$Wsub1, Wsub2 := \frac{-12U^2+\sqrt{108U^4-192U^3+108U^2-20U+1}+12U-2}{6U^2-6U+1}, -\frac{12U^2+\sqrt{108U^4-192U^3+108U^2-20U+1}-12U+2}{6U^2-6U+1} \quad \textbf{(10.2.1.4)}$$

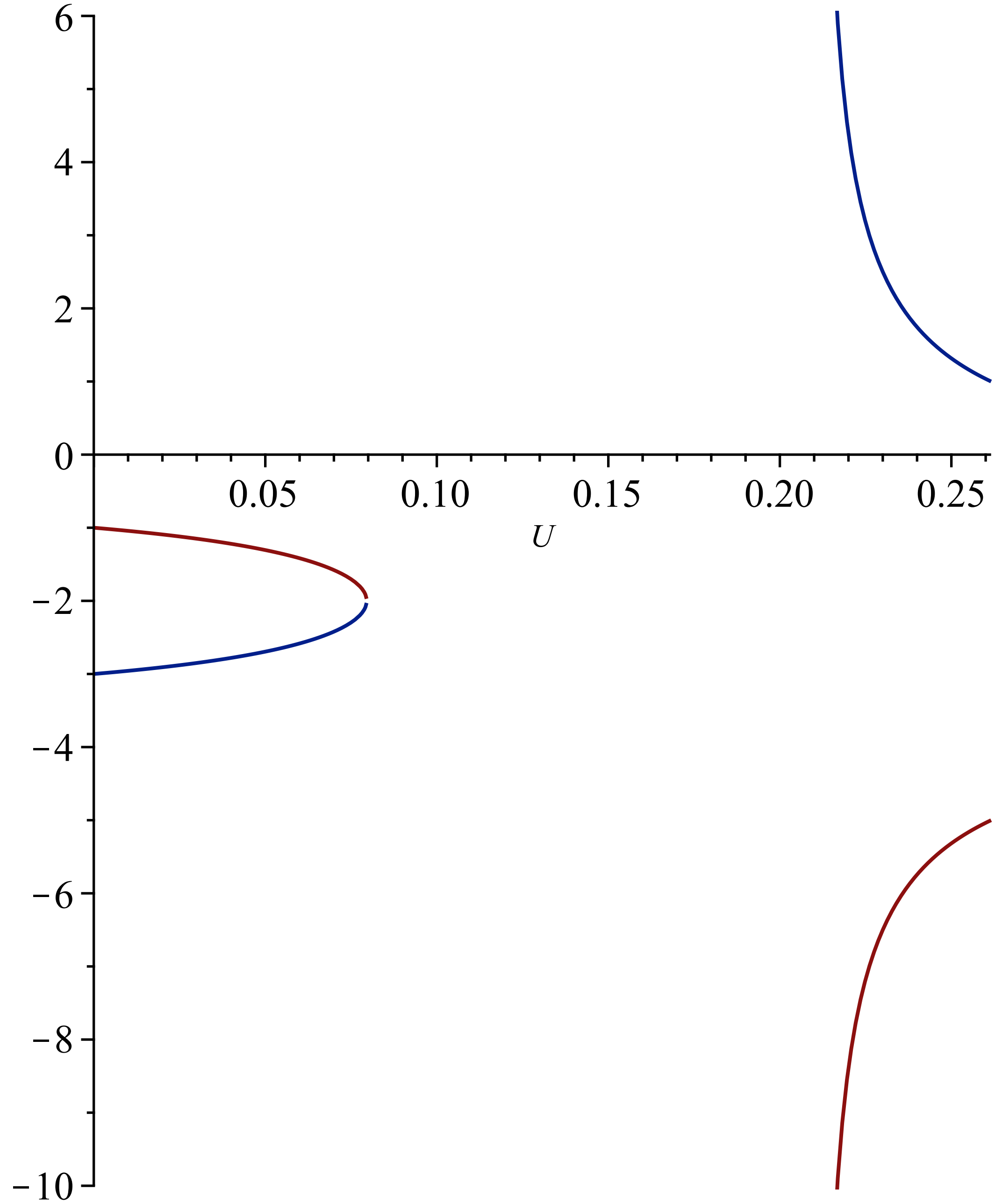
We first have to check that these two roots are never in [0,1]. (see Chen-Turunen prop 21)

```
> factor(discrim(BadPol, W));fsolve(%);
```

$$4(18U^2-14U+1)(6U^2-6U+1)$$

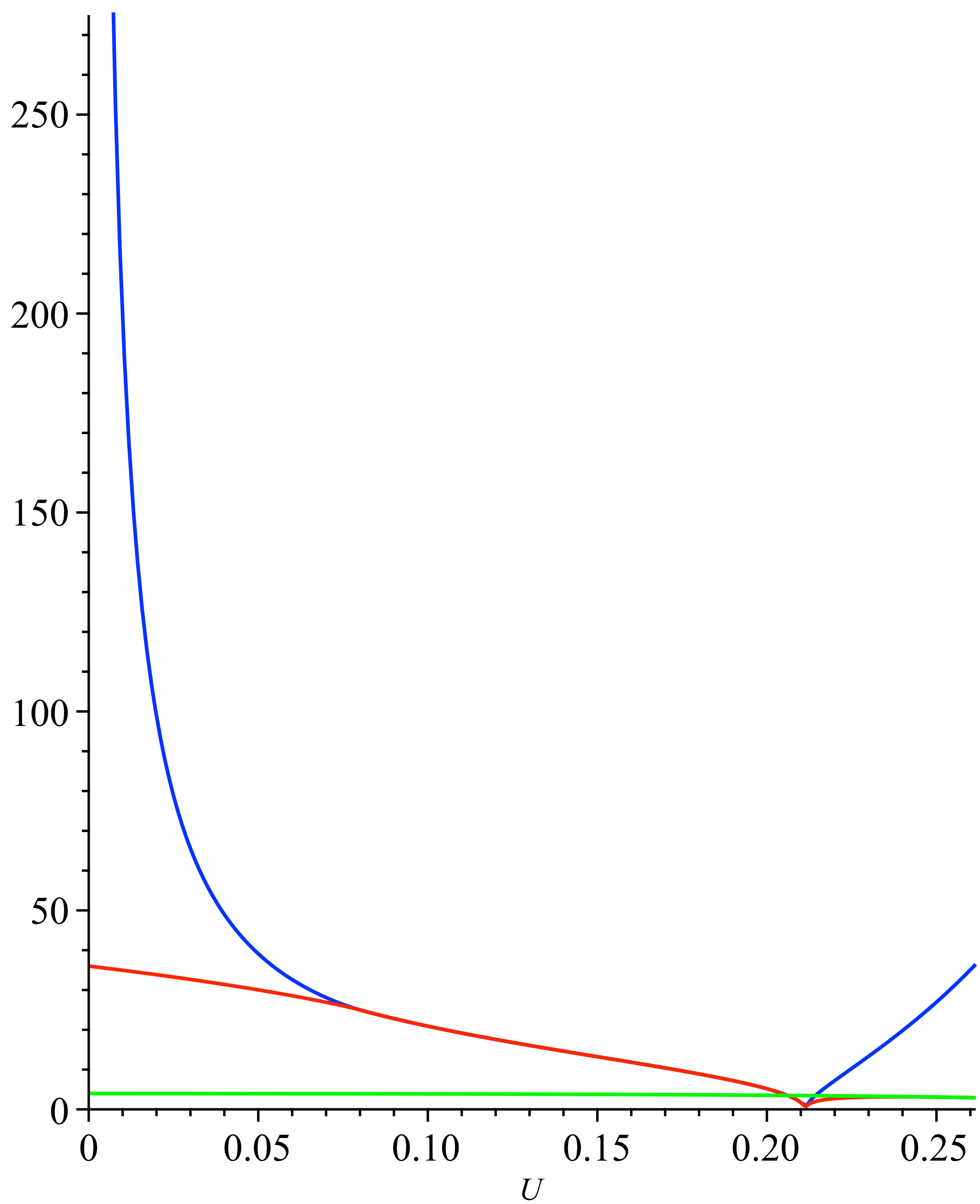
$$0.07956864651, 0.2113248654, 0.6982091313, 0.7886751346 \quad \textbf{(10.2.1.5)}$$

```
> plot({Wsub1, Wsub2}, U = 0 ..Uc);
```

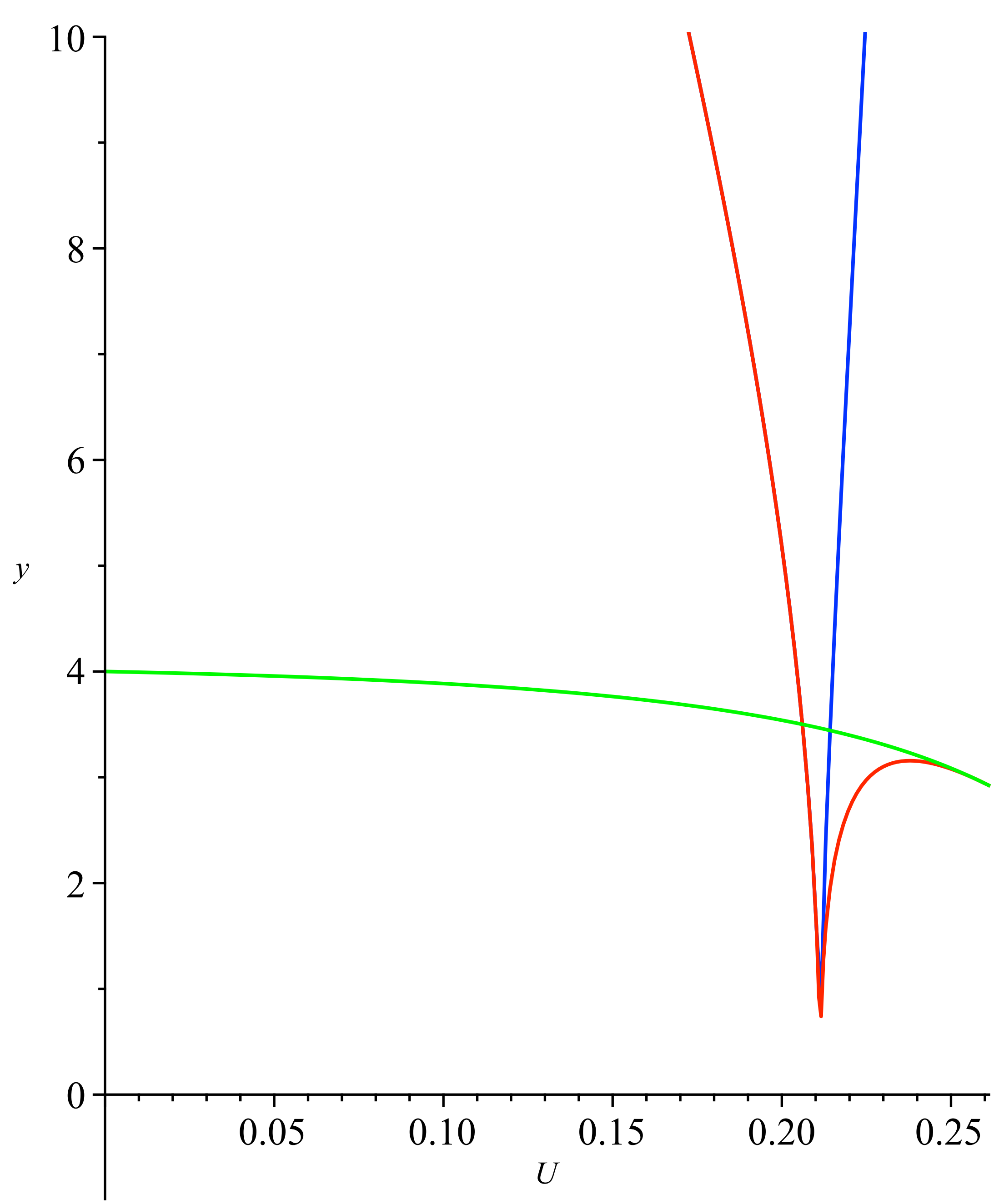


The radius of convergence is indeed given by W = 1. We then have to check that the roots of the polynomial of degree 2 don't give other dominant singularities. We can directly excluse them when they are real since none of them are in (-1,1) in this case ans \hat X is bijective on this interval. When the roots are imaginary, only one value of U gives |\hat X(Wi)| = \hat X(1):

```
> plot([abs(subs(W = Wsub1, XUWsub)), abs(subs(W = Wsub2, XUWsub)), subs(W
    = 1 , XUWsub)], U = 0 ..Uc, color = [blue, red, green]); plot({abs(subs(W
    = Wsub1, XUWsub)), abs(subs(W = Wsub2, XUWsub)), subs(W = 1 ,
   XUWsub)}, U = 0 ..Uc, y = -1 ..10, color = [blue, red, green]);
```



250
200
150
100
50
0
0
0.05
0.10
0.15
0.20
0.25
U



We cannot compute explicitely the corresponding value of U

> $factor\Big(rationalize\Big(factor\Big(subs(W = Wsub1, XUWsub) \cdot subs(W = Wsub2, XUWsub) - subs(W = 1, XUWsub)^2\Big)\Big)\Big); fsolve\big(243\,U^6 - 1026\,U^5 + 1686\,U^4 - 1364\,U^3 + 569\,U^2 - 116\,U + 9\big);$

$$-\frac{1}{U\,(-2+3\,U)^3\,(6\,U^2-10\,U+3)^2}\left(576\,\left(243\,U^6-1026\,U^5+1686\,U^4-1364\,U^3+569\,U^2-116\,U+9\right)\,(U-1)^2\right)$$

$$0.2060759672,\ 0.7835199713 \qquad \textbf{(10.2.1.6)}$$

Fortunately, we can still prove that Q is non singular at the corresponding values of \hat X when U is close to 0.2. We start by computing the values of \hat X:

> $Xsubbad1 := factor(expand(rationalize(factor(subs(W = Wsub1, XUWsub)))));$
$Xsubbad2 := factor(expand(rationalize(factor(subs(W = Wsub2, XUWsub)))));$

$$Xsubbad1 := -\frac{1}{U(-2+3U)^2(6U^2-10U+3)}\Big(12(U-1)\Big(-180U^4 + 18\sqrt{(18U^2-14U+1)(6U^2-6U+1)}\,U^2 + 288U^3 - 14\sqrt{(18U^2-14U+1)(6U^2-6U+1)}\,U - 132U^2 + \sqrt{(18U^2-14U+1)(6U^2-6U+1)} + 12U + 1\Big)\Big)$$

$$Xsubbad2 := \frac{1}{U(-2+3U)^2(6U^2-10U+3)}\Big(12(U-1)\Big(180U^4 + 18\sqrt{(18U^2-14U+1)(6U^2-6U+1)}\,U^2 - 288U^3 - 14\sqrt{(18U^2-14U+1)(6U^2-6U+1)}\,U + 132U^2 + \sqrt{(18U^2-14U+1)(6U^2-6U+1)} - 12U - 1\Big)\Big) \quad \textbf{(10.2.1.7)}$$

And we directly calculate the development of Z at these values (we only do one, the other is the complex conjugate).

> $eqZtUxsub := op(5, factor(subs(\mathrm{nu} = nuUsub, eqZtUx))) : indets(\%);$

$$\{U, Zt, x\} \quad \textbf{(10.2.1.8)}$$

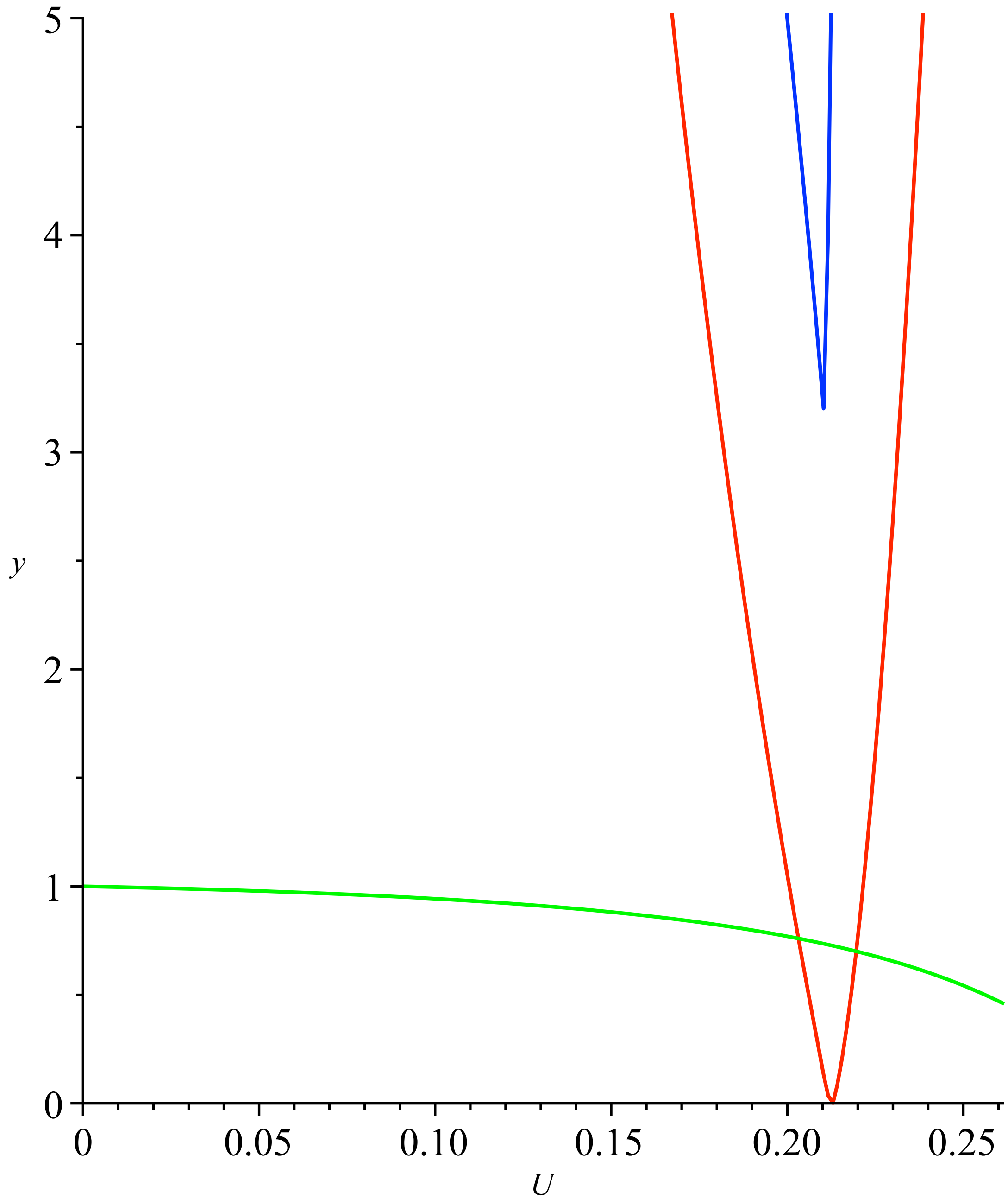
> $algeqtoseries(subs(x = Xsubbad1 \cdot (1 - XX), eqZtUxsub), XX, Zt, 1);$

$$\Big[\Big(-12528U^6 + 1188\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^4 + 31320U^5 - 1896\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^3 - 29268U^4 + 924\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^2 + 12456U^3 - 128\sqrt{108U^4-192U^3+108U^2-20U+1}\,U - 2328U^2 + \sqrt{108U^4-192U^3+108U^2-20U+1} + 150U + 1\Big)\Big/\Big(2(54U^4 - 162U^3 + 171U^2 - 76U + 12)U^2\Big) + \mathrm{O}(XX), -\Big(-3942U^6 + 378\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^4 + 9450U^5 - 564\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^3 - 8091U^4 + 216\sqrt{108U^4-192U^3+108U^2-20U+1}\,U^2 + 2868U^3 \quad \textbf{(10.2.1.9)}$$

$$- 2\sqrt{108\,U^4 - 192\,U^3 + 108\,U^2 - 20\,U + 1}\;U - 360\,U^2$$
$$- 2\sqrt{108\,U^4 - 192\,U^3 + 108\,U^2 - 20\,U + 1} + 18\,U - 2\Big) \Big/ \Big(\big(54\,U^4$$
$$- 162\,U^3 + 171\,U^2 - 76\,U + 12\big)\,U^2\Big) + O\big(\sqrt{XX}\big)\Big]$$

To decide which branch is the right one: if |\hat X| = \hat X(1), the the modulus of Zt at this value of x has to be smaller than the value at the radius of convergence (the series Zt has positive coefficients). We see that the right branch is the first one, which is non singular.

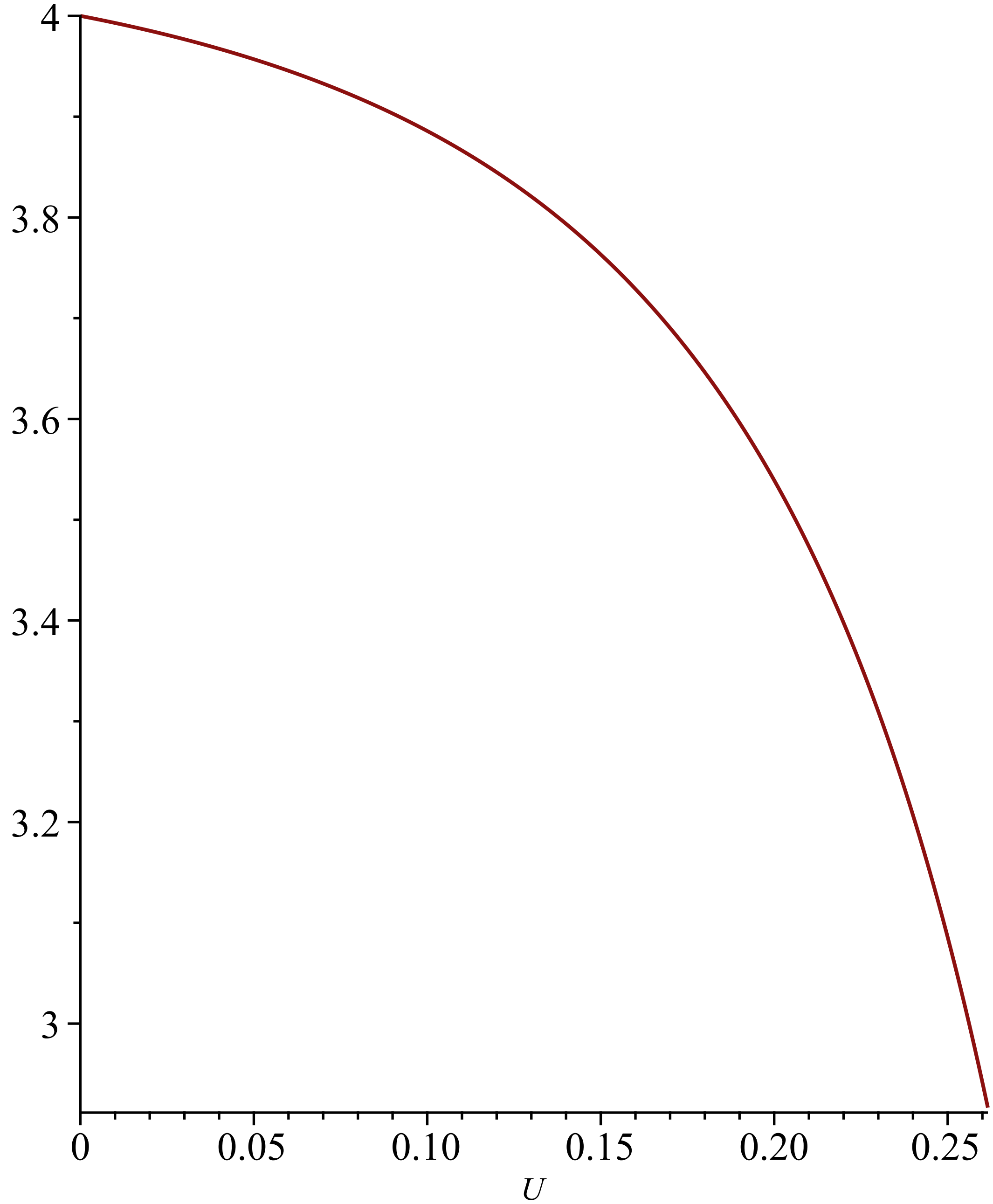
```
> plot([abs(convert(op(1, (10.2.1.9)), polynom)), abs(convert(op(2, (10.2.1.9)),
    polynom)), subs(W = 1, ZtUWsub)], U = 0..Uc, y = 0..5, color = [red, blue,
    green]);
```



Now we can produce the expansion of Zt at its unique dominant singularity:

> $xcritsub := factor(subs(W = 1, XUWsub)); plot(\%, U = 0..Uc);$

$$xcritsub := \frac{24\,(3\,U-1)\,(U-1)^2}{(-2+3\,U)\,(6\,U^2-10\,U+3)}$$



> $algeqtoseries(numer(xcritsub \cdot (1-XX) - XUWsub), XX, W, 2)$

$$\left[-\frac{3\,U^2-4\,U+1}{6\,U^2-6\,U+1} + \frac{(9\,U^2-12\,U+4)\,U^2\,(3\,U^2-4\,U+1)}{(81\,U^4-180\,U^3+136\,U^2-40\,U+4)\,(6\,U^2-6\,U+1)}\,XX + \right. \qquad \textbf{(10.2.1.10)}$$

$$\mathrm{O}\left(XX^2\right), 1 + RootOf\left(\left(9\,U^2 - 10\,U + 2\right)\_Z^2 - 12\,U^2 + 16\,U - 4\right)\sqrt{XX} + \mathrm{O}(XX)\Big]$$

> $Wsersubc := convert(op(2, algeqtoseries(numer(xcritsub \cdot (1 - XX) - XUWsub), XX, W, 4)), polynom);$

$$Wsersubc := 1 + RootOf\left(\left(9\,U^2 - 10\,U + 2\right)\_Z^2 - 12\,U^2 + 16\,U - 4\right)\sqrt{XX} + \frac{2\,(3\,U-1)\left(3\,U^3 - 7\,U^2 + 5\,U - 1\right)XX}{\left(9\,U^2 - 10\,U + 2\right)^2} + \frac{1}{2\left(9\,U^2 - 10\,U + 2\right)^3}\left(RootOf\left(\left(9\,U^2 - 10\,U + 2\right)\_Z^2 - 12\,U^2 + 16\,U - 4\right)\left(54\,U^6 - 180\,U^5 + 240\,U^4 - 164\,U^3 + 61\,U^2 - 12\,U + 1\right)XX^{3/2}\right) \quad \textbf{(10.2.1.11)}$$

> $Zplusubcser := collect(expand(rationalize(simplify(convert(series(subs(W = Wsersubc, ZtUWsub), XX, 2), polynom)))), XX, factor);$

$$Zplusubcser := -\left(2\,(3\,U-1)^2\,(U-1)^2\,RootOf\left(\left(9\,U^2 - 10\,U + 2\right)\_Z^2 - 12\,U^2 + 16\,U - 4\right)XX^{3/2}\right)\Big/\left(\left(9\,U^2 - 10\,U + 2\right)\left(6\,U^2 - 10\,U + 3\right)\right) - \frac{12\,(3\,U-1)\,(U-1)^2\,XX}{(-2 + 3\,U)\left(6\,U^2 - 10\,U + 3\right)} + \frac{18\,U^3 - 42\,U^2 + 31\,U - 6}{(-2 + 3\,U)\left(6\,U^2 - 10\,U + 3\right)} \quad \textbf{(10.2.1.12)}$$

The corresponding mean for mu:

> $factor\left(\frac{12\,(U-1)^2\,(3\,U-1)}{(3\,U-2)\left(6\,U^2 - 10\,U + 3\right)} \cdot \frac{1}{1 + \frac{18\,U^3 - 42\,U^2 + 31\,U - 6}{(3\,U-2)\left(6\,U^2 - 10\,U + 3\right)}}\right);$

$$1 \quad \textbf{(10.2.1.13)}$$

Expansion of the singular term with NN = (nuc -nu):

> $algeqtoseries\left(subs\left(U = Uc - UU, \mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7} - NN, numer(nuUsub - \mathrm{nu})\right), NN, UU, 1\right);$

$$\left[\frac{1}{9} - \frac{2\sqrt{7}}{9} + \mathrm{O}(NN), \left(-\frac{7}{243} + \frac{14\sqrt{7}}{243}\right)NN + \mathrm{O}\left(NN^2\right)\right] \quad \textbf{(10.2.1.14)}$$

> $factor\left(expand\left(rationalize\left(simplify\left(convert\left(series\left(subs\left(U = Uc - \left(-\frac{7}{243}\right.\right.\right.\right.\right.\right.\right.$

$$+ \frac{14\sqrt{7}}{243}\Big) NN,$$

$$-\frac{1}{(9U^2-10U+2)(6U^2-10U+3)}\Big(2(3U-1)^2(U-1)^2 \mathit{RootOf}\big((9U^2-10U+2)\_Z^2-12U^2+16U-4\big)\Big), NN, 2\Big),$$

$$\mathit{polynom}\Big)\Big)\Big)\Big)\Big);$$

$$\frac{\sqrt{-392+406\sqrt{7}}\,(2\sqrt{7}+1)}{1715\, NN^{3/2}} \qquad \textbf{(10.2.1.15)}$$

nu = nuc

> $xcritcrit := expand(rationalize(subs(U = Uc, xcritsub)));$

$$xcritcrit := \frac{4}{5} + \frac{4\sqrt{7}}{5} \qquad \textbf{(10.2.2.1)}$$

with XX = 1-x/xcrit

> $algeqtoseries(numer(\ xcritcrit \cdot (1 - XX) - subs(U = Uc, XUWsub)), XX, W, 5);$

$$\left[1 + \mathit{RootOf}(\_Z^3+4)\, XX^{1/3} + \frac{\mathit{RootOf}(\_Z^3+4)^2\, XX^{2/3}}{3} - \frac{XX}{3} - \frac{5\, \mathit{RootOf}(\_Z^3+4)\, XX^{4/3}}{81} + \mathrm{O}(XX^{5/3})\right] \qquad \textbf{(10.2.2.2)}$$

> $Wsercrit := convert(op(1, algeqtoseries(numer(\ xcritcrit \cdot (1 - XX) - subs(U = Uc, XUWsub)), XX, W, 5)), polynom);$

$$Wsercrit := 1 + \mathit{RootOf}(\_Z^3+4)\, XX^{1/3} + \frac{\mathit{RootOf}(\_Z^3+4)^2\, XX^{2/3}}{3} - \frac{XX}{3} - \frac{5\, \mathit{RootOf}(\_Z^3+4)\, XX^{4/3}}{81} \qquad \textbf{(10.2.2.3)}$$

> $expand(rationalize(simplify(convert(series(subs(W = Wsercrit, U = Uc, ZtUWsub), XX, 2), polynom))));$

$$\frac{2\, XX^{5/3}\, \mathit{RootOf}(\_Z^3+4)^2}{15} - \frac{\mathit{RootOf}(\_Z^3+4)\, XX^{4/3}}{5} - \frac{2\sqrt{7}\, XX}{5} + \frac{2\sqrt{7}}{5} - \frac{3}{5} \qquad \textbf{(10.2.2.4)}$$

The corresponding mean for the offspring distribution mu:

> $simplify\left(\frac{2\sqrt{7}}{5} \cdot \frac{1}{1 + \frac{2\sqrt{7}}{5} - \frac{3}{5}}\right); rationalize(expand(\%));$

$$\frac{\sqrt{7}}{1+\sqrt{7}}$$

$$\frac{\sqrt{7}\left(-1+\sqrt{7}\right)}{6} \qquad \textbf{(10.2.2.5)}$$

nu > nuc

> $XUWsupc := factor(subs(\mathrm{nu} = nusupK, U = UsupK, XUW));$

$$XUWsupc := -\left(8\left(K\sim^3+3K\sim^2+9K\sim+11\right)(K\sim+1)W\left(K\sim^4W^2-2K\sim^4W-K\sim^4-24K\sim^3W-6K\sim^2W^2-8K\sim^3-68K\sim^2W-10K\sim^2-56K\sim W+9W^2+24K\sim-2W+39\right)\right)\Big/\left(\left(K\sim^2+8K\sim+13\right)(W+1)^2\left(K\sim^2-3\right)^3\right) \qquad \textbf{(10.2.3.1)}$$

We start by locating the critical points of \hat X:

> $factor(diff(XUWsupc, W));$

$$-\left(8(K\sim+1)\left(K\sim^3+3K\sim^2+9K\sim+11\right)\left(K\sim^2W-K\sim^2-8K\sim-3W-13\right)\left(K\sim^2W^2+4K\sim^2W+K\sim^2+8K\sim W-3W^2+4W-3\right)\right)\Big/\left(\left(K\sim^2+8K\sim+13\right)(W+1)^3\left(K\sim^2-3\right)^3\right) \qquad \textbf{(10.2.3.2)}$$

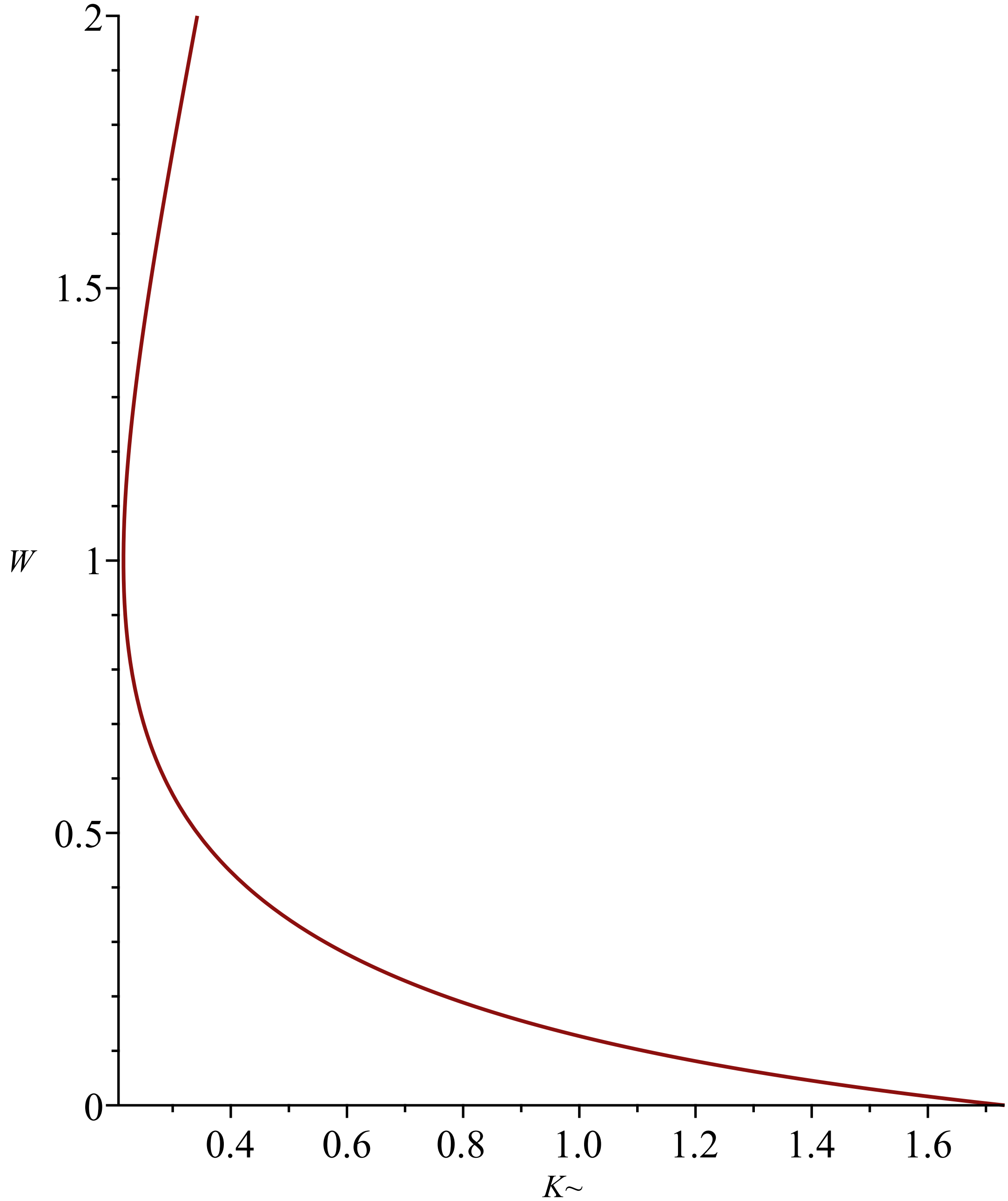
The root of the polynomial of degree 1 is < -1 (which we recall is the pole of \hat X):

> $factor\left(solve\left(K^2W-K^2-8K-3W-13, W\right)+1\right);$

$$\frac{2\left(K\sim^2+4K\sim+5\right)}{K\sim^2-3} \qquad \textbf{(10.2.3.3)}$$

The roots of the polynomial of degree 2 are positive and the smallest gives the radius of convergence. There is no other non real singularity.

> $implicitplot\left(\left(\left(K^2W^2+4K^2W+K^2+8KW-3W^2+4W-3\right)\right), K = Kc..Kinfini, W = -2..2\right);$



> $WKsupccrit := -\frac{2K^2+4K-\sqrt{(K^2+4K+5)(3K^2+4K-1)}+2}{K^2-3}$:

> $simplify\left(subs\left(W = WKsupccrit, \left(\left(K^2 W^2 + 4K^2 W + K^2 + 8KW - 3W^2 + 4W - 3\right)\right)\right)\right);$

$$0 \tag{10.2.3.4}$$

The corresponding radius of convergence:

> $XWsupccrit := simplify(subs(W = WKsupccrit, XUWsupc), symbolic);$

$$XWsupccrit := \Big(16\left(K\sim^3 + 3K\sim^2 + 9K\sim + 11\right)\left(\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 2\left(K\sim+1\right)^2\right)\big(K\sim^2 \tag{10.2.3.5}$$

$$+ 4 K\sim + 5) (K\sim + 1) \Big((3 K\sim^2 + 4 K\sim - 1)^{3/2} \sqrt{K\sim^2 + 4 K\sim + 5}$$

$$- 5 K\sim^4 - 20 K\sim^3 - 26 K\sim^2 - 4 K\sim + 11\Big)\Big) \Big/ \Big(\big(-K\sim^2 - 4 K\sim$$

$$+ \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 5\big)^2 (K\sim^2 + 8 K\sim$$

$$+ 13) (K\sim^2 - 3)^3\Big)$$

We compute the development of W at the radius of convergence. (XX = (1-x/xc)^1/2

> $simplify(algeqtoseries(numer(XWsupccrit \cdot (1 - XX^2) - XUWsupc), XX, W, 4))$;

$$\Big[\frac{1}{(K\sim^2 - 3)^2} \Big((K\sim^2 + 4 K\sim + 5)^{3/2} \sqrt{3 K\sim^2 + 4 K\sim - 1} + K\sim^4 + 12 K\sim^3$$

$$+ 34 K\sim^2 + 28 K\sim + 1\Big) + 44 \Bigg(\Bigg(\sqrt{K\sim^2 + 4 K\sim + 5}\, (K\sim$$

$$+ 1)^2 \sqrt{3 K\sim^2 + 4 K\sim - 1} + \frac{7 K\sim^4}{4} + 8 K\sim^3 + \frac{27 K\sim^2}{2} + 8 K\sim - \frac{1}{4}\Bigg)$$

$$(K\sim^2 + 4 K\sim + 5) \Bigg(\frac{1}{11} \Big(\sqrt{3 K\sim^2 + 4 K\sim - 1}\, (11 K\sim^4 + 40 K\sim^3$$

$$+ 46 K\sim^2 + 8 K\sim - 13) \sqrt{K\sim^2 + 4 K\sim + 5}\Big) - \frac{19 K\sim^6}{11} - \frac{120 K\sim^5}{11}$$

$$- \frac{293 K\sim^4}{11} - \frac{304 K\sim^3}{11} - \frac{65 K\sim^2}{11} + \frac{72 K\sim}{11} + \frac{17}{11}\Bigg)\Bigg) \Bigg/ \Big((K\sim^2$$

$$- 3)^4 (3 K\sim^2 + 4 K\sim - 1) \Big(3 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1}$$

$$- 6\,K\sim^2 - 20\,K\sim - 22\big)\big)\,XX^2 + O\big(XX^4\big),\ \Big(27\,\Big(K\sim^2 + \frac{16}{3}\,K\sim + \frac{23}{3}\Big)^2\,\big(K\sim^2 - 3\big)^2\,\Big(K\sim^2 +$$

$$+ 936\,K\sim^5 + 3474\,K\sim^4 + 432\,K\sim^3 - 4131\,K\sim^2 - 2052\,K\sim + 621\big)$$

$$+ 114\,K\sim^{10} - 66\,K\sim^8\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 1592\,K\sim^9$$

$$- 744\,K\sim^7\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 9634\,K\sim^8$$

$$- 3480\,K\sim^6\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 32608\,K\sim^7$$

$$- 8536\,K\sim^5\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 66164\,K\sim^6$$

$$- 11228\,K\sim^4\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 78672\,K\sim^5$$

$$- 6520\,K\sim^3\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 45700\,K\sim^4$$

$$+ 520\,K\sim^2\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 672\,K\sim^3$$

$$+ 1784\,K\sim\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 13830\,K\sim^2$$

$$+ 142\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 3272\,K\sim + 1210\big)$$

$$- 39\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\Big(\Big(XX^2 - \frac{9}{13}\Big)\,K\sim^8 + \Big(\frac{524\,XX^2}{39} - \frac{108}{13}\Big)\,K\sim^7$$

$$+ \Big(\frac{972\,XX^2}{13} - \frac{492}{13}\Big)\,K\sim^6 + \Big(\frac{8468\,XX^2}{39} - \frac{2716}{39}\Big)\,K\sim^5$$

$$+ \Big(\frac{13090\,XX^2}{39} + \frac{178}{13}\Big)\,K\sim^4 + \Big(\frac{9028\,XX^2}{39} + \frac{3228}{13}\Big)\,K\sim^3 + \Big($$

$$-\frac{68\,XX^2}{13} + \frac{12940}{39}\Big)\,K\sim^2 + \Big(-\frac{212\,XX^2}{3} + \frac{1380}{13}\Big)\,K\sim - \frac{433\,XX^2}{39}$$

$$- \frac{529}{13}\Big)\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 27\,\Big(K\sim^2 + \frac{16}{3}\,K\sim + \frac{23}{3}\Big)^2\,\big(K\sim^2$$

$$- 3\big)^2\,\Big(K\sim^2 + \frac{4}{3}\,K\sim - \frac{1}{3}\Big)\,O\big(XX^{5/2}\big) + \big(69\,XX^2 - 54\big)\,K\sim^{10}$$

$$+ \big(1100\,XX^2 - 756\big)\,K\sim^9 + \big(7581\,XX^2 - 4302\big)\,K\sim^8 + \big(29232\,XX^2$$

$$- 11984) K\sim^7 + (67794\, XX^2 - 12748) K\sim^6 + (92872\, XX^2 + 16072) K\sim^5$$
$$+ (63418\, XX^2 + 65684) K\sim^4 + (1072\, XX^2 + 79408) K\sim^3 + (-23527\, XX^2$$
$$+ 39266) K\sim^2 + (-5300\, XX^2 + 1932) K\sim + 4057\, XX^2 - 3174) \Big/$$
$$\left(27 \left(K\sim^2 + \frac{16}{3} K\sim + \frac{23}{3}\right)^2 (K\sim^2 - 3)^2 \left(K\sim^2 + \frac{4}{3} K\sim - \frac{1}{3}\right)\right)\Bigg]$$

The singular branch is the second one:

> $devWsupc := collect(convert(op(2, simplify(algeqtoseries(numer(XWsupccrit \cdot (1 - XX^2) - XUWsupc), XX, W, 6))), polynom), XX, factor); degree(\%, XX);$

$$devWsupc := -((-224605 + 57053\, K\sim^{12} + 374712\, K\sim^{11} + 1600211\, K\sim^{10}$$
$$- 1460700\, K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 285028\, K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 346046\, K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 161028\, K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 207\, K\sim^{14}$$
$$+ 5124\, K\sim^{13} + 4606300\, K\sim^9 + 8898169\, K\sim^8 + 10895504\, K\sim^7$$
$$- 489861\, K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 1051512\, K\sim^7 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 150316\, K\sim^9 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 3372\, K\sim^{11} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 29506\, K\sim^{10} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 171\, K\sim^{12} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 568996\, K\sim$$
$$- 1194264\, K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 355845\, K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 37589 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 655928\, K\sim^3$$
$$+ 2157313\, K\sim^2 + 6883741\, K\sim^6 - 168324\, K\sim^5 - 2447289\, K\sim^4)$$
$$RootOf(\_Z^2\, (9\, K\sim^{10} + 60\, K\sim^9 + 49\, K\sim^8 - 464\, K\sim^7 - 950\, K\sim^6 + 936\, K\sim^5$$
$$+ 3474\, K\sim^4 + 432\, K\sim^3 - 4131\, K\sim^2 - 2052\, K\sim + 621) + 114\, K\sim^{10}$$
$$- 66\, K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1592\, K\sim^9$$

$$- 744\, K\sim^{7} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 9634\, K\sim^{8}$$
$$- 3480\, K\sim^{6} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 32608\, K\sim^{7}$$
$$- 8536\, K\sim^{5} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 66164\, K\sim^{6}$$
$$- 11228\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 78672\, K\sim^{5}$$
$$- 6520\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 45700\, K\sim^{4}$$
$$+ 520\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 672\, K\sim^{3}$$
$$+ 1784\, K\sim \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 13830\, K\sim^{2}$$
$$+ 142 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 3272\, K\sim + 1210\Big)\, XX^{3}\Big) \Big/$$
$$\Big(4\, (K\sim^{2} - 3)\, (3\, K\sim^{2} + 16\, K\sim + 23)^{3}\, (3\, K\sim^{2} + 4\, K\sim - 1)^{2}\, (K\sim^{2} + 4\, K\sim$$
$$+ 5)\Big) + \Big(\Big(69\, K\sim^{10} - 39\, K\sim^{8} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 1100\, K\sim^{9} - 524\, K\sim^{7} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 7581\, K\sim^{8} - 2916\, K\sim^{6} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 29232\, K\sim^{7} - 8468\, K\sim^{5} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 67794\, K\sim^{6} - 13090\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 92872\, K\sim^{5} - 9028\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 63418\, K\sim^{4} + 204\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 1072\, K\sim^{3} + 2756\, K\sim \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$- 23527\, K\sim^{2} + 433 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 5300\, K\sim$$
$$+ 4057\Big)\, XX^{2}\Big) \Big/ \Big(\Big((3\, K\sim^{2} + 16\, K\sim + 23)^{2}\, (3\, K\sim^{2} + 4\, K\sim - 1)\, \big(K\sim^{2}$$
$$- 3\big)^{2}\Big) + \mathit{RootOf}\Big(\_Z^{2}\, (9\, K\sim^{10} + 60\, K\sim^{9} + 49\, K\sim^{8} - 464\, K\sim^{7} - 950\, K\sim^{6}$$
$$+ 936\, K\sim^{5} + 3474\, K\sim^{4} + 432\, K\sim^{3} - 4131\, K\sim^{2} - 2052\, K\sim + 621)$$
$$+ 114\, K\sim^{10} - 66\, K\sim^{8} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 1592\, K\sim^{9}$$
$$- 744\, K\sim^{7} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 9634\, K\sim^{8}$$
$$- 3480\, K\sim^{6} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 32608\, K\sim^{7}$$
$$- 8536\, K\sim^{5} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 66164\, K\sim^{6}$$
$$- 11228\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 78672\, K\sim^{5}$$
$$- 6520\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 45700\, K\sim^{4}$$

$$+ 520\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 672\,K\sim^3$$
$$+ 1784\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 13830\,K\sim^2$$
$$+ 142 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 3272\,K\sim + 1210\Big)\,XX$$
$$- \frac{2\,K\sim^2 + 4\,K\sim - \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 2}{K\sim^2 - 3}$$
$$3 \qquad \textbf{(10.2.3.7)}$$

> *Zplussupcser* := *collect*(*expand*(*rationalize*(*convert*(*simplify*(*series*(*subs*(*W* = *devWsupc*, *subs*(nu = *nusupK*, *U* = *UsupK*, *ZtUW*)), *XX*, 4)), *polynom*))), *XX*, *factor*); *degree*(%, *XX*);

$$Zplussupcser := \Big(8\,(K\sim^2 + 4\,K\sim + 5)\,\Big(3\,K\sim^6 + 40\,K\sim^5$$
$$+ 4\,K\sim^3 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 213\,K\sim^4$$
$$+ 32\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 560\,K\sim^3$$
$$+ 84\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 721\,K\sim^2$$
$$+ 72 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 360\,K\sim - 1\Big)$$
$$RootOf\Big(\_Z^2\,(9\,K\sim^{10} + 60\,K\sim^9 + 49\,K\sim^8 - 464\,K\sim^7 - 950\,K\sim^6 + 936\,K\sim^5$$
$$+ 3474\,K\sim^4 + 432\,K\sim^3 - 4131\,K\sim^2 - 2052\,K\sim + 621) + 114\,K\sim^{10}$$
$$- 66\,K\sim^8 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 1592\,K\sim^9$$
$$- 744\,K\sim^7 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 9634\,K\sim^8$$
$$- 3480\,K\sim^6 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 32608\,K\sim^7$$
$$- 8536\,K\sim^5 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 66164\,K\sim^6$$
$$- 11228\,K\sim^4 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 78672\,K\sim^5$$
$$- 6520\,K\sim^3 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 45700\,K\sim^4$$
$$+ 520\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 672\,K\sim^3$$
$$+ 1784\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 13830\,K\sim^2$$
$$+ 142 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 3272\,K\sim + 1210\Big)\,XX^3\Big) \Big/$$
$$\Big((K\sim^2 - 3)\,(K\sim^2 + 8\,K\sim + 13)\,(3\,K\sim^2 + 16\,K\sim + 23)^2\Big)$$
$$- \Big(2\,\Big(183\,K\sim^{10} - 105\,K\sim^8 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 2440\,K\sim^9 - 1120\,K\sim^7 \sqrt{}$$
$$- 11984\,K\sim^5 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 93606\,K\sim^6$$
$$- 16354\,K\sim^4 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 116608\,K\sim^5$$

$$- 11776\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 84070\, K\sim^{4}$$
$$- 2576\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 25456\, K\sim^{3}$$
$$+ 1392\, K\sim \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 8045\, K\sim^{2}$$
$$+ 435 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 10248\, K\sim - 3273\Big)\, XX^{2}\Big)$$
$$\Big/ \Big(\big(3\, K\sim^{2} + 16\, K\sim + 23\big) \big(K\sim^{2} + 8\, K\sim + 13\big) \big(K\sim^{2} - 3\big)^{3} \Big) + \Big($$
$$-5212350 - 55118056\, K\sim^{12} + 116275392\, K\sim^{11} + 269553168\, K\sim^{10}$$
$$- 5569280751\, K\sim^{6} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 63446596\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 75376799\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 6251410\, K\sim \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 405350\, \big(K\sim^{2}$$
$$+ 4\, K\sim + 5\big)^{3/2} \big(3\, K\sim^{2} + 4\, K\sim - 1\big)^{3/2} - 1158360\, K\sim^{18} + 1348704\, K\sim^{17}$$
$$+ 9795262\, K\sim^{16} + 12563776\, K\sim^{15} - 18628784\, K\sim^{14} - 74766528\, K\sim^{13}$$
$$+ 146\, K\sim^{24} + 2336\, K\sim^{23} + 14568\, K\sim^{22} + 34336\, K\sim^{21} + 107911872\, K\sim^{9}$$
$$- 298256850\, K\sim^{8} - 455494752\, K\sim^{7}$$
$$+ 121675125666\, K\sim^{14} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 75985384274\, K\sim^{8} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 14360027568\, K\sim^{7} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 176278975876\, K\sim^{9} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 319697376664\, K\sim^{11} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 274849334678\, K\sim^{10} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 209374781412\, K\sim^{13} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$+ 289880536806\, K\sim^{12} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$- 22044960\, K\sim - 63756\, K\sim^{20} - 595104\, K\sim^{19}$$
$$- 4753091686\, K\sim^{5} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$- 1152681213\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1}$$
$$- 1155275 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 133786080\, K\sim^{3}$$
$$- 5406264\, K\sim^{2} - 135842616\, K\sim^{6} + 263816352\, K\sim^{5} + 323169588\, K\sim^{4}$$

$$+ 92541 K\sim^{22} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 2488834 K\sim^{21} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 31903335 K\sim^{20} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 258673604 K\sim^{19} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 1484891035 K\sim^{18} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 6401006954 K\sim^{17} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 1495 K\sim^{14} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 23038 K\sim^{13} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 164537 K\sim^{12} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 382507 K\sim^{4} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$+ 98655 K\sim^{2} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 26390 K\sim^{18} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 572284 K\sim^{17} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 5872022 K\sim^{16} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 37754304 K\sim^{15} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 169865240 K\sim^{14} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 565728496 K\sim^{13} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 6650365 K\sim^{8} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 5298341 K\sim^{6} \left(K\sim^2 + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{5/2}$$
$$- 1432734776 K\sim^{6} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 312185520 K\sim^{5} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$+ 61554600 K\sim^{4} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$+ 63731584 K\sim^{3} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$+ 14858810 K\sim^{2} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$- 186780 K\sim \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$
$$+ 21446541609 K\sim^{16} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 1438617976 K\sim^{12} \left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2}$$

$$+ 57016140464 K\sim^{15} \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$- 2838897152 K\sim^{11} (K\sim^2 + 4 K\sim + 5)^{3/2} (3 K\sim^2 + 4 K\sim - 1)^{3/2}$$
$$- 4370666708 K\sim^{10} (K\sim^2 + 4 K\sim + 5)^{3/2} (3 K\sim^2 + 4 K\sim - 1)^{3/2}$$
$$- 5224726856 K\sim^{9} (K\sim^2 + 4 K\sim + 5)^{3/2} (3 K\sim^2 + 4 K\sim - 1)^{3/2}$$
$$- 4765934388 K\sim^{8} (K\sim^2 + 4 K\sim + 5)^{3/2} (3 K\sim^2 + 4 K\sim - 1)^{3/2}$$
$$- 3191389632 K\sim^{7} (K\sim^2 + 4 K\sim + 5)^{3/2} (3 K\sim^2 + 4 K\sim - 1)^{3/2}$$
$$- 716996 K\sim^{11} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$- 2113699 K\sim^{10} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$- 4416402 K\sim^{9} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$- 7162488 K\sim^{7} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$- 2393538 K\sim^{5} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$+ 191868 K\sim^{3} (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$+ 5522 K\sim (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}$$
$$- 2783 (K\sim^2 + 4 K\sim + 5)^{5/2} (3 K\sim^2 + 4 K\sim - 1)^{5/2}\Big) \Big/ \Big(2 (K\sim^2 + 8 K\sim + 13) (K\sim^2 + 4 K\sim + 5)^2 (K\sim^2 - 3)^9\Big)$$

$$3 \qquad \textbf{(10.2.3.8)}$$

The singularity is in XX^3 = (1-x/xc)^3/2:

> *coeff*(*Zplussupcser*, *XX*, 1);

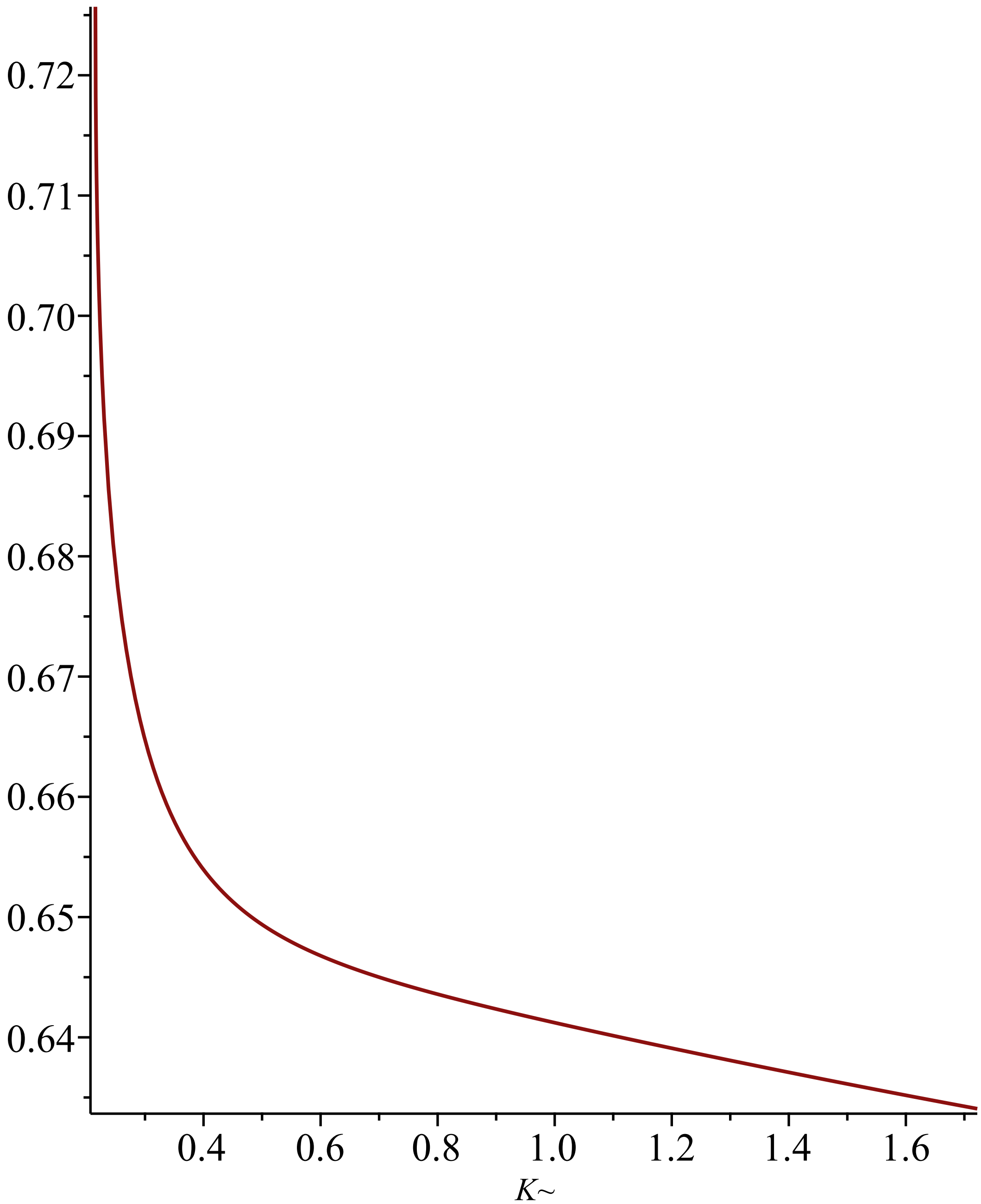
$$0 \qquad \textbf{(10.2.3.9)}$$

The corresponding mean for mu:

> *dermuK* := *factor*(*simplify*(*rationalize*(*simplify*(*factor*($\frac{-coeff(Zplussupcser, XX, 2)}{1 + coeff(Zplussupcser, XX, 0)}$), *symbolic*)), *symbolic*)); *plot*(%, *K* = *Kc* ..*Kinfini* − 0.01);

$$dermuK := \Big(\Big(183 K\sim^{10} - 105 K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1}$$
$$+ 2440 K\sim^9 - 1120 K\sim^7 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 14163 K\sim^8$$
$$- 5000 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 46480 K\sim^7$$
$$- 11984 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 93606 K\sim^6$$
$$- 16354 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 116608 K\sim^5$$
$$- 11776 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 84070 K\sim^4$$

$$- 2576 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 25456 K\sim^3$$
$$+ 1392 K\sim \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 8045 K\sim^2$$
$$+ 435 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 10248 K\sim - 3273\big) \big(37 K\sim^8$$
$$+ 21 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 296 K\sim^7$$
$$+ 112 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1016 K\sim^6$$
$$+ 255 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1896 K\sim^5$$
$$+ 304 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 2058 K\sim^4$$
$$+ 159 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1304 K\sim^3 + 320 K\sim^2$$
$$- 11 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 296 K\sim - 247\big)\big) \Big/ \big(2 \big(K\sim^2$$
$$+ 6 K\sim + 7\big) \big(K\sim^2 + 2 K\sim - 1\big) \big(23 K\sim^6 + 184 K\sim^5 + 593 K\sim^4 + 1008 K\sim^3$$
$$+ 989 K\sim^2 + 568 K\sim + 163\big) \big(K\sim^2 - 3\big)^3 \big(3 K\sim^2 + 16 K\sim + 23\big)\big)$$



The limit at Kc is the right one:

> $expand(rationalize(limit(dermuK, K = Kc, right)))$; $evalf(\%)$;

$$\frac{7}{6} - \frac{\sqrt{7}}{6}$$

$$0.7257081151 \qquad \textbf{(10.2.3.10)}$$

> $expand(rationalize(limit(dermuK, K = Kinfini, left)))$; $evalf(\%)$;

$$-\frac{\sqrt{3}}{2} + \frac{3}{2}$$

$$0.6339745960 \qquad \textbf{(10.2.3.11)}$$

When nu -> nuc, the derivative has a behavior in (nu-nuc)^1/2:

> $algeqtoseries(subs(\mathrm{nu} = nuc + NN, K = Kc + KK, numer(nusupK - \mathrm{nu})), NN, KK, 2);$

$$\left[RootOf\left(9\_Z^2 + \left(9\sqrt{7}+9\right)\_Z + 2\sqrt{7} + 26\right) + \left(-\frac{28\sqrt{7}}{81} + \frac{14}{81} + \frac{7\,RootOf\left(9\_Z^2 + \left(9\sqrt{7}+9\right)\_Z + 2\sqrt{7} + 26\right)\sqrt{7}}{27} - \frac{7\,RootOf\left(9\_Z^2 + \left(9\sqrt{7}+9\right)\_Z + 2\sqrt{7} + 26\right)}{9}\right) NN + \mathrm{O}\left(NN^2\right), \left(\frac{56}{81} + \frac{14\sqrt{7}}{81}\right) NN - \frac{196}{729}\,\frac{\left(4+\sqrt{7}\right)\left(2\sqrt{7}+1\right)}{7+\sqrt{7}}\,NN^2 + \mathrm{O}\left(NN^3\right)\right] \tag{10.2.3.12}$$

> $collect\left(subs\left(KK = \left(\frac{56}{81} + \frac{14\sqrt{7}}{81}\right) NN, expand(rationalize(convert(simplify(series(subs(K = Kc + KK, dermuK), KK, 1)), polynom)))\right), NN, factor\right);$

$$\frac{7^{3/4}\sqrt{14}\sqrt{\left(4+\sqrt{7}\right) NN}\sqrt{2}}{81} - \frac{4\,7^{1/4}\sqrt{14}\sqrt{\left(4+\sqrt{7}\right) NN}\sqrt{2}}{81} - \frac{\sqrt{7}}{6} + \frac{7}{6} \tag{10.2.3.13}$$

Proof of proposition 4.14 : asymptotic behavior of the weights q_k

We start from the algebraic equation satisfied by Q(t,ty):

> $eqQt := Qt^3\nu^2 w^2 y^5 - w\nu y^2\left(\nu y^2 - 2\nu y - y^2 - \nu + 3y\right) Qt^2 - \left(2\nu^2 w y^3 tZ1 + \nu^2 w y^3 - 2\nu w y^3 + \nu^2 y - \nu y^2 - 2\nu^2 + \nu y + y^2 + 2\nu - 2y\right) Qt - \left(2\nu^2 tZ1^2 + 2\nu^2 t2Z2 - \nu^2 tZ1 + w\nu^2 - 2\nu tZ1^2 - 2\nu t2Z2 - \nu tZ1 + \nu + 2tZ1 - 1\right) y^2 - \left(2\nu tZ1 - \nu - 2\right)\left(\nu - 1\right) y - 2\left(\nu - 1\right)\nu;$

$$eqQt := Qt^3\nu^2 w^2 y^5 - w\nu y^2\left(\nu y^2 - 2\nu y - y^2 - \nu + 3y\right) Qt^2 - \left(2\nu^2 w y^3 tZ1 + \nu^2 w y^3 - 2\nu w y^3 + \nu^2 y - \nu y^2 - 2\nu^2 + \nu y + y^2 + 2\nu - 2y\right) Qt - \left(2\nu^2 tZ1^2 + 2\nu^2 t2Z2 - \nu^2 tZ1 + \nu^2 w - 2\nu tZ1^2 - 2\nu t2Z2 - \nu tZ1 + \nu + 2tZ1 - 1\right) y^2 - \left(\nu - 1\right)\left(2\nu tZ1 - \nu - 2\right) y - 2\nu\left(\nu - 1\right) \tag{11.1}$$

By a change of varibles, this readily gives a algebraic equation for \tilde F:

> $eqtildeF := collect\left(factor\left(numer\left(factor\left(subs\left(Qt = FF\cdot(1-z), y = \frac{1}{1-z}, eqQt\right)\right)\right)\right), z, factor\right);$

$$eqtildeF := -2\,FF\,\nu\,(\nu-1)\,z^3 + \left(FF^2\,\nu^2\,w + 5\,FF\,\nu^2 - 7\,FF\,\nu - 2\,\nu^2 + 2\,FF + 2\,\nu\right)z^2 + \left(-4\,FF^2\,\nu^2\,w + 3\,w\,\nu\,FF^2 - 4\,FF\,\nu^2 + 2\,\nu^2\,tZ1 + 7\,FF\,\nu + 3\,\nu^2 - 2\,\nu\,tZ1 - 3\,FF - 5\,\nu + 2\right)z + FF^3\,\nu^2\,w^2 + 2\,FF^2\,\nu^2\,w - 2\,FF\,\nu^2\,tZ1\,w - 2\,w\,\nu\,FF^2 - FF\,\nu^2\,w - 2\,\nu^2\,tZ1^2 + FF\,\nu^2 + 2\,FF\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2\,w + 2\,\nu\,tZ1^2 - 2\,FF\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + FF + 2\,\nu - 2\,tZ1 - 1 \quad \textbf{(11.2)}$$

From the definition of \tilde F, we know that its constant term (as a formal power series in z) is equal to Q(t,t). We would like to get from the previous equation and equation of the form (\tilde F-Q(t,t)) Pol_1 = z* Pol_2.
We start from the algebraic equation satisfied by Q(t,t) (=Qty1)

> $eqQty1 := collect(subs(y = 1, Qt = Qty1, eqQt), Qt, factor):$

we check whether there exists another solution of the previous equation which is also a formal power series in z but with a different constant term. Its constant term FFz0 should be solution of the following equation:

> $subs(FF = FFz0, coeff(eqtildeF, z, 0));$

$$FFz0^3\,\nu^2\,w^2 + 2\,FFz0^2\,\nu^2\,w - 2\,FFz0\,\nu^2\,tZ1\,w - 2\,FFz0^2\,\nu\,w - FFz0\,\nu^2\,w - 2\,\nu^2\,tZ1^2 + FFz0\,\nu^2 + 2\,FFz0\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2\,w + 2\,\nu\,tZ1^2 - 2\,FFz0\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + FFz0 + 2\,\nu - 2\,tZ1 - 1 \quad \textbf{(11.3)}$$

And Q(t,t) is solution to the following algebraic equation:

> $collect(subs(y = 1, Qt = Qty1, eqQt), Qt, factor);$

$$Qty1^3\,\nu^2\,w^2 + 2\,w\,\nu^2\,Qty1^2 - 2\,Qty1\,\nu^2\,tZ1\,w - 2\,w\,\nu\,Qty1^2 - Qty1\,\nu^2\,w - 2\,\nu^2\,tZ1^2 + Qty1\,\nu^2 + 2\,Qty1\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2\,w + 2\,\nu\,tZ1^2 - 2\,Qty1\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + Qty1 + 2\,\nu - 2\,tZ1 - 1 \quad \textbf{(11.4)}$$

Hence the constant term FFz0 of a solution of the algebraic equation satisfied by eqtilde F, must be solution of:

> $factor(simplify(\textbf{(11.3)} - \textbf{(11.4)}));$

$$(FFz0 - Qty1)\left(FFz0^2\,\nu^2\,w^2 + FFz0\,Qty1\,\nu^2\,w^2 + Qty1^2\,\nu^2\,w^2 + 2\,FFz0\,\nu^2\,w + 2\,Qty1\,\nu^2\,w - 2\,\nu^2\,tZ1\,w - 2\,FFz0\,\nu\,w - 2\,Qty1\,\nu\,w - \nu^2\,w + \nu^2 + 2\,\nu\,w - 2\,\nu + 1\right) \quad \textbf{(11.5)}$$

The first factor when FFz0 = Qty1 is the derivative of the equation satistied by Qt:

> $simplify(subs(FFz0 = Qty1, op(1, \textbf{(11.5)}))) - factor(subs(y = 1, Qt = Qty1, diff(eqQt, Qt)));$

$$-1 + \left(-1 - 3\,w^2\,Qty1^2 + (2\,tZ1 - 4\,Qty1 + 1)\,w\right)\nu^2 + (2 + (4\,Qty1 - 2)\,w)\,\nu \quad \textbf{(11.6)}$$

The series Delta when nu< nu_c

The singular term in the asymptotic expansion of Q(t,ty)

```
> subs(Usubc = U, coeff(Qtsubcsing3, XX, 3));
```

$$-\left(4\,(Vsub+1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\,\left((Vsub^3-7\,Vsub^2-Vsub-1)\,U-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)\right)\Big/\left(9\left(U-\frac{2}{3}\right)(Vsub^2+4\,Vsub+1)\,(Vsub-1)^4\right) \quad \textbf{(11.1.1)}$$

The singular term in the asymptotic expansion of the partition function \mathcal{Z}:

```
> coeff(Zpsubcdevt, XX, 3);
```

$$\frac{12\sqrt{\frac{6U^2-10U+3}{9U^2-10U+2}}\,\sqrt{6}\left(U^2-U+\frac{1}{3}\right)(U-2)}{54U^3-126U^2+87U-18} \quad \textbf{(11.1.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vsub:

```
> AlephDeltaSubc := factor(simplify((11.1.1)/(11.1.2)));
```

$$AlephDeltaSubc := -\left((6U^2-10U+3)\,(3\,U\,Vsub^3-21\,U\,Vsub^2-2\,Vsub^3-3\,U\,Vsub+18\,Vsub^2-3U+6\,Vsub+2)\,Vsub\,(Vsub+1)\right)\Big/\left(3\,(U-2)\,(3U^2-3U+1)\,(Vsub-1)^4\,(Vsub^2+4\,Vsub+1)\right) \quad \textbf{(11.1.3)}$$

The series Delta and B when n = nu_c

The singular term in the asymptotic expansion of Q(t,ty)

```
> coeff(Qtcsing4, XX, 4);
```

$$\frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\left((1240\sqrt{7}-1700)^{1/3}\,(2\sqrt{7}+1)\,(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1)\,Vc\,(Vc+1)\right) \quad \textbf{(11.2.1)}$$

The singular term in the asymptotic expansion of the partition function \mathcal{Z}:

```
> coeff(Zpscritdevt, XX, 4);
```

$$\frac{3\,(1240\sqrt{7}-1700)^{1/3}\sqrt{7}}{20} \quad \textbf{(11.2.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vc:

```
> AlephDeltaCrit := factor(simplify((11.2.1)/(11.2.2)));
```

(11.2.3)

$$AlephDeltaCrit := \frac{5\left(14+\sqrt{7}\right)\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right) Vc\,(Vc+1)}{189\,(Vc-1)^4\left(Vc^2 + 4\,Vc + 1\right)} \quad \textbf{(11.2.3)}$$

We can verify that it is the same expression as the subcritical one:

$$> factor\left(\frac{subs(U = Uc, Vsub = Vc, AlephDeltaSubc)}{AlephDeltaCrit}\right);$$

$$1 \quad \textbf{(11.2.4)}$$

The series Delta when n > nu_c

$$> coeff(Qtsupsing, XX, 3);$$

$$\Big(32\,RootOf\big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right)\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,Vsup\,\Big(\left(K\sim^2-3\right)^2 Vsup^3 + \left(-7\,K\sim^4 - 40\,K\sim^3 - 110\,K\sim^2 - 136\,K\sim - 55\right) Vsup^2 - \left(K\sim^2 - 8\,K\sim - 11\right)\left(K\sim^2-3\right) Vsup - \left(K\sim^2-3\right)^2\Big)\Big(\left(K\sim + 1\right)^2 Vsup^2 + \left(K\sim^2-3\right) Vsup + \left(K\sim+1\right)^2\Big)\left(K\sim + \frac{5}{3}\right)(Vsup+1)\Big) \Big/ \Big(\left(K\sim+1\right)\Big(\left(K\sim^2-3\right) Vsup^2 + \left(-2\,K\sim^2 - 8\,K\sim - 10\right) Vsup + K\sim^2 - 3\Big)\Big(\left(K\sim^2-3\right) Vsup^2 + 4\,(K\sim+1)^2\,Vsup + K\sim^2 - 3\Big)^3\Big) \quad \textbf{(11.3.1)}$$

$$> coeff(Zpsupcdevt, XX, 3);$$

$$\Big(8\left(21\,K\sim^6 + 242\,K\sim^5 + 1083\,K\sim^4 + 2388\,K\sim^3 + 2695\,K\sim^2 + 1410\,K\sim + 225\right) RootOf\big(\left(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\right)\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\Big) \Big/ \left(3\left(K\sim^2 + 8\,K\sim + 13\right)(K\sim+1)^4\left(K\sim^2-3\right)\right) \quad \textbf{(11.3.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vsup:

$$> AlephDeltaSupc := factor\left(simplify\left(\frac{\textbf{(11.3.1)}}{\textbf{(11.3.2)}}\right)\right);$$

$$AlephDeltaSupc := \Big(4\left(K\sim^2-3\right)\left(K\sim^2 + 8\,K\sim + 13\right)(K\sim+1)^3\,(Vsup+1)\left(K\sim^2\,Vsup^2 + K\sim^2\,Vsup + 2\,K\sim\,Vsup^2 + K\sim^2 + Vsup^2 + 2\,K\sim - 3\,Vsup + 1\right)\left(K\sim^4\,Vsup^3 - 7\,K\sim^4\,Vsup^2 - K\sim^4\,Vsup - 40\,K\sim^3\,Vsup^2 - 6\,K\sim^2\,Vsup^3 - K\sim^4 + 8\,K\sim^3\,Vsup - 110\,K\sim^2\,Vsup^2 + 14\,K\sim^2\,Vsup - 136\,K\sim\,Vsup^2 + 9\,Vsup^3 + 6\,K\sim^2 - 24\,K\sim\,Vsup - 55\,Vsup^2 - 33\,Vsup - 9\right) Vsup\Big) \Big/ \Big((K\sim+3)\left(K\sim^2\,Vsup^2 + 4\,K\sim^2\,Vsup + K\sim^2 + 8\,K\sim\,Vsup - 3\,Vsup^2 + 4\,Vsup \right. \quad \textbf{(11.3.3)}$$

$$-3)^3\,(K\sim^2+4\,K\sim+1)\,(7\,K\sim^2+20\,K\sim+15)\,(K\sim^2\,Vsup^2-2\,K\sim^2\,Vsup$$
$$+K\sim^2-8\,K\sim\,Vsup-3\,Vsup^2-10\,Vsup-3))$$

The value for Kc is also the critical value:

> $factor\left(\frac{subs(K=Kc,\,Vsup=Vc,\,AlephDeltaSupc)}{AlephDeltaCrit}\right);$

$$1 \tag{11.3.4}$$

Hypergeometric functions and their singular expansions in Lemma 4.17

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{-\frac{1}{2}}\cdot(1-z\cdot x)^{\frac{2}{3}},\,x=0..1\right)\right)$ assuming $z<1$ **and** $z>0$;
> $series(\%,\,z=1,\,2);$

$$\mathrm{hypergeom}\left(\left[-\frac{2}{3},\,\frac{1}{2}\right],\,[1],\,z\right)$$

$$\frac{\sqrt{\pi}}{2\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}-\frac{\sqrt{\pi}\,(-1+z)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}+\frac{12\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}\,(-1+z)^{7/6}}{7\,\pi^{3/2}}+\mathrm{O}\left((-1+z)^2\right) \tag{12.1}$$

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{-\frac{1}{2}}\cdot(1-z\cdot x)^{\frac{4}{3}},\,x=0..1\right)\right)$ assuming $z<1$ **and** $z>0$;
> $series(\%,\,z=1,\,2);$

$$\mathrm{hypergeom}\left(\left[-\frac{4}{3},\,\frac{1}{2}\right],\,[1],\,z\right)$$

$$\frac{15\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{16\,\pi^{3/2}}-\frac{3\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1+z)}{4\,\pi^{3/2}}-\frac{32\sqrt{\pi}\,(-1)^{5/6}\,(-1+z)^{11/6}}{55\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}+\mathrm{O}\left((-1+z)^2\right) \tag{12.2}$$

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{-\frac{1}{2}}\cdot(1-z\cdot x)^{\frac{5}{3}},\,x=0..1\right)\right)$ assuming $z<1$ **and** $z>0$;
> $series(\%,\,z=1,\,2);$

$$\mathrm{hypergeom}\left(\left[-\frac{5}{3},\,\frac{1}{2}\right],\,[1],\,z\right)$$

$$\frac{7\sqrt{\pi}}{20\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{1}{4}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}\,(-1+z) + \mathrm{O}\left((-1+z)^2\right) \qquad \textbf{(12.3)}$$

Cluster volume expectation (proof of Theorem 1.4)

The numerator of the expected volume as calculated in the paper:

$$> EVol := \frac{1}{wU}\cdot\left(\frac{3}{8}\cdot\left(\frac{1}{yp^2}+\frac{1}{ym^2}\right)+\frac{1}{4}\cdot\frac{1}{yp\cdot ym}\right);$$

$$EVol := \left(32\,(-1+2\,U)^2\,\nu^3\left(\frac{3}{8\,yp^2}+\frac{3}{8\,ym^2}+\frac{1}{4\,yp\,ym}\right)\right)\Big/\Big(\big((U\,(\nu+1) - 2)\,U\,\big(8\,(\nu+1)^2\,U^3-(11\,\nu+13)\,(\nu+1)\,U^2+2\,(\nu+3)\,(2\,\nu+1)\,U - 4\,\nu\big)\Big) \qquad \textbf{(13.1)}$$

nu < nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

$$> ypsubsing;\ ymsubsing;$$

$$2-\frac{1}{9\,Usubc-9}\left(3\left(-\frac{2}{3}+Usubc\right)\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)XX^{3/2}\right)+\mathrm{O}\left(XX^2\right)$$

$$-\frac{4\,(Usubc-1)\,(-2+\sqrt{3})\,(\sqrt{3}-1)}{(21\,Usubc-16)\sqrt{3}-37\,Usubc+28} - 4\left((6\,Usubc^2-10\,Usubc+3)\,(-2+3\,Usubc)\,(780\sqrt{3}-1351)\,(Usubc-1)\right)\Big/\left((21\sqrt{3}\,Usubc-16\sqrt{3}-37\,Usubc+28)^2\,(2\,Usubc-1)\,(2\sqrt{3}-3)^3\right)XX^2 - \frac{8}{9}\left((6\,Usubc^2-10\,Usubc+3)\sqrt{2}\,(-2+3\,Usubc)\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,(Usubc-1)\,(1380661\sqrt{3}\,Usubc-2391375\,Usubc-1048348\sqrt{3}+1815792)\right)\Big/\left((2\,Usubc-1)\,(21\sqrt{3}\,Usubc-16\sqrt{3}-37\,Usubc+28)^3\,(2\sqrt{3}-3)^5\right)XX^3+\mathrm{O}\left(XX^4\right) \qquad \textbf{(13.1.1)}$$

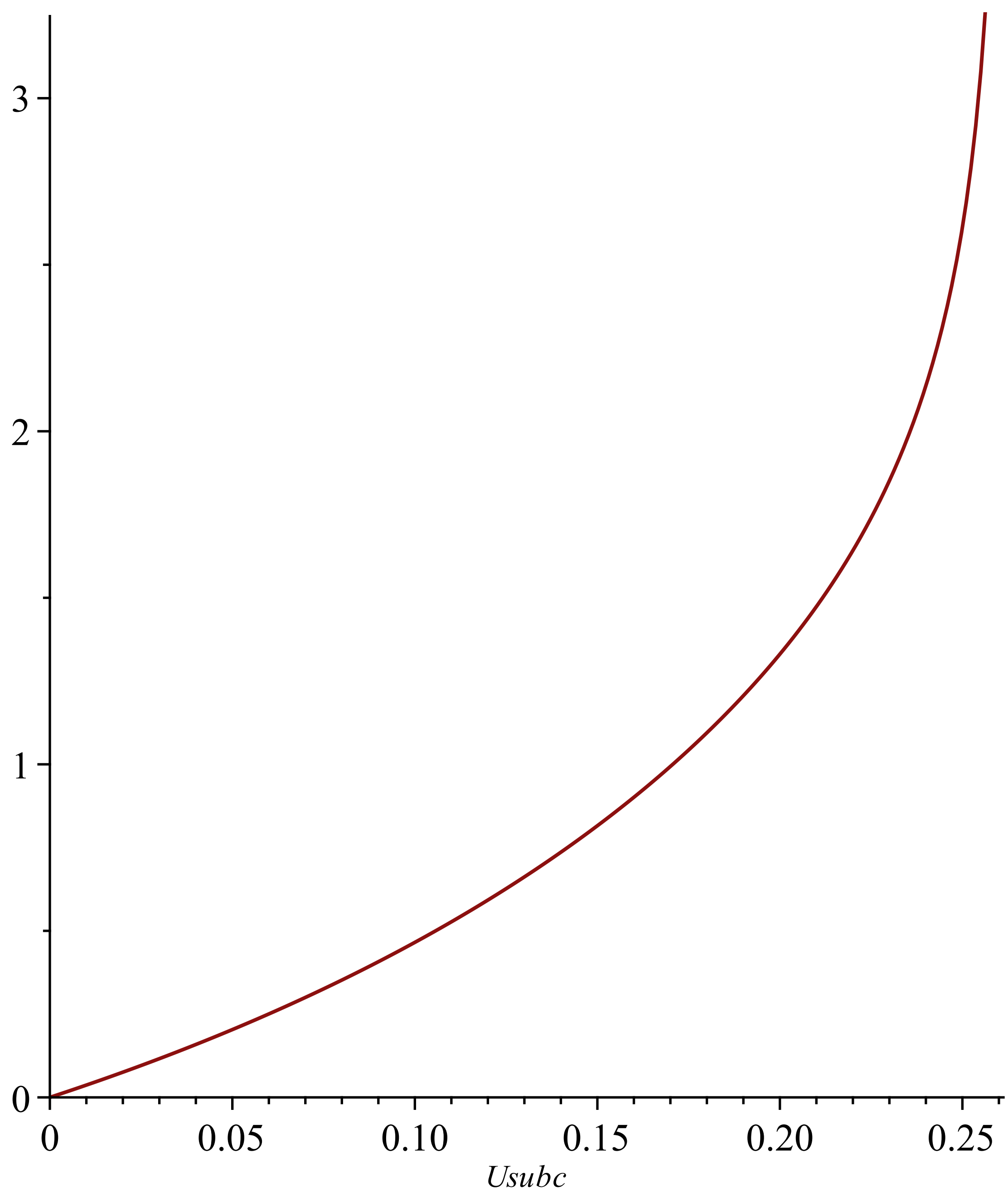
> $Evolsubcnumser := simplify\Big(series\Big(subs\big(subs\big(U = Usubc, subs(\mathrm{nu} = nuUsub)\big), U = Usubcsing3, yp = ypsubsing, ym = ymsubsing, XX = XX^2, EVol\big), XX, 4\Big)\Big)$ assuming $XX > 0$;

$$Evolsubcnumser := \Big(27\,\big(2943\sqrt{3}\,Usubc^2 - 4660\sqrt{3}\,Usubc - 5098\,Usubc^2 + 1852\sqrt{3} + 8072\,Usubc - 3208\big)\,Usubc\,(Usubc-1)\Big)\Big/\Big(2\,\big(\sqrt{3}-1\big)^2(-2+3\,Usubc)^2\big(-2+\sqrt{3}\big)^2\big(6\,Usubc^2-10\,Usubc+3\big)\Big) + \frac{3}{2}\left(Usubc\,(Usubc-1)\sqrt{3}\sqrt{2}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\big(39\sqrt{3}\,Usubc-34\sqrt{3}-67\,Usubc+58\big)\right)\Big/\Big(\big(18\,Usubc^3-42\,Usubc^2+29\,Usubc-6\big)\big(\sqrt{3}-1\big)\big(-2+\sqrt{3}\big)\Big)\,XX^3 + \mathrm{O}\big(XX^4\big) \quad \textbf{(13.1.2)}$$

The constant in the asymptotics of the Expected volume:

> $simplify\left(\frac{coeff(Evolsubcnumser, XX, 3)}{subs(U = Usubc, coeff(Zpsubcdevt, XX, 3))}\right); plot(\%, Usubc = 0..Uc);$

$$\left(3\,Usubc\,(Usubc-1)\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\big(39\sqrt{3}\,Usubc-34\sqrt{3}-67\,Usubc+58\big)\right)\Big/\left(8\,\big(\sqrt{3}-1\big)\big(-2+\sqrt{3}\big)\left(Usubc^2-Usubc+\frac{1}{3}\right)(Usubc-2)\right)$$



nu =nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

> *ypcsing*; *ymcsing*;

$$2+\frac{1}{\left(-5+\sqrt{7}\right)^2\left(7+\sqrt{7}\right)^2\left(7+13\sqrt{7}\right)}\left(\left(-812\sqrt{7}+784\right)XX^{3/2}\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\right)+\mathrm{O}\left(XX^2\right)$$

$$-\frac{4\left(-\frac{1}{2}+\sqrt{7}\right)\left(7+\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)}{\left(24\sqrt{7}+231\right)\sqrt{3}-38\sqrt{7}-413}+\frac{27440}{3}\left(\left(362\sqrt{3}-627\right)\left(78806\sqrt{7}-181693\right)\right)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^{2}\left(7+\sqrt{7}\right)^{2}\left(-5+\sqrt{7}\right)^{3}\left(-14+\sqrt{7}\right)^{2}\left(2\sqrt{3}-3\right)^{2}\right)XX^{3}$$
$$+\frac{1}{8}\Bigg(\Big(\left(330082753200510240\sqrt{3}-571720105508325550\right)\sqrt{7}-1024391999457256185\sqrt{3}+1774299006738717515\Big)\left(1240\sqrt{7}-1700\right)^{1/3}+15704812680490206\left(\left(\sqrt{3}-\frac{33518496652}{19351912887}\right)\sqrt{7}-\frac{8999600785\sqrt{3}}{4300425086}+\frac{140289893863}{38703825774}\right)50^{1/3}\left(1240\sqrt{7}-1700\right)^{2/3}+\left(-586902892127647914\,50^{2/3}\sqrt{3}+1016545638114735092\,50^{2/3}\right)\sqrt{7}+1392843856906107333\,50^{2/3}\sqrt{3}-2412476352951155491\,50^{2/3}\Bigg)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^{3}\left(7+\sqrt{7}\right)^{3}\left(-14+\sqrt{7}\right)^{3}\left(2\sqrt{3}-3\right)^{3}\left(-5+\sqrt{7}\right)^{4}\right)XX^{4}+O\left(XX^{5}\right) \quad (13.2.1)$$

> *Evolcnumser* := *simplify*(*series*(*subs*(nu = *nuc*, *U* = *Ucsing4*, *yp* = *ypcsing*, *ym* = *ymcsing*, *XX* = *XX*^2, *EVol*), *XX*, 4))assuming *XX* > 0;

$$Evolcnumser := \frac{\left(13860\sqrt{7}+56979\right)\sqrt{3}-23984\sqrt{7}-98762}{10\left(-\frac{1}{2}+\sqrt{7}\right)\left(\sqrt{3}-1\right)^{2}\left(-5+\sqrt{7}\right)\left(-2+\sqrt{3}\right)^{2}}+\frac{1}{5}\Big(\Big(\left(\left(192825780\sqrt{3}-332594948\right)\sqrt{7}+564274788\sqrt{3}-947353652\right)\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^{2}-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\Big)\Big/\Big(\left(7+\sqrt{7}\right)^{4}\left(-5+\sqrt{7}\right)^{3}\left(-1+2\sqrt{7}\right)^{2}\left(7+13\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)\Big)\Big)XX^{3}+O\left(XX^{4}\right) \quad (13.2.2)$$

The constant in the asymptotics of the Expected volume:

> *Zpscritdevt*

$$\frac{3\sqrt{7}\left(1240\sqrt{7}-1700\right)^{1/3}XX^{4}}{20}+\left(-\frac{476}{25}+\frac{148\sqrt{7}}{25}\right)XX^{3}+\frac{263\sqrt{7}}{50}-\frac{308}{25} \quad (13.2.3)$$

> $simplify\left(expand\left(rationalize\left(simplify\left(\frac{coeff(Evolcnumser, XX, 3)}{coeff(Zpscritdevt, XX, 3)}\right)\right)\right)\right)$; $evalf(\%)$;

$$\frac{1}{109872}\left(\left(\left(760\sqrt{7}+2135\right)\sqrt{3}-3860\sqrt{7}-9940\right)\left(1240\sqrt{7}-1700\right)^{1/3} RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\right)$$
$$-2.265903514 \quad \textbf{(13.2.4)}$$

nu>nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

> $ypsupsing$; $ymsupsing$;

$$-\left(16\left(3K\sim+5\right)\left(3K\sim^2+8K\sim+7\right)\left(K\sim^3+3K\sim^2+9K\sim+11\right)\left(36K\sim^{10} + 31K\sim^8\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 384K\sim^9 + 248K\sim^7\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 1996K\sim^8 + 844K\sim^6\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 6624K\sim^7 + 1544K\sim^5\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 14952K\sim^6 + 1818K\sim^4\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 22176K\sim^5 + 2088K\sim^3\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 19160K\sim^4 + 2508K\sim^2\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 6560K\sim^3 + 1816K\sim\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} - 1804K\sim^2 + 479\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} - 1184K\sim + 220\right) RootOf\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2 - K\sim^8 - 10K\sim^7 - 24K\sim^6 + 26K\sim^5 + 158K\sim^4 + 114K\sim^3 - 192K\sim^2 - 306K\sim - 117\right) XX\right) \Big/ \left(\left(K\sim^2+4K\sim+5\right)\left(23K\sim^6+184K\sim^5+593K\sim^4 + 1008K\sim^3+989K\sim^2+568K\sim+163\right)^2\left(K\sim^2-3\right)^2\right) - \left(4\left(K\sim + 1\right)\left(K\sim^3+3K\sim^2+9K\sim+11\right)\left(2K\sim^4 + 3K\sim^2\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 8K\sim^3 + 4K\sim\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 24K\sim^2 - \sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} + 40K\sim + 22\right)\right) \Big/ \left(\left(23K\sim^6 + 184K\sim^5+593K\sim^4+1008K\sim^3+989K\sim^2+568K\sim+163\right)\left(K\sim^2-3\right)\right)$$

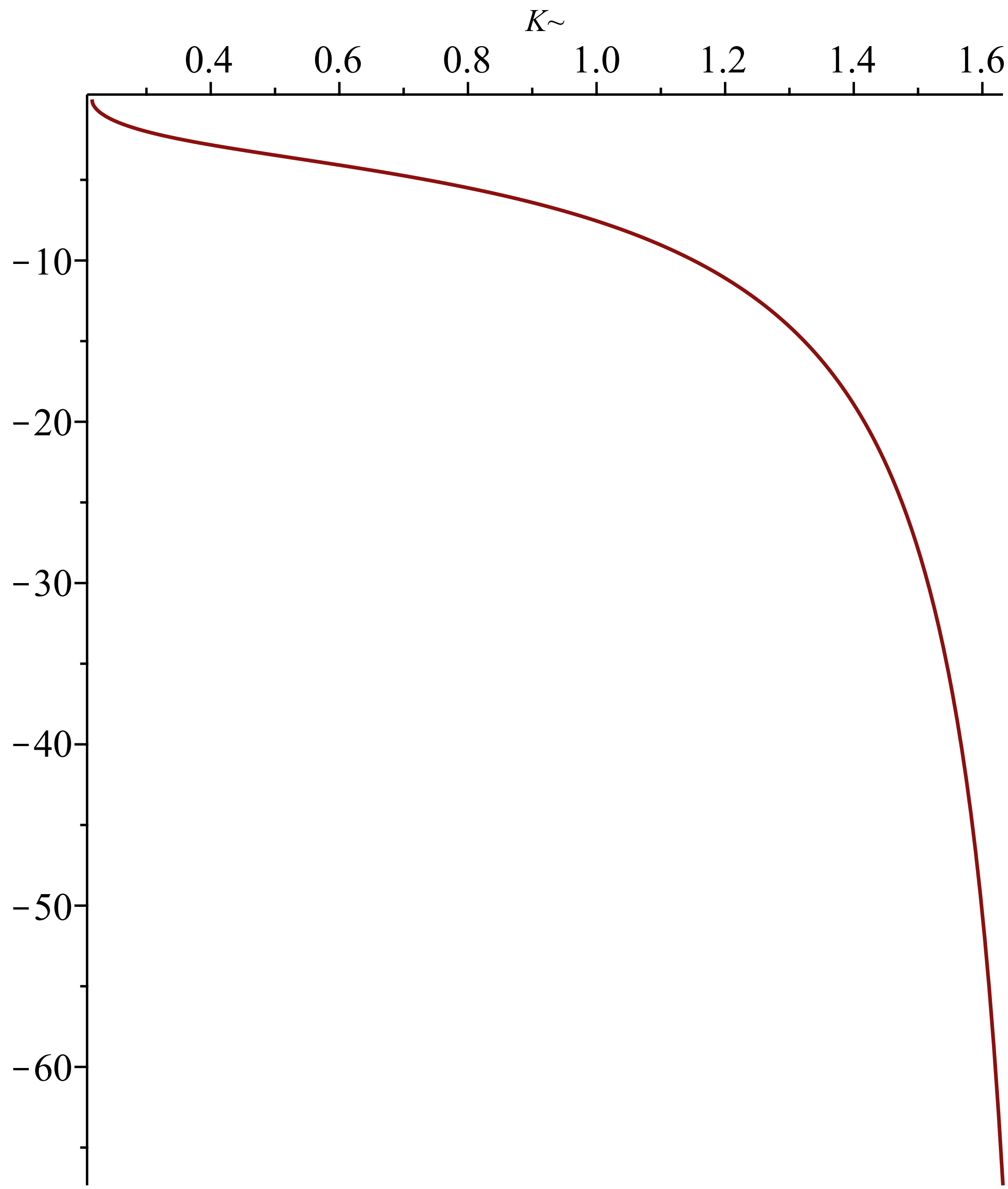
$$\left(4\,RootOf\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2 - K\sim^8 \right.\right. \quad \textbf{(13.3.1)}$$

$$- 10 K\sim^{7} - 24 K\sim^{6} + 26 K\sim^{5} + 158 K\sim^{4} + 114 K\sim^{3} - 192 K\sim^{2} - 306 K\sim$$
$$- 117\big) (3 K\sim + 5) (K\sim + 1) \big(K\sim^{2} + 8 K\sim + 13\big) \big(K\sim^{3} + 3 K\sim^{2} + 9 K\sim$$
$$+ 11\big) \big(3 K\sim^{4} + 12 K\sim^{3} + 22 K\sim^{2} + 28 K\sim + 19\big) \big(K\sim^{6} \sqrt{2} + 6 K\sim^{5} \sqrt{2}$$
$$+ 27 K\sim^{4} \sqrt{2} + 16 K\sim^{4} \sqrt{2 + K\sim} + 108 K\sim^{3} \sqrt{2} + 80 K\sim^{3} \sqrt{2 + K\sim}$$
$$+ 339 K\sim^{2} \sqrt{2} + 240 K\sim^{2} \sqrt{2 + K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2 + K\sim}\, K\sim$$
$$+ 377 \sqrt{2} + 352 \sqrt{2 + K\sim}\big) XX^{3}\big) \Big/ \big(3 \sqrt{2 + K\sim} \big(3 K\sim^{2} + 8 K\sim$$
$$+ 7\big)^{3} \big(K\sim^{2} - 3\big)^{2} \big(K\sim^{4} + 6 K\sim^{3} + 30 K\sim^{2} + 62 K\sim + 45\big)^{2}\big) + \big(\big(K\sim^{2}$$
$$+ 8 K\sim + 13\big) \big(K\sim^{3} + 3 K\sim^{2} + 9 K\sim + 11\big) (K\sim + 1)^{2} \big(K\sim^{6} \sqrt{2} + 6 K\sim^{5} \sqrt{2}$$
$$+ 27 K\sim^{4} \sqrt{2} + 16 K\sim^{4} \sqrt{2 + K\sim} + 108 K\sim^{3} \sqrt{2} + 80 K\sim^{3} \sqrt{2 + K\sim}$$
$$+ 339 K\sim^{2} \sqrt{2} + 240 K\sim^{2} \sqrt{2 + K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2 + K\sim}\, K\sim$$
$$+ 377 \sqrt{2} + 352 \sqrt{2 + K\sim}\big) XX^{2}\big) \Big/ \big(2 \sqrt{2 + K\sim} \big(K\sim^{2} - 3\big) \big(K\sim^{4} + 6 K\sim^{3}$$
$$+ 30 K\sim^{2} + 62 K\sim + 45\big)^{2} \big(3 K\sim^{2} + 8 K\sim + 7\big)\big) + \big(2 \big(K\sim^{3} + 3 K\sim^{2}$$
$$+ 9 K\sim + 11\big) \big(K\sim^{3} + 4 \sqrt{2} \sqrt{2 + K\sim}\, K\sim + 3 K\sim^{2} + 8 \sqrt{2} \sqrt{2 + K\sim} + 9 K\sim$$
$$+ 11\big)\big) \Big/ \big(\big(K\sim^{2} - 3\big) \big(K\sim^{4} + 6 K\sim^{3} + 30 K\sim^{2} + 62 K\sim + 45\big)\big)$$

```
> Evolsupcnumser := simplify(series(subs(nu = nusupK, U = Usupcsing, yp = ypsupsing,
      ym = ymsupsing, EVol), XX, 4)) :
```

The constant in the asymptotics of the Expected volume:

```
> simplify( coeff(Evolsupcnumser, XX, 3) / coeff(Zpsupcdevt, XX, 3) ) : plot(%, K = Kc ..Kinfini - 0.1);
```



Percolation probability (proof of Theorem 1.1)

Finite clusters in the high temperature regime (nu \leq nu_c)

We start from the rational parametrization of y in terms of U_nu and V (in which nu has been replaced by its expression in terms of U_\nu).

> *yUVsubc*;

(14.1.1)

$$-\frac{24\,(U-1)\,V\,(V+1)}{3\,U\,V^3-21\,U\,V^2-2\,V^3-3\,V\,U+18\,V^2-3\,U+6\,V+2} \quad \textbf{(14.1.1)}$$

The value of the negative singularity y- corresponds to V equal to Vsubl (computed in **(4.1.6)**):

> $ymsub := factor(subs(V=-2+\mathrm{sqrt}(3), yUVsubc));$

$$ymsub := -\frac{4\,(3\sqrt{3}+2)\,(U-1)}{23\,U-14+2\sqrt{3}} \quad \textbf{(14.1.2)}$$

To perform the change of variables in the integral, we compute the new bounds by solving the following equations, (recall that y- = ymsub and y+ = 2)

> $factor\left(\frac{yUVsubc-1}{yUVsubc}-\frac{1}{ymsub}\right);$

$$\frac{(-2+3\,U)\,(V-7+4\sqrt{3})\,(V+2+\sqrt{3})^2}{24\,(U-1)\,V\,(V+1)} \quad \textbf{(14.1.3)}$$

> $factor\left(\frac{1}{2}-\frac{yUVsubc-1}{yUVsubc}\right);$

$$-\frac{(V-1)^3\,(-2+3\,U)}{24\,(U-1)\,V\,(V+1)} \quad \textbf{(14.1.4)}$$

In the integral, V varies between -2+sqrt(3) and 1 so the square root factor is given by:

> $rootfactorsubc := \frac{(2-3\,U)}{24\cdot(1-U)}\cdot\frac{(V+2+\mathrm{sqrt}(3))\cdot(1-V)}{V\cdot(V+1)}\cdot\mathrm{sqrt}\big((1-V)\cdot\big(V-7$
> $+4\sqrt{3}\big)\big);$

$$rootfactorsubc := \frac{(2-3\,U)\,(V+2+\sqrt{3})\,(1-V)\,\sqrt{(1-V)\,(V-7+4\sqrt{3})}}{(-24\,U+24)\,V\,(V+1)} \quad \textbf{(14.1.5)}$$

Recall that our expression for AlephDeltaSubc is also valid at nuc :

> $factor\left(\frac{1}{yUVsubc}\cdot subs(Usubc=U, Vsub=V, AlephDeltaSubc)\cdot diff(yUVsubc, V)\right);$

$$\frac{(6\,U^2-10\,U+3)\,(-2+3\,U)}{3\,(V-1)^2\,(3\,U^2-3\,U+1)\,(U-2)} \quad \textbf{(14.1.6)}$$

> $factor\left(\frac{yUVsubc-1}{yUVsubc}+\frac{1}{2}\cdot\left(\frac{1}{ymsub}+\frac{1}{2}\right)\right);$

$$-\frac{1}{24\,(U-1)\,V\,(V+1)}\big(9\sqrt{3}\,U\,V^2-3\,U\,V^3+9\sqrt{3}\,U\,V-6\sqrt{3}\,V^2-15\,U\,V^2$$
$$+2\,V^3-6\sqrt{3}\,V-33\,V\,U+18\,V^2+3\,U+30\,V-2\big) \quad \textbf{(14.1.7)}$$

The following is the probability that the cluster is finite:

> $simplify\left(\frac{1}{2\cdot\mathrm{Pi}\cdot nuUsub\cdot subs(\mathrm{nu}=nuUsub, wU)}\; int(\textbf{(14.1.6)}\cdot\textbf{(14.1.7)}\cdot\textbf{(14.1.5)}, V=7\right.$
> $\left.-4\,\mathrm{sqrt}(3)\,..1)\right);$

$$1 \quad \textbf{(14.1.8)}$$

Percolation probability when nu>nuc and critical exponent beta:

The symmetry in 1/V of \hat y:

> $simplify\left(subs\left(V=\frac{1}{V}, yUV\right) - \frac{yUV}{yUV-1}\right);$

$$0 \tag{14.2.1}$$

The values of V for the singularities of Q(t,ty) at t_\nu were computed in Section 4 (equations **(4.2.11)** and below).

> $nusupK; UsupK; factor(numer(diff(yUVsupc, V)));$

$$-\frac{K\sim^3 + 3K\sim^2 + 9K\sim + 11}{(K\sim+3)(K\sim^2-3)}$$

$$-\frac{K\sim^2-3}{6K\sim+10}$$

$$8(K\sim+1)(K\sim^3+3K\sim^2+9K\sim+11)(V^2K\sim^2+4K\sim^2V+K\sim^2+8K\sim V-3V^2+4V-3)(V^2K\sim^2-2K\sim^2V+K\sim^2-8K\sim V-3V^2-10V-3) \tag{14.2.2}$$

> $collect((K\sim^2V^2+4K\sim^2V+K\sim^2+8K\sim V-3V^2+4V-3), V, factor);$
> $collect((K\sim^2V^2-2K\sim^2V+K\sim^2-8K\sim V-3V^2-10V-3), V, factor);$

$$(K\sim^2-3)V^2+4(K\sim+1)^2V+K\sim^2-3$$

$$(K\sim^2-3)V^2+(-2K\sim^2-8K\sim-10)V+K\sim^2-3 \tag{14.2.3}$$

> $VK11; VK22;$

$$-\frac{2K\sim^2+4K\sim-\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}+2}{K\sim^2-3}$$

$$\frac{K\sim^2+4K\sim-2\sqrt{2}(K\sim+1)\sqrt{2+K\sim}+5}{K\sim^2-3} \tag{14.2.4}$$

We first compute the bounds for the integral. We have to factorize:

> $factor\left(numer\left(factor\left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V=\frac{1}{VK11}, yUVsupc\right)}\right)\right)\right);$

$$45+15V-1188K\sim V+3132V^2K\sim^2+5K\sim^8+28K\sim^7 \tag{14.2.5}$$
$$-3K\sim^6\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}$$
$$-8K\sim^5\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}$$
$$+11K\sim^4\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}$$
$$+48K\sim^3\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}$$
$$+15K\sim^2\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}$$
$$-72K\sim\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)}+396K\sim+3((K\sim^2$$

$$+ 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1))^{3/2} V^2 + 3 ((\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2$$
$$+ 4 \tilde{K} - 1))^{3/2} V + 63 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$- 52 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2$$
$$+ 102 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V - 12 \tilde{K}^3 + 492 \tilde{K}^2$$
$$+ 28 \tilde{K}^6 - 124 \tilde{K}^5 - 298 \tilde{K}^4 + 298 \tilde{K}^4 V^3 + 10950 \tilde{K}^4 V^2 - 1630 \tilde{K}^4 V$$
$$+ 8620 \tilde{K}^3 V^2 - 492 \tilde{K}^2 V^3 - 45 V^3$$
$$- 63 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} + 125 V^2 - 4188 \tilde{K}^3 V$$
$$+ 12 \tilde{K}^3 V^3 - 396 \tilde{K} V^3 - 3692 \tilde{K}^2 V + 340 \tilde{K} V^2 - 5 \tilde{K}^8 V^3 + 69 \tilde{K}^8 V^2$$
$$- 28 \tilde{K}^7 V^3 + 39 \tilde{K}^8 V + 708 \tilde{K}^7 V^2 - 28 \tilde{K}^6 V^3 + 300 \tilde{K}^7 V$$
$$+ 3148 \tilde{K}^6 V^2 + 124 \tilde{K}^5 V^3 + 836 \tilde{K}^6 V + 7708 \tilde{K}^5 V^2 + 628 \tilde{K}^5 V$$
$$+ 3 \tilde{K}^6 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$- 36 \tilde{K}^6 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2$$
$$+ 8 \tilde{K}^5 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$- 18 \tilde{K}^6 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V$$
$$- 288 \tilde{K}^5 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2$$
$$- 11 \tilde{K}^4 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$- 96 \tilde{K}^5 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V$$
$$- 940 \tilde{K}^4 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2$$
$$- 48 \tilde{K}^3 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3 - \tilde{K}^2 ((\tilde{K}^2$$
$$+ 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1))^{3/2} V^2$$
$$- 142 \tilde{K}^4 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V$$
$$- 1536 \tilde{K}^3 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2 - \tilde{K}^2 ((\tilde{K}^2$$
$$+ 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1))^{3/2} V$$
$$- 15 \tilde{K}^2 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$+ 128 \tilde{K}^3 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V$$
$$- 1276 \tilde{K}^2 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^2$$
$$+ 72 \tilde{K} \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V^3$$
$$+ 554 \tilde{K}^2 \sqrt{(\tilde{K}^2 + 4 \tilde{K} + 5) (3 \tilde{K}^2 + 4 \tilde{K} - 1)} V$$

$- 480\, K\sim \sqrt{\left(K\sim^2 + 4\, K\sim + 5\right) \left(3\, K\sim^2 + 4\, K\sim - 1\right)}\; V^2$

$+ 480\, K\sim \sqrt{\left(K\sim^2 + 4\, K\sim + 5\right) \left(3\, K\sim^2 + 4\, K\sim - 1\right)}\; V$

1/V+ is a double root:

> $simplify\left(rem\left(\mathbf{(14.2.5)}, \left(V - \frac{1}{VK11}\right)^2, V\right)\right);$

$$0 \qquad \mathbf{(14.2.6)}$$

VK11^2 is the third root:

> $simplify\left(subs\left(V = VK11^2, \mathbf{(14.2.5)}\right)\right);$

$$0 \qquad \mathbf{(14.2.7)}$$

Same for the other bound, we want to factorize:

> $factor\left(numer\left(factor\left(\left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V = \frac{1}{VK22}, yUVsupc\right)}\right)\right)\right)\right);$

$$\begin{aligned}
&117 + 753\, V + 8\, K\sim^4 \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2 + 3\, K\sim^6 \sqrt{2} \sqrt{2 + K\sim}\, V^3 \qquad \mathbf{(14.2.8)}\\
&\quad + 8\, K\sim^4 \sqrt{2}\, (2 + K\sim)^{3/2}\, V + 16\, K\sim^3 \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2\\
&\quad - 9\, K\sim^6 \sqrt{2} \sqrt{2 + K\sim}\, V^2 + 8\, K\sim^5 \sqrt{2} \sqrt{2 + K\sim}\, V^3 + 16\, K\sim^3 \sqrt{2}\, (2\\
&\quad + K\sim)^{3/2}\, V - 16\, K\sim^2 \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2 + 9\, K\sim^6 \sqrt{2} \sqrt{2 + K\sim}\, V\\
&\quad - 120\, K\sim^5 \sqrt{2} \sqrt{2 + K\sim}\, V^2 - 11\, K\sim^4 \sqrt{2} \sqrt{2 + K\sim}\, V^3 - 16\, K\sim^2 \sqrt{2}\, (2\\
&\quad + K\sim)^{3/2}\, V - 48\, K\sim \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2 + 72\, K\sim^5 \sqrt{2} \sqrt{2 + K\sim}\, V\\
&\quad - 623\, K\sim^4 \sqrt{2} \sqrt{2 + K\sim}\, V^2 - 48\, K\sim^3 \sqrt{2} \sqrt{2 + K\sim}\, V^3 - 48\, K\sim \sqrt{2}\, (2\\
&\quad + K\sim)^{3/2}\, V + 175\, K\sim^4 \sqrt{2} \sqrt{2 + K\sim}\, V - 1648\, K\sim^3 \sqrt{2} \sqrt{2 + K\sim}\, V^2\\
&\quad - 15\, K\sim^2 \sqrt{2} \sqrt{2 + K\sim}\, V^3 + 16\, K\sim^3 \sqrt{2} \sqrt{2 + K\sim}\, V\\
&\quad - 2339\, K\sim^2 \sqrt{2} \sqrt{2 + K\sim}\, V^2 + 72\, K\sim \sqrt{2} \sqrt{2 + K\sim}\, V^3\\
&\quad - 509\, K\sim^2 \sqrt{2} \sqrt{2 + K\sim}\, V - 1656\, K\sim \sqrt{2} \sqrt{2 + K\sim}\, V^2\\
&\quad - 696\, K\sim \sqrt{2} \sqrt{2 + K\sim}\, V + 1857\, K\sim\, V + 5633\, V^2 K\sim^2 + 11\, K\sim^4 \sqrt{2} \sqrt{2 + K\sim}\\
&\quad + 48\, K\sim^3 \sqrt{2} \sqrt{2 + K\sim} + 15\, K\sim^2 \sqrt{2} \sqrt{2 + K\sim} + K\sim^7 - 8\, K\sim^5 \sqrt{2} \sqrt{2 + K\sim}\\
&\quad + 189\, K\sim - 117\, K\sim^3 + 3\, K\sim^2 - 24 \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2 - 24 \sqrt{2}\, (2\\
&\quad + K\sim)^{3/2}\, V + 63 \sqrt{2} \sqrt{2 + K\sim}\, V^3 - 445 \sqrt{2} \sqrt{2 + K\sim}\, V^2\\
&\quad - 291 \sqrt{2} \sqrt{2 + K\sim}\, V - 63 \sqrt{2} \sqrt{2 + K\sim} + 9\, K\sim^6 + 15\, K\sim^5 - 41\, K\sim^4\\
&\quad + 41\, K\sim^4\, V^3 + 1893\, K\sim^4\, V^2 - 421\, K\sim^4\, V + 4401\, K\sim^3\, V^2 - 3\, K\sim^2\, V^3 - 117\, V^3\\
&\quad + 1039\, V^2 - 72 \sqrt{2} \sqrt{2 + K\sim}\, K\sim + 143\, K\sim^3\, V + 117\, K\sim^3\, V^3 - 189\, K\sim\, V^3\\
&\quad + 1471\, K\sim^2\, V + 3775\, K\sim\, V^2 - K\sim^7\, V^3 + 3\, K\sim^7\, V^2 - 9\, K\sim^6\, V^3 - 3\, K\sim^7\, V\\
&\quad + 51\, K\sim^6\, V^2 - 15\, K\sim^5\, V^3 - 51\, K\sim^6\, V + 437\, K\sim^5\, V^2 - 245\, K\sim^5\, V
\end{aligned}$$

$$-3\,K\sim^{6}\,\sqrt{2}\,\sqrt{2+K\sim}$$

1/V- is a double root:

$$> simplify\left(rem\left(\mathbf{(14.2.8)},\ \left(V-\frac{1}{VK22}\right)^{2},\ V\right)\right);$$

$$0 \qquad \mathbf{(14.2.9)}$$

and V-^2 is the third root:

$$> simplify\left(subs\left(V=VK22^{2},\ \mathbf{(14.2.8)}\right)\right);$$

$$0 \qquad \mathbf{(14.2.10)}$$

Since we identify all the roots, the numerator of the terms under the square root is fully factorized. We now look at its denominator:

$$> factor\left(denom\left(factor\left(\left(\frac{1}{yUVsupc}-\frac{1}{subs\left(V=\frac{1}{VK22},yUVsupc\right)}\right)\cdot\left(\frac{1}{yUVsupc}-\frac{1}{subs\left(V=\frac{1}{VK11},yUVsupc\right)}\right)\right)\right)\right);$$

$$64\left(2K\sim^{2}+4K\sim-\sqrt{\left(K\sim^{2}+4K\sim+5\right)\left(3K\sim^{2}+4K\sim-1\right)}+2\right)\left(K\sim^{2}+4K\sim-\sqrt{\left(K\sim^{2}+4K\sim+5\right)\left(3K\sim^{2}+4K\sim-1\right)}+5\right)\left(-2\sqrt{2}\,\sqrt{2+K\sim}\,K\sim+K\sim^{2}-2\sqrt{2}\,\sqrt{2+K\sim}+4K\sim+5\right)\left(K\sim+1\right)^{2}\left(-\sqrt{2}\,\sqrt{2+K\sim}+K\sim+1\right)\left(V+1\right)^{2}V^{2}\left(K\sim^{3}+3K\sim^{2}+9K\sim+11\right)^{2} \qquad \mathbf{(14.2.11)}$$

We can hence write that the root factor in the integral is equal to f1K * rootfactorsupc, where f1K depends only on K (and not on V) and:

$$> rootfactorsupc := \frac{\left(\frac{1}{Vp}-V\right)\cdot\left(V-\frac{1}{Vm}\right)\cdot\mathrm{sqrt}\left(\left(Vp^{2}-V\right)\cdot\left(V-Vm^{2}\right)\right)}{V\cdot\left(V+1\right)};$$

$$rootfactorsupc := \frac{\left(\frac{1}{Vp}-V\right)\left(V-\frac{1}{Vm}\right)\sqrt{\left(Vp^{2}-V\right)\left(-Vm^{2}+V\right)}}{V\left(V+1\right)} \qquad \mathbf{(14.2.12)}$$

To fully factorize the square root in the integral, we want to compute f1K:

$$> factor\left(expand\left(rationalize\left(factor\left(\left(coeff\left(numer\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V},yUVsupc\right)}-\frac{1}{subs\left(V=VK22,yUVsupc\right)}\right)\right),V,3\right)\right)\Big/\left(coeff\left(denom\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V},yUVsupc\right)}\right.\right.\right.\right.\right.\right.\right.$$

$$- \frac{1}{subs(V = VK22, yUVsupc)}\Bigg)\Bigg), V, 2\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg);$$

$$factor\Bigg(expand\Bigg(rationalize\Bigg(factor\Bigg(\Bigg(coeff\Bigg(numer\Bigg(factor\Bigg(1 \Bigg/ \Bigg(subs\Bigg(V = \frac{1}{V}, yUVsupc\Bigg)\Bigg) - \frac{1}{subs(V = VK11, yUVsupc)}\Bigg)\Bigg), V, 3\Bigg)\Bigg) \Bigg/ \Bigg(coeff\Bigg(denom\Bigg(factor\Bigg(\frac{1}{subs\left(V = \frac{1}{V}, yUVsupc\right)} - \frac{1}{subs(V = VK11, yUVsupc)}\Bigg)\Bigg), V, 2\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg);$$

$$\frac{(K\sim^2 - 3)^2}{8\,(K\sim + 1)\,(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11)}$$

$$\frac{(K\sim^2 - 3)^2}{8\,(K\sim + 1)\,(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11)} \qquad \textbf{(14.2.13)}$$

> $f1K := \dfrac{(3 - K^2)^2}{8\,(K + 1)\,(K^3 + 3\,K^2 + 9\,K + 11)};$

$$f1K := \frac{(-K\sim^2 + 3)^2}{8\,(K\sim + 1)\,(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11)} \qquad \textbf{(14.2.14)}$$

We turn our attention to the other factors of the integral.
Another factor of the integral will give poles at V+ and 1/V+:

> $factor\left(\dfrac{diff(yUVsupc, V)}{yUVsupc} \cdot subs(Vsup = V, AlephDeltaSupc)\right);$

$$-\big(4\,(K\sim + 1)^3\,(K\sim^2 - 3)\,(K\sim^2 + 8\,K\sim + 13)\,(V^2\,K\sim^2 + K\sim^2\,V + 2\,K\sim\,V^2 + K\sim^2 + V^2 + 2\,K\sim - 3\,V + 1)\big) \Big/ \big((7\,K\sim^2 + 20\,K\sim + 15)\,(K\sim^2 + 4\,K\sim + 1)\,(V^2\,K\sim^2 + 4\,K\sim^2\,V + K\sim^2 + 8\,K\sim\,V - 3\,V^2 + 4\,V - 3)^2\,(K\sim + 3)\big) \qquad \textbf{(14.2.15)}$$

The roots of the polynomial (in V) of the denominator are VK11 and 1/VK11. Indeed, we have:

> $simplify(subs(V = VK11, (K^2\,V^2 + 4\,K^2\,V + K^2 + 8\,K\,V - 3\,V^2 + 4\,V - 3)));$
> $simplify\left(subs\left(V = \dfrac{1}{VK11}, (K^2\,V^2 + 4\,K^2\,V + K^2 + 8\,K\,V - 3\,V^2 + 4\,V - 3)\right)\right);$

$$0$$

$$0 \qquad \textbf{(14.2.16)}$$

We can also rewrite the polynomial (in V) of the numerator as:

> $collect(V^2\,K^2 + K^2\,V + 2\,K\,V^2 + K^2 + V^2 + 2\,K - 3\,V + 1, V, factor);$
> $factor(\,solve(\%, V)\,[1]);$

$$(K\sim+1)^2 V^2 + (K\sim^2-3) V + (K\sim+1)^2$$

$$-\frac{K\sim^2 - \mathrm{I}\sqrt{(K\sim^2+4K\sim+5)(3K\sim^2+4K\sim-1)} - 3}{2(K\sim+1)^2} \quad \textbf{(14.2.17)}$$

So that the expression in **(14.2.15)** can be factorized as f2K * AlephFactor, where:

$$> f2K := \frac{4(K+1)^5(K^2+8K+13)}{(3-K^2)(7K^2+20K+15)(K^2+4K+1)(K+3)};$$

$$f2K := \frac{4(K\sim+1)^5(K\sim^2+8K\sim+13)}{(-K\sim^2+3)(7K\sim^2+20K\sim+15)(K\sim^2+4K\sim+1)(K\sim+3)} \quad \textbf{(14.2.18)}$$

$$> AlephFactor := \frac{\left(V^2 - \frac{3-K^2}{(K+1)^2}V + 1\right)}{\left((V-Vp)\cdot\left(V-\frac{1}{Vp}\right)\right)^2};$$

$$AlephFactor := \frac{V^2 - \frac{(-K\sim^2+3)V}{(K\sim+1)^2} + 1}{(V-Vp)^2\left(V-\frac{1}{Vp}\right)^2} \quad \textbf{(14.2.19)}$$

The last factor is equal to:

$$> lastfactor := \left(factor\left(\frac{yUVsupc-1}{yUVsupc} + \frac{1}{2}\cdot\left(subs\left(V=Vp, \frac{1}{yUVsupc}\right) + subs\left(V=Vm, \frac{1}{yUVsupc}\right)\right)\right)\right):$$

$$denom(lastfactor);$$

$$16(Vm+1)Vm(Vp+1)Vp(K\sim^3+3K\sim^2+9K\sim+11)(K\sim+1)V(V+1) \quad \textbf{(14.2.20)}$$

\>

\>

To compute the integral, we perform a partial fraction decomposition of the denominator:

$$> convert\left(\frac{1}{V^2\cdot(V+1)^2\cdot(V-Vp)^2\cdot\left(\frac{1}{Vp}-V\right)}, fullparfrac, V, factor\right);$$

$$\frac{-Vp^3+Vp}{(Vp^2-1)^2 Vp^2 (Vp+1)^2 (V-Vp)^2} + \frac{5Vp^2-2Vp-2}{(Vp^2-1)^2 Vp^2 (Vp+1)^2 (V-Vp)} + \frac{1}{Vp V^2} + \frac{Vp^2-2Vp+2}{Vp^2 V} + \frac{Vp}{(Vp^3+3Vp^2+3Vp+1)(V+1)^2} + \frac{Vp(4+3Vp)}{(Vp^4+4Vp^3+6Vp^2+4Vp+1)(V+1)} \quad \textbf{(14.2.21)}$$

$$-\frac{Vp^6}{\left(Vp^2-1\right)^2 (Vp+1)^2 \left(V-\frac{1}{Vp}\right)}$$

> $coefVp2 := \frac{-Vp^3+Vp}{\left(Vp^2-1\right)^2 Vp^2 (1+Vp)^2}:$

$coefVp1 := \frac{5\,Vp^2-2\,Vp-2}{\left(Vp^2-1\right)^2 Vp^2 (1+Vp)^2}:$

$coefInvVp := -\frac{Vp^6}{(1+Vp)^2 \left(Vp^2-1\right)^2}:$

$coef02 := \frac{1}{Vp}:$

$coef01 := \frac{Vp^2-2\,Vp+2}{Vp^2}:$

$coefMinus12 := \frac{Vp}{\left(Vp^3+3\,Vp^2+3\,Vp+1\right)}:$

$coefMinus11 := \frac{Vp\,(4+3\,Vp)}{\left(Vp^4+4\,Vp^3+6\,Vp^2+4\,Vp+1\right)}:$

The elementary integrals that appear in the calculation (For some reason Maple simplifies better if we tell it z>0, but the result is the same if z<0):

> $\mathrm{psi} := \left[seq\left(\left[seq\left(simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left(\frac{\mathrm{sqrt}(x\cdot(1-x))}{(1-z\cdot x)^i}\cdot x^j, x=0..1\right)\right), j=0..6\right)\right], i=1..2\right)\right]$ assuming $z<1$ **and** $z>0$:

> $psineg := \left[seq\left(\left[seq\left(simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left(\frac{\mathrm{sqrt}(x\cdot(1-x))}{(1-z\cdot x)^i}\cdot x^j, x=0..1\right)\right), j=0..6\right)\right], i=1..2\right)\right]$ assuming $z<1$ **and** $z<0$:

> $\mathrm{psi}-psineg;$

$$[[0,0,0,0,0,0,0],[0,0,0,0,0,0,0]] \qquad \textbf{(14.2.22)}$$

For each pole with multiplicity, we will replace z by:

> $zpole := \frac{Vp^2-Vm^2}{pole-Vm^2};$

$$zpole := \frac{-Vm^2+Vp^2}{-Vm^2+pole} \qquad \textbf{(14.2.23)}$$

After the change of variables, the numerator becomes:

> $numerproba := collect\Big(simplify\Big(subs\Big(V=Vm^2+\left(Vp^2-Vm^2\right)\cdot x, \Big(V^2$

$$- \frac{3 - K^2}{(K+1)^2} V + 1 \Big) \cdot \Big(V - \frac{1}{Vm} \Big) \cdot numer(lastfactor) \cdot \big(Vp^2 - Vm^2 \big)^2 \Big) \Big), x, factor \Big)$$

:

The contributions of each monomial for the integral:

$$> psiint := \Big[seq\Big(simplify\Big(coefVp2 \cdot subs\Big(z = zpole, pole = Vp, \frac{\mathrm{psi}[2][j]}{(Vm^2 - pole)^2} \Big) + coefVp1 \cdot subs\Big(z = zpole, pole = Vp, \frac{\mathrm{psi}[1][j]}{(Vm^2 - pole)} \Big) + coefInvVp \cdot subs\Big(z = zpole, pole = \frac{1}{Vp}, \frac{\mathrm{psi}[1][j]}{(Vm^2 - pole)} \Big) + coefMinus12 \cdot subs\Big(z = zpole, pole = -1, \frac{\mathrm{psi}[2][j]}{(Vm^2 - pole)^2} \Big) + coefMinus11 \cdot subs\Big(z = zpole, pole = -1, \frac{\mathrm{psi}[1][j]}{(Vm^2 - pole)} \Big) + coef02 \cdot subs\Big(z = zpole, pole = 0, \frac{\mathrm{psi}[2][j]}{(Vm^2 - pole)^2} \Big) + coef01 \cdot subs\Big(z = zpole, pole = 0, \frac{\mathrm{psi}[1][j]}{(Vm^2 - pole)} \Big) \Big), j = 1..7 \Big) \Big] :$$

We just have to compute the global prefactor of the integral (not forgetting the term nu t^3 in the denominator and the denomiator of 'lastfactor':

$$> factor(subs(\mathrm{nu} = nusupK, U = UsupK, \mathrm{nu} \cdot wU));$$

$$\frac{(K\sim^2 - 3)^2 (K\sim + 1)(K\sim^2 + 8K\sim + 13)}{16 (K\sim + 3)(K\sim^3 + 3K\sim^2 + 9K\sim + 11)^2} \qquad \textbf{(14.2.24)}$$

$$> factor\Big(\frac{denom(lastfactor)}{V \cdot (V+1)} \Big);$$

$$16 (Vm + 1) Vm (Vp + 1) Vp (K\sim^3 + 3K\sim^2 + 9K\sim + 11)(K\sim + 1) \qquad \textbf{(14.2.25)}$$

$$> prefactor := factor\Big(\frac{f1K \cdot f2K}{\textbf{(14.2.25)} \cdot 2 \cdot \textbf{(14.2.24)}} \Big);$$

$$prefactor := -(K\sim + 1)^2 / \big(4 (K\sim^2 - 3) Vp (Vp + 1) Vm (Vm + 1)(K\sim^2 + 4K\sim + 1)(7K\sim^2 + 20K\sim + 15)\big) \qquad \textbf{(14.2.26)}$$

We finally have the probability that the cluster is infinite:

$$> Probaperco := 1 - (prefactor \cdot add(coeff(numerproba, x, j-1) \cdot psiint[j], j = 1..7)) :$$

$$> Probaperco2 := simplify(Probaperco) :$$

$$> Probaperco3 := simplify(expand(rationalize(simplify(subs(Vp = VK11, Vm = VK22, Probaperco2)))));$$

$$Probaperco3 := \Bigg(\Bigg(-24 \Big(-\frac{1}{8} \Big(51 \Big(K\sim^8 + \frac{704}{51} K\sim^7 + \frac{4204}{51} K\sim^6 + \frac{4800}{17} K\sim^5$$

$$+ 614 K\sim^4 + \frac{134080}{153} K\sim^3 + \frac{124292}{153} K\sim^2 + \frac{69440}{153} K\sim + \frac{17977}{153}\Big) \sqrt{2}\, (K\sim$$

$$+ 1) \sqrt{2 + K\sim}\Big) + K\sim^{10} + \frac{475 K\sim^9}{12} + \frac{2453 K\sim^8}{6} + \frac{6467 K\sim^7}{3} + 7048 K\sim^6$$

$$+ \frac{93281 K\sim^5}{6} + \frac{71735 K\sim^4}{3} + 25489 K\sim^3 + \frac{53953 K\sim^2}{3} + \frac{90683 K\sim}{12}$$

$$+ \frac{8629}{6}\Big) \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 81 \big(K\sim^2 + 4 K\sim$$

$$+ 5\big)\, (K\sim + 1) \Bigg(-\frac{1}{9} \Bigg(35 \Bigg(K\sim^8 + \frac{512}{45} K\sim^7 + \frac{17452}{315} K\sim^6 + \frac{16384}{105} K\sim^5$$

$$+ \frac{18106}{63} K\sim^4 + \frac{112384}{315} K\sim^3 + \frac{29924}{105} K\sim^2 + \frac{5888}{45} K\sim + \frac{8443}{315}\Bigg)$$

$$\sqrt{2}\, \sqrt{2 + K\sim}\Bigg) + K\sim^9 + \frac{67 K\sim^8}{3} + \frac{4540 K\sim^7}{27} + \frac{18332 K\sim^6}{27} + \frac{15430 K\sim^5}{9}$$

$$+ \frac{25894 K\sim^4}{9} + \frac{264596 K\sim^3}{81} + \frac{66172 K\sim^2}{27} + \frac{29611 K\sim}{27} + \frac{17377}{81}\Bigg)\Bigg)$$

$$\Bigg(\Bigg(\big(-15 K\sim^4 - 64 K\sim^3 - 102 K\sim^2 - 64 K\sim - 7\big) \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 27 \Bigg(K$$

$$+ 58 K\sim^2 + 80 K\sim + 41\Big) \sqrt{K\sim^2 + 4 K\sim + 5}\,\sqrt{3 K\sim^2 + 4 K\sim - 1} + 3 \Bigg($$

$$- \frac{8\sqrt{2}\,(K\sim + 1)^3 \sqrt{2 + K\sim}}{3} + \left(K\sim^2 + 4 K\sim + 1\right)\left(K\sim^2 + 4 K\sim + \frac{11}{3}\right)\Bigg)$$

$$\left(K\sim^2 + 4 K\sim + 5\right)\Bigg)\Bigg)^{1/2} - 96\left(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\right)\Bigg(\frac{1}{3}\Bigg(4\Bigg($$

$$- \frac{\left(-2\sqrt{2 + K\sim} + \sqrt{2}\,(K\sim + 1)\right)\sqrt{K\sim^2 + 4 K\sim + 5}\,\sqrt{3 K\sim^2 + 4 K\sim - 1}}{4}$$

$$+ \left(\left(-K\sim - 1\right)\sqrt{2 + K\sim} + \sqrt{2}\,(2 + K\sim)\right)(K\sim + 1)\Bigg)$$

$$\sqrt{-\sqrt{K\sim^2 + 4 K\sim + 5}\,\sqrt{3 K\sim^2 + 4 K\sim - 1} + 2\,(K\sim + 1)^2}$$

$$\sqrt{\left(K\sim^2 + 4 K\sim - 2\sqrt{2}\,(K\sim + 1)\sqrt{2 + K\sim} + 5\right)\left(K\sim^2 + 4 K\sim + 5\right)}\Bigg)$$

$$- \frac{1}{3}\Big(\Big(-4\sqrt{2}\,(K\sim + 1)\left(K\sim^2 + 4 K\sim + 5\right)\sqrt{2 + K\sim} + K\sim^4 + 16 K\sim^3$$

$$+ 58 K\sim^2 + 80 K\sim + 41\Big)\sqrt{K\sim^2 + 4 K\sim + 5}\,\sqrt{3 K\sim^2 + 4 K\sim - 1}\Big) + \Bigg($$

$$- \frac{8\sqrt{2}\,(K\sim + 1)^3 \sqrt{2 + K\sim}}{3} + \left(K\sim^2 + 4 K\sim + 1\right)\left(K\sim^2 + 4 K\sim + \frac{11}{3}\right)\Bigg)$$

$$\left(K\sim^2 + 4 K\sim + 5\right)\Bigg)(K\sim + 1)^3\Bigg) \Bigg/ \Bigg(84\left(K\sim^2 + 4 K\sim + 1\right)\big(K\sim^2$$

$$- 3\big)\left(K\sim^2 + \frac{20}{7} K\sim + \frac{15}{7}\right)\Bigg(-\frac{1}{3}\Big(\Big(-4\sqrt{2}\,(K\sim + 1)\big(K\sim^2 + 4 K\sim$$

$$+ 5\big)\sqrt{2 + K\sim} + K\sim^4 + 16 K\sim^3 + 58 K\sim^2 + 80 K\sim + 41\Big)$$

$$\sqrt{K\sim^2 + 4 K\sim + 5}\,\sqrt{3 K\sim^2 + 4 K\sim - 1}\Big) + \Bigg(- \frac{8\sqrt{2}\,(K\sim + 1)^3 \sqrt{2 + K\sim}}{3}$$

$$+ \left(K\sim^2 + 4 K\sim + 1\right)\left(K\sim^2 + 4 K\sim + \frac{11}{3}\right)\Bigg)\left(K\sim^2 + 4 K\sim + 5\right)\Bigg)\Bigg)$$

> *Probaperco4* := *simplify*(*expand*(*rationalize*(*Probaperco3*)));

$$Probaperco4 := \Bigg(57\,(K\sim^2 - 3)\Bigg(\frac{1}{19}\Bigg(\sqrt{K\sim^2+4\,K\sim+5}\,\Bigg(\frac{1}{3}\Big(16\sqrt{2}\,(K\sim+1)\,\big(K\sim^4+K\sim^3+2\,K\sim^2 + 13\,K\sim+13\big)\sqrt{2+K\sim}\Big) + K\sim^6 + \frac{16\,K\sim^5}{3} - \frac{107\,K\sim^4}{3} - \frac{544\,K\sim^3}{3} - 261\,K\sim^2 - 144\,K\sim - \frac{97}{3}\Bigg)\sqrt{3\,K\sim^2+4\,K\sim-1}\Bigg) - \frac{1}{19}\Bigg(18\sqrt{2}\,\Bigg(K\sim^6 + \frac{136}{27}\,K\sim^5 + 7\,K\sim^4 - \frac{368}{27}\,K\sim^3 - \frac{1639}{27}\,K\sim^2 - \frac{680}{9}\,K\sim - \frac{841}{27}\Bigg)(K\sim + 1)\sqrt{2+K\sim}\Bigg) - \frac{4516\,K\sim^2}{19} + \frac{104\,K\sim^5}{19} - \frac{8920\,K\sim}{57} + 8\,K\sim^7 - \frac{2711}{57} + K\sim^8 - \frac{11624\,K\sim^3}{57} - \frac{254\,K\sim^4}{3} + \frac{1204\,K\sim^6}{57}\Bigg)\big(K\sim^2 + 4\,K\sim$$

$$+ 5)$$

$$\Bigg(\Big(\big(-15\,K\sim^4 - 64\,K\sim^3 - 102\,K\sim^2 - 64\,K\sim - 7\big)\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 27\,\Big(K\sim + \frac{7}{3}\Big)^2\Big) \Big/ \Big(-\big(-4\sqrt{2}\,(K\sim + 1)\,(K\sim^2 + 4\,K\sim + 5)\sqrt{2 + K\sim} + K\sim^4 + 16\,K\sim^3 + 58\,K\sim^2 + 80\,K\sim + 41\big)\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 3\,\Big(-\frac{8\sqrt{2}\,(K\sim + 1)^3\sqrt{2 + K\sim}}{3} + (K\sim^2 + 4\,K\sim + 1)\,\Big(K\sim^2 + 4\,K\sim + \frac{11}{3}\Big)\Big)\,(K\sim^2 + 4\,K\sim + 5)\Big)\Bigg)^{1/2} - 64\,(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11)\,(K\sim + 1)^3\,\Big(-\frac{1}{2}\Big(\big(2\sqrt{2 + K\sim} + \sqrt{2}\,(K\sim + 1)\big)\sqrt{-\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 2\,(K\sim + 1)^2}\,\big(K\sim^2 + \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 4\,K\sim + 5\big)\sqrt{\big(K\sim^2 + 4\,K\sim - 2\sqrt{2}\,(K\sim + 1)\sqrt{2 + K\sim} + 5\big)\,(K\sim^2 + 4\,K\sim + 5)}\Big) + (K\sim^2 + 4\,K\sim + 5)\,(K\sim^2 - 3)^2\Big)\Big) \Big/ \Big(56\,(K\sim^2 - 3)^3\,(K\sim^2 + 4\,K\sim + 1)\,\Big(K\sim^2 + \frac{20}{7}\,K\sim + \frac{15}{7}\Big)\,(K\sim^2 + 4\,K\sim + 5)\Big)$$

> *den4* := *denom*(*Probaperco4*);

$$den4 := 8\,(K\sim^2 - 3)^3\,(K\sim^2 + 4\,K\sim + 1)\,(7\,K\sim^2 + 20\,K\sim + 15)\,(K\sim^2 + 4\,K\sim + 5) \qquad \textbf{(14.2.29)}$$

> $num4 := simplify(numer(Probaperco4)); nops(\%);$

$$num4 := 57\left((K\sim^2 - 3)\left(\frac{1}{19}\left(\sqrt{K\sim^2+4K\sim+5}\left(\frac{1}{3}\left(16\sqrt{2}\,(K\sim+1)\,(K\sim^4+K\sim^3+2K\sim^2+13K\sim+13)\right)\sqrt{2+K\sim}\right)+K\sim^6+\frac{16K\sim^5}{3}-\frac{107K\sim^4}{3}-\frac{544K\sim^3}{3}-261K\sim^2-144K\sim-\frac{97}{3}\right)\sqrt{3K\sim^2+4K\sim-1}\right)-\frac{1}{19}\left(18\sqrt{2}\left(K\sim^6+\frac{136}{27}K\sim^5+7K\sim^4-\frac{368}{27}K\sim^3-\frac{1639}{27}K\sim^2-\frac{680}{9}K\sim-\frac{841}{27}\right)(K\sim+1)\sqrt{2+K\sim}\right)-\frac{4516K\sim^2}{19}+\frac{104K\sim^5}{19}-\frac{8920K\sim}{57}+8K\sim^7-\frac{2711}{57}+K\sim^8-\frac{11624K\sim^3}{57}-\frac{254K\sim^4}{3}+\frac{1204K\sim^6}{57}\right)$$

$$\Big(\Big(\big(-15\,K\sim^4 - 64\,K\sim^3 - 102\,K\sim^2 - 64\,K\sim - 7\big)\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 27\,\big(K$$

$$+ 5\big)\sqrt{2 + K\sim} + K\sim^4 + 16\,K\sim^3 + 58\,K\sim^2 + 80\,K\sim + 41\big)$$

$$\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 3\,\Big(-\frac{8\sqrt{2}\,(K\sim + 1)^3\sqrt{2 + K\sim}}{3}$$

$$+ \big(K\sim^2 + 4\,K\sim + 1\big)\Big(K\sim^2 + 4\,K\sim + \frac{11}{3}\Big)\Big)\big(K\sim^2 + 4\,K\sim + 5\big)\Big)^{1/2}\Big)$$

$$- \frac{1}{57}\Big(64\,\big(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11\big)\Big(-\frac{1}{2}\Big(\big(2\sqrt{2 + K\sim} + \sqrt{2}\,(K\sim$$

$$+ 1)\big)\sqrt{K\sim^2 + 4\,K\sim - 2\sqrt{2}\,(K\sim + 1)\sqrt{2 + K\sim} + 5}\,\Big(\sqrt{K\sim^2 + 4\,K\sim + 5}$$

$$+ \sqrt{3\,K\sim^2 + 4\,K\sim - 1}\Big)$$

$$\sqrt{-\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 2\,(K\sim + 1)^2}\Big) + \big(K\sim^2 - 3\big)^2\Big)$$

$$(K\sim + 1)^3\Big)\Big)\big(K\sim^2 + 4\,K\sim + 5\big)$$

$$3 \qquad \textbf{(14.2.30)}$$

The secondterm in the big factor is actually 0:

> *num42* := *simplify*(*op*(2, *op*(2, *num4*))) assuming $K > Kc$ **and** $K < Kinfini$;

$$num42 := \frac{1}{57}\Big(32\,\Big(\big(2\sqrt{2 + K\sim} + \sqrt{2}\,(K\sim \qquad \textbf{(14.2.31)}$$

$$+ 1)\big)\sqrt{K\sim^2 + 4\,K\sim - 2\sqrt{2}\,(K\sim + 1)\sqrt{2 + K\sim} + 5}\,\Big(\sqrt{K\sim^2 + 4\,K\sim + 5}$$

$$+ \sqrt{3\,K\sim^2 + 4\,K\sim - 1}\Big)$$

$$\sqrt{-\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 2\,(K\sim + 1)^2} - 2\,\big(K\sim^2 - 3\big)^2\Big)$$

$$\big(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11\big)\,(K\sim + 1)^3\Big)$$

> $factor\Big(expand\Big(simplify\Big(\Big(\big(2\sqrt{2 + K} + \sqrt{2}\,(K + 1)\big)\Big(\sqrt{K^2 + 4\,K + 5}$

$+ \sqrt{3\,K^2 + 4\,K - 1}\Big)$

$$\sqrt{K^2+4K-2\sqrt{2}\,(K+1)\sqrt{2+K}+5}$$
$$\sqrt{-\sqrt{K^2+4K+5}\,\sqrt{3K^2+4K-1}+2(K+1)^2}\,\Big)^2\Big)\Big)\Big);$$

$$4\left(K\sim^2-3\right)^4 \qquad \textbf{(14.2.32)}$$

Let's look at the first factor:

> *num41* := *simplify*(*op*(1, *num4*)·*op*(3, *num4*)·*op*(1, *op*(2, *num4*))) assuming $K > Kc$ **and** $K < Kinfini$;

$$num41 := \left(K\sim^2+4K\sim+5\right)\left(K\sim^2-3\right)\Bigg(3\sqrt{K\sim^2+4K\sim+5}\,\Bigg(\frac{1}{3}\Big(16\sqrt{2}\,(K\sim+1)\left(K\sim^4+K\sim^3+2K\sim^2+13K\sim+13\right)\sqrt{2+K\sim}\Big)+K\sim^6+\frac{16K\sim^5}{3}-\frac{107K\sim^4}{3}-\frac{544K\sim^3}{3}-261K\sim^2-144K\sim-\frac{97}{3}\Bigg)\sqrt{3K\sim^2+4K\sim-1}-54\sqrt{2}\left(K\sim^6+\frac{136}{27}K\sim^5+7K\sim^4-\frac{368}{27}K\sim^3-\frac{1639}{27}K\sim^2-\frac{680}{9}K\sim-\frac{841}{27}\right)(K\sim+1)\sqrt{2+K\sim}+57K\sim^8+456K\sim^7+1204K\sim^6+312K\sim^5-4826K\sim^4-11624K\sim^3-13548K\sim^2-8920K\sim-2711\Bigg)$$
$$\Bigg(\Bigg(\left(-15K\sim^4-64K\sim^3-102K\sim^2-64K\sim-7\right)\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+27\left(K\sim+1\right)\left(K\sim^2+4K\sim+5\right)\sqrt{2+K\sim}+K\sim^4+16K\sim^3+58K\sim^2+80K\sim+41\Bigg)\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+3\left(-\frac{8\sqrt{2}\,(K\sim+1)^3\sqrt{2+K\sim}}{3}+\left(K\sim^2+4K\sim+1\right)\left(K\sim^2+4K\sim+\frac{11}{3}\right)\right)\left(K\sim^2+4K\sim+5\right)\Bigg)^{1/2}$$

> *rootProba* := *simplify*(*expand*(*rationalize*($num41^2$)));

$$rootProba := -6960\left(K\sim^2-3\right)^3\Bigg(\Bigg(\frac{1}{145}\Big((2+K\sim)^{3/2}\left(145K\sim^8+1160K\sim^7\right. \qquad \textbf{(14.2.34)}$$

$$+ 4612 K\sim^6 + 11832 K\sim^5 + 23430 K\sim^4 + 39000 K\sim^3 + 46916 K\sim^2 + 31912 K\sim$$

$$+ 8753) \sqrt{2}) + \frac{1}{145}\Big(162 (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11) \Big(K\sim^8 + 8 K\sim^7$$

$$+ \frac{76}{3} K\sim^6 + \frac{2872}{81} K\sim^5 + \frac{518}{81} K\sim^4 - \frac{3752}{81} K\sim^3 - \frac{5308}{81} K\sim^2 - \frac{3416}{81} K\sim$$

$$- \frac{1039}{81}\Big)\Big)\Big) \sqrt{3 K\sim^2 + 4 K\sim - 1}\ (K\sim + 1) \sqrt{K\sim^2 + 4 K\sim + 5}$$

$$- \frac{1}{145}\Big(288 (K\sim^2 + 4 K\sim + 5) \left(K\sim^2 + \frac{4}{3} K\sim - \frac{1}{3}\right)^2 \Big(\sqrt{2}\ (2 + K\sim)^{3/}$$

$$^{2} (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11) (K\sim + 1)^2 + \frac{307 K\sim^8}{256} + \frac{307 K\sim^7}{32}$$

$$+ \frac{2179 K\sim^6}{64} + \frac{2133 K\sim^5}{32} + \frac{10185 K\sim^4}{128} + \frac{2337 K\sim^3}{32} + \frac{4211 K\sim^2}{64}$$

$$+ \frac{1343 K\sim}{32} + \frac{2579}{256}\Big)\Big)\Big) \left(K\sim^2 + \frac{8}{3} K\sim + \frac{7}{3}\right) (K\sim^2 + 4 K\sim + 5)$$

> *den4*;

$$8 (K\sim^2 - 3)^3 (K\sim^2 + 4 K\sim + 1) (7 K\sim^2 + 20 K\sim + 15) (K\sim^2 + 4 K\sim + 5) \qquad \textbf{(14.2.35)}$$

> $Probapercosimple := \Big(\mathrm{sqrt}\Big(-6960 \Big(\sqrt{3 K\sim^2 + 4 K\sim - 1} \Big(\frac{1}{145} ((2 + K\sim)^{3/}$

$$^{2} (145 K\sim^8 + 1160 K\sim^7 + 4612 K\sim^6 + 11832 K\sim^5 + 23430 K\sim^4 + 39000 K\sim^3$$

$$+ 46916 K\sim^2 + 31912 K\sim + 8753) \sqrt{2}) + \frac{1}{145}\Big(162 (K\sim^3 + 3 K\sim^2 + 9 K\sim$$

$$+ 11) \Big(K\sim^8 + 8 K\sim^7 + \frac{76}{3} K\sim^6 + \frac{2872}{81} K\sim^5 + \frac{518}{81} K\sim^4 - \frac{3752}{81} K\sim^3$$

$$- \frac{5308}{81} K\sim^2 - \frac{3416}{81} K\sim - \frac{1039}{81}\Big)\Big)\Big) (K\sim + 1) \sqrt{K\sim^2 + 4 K\sim + 5}$$

$$- \frac{1}{145}\Big(288 (K\sim^2 + 4 K\sim + 5) \Big(\sqrt{2}\ (2 + K\sim)^{3/2} (K\sim^3 + 3 K\sim^2 + 9 K\sim$$

$$+ 11) (K\sim + 1)^2 + \frac{307 K\sim^8}{256} + \frac{307 K\sim^7}{32} + \frac{2179 K\sim^6}{64} + \frac{2133 K\sim^5}{32}$$

$$+\frac{10185\,K\sim^4}{128}+\frac{2337\,K\sim^3}{32}+\frac{4211\,K\sim^2}{64}+\frac{1343\,K\sim}{32}+\frac{2579}{256}\Big)\Big(K\sim^2+\frac{4}{3}K\sim-\frac{1}{3}\Big)^2\Big)\Big)\left(K\sim^2+4\,K\sim+5\right)\left(K\sim^2-3\right)^3\left(K\sim^2+\frac{8}{3}K\sim+\frac{7}{3}\right)\Big)\Big)\Big/\left(8\left(K\sim^2+4\,K\sim+5\right)\left(K\sim^2+4\,K\sim+1\right)\left(7\,K\sim^2+20\,K\sim+15\right)\left(3-K\sim^2\right)^3\right):$$

We do an expansion at Kc:

> *Vpser* := *collect*(*map*(*expand*, *map*(*rationalize*, *convert*(*series*(*subs*(*K* = *Kc* + *KK*^4, *VK11*), *KK*, 9), *polynom*))), *KK*, *factor*) assuming *KK* > 0;

$$Vpser := 1+\left(\frac{103}{4}-\frac{37\sqrt{7}}{4}\right)KK^8+\left(\frac{65\sqrt{7+4\sqrt{7}}}{12}-\frac{367\sqrt{7+4\sqrt{7}}\sqrt{7}}{168}\right)KK^6+\left(2\sqrt{7}-\frac{7}{2}\right)KK^4+\left(-\frac{4\sqrt{7+4\sqrt{7}}}{3}+\frac{\sqrt{7+4\sqrt{7}}\sqrt{7}}{3}\right)KK^2 \qquad \textbf{(14.2.36)}$$

> *Vmser* := *collect*(*map*(*expand*, *map*(*rationalize*, *convert*(*series*(*subs*(*K* = *Kc* + *KK*^4, *VK22*), *KK*, 9), *polynom*))), *KK*, *factor*) assuming *KK* > 0;

$$Vmser := \left(-\frac{97\sqrt{7}\sqrt{3}}{18}+\frac{1075\sqrt{3}}{72}-\frac{103}{4}+\frac{37\sqrt{7}}{4}\right)KK^8+\left(-\frac{7\sqrt{3}}{3}+\frac{4\sqrt{7}\sqrt{3}}{3}-2\sqrt{7}+\frac{7}{2}\right)KK^4-2+\sqrt{3} \qquad \textbf{(14.2.37)}$$

> *map*(*simplify*, *series*(*subs*(*Vp* = *Vpser*, *Vm* = *Vmser*, *K* = *Kc* + *KK*^4, *Probapercosimple*), *KK*, 7)) assuming *KK* > 0;

$$\left(3\,7^{1/8}\,2^{3/4}\sqrt{\left(2140679\sqrt{7}+5663177\right)\sqrt{3}-2344320\sqrt{7}-6203376}\,\left(1+\sqrt{7}\right)^2\sqrt{3}\right)\Big/\left(32\left(8\sqrt{7}+23\right)\left(-1+2\sqrt{7}\right)\left(4+\sqrt{7}\right)^2\right)KK-\frac{9}{1792}\left(\sqrt{3}\,7^{1/8}\,2^{3/4}\left(86730850565683\sqrt{7}\sqrt{3}-104983600061232\sqrt{7}+229468259516419\sqrt{3}-277760499103584\right)\right)\Big/\left(\sqrt{\left(2140679\sqrt{7}+5663177\right)\sqrt{3}-2344320\sqrt{7}-6203376}\,\left(4+\sqrt{7}\right)^6\left(-1+2\sqrt{7}\right)^2\left(8\sqrt{7}+23\right)^2\right)KK^5+O\left(KK^7\right) \qquad \textbf{(14.2.38)}$$

> *collect*(*simplify*(*expand*(*rationalize*(*convert*(**(14.2.38)**, *polynom*)))), *KK*, *factor*);

$$\Big(\frac{1}{35659927296}\Big(1706786219\sqrt{3}\,7^{5/8}\,2^{3/4} \qquad \textbf{(14.2.39)}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)$$

$$- \frac{1}{5896152}\Big(454837\, 7^{5/8}\, 2^{3/4}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)$$

$$- \frac{1}{636784416}\Big(80677003\sqrt{3}\, 7^{1/8}\, 2^{3/4}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)$$

$$+ \frac{1}{1474038}\Big(300865\, 7^{1/8}\, 2^{3/4}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)\Big) KK^5$$

$$+ \Bigg(\frac{1}{104976}\Big(241\sqrt{3}\, 7^{5/8}\, 2^{3/4}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)$$

$$- \frac{1}{52488}\Big(311\sqrt{3}\, 7^{1/8}\, 2^{3/4}$$

$$\sqrt{2140679\sqrt{7}\sqrt{3} - 2344320\sqrt{7} + 5663177\sqrt{3} - 6203376}\Big)\Bigg) KK$$

We have the expansion of nu in terms of KK:

> *expand*(*map*(*rationalize*, *series*(*subs*($K = Kc + KK^4$, *nusupK*), *KK*, 5)))

$$1 + \frac{\sqrt{7}}{7} + \left(-\frac{9\sqrt{7}}{14} + \frac{18}{7}\right) KK^4 + O\left(KK^8\right) \qquad \textbf{(14.2.40)}$$

The coefficient in front of (nu - nu_c)^{1/4} in the expansion:

> *simplify*$\left(\textit{expand}\left(\textit{rationalize}\left(\frac{\textit{coeff}(\textbf{(14.2.39)}, KK, 1)}{\left(\frac{18}{7} - \frac{9\sqrt{7}}{14}\right)^{\frac{1}{4}}}\right)\right)\right)$;

$$\frac{1}{4408992}\Big(\sqrt{3}\, 7^{1/8}\, 2^{3/4} \qquad \textbf{(14.2.41)}$$

$$\sqrt{\left(2140679\sqrt{7} + 5663177\right)\sqrt{3} - 2344320\sqrt{7} - 6203376}\,\sqrt{294 - 42\sqrt{7}}$$

$$\left(355\sqrt{7} - 889\right)\Big)$$

A simpler expression for this coefficient:

> $simplify\Big($ **(14.2.41)**

$$-\frac{\sqrt{3+2\sqrt{3}}\,2^{3/4}\,7^{3/8}\left(-10\sqrt{3}\sqrt{7}+18\sqrt{7}-23\sqrt{3}+63\right)}{144}\Big);$$

$$0 \qquad \textbf{(14.2.42)}$$

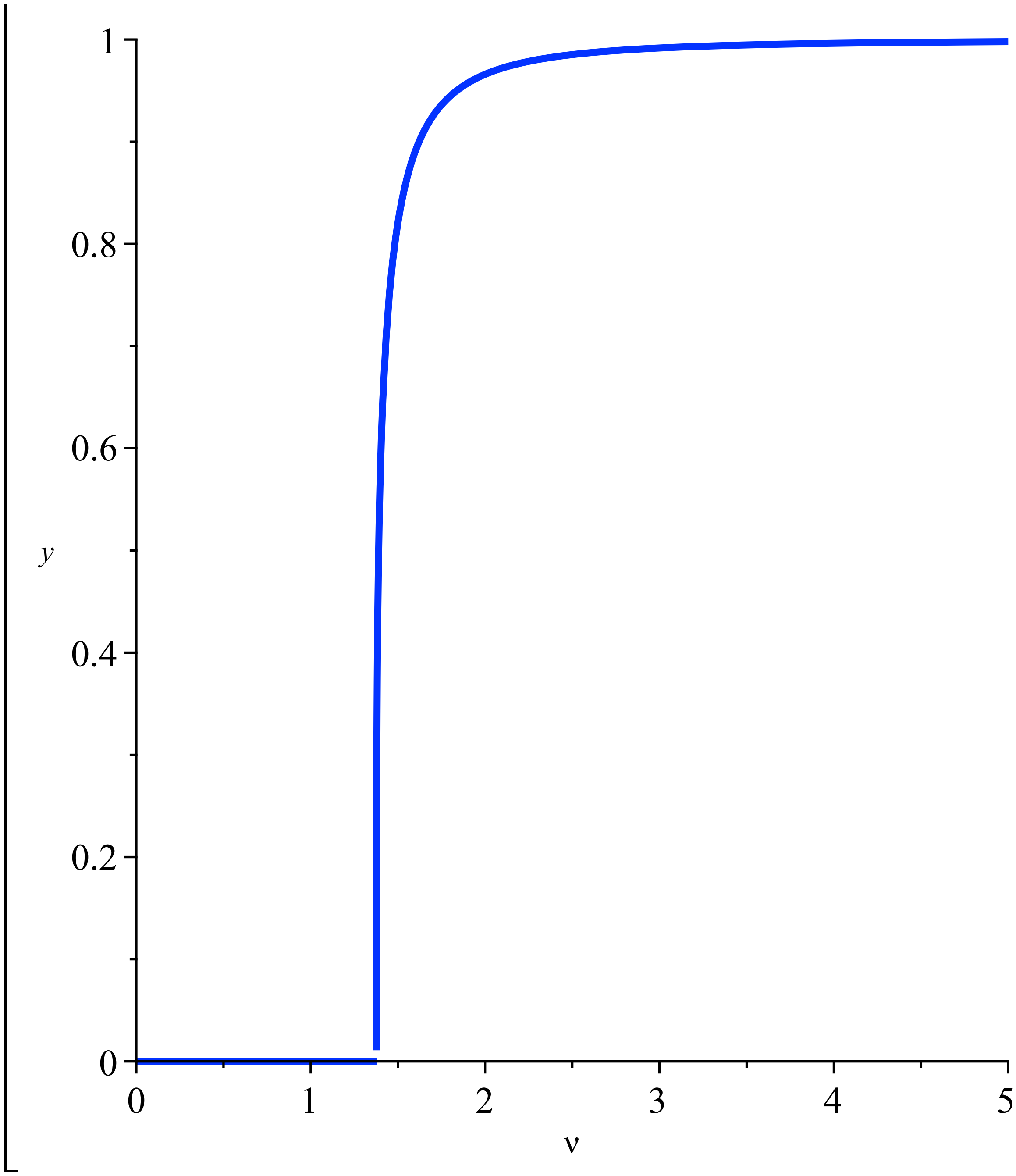
We want a plot of the probability in terms of nu, so we need K in terms of nu:

> $Knu := RootOf(numer(nusupK - \mathrm{nu}), K);$

$$Knu := RootOf\left((\nu+1)\,\_Z^3 + (3\,\nu+3)\,\_Z^2 + (-3\,\nu+9)\,\_Z - 9\,\nu + 11\right) \qquad \textbf{(14.2.43)}$$

> $Plotnusupc := plot(subs(Vp = VK11, Vm = VK22, K = Knu, Probapercosimple), \mathrm{nu} = nuc\,..5, y = 0\,..1, color = \text{"Blue"}, thickness = 3) : Plotnusubc := plot(0, \mathrm{nu} = 0\,..\,nuc, color = \text{"Blue"}, thickness = 3) :$

> $display(\{Plotnusupc, Plotnusubc\});$



▶ Hypergeometric functions and their singular expansion in Theorem 1.2

▼ Boltzmann maps of the cylinder (Appendix A.1)

The rational parametrization of z in terms of x in Eynard's book is given via the following relation (where zd = z^\diamond and zp = \sqrt{z^+})

> $zx := zd + zp \cdot \left(x + \frac{1}{x}\right)$:

Then the expression for the cylinder generating function in terms of x1 and x2 given in Eynard's book is equal to:

> $simplify\left(-\frac{1}{(x1\cdot x2-1)^2}\cdot\frac{1}{subs(x=x1, diff(zx,x))\cdot subs(x=x2, diff(zx,x))}\right);$

$$-\frac{x1^2\,x2^2}{(x1\,x2-1)^2\,zp^2\,\left(x1^2-1\right)\left(x2^2-1\right)} \quad \textbf{(16.1)}$$

The formula appearing in the Proposition 4.6 of the paper reads:

> $Wprop := \frac{1}{2\cdot(z1-z2)^2}\cdot\left(Wqz1\cdot Wqz2\cdot\left(z1\cdot z2-\frac{cp+cm}{2}\cdot(z1+z2)+cp\cdot cm\right)-1\right):$

Recall from (4.8) that:

> $Wqz := \frac{1}{\mathrm{sqrt}((z-cp)\cdot(z-cm))}:$

Replacing cp and cm by their value in terms of zd and zp, and using the rational parametrization for z, we get the following expression for Wqz:

> $Wqzx := simplify(subs(z=zx, subs(cp=zd+2\cdot zp, cm=zd-2\cdot zp, Wqz)), symbolic);$

$$Wqzx := \frac{x}{zp\,\left(x^2-1\right)} \quad \textbf{(16.2)}$$

Finally, replacing all quantities in Wprop by their expression in the variables x1 and x2, we get:

> $simplify(subs(z1=subs(x=x1, zx), z2=subs(x=x2, zx), cp=zd+2\cdot zp, cm=zd-2\cdot zp,$
> $Wqz1=subs(x=x1, Wqzx), Wqz2=subs(x=x2, Wqzx), Wprop));$

$$\frac{x2^2\,x1^2}{zp^2\,\left(x2^2-1\right)\left(x1^2-1\right)(x1\,x2-1)^2} \quad \textbf{(16.3)}$$