```
> restart;
> with(algcurves) : with(gfun) : with(plots) : with(CurveFitting) :
```

Quantities, equations and parametrizations from the previous paper [AMS 21] (Section 2.2)

Equation for simple monochromatic boundary condition Z(t,y), equation 2.23 of [AMS 21]

$$> eqZ := 2 t^2 \nu (\nu-1) Z^3 + t\left(\nu^2 t^2 y^2 + (4\nu^2 - 4\nu) t - y(\nu+2)(\nu-1)\right) Z^2 + \left(2\nu^2 t^3 y^2 + 2\left((y^3 - Z1\, y + 1)\nu - \frac{3}{2} y^3 + Z1\, y - 1\right)\nu t^2 - y t(\nu+2)(\nu-1) + y^2(\nu-1)\right) Z + y t\left(y^2 \nu^2 (y^2 - 2 Z1) t^2 + (\nu-1)\left(y^2 + (-2 Z1^2 - 2 Z2) y - 2 Z1\right)\nu t - y(\nu-1)\left((y^2 - Z1)\nu - 2 Z1\right)\right) :$$

Rational parametrization

$$> P := collect\left((8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4\nu), U, factor\right);$$

$$P := 8(\nu+1)^2 U^3 - (11\nu+13)(\nu+1) U^2 + 2(\nu+3)(2\nu+1) U - 4\nu \qquad (1.1)$$

Since all the generating series in t that we consider are actually series in t^3, we set **w := t^3.**
We have the following parametrization of w in terms of U:

$$> wU := \frac{1}{32} \frac{(U\cdot(\nu+1)-2)\, U\cdot P}{(-1+2U)^2 \nu^3} :$$

$$> tZ1U := \frac{1}{2}\left((6 U^3 \nu^2 + 12 U^3 \nu - 8 U^2 \nu^2 + 6 U^3 - 16 U^2 \nu + 3 U \nu^2 - 8 U^2 + 7 U \nu + 4 U - 2\nu) U (\nu+1)\right) \Big/ \left((8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4\nu)\nu\right) :$$

$$> t2Z2U := -\frac{1}{64}\left(U(U\nu + U - 2)\left(1152 U^8 \nu^5 + 5760 U^8 \nu^4 - 4872 U^7 \nu^5 + 11520 U^8 \nu^3 - 23832 U^7 \nu^4 + 8589 U^6 \nu^5 + 11520 U^8 \nu^2 - 46608 U^7 \nu^3 + 42693 U^6 \nu^4 - 8084 U^5 \nu^5 + 5760 U^8 \nu - 45552 U^7 \nu^2 + 83158 U^6 \nu^3 - 43450 U^5 \nu^4 + 4288 U^4 \nu^5 + 1152 U^8 - 22248 U^7 \nu + 79206 U^6 \nu^2 - 85872 U^5 \nu^3 + 27556 U^4 \nu^4 - 1216 U^3 \nu^5 - 4344 U^7 + 36765 U^6 \nu - 78884 U^5 \nu^2 + 56384 U^4 \nu^3 - 11072 U^3 \nu^4 + 144 U^2 \nu^5 + 6613 U^6 - 33532 U^5 \nu + 49088 U^4 \nu^2 - 24320 U^3 \nu^3 + 2640 U^2 \nu^4 - 5154 U^5 + 18048 U^4 \nu - 19480 U^3 \nu^2 + 6928 U^2 \nu^3 - 288 U \nu^4 + 2076 U^4 - 5520 U^3 \nu + 4704 U^2 \nu^2 - 1280 U \nu^3 - 344 U^3 + 752 U^2 \nu - 544 U \nu^2 + 128 \nu^3\right)\right) \Big/ \left(\nu^2 (8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4\nu)^2 (-1+2U)^2\right) :$$

```
> algU := collect(numer(wU - w), U, factor) :
```

The two polynoms giving the radius of convergence rho

> $algrhosubc := 27648\,\nu^4 w^2 + 864\,\nu\,(\nu-1)\left(\nu^2-2\,\nu-1\right) w + \left(7\,\nu^2-14\,\nu-9\right)(-2+\nu)^2;$

$$algrhosubc := 27648\,\nu^4 w^2 + 864\,\nu\,(\nu-1)\left(\nu^2-2\,\nu-1\right) w + \left(7\,\nu^2-14\,\nu-9\right)(-2+\nu)^2 \quad \textbf{(1.2)}$$

> $algrhosupc := 131072\,\nu^9 w^3 - 192\,\nu^6\,(3\,\nu+5)\,(\nu-1)\,(3\,\nu-11)\,w^2 - 48\,\nu^3\,(\nu-1)^2\,w + (\nu-1)\left(4\,\nu^2-8\,\nu-23\right);$

$$algrhosupc := 131072\,\nu^9 w^3 - 192\,\nu^6\,(3\,\nu+5)\,(\nu-1)\,(3\,\nu-11)\,w^2 - 48\,\nu^3\,(\nu-1)^2\,w + (\nu-1)\left(4\,\nu^2-8\,\nu-23\right) \quad \textbf{(1.3)}$$

Phase transition at

> $nuc := 1 + \frac{1}{\mathrm{sqrt}(7)};\ Uc := \frac{5}{9} - \frac{\sqrt{7}}{9};$

$$nuc := 1 + \frac{\sqrt{7}}{7}$$

$$Uc := \frac{5}{9} - \frac{\sqrt{7}}{9} \quad \textbf{(1.4)}$$

> $rhoc := simplify(rationalize(subs(\mathrm{nu} = nuc, rhosubc)));$
$simplify(rationalize(subs(U = Uc, \mathrm{nu} = nuc, wU)));$

$$rhoc := rhosubc$$

$$-\frac{55}{864} + \frac{25\sqrt{7}}{864} \quad \textbf{(1.5)}$$

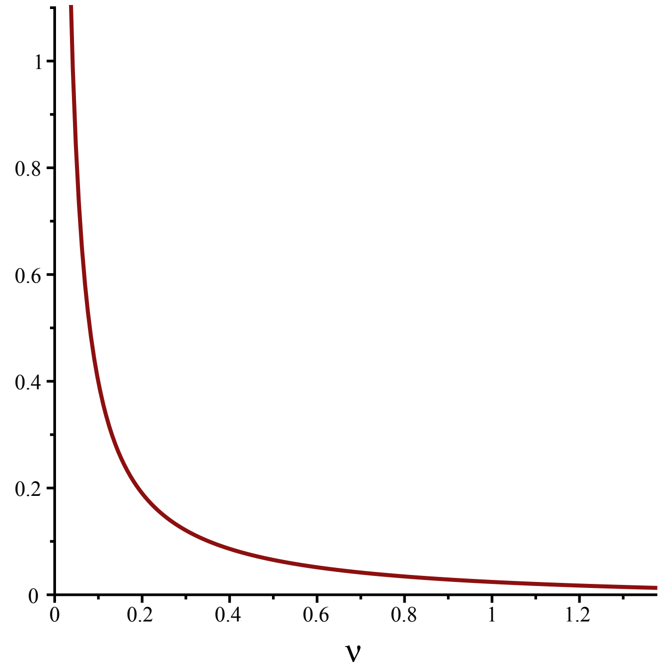
For nu subcritical, we can obtain an explicit expression for the radius of convergence:

> $rhosubc1, rhosubc2 := solve(algrhosubc, w) :$
$simplify(rhosubc1); simplify(rhosubc2);$

$$\frac{-9\,\nu^3 + 27\,\nu^2 + \sqrt{3}\,\sqrt{-\left(\nu^2-2\,\nu-3\right)^3} - 9\,\nu - 9}{576\,\nu^3}$$

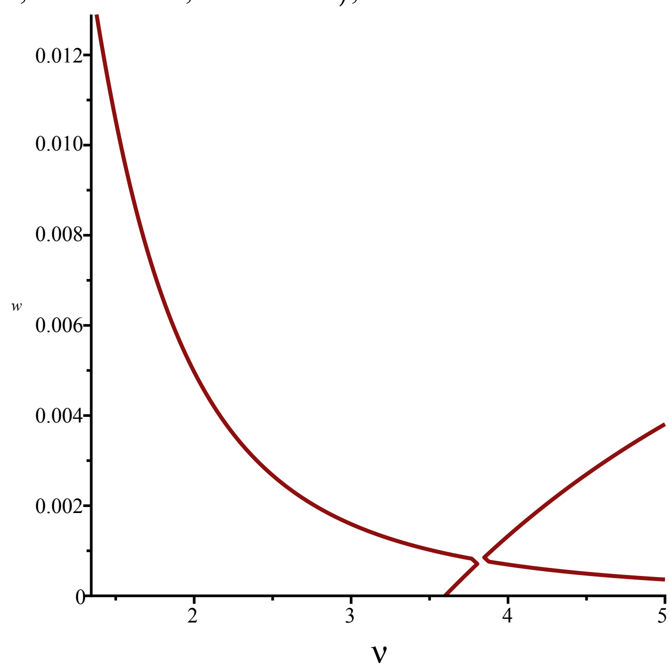
$$\frac{-9\,\nu^3 + 27\,\nu^2 - \sqrt{3}\,\sqrt{-\left(\nu^2-2\,\nu-3\right)^3} - 9\,\nu - 9}{576\,\nu^3} \quad \textbf{(1.6)}$$

> $rhosubc := \frac{-9\,\nu^3 + 27\,\nu^2 + \sqrt{3}\,\sqrt{-\left(\nu^2-2\,\nu-3\right)^3} - 9\,\nu - 9}{576\,\nu^3} : plot(rhosubc, \mathrm{nu} = 0..nuc);$



For nu > nuc, the radius of convergence is the positive decreasing branch of algrosupc:

> *implicitplot*(*algrhosupc*, nu = *nuc* ..5, *w* = 0 ..0.1);



Critical values U(nu,tnu^3) (Proposition 2.2)

Equations for U at criticality and parametrization of the critical line

> *numer*(*factor*(diff(wU, U)));

$$\left(3U^2\nu+3U^2-3U\nu-3U+\nu\right)\left(4U^3\nu^2+8U^3\nu-3U^2\nu^2+4U^3-12U^2\nu-9U^2+6U\nu+6U-2\right) \quad \textbf{(1.1.1)}$$

First factor is for nu < nuc, second for nu>nuc

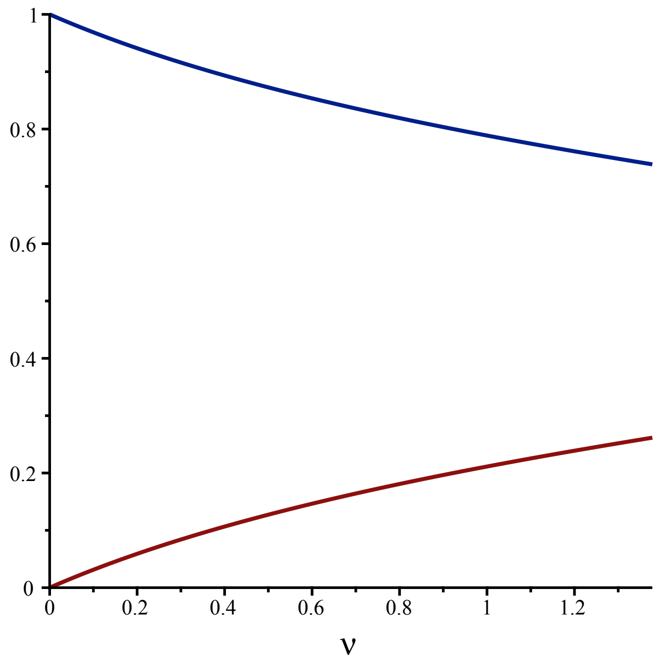
> $algUsubcrit := \left(3\,U^2\,\nu - 3\,U\,\nu + \nu - 3\,U + 3\,U^2\right):$

> $algUsupcrit := \left(4\,U^3\,\nu^2 + 8\,U^3\,\nu + 4\,U^3 - 3\,U^2\,\nu^2 - 12\,U^2\,\nu - 9\,U^2 + 6\,U\,\nu + 6\,U - 2\right):$

For nu <nuc, the value of Unu is the smallest positive branch of algUsubcrit

> $solve(algUsubcrit, U);\ plot(\{\%\}, \mathrm{nu} = 0\,..nuc);$

$$\frac{3\,\nu + 3 + \sqrt{-3\,\nu^2 + 6\,\nu + 9}}{6\,(\nu + 1)},\ -\frac{-3\,\nu - 3 + \sqrt{-3\,\nu^2 + 6\,\nu + 9}}{6\,(\nu + 1)}$$



> $Unusubc := \frac{3\,\nu + 3 - \sqrt{-3\,\nu^2 + 6\,\nu + 9}}{6\,(\nu + 1)};\ nuUsub := solve(algUsubcrit, \mathrm{nu});$

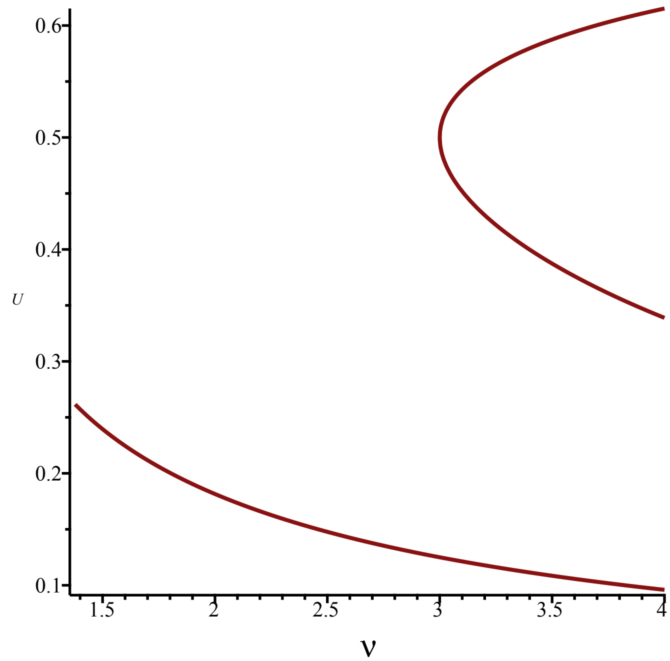
$$Unusubc := \frac{3\,\nu + 3 - \sqrt{-3\,\nu^2 + 6\,\nu + 9}}{6\,\nu + 6}$$

$$nuUsub := -\frac{3\,U\,(U - 1)}{3\,U^2 - 3\,U + 1} \qquad \textbf{(1.1.2)}$$

When nu >nuc it is the smallest positive root of algUsupc:

We look at the solutions of algUsubcrit and algUsupcrit, and identify the right branches (in PUsub and PUsur):

> $implicitplot(algUsupcrit, \mathrm{nu} = nuc\,..4, U = 0\,..1);$

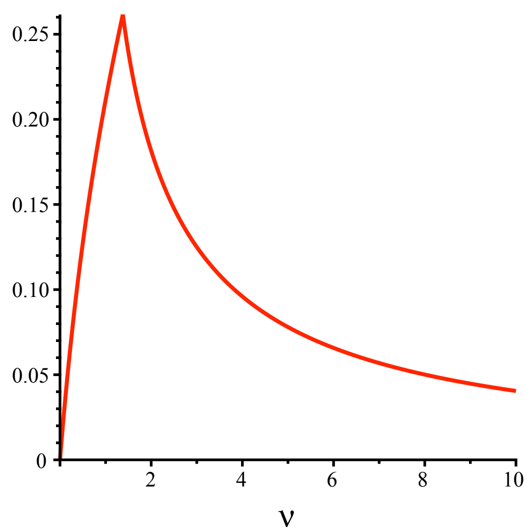


> $PUsub := plot\left(\frac{3\nu + 3 - \sqrt{-3\nu^2 + 6\nu + 9}}{6(\nu + 1)}, nu = 0..nuc, color = red\right)$:

> $PUsur := plot(op(3, \{solve(algUsupcrit, U)\}), nu = nuc..10, color = red)$:

The values of Uc

> $display([PUsub, PUsur]);$



We can parametrize the values of (nu,Unu) with rational functions:

> $UsupK := -\frac{K^2 - 3}{2(3K + 5)}; nusupK := factor\left(-\frac{K^3 + 3K^2 + 9K + 11}{K^3 + 3K^2 - 3K - 9}\right); simplify(subs(U = UsupK, nu = nusupK, algUsupcrit));$

$$UsupK := -\frac{K^2 - 3}{6K + 10}$$

$$nusupK := -\frac{K^3 + 3K^2 + 9K + 11}{(K + 3)(K^2 - 3)}$$

$$0 \qquad \textbf{(1.1.3)}$$

UsupK is clearly a decreasing function of K and to determine if nusupK is decreasing or increasing, we compute its derivative with respect to K:

```
> simplify(diff(nusupK, K));
```

$$\frac{24\,(K+2)\,(K+1)^2}{(K+3)^2\,\left(K^2-3\right)^2} \qquad \textbf{(1.1.4)}$$

So that nusupK is increasing for K>-2 and decreasing otherwise.

We determine the range of values of K which are of interest, Kc=value of K corresponding to Uc and nuc, Kinfini=value of K corresponding to U=0 and nu=infinity.

> $solve(\{UsupK = Uc, nusupK = nuc\}); evalf(\%); Kc := -\frac{2}{3} + \frac{1}{3}\sqrt{7}; Kinfini := \mathrm{sqrt}(3);$

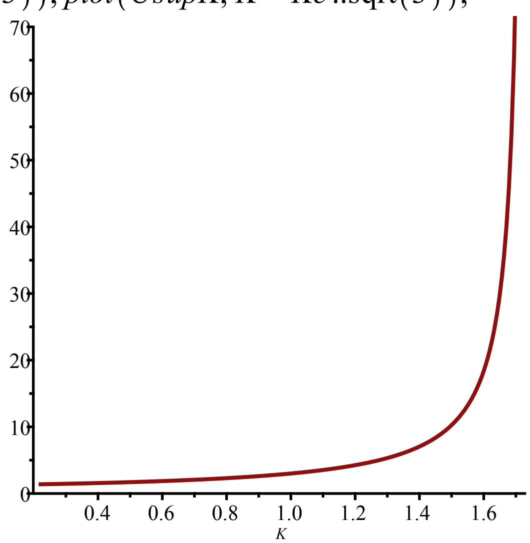
$$\left\{K = -\frac{2}{3} + \frac{\sqrt{7}}{3}\right\}$$

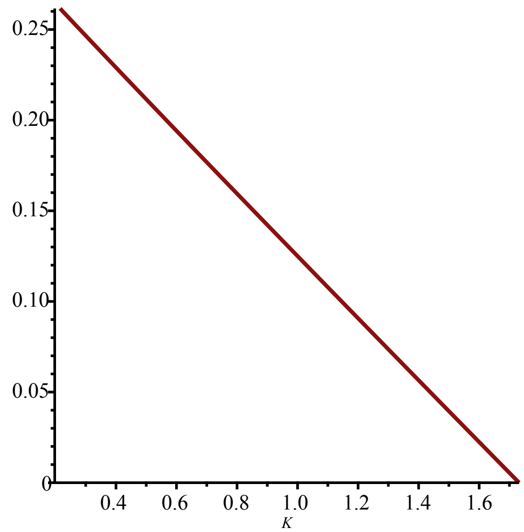
$$\{K = 0.2152504369\}$$

$$Kc := -\frac{2}{3} + \frac{\sqrt{7}}{3}$$

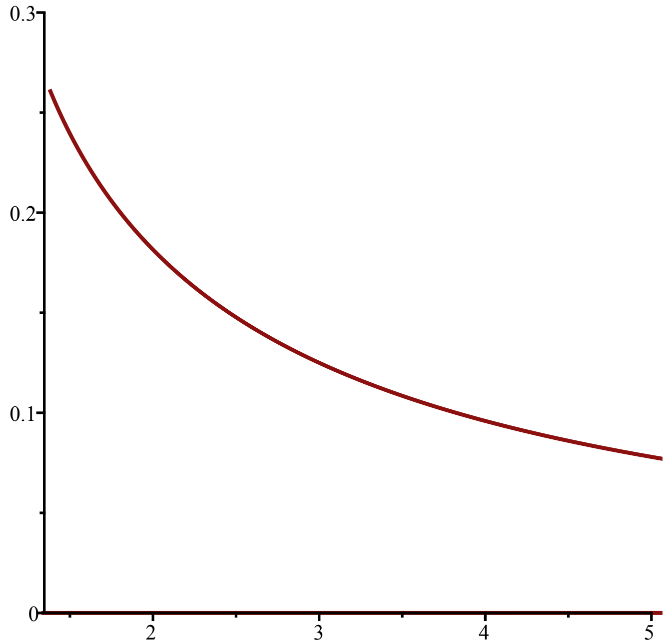
$$Kinfini := \sqrt{3} \qquad \textbf{(1.1.5)}$$

```
> plot(nusupK, K = Kc..sqrt(3)); plot(UsupK, K = Kc..sqrt(3));
```





> *plot*([*nusupK*, *UsupK*, *K* = *Kc* ..*Kinfini*], *nuc* ..5, 0 ..0.3);



We always assume Kc<K<Kinfini:

> *assume*(*K* > *Kc* **and** *K* < *Kinfini*);

> *about*(*K*);

```
Originally K, renamed K~:
  is assumed to be: RealRange(Open(-2/3+1/3*7^(1/2)),Open(3^(1/2)))
```

Development in t of U (Lemma 2.3)

Subcritical regime nu < nuc

We start with the equation for U(nu,t^3) here with w=t^3:

> *algU*;

$$8\,(\nu+1)^3\,U^5-(11\,\nu+29)\,(\nu+1)^2\,U^4+4\,(\nu+8)\,(\nu+1)^2\,U^3+\left(-128\,w\,\nu^3-12\,\nu^2-32\,\nu-12\right)U^2+8\,\nu\left(16\,\nu^2\,w+1\right)U-32\,w\,\nu^3 \tag{1.2.1.1}$$

We replace w by the value of the radius of convergence (=t_\nu^3 in the paper) and compute the corresponding singular behavior of U, (with XX=(1-w/rho)^1/2)

```
> op(2, algeqtoseries(subs(w = rhosubc·(1 − XX^2), algU), XX, U, 7));
```

$$\frac{3\,\nu^3+\sqrt{3}\,\sqrt{-\nu^6+6\,\nu^5-3\,\nu^4-28\,\nu^3+9\,\nu^2+54\,\nu+27}-3\,\nu^2-15\,\nu-9}{6\,(\nu^3-\nu^2-5\,\nu-3)}+\mathit{RootOf}\Big(\big(252\,\nu^6-504\,\nu^5-1296\,\nu^4+1008\,\nu^3+1980\,\nu^2-216\,\nu-648\big)\,\_Z^2-13\,\nu^6+3\,\sqrt{3}\,\sqrt{-\nu^6+6\,\nu^5-3\,\nu^4-28\,\nu^3+9\,\nu^2+54\,\nu+27}\,\nu^3+78\,\nu^5-9\,\sqrt{3}\,\sqrt{-\nu^6+6\,\nu^5-3\,\nu^4-28\,\nu^3+9\,\nu^2+54\,\nu+27}\,\nu^2-120\,\nu^4-40\,\nu^3+6\,\sqrt{3}\,\sqrt{-\nu^6+6\,\nu^5-3\,\nu^4-28\,\nu^3+9\,\nu^2+54\,\nu+27}+171\,\nu^2-54\,\nu-54\Big)\,XX-\frac{1}{108}\Big(297\,\nu^9+233\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\sqrt{3}\,\nu^6-2673\,\nu^8-1398\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^5+7578\,\nu^7+3237\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^4-3150\,\nu^6-3628\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^3-17415\,\nu^5+1440\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^2+18783\,\nu^4+648\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\sqrt{3}\,\nu+10512\,\nu^3-540\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}-17820\,\nu^2-972\,\nu+4860\Big)\Big/\Big(\big(7\,\nu^4-28\,\nu^3+13\,\nu^2+30\,\nu-18\big)\,\big(7\,\nu^2-14\,\nu+6\big)\,\big(\nu^3-\nu^2-5\,\nu-3\big)\Big)\,XX^2+\frac{5}{216}\Big(\mathit{RootOf}\Big(\big(252\,\nu^6-504\,\nu^5-1296\,\nu^4+1008\,\nu^3+1980\,\nu^2-216\,\nu-648\big)\,\_Z^2-13\,\nu^6+3\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^3+78\,\nu^5-9\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^2-120\,\nu^4-40\,\nu^3+6\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}+171\,\nu^2-54\,\nu-54\Big)\Big(1334\,\nu^{10}+789\,\sqrt{3}\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\nu^7-13340\,\nu^9-5523\,\sqrt{(\nu^2-2\,\nu-3)^2\,(-\nu^2+2\,\nu+3)}\,\sqrt{3}\,\nu^6+46310\,\nu^8 \tag{1.2.1.2}$$

$$+ 16158 \sqrt{3} \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)}\, \nu^5 - 50320 \nu^7$$
$$- 25560 \sqrt{3} \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)}\, \nu^4 - 59060 \nu^6$$
$$+ 20754 \sqrt{3} \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)}\, \nu^3 + 162392 \nu^5$$
$$- 4206 \sqrt{3} \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)}\, \nu^2 - 48486 \nu^4$$
$$- 4896 \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)} \sqrt{3}\, \nu - 120456 \nu^3$$
$$+ 2484 \sqrt{3} \sqrt{(\nu^2 - 2\nu - 3)^2 (-\nu^2 + 2\nu + 3)} + 82242 \nu^2 + 21708 \nu - 22356))$$
$$\Big/ ((343 \nu^9 - 2401 \nu^8 + 5341 \nu^7 - 2009 \nu^6 - 8260 \nu^5 + 10570 \nu^4 - 876 \nu^3$$
$$- 5508 \nu^2 + 3456 \nu - 648) (\nu - 3)) XX^3 + O(XX^4)$$

We can now replace nu by its value in terms of Usubc (=value of U corresponding to the radius of convergence rhosubc) in the development:

> $simplify(subs(\mathrm{nu} = nuUsub, U = Usubc, \mathbf{(1.2.1.2)}))$ assuming $\left(Usubc < \frac{1}{2}\right)$ **and** $(Usubc > 0)$;

$$Usubc + RootOf\big((54\, Usubc^2 - 60\, Usubc + 12)\, \_Z^2 - 54\, Usubc^6 + 162\, Usubc^5 - 171\, Usubc^4 + 76\, Usubc^3 - 12\, Usubc^2\big)\, XX + \frac{1}{18} \frac{1}{(2\, Usubc - 1)(9\, Usubc^2 - 10\, Usubc + 2)^2} \big((1458\, Usubc^6 - 5778\, Usubc^5 + 9045\, Usubc^4 - 7146\, Usubc^3 + 2984\, Usubc^2 - 616\, Usubc + 48)\, Usubc^2\big)\, XX^2 + \frac{5}{216} \frac{1}{(9\, Usubc^2 - 10\, Usubc + 2)^3 (2\, Usubc - 1)} \big(Usubc^2 (135\, Usubc^2 - 134\, Usubc + 22)(-2 + 3\, Usubc)^2 RootOf\big((54\, Usubc^2 - 60\, Usubc + 12)\, \_Z^2 - 54\, Usubc^6 + 162\, Usubc^5 - 171\, Usubc^4 + 76\, Usubc^3 - 12\, Usubc^2\big)(6\, Usubc^2 - 10\, Usubc + 3)\big)\, XX^3 + O(XX^4) \quad \mathbf{(1.2.1.3)}$$

To get rid of the RootOf in the previous display, we factorize the polynom, and identify as many square terms as possible:

> $factor\left(-\frac{(-54\, U^6 + 162\, U^5 - 171\, U^4 + 76\, U^3 - 12\, U^2)}{(54\, U^2 - 60\, U + 12)}\right)$;

$$\frac{U^2 (6\, U^2 - 10\, U + 3)(-2 + 3\, U)^2}{6\, (9\, U^2 - 10\, U + 2)} \quad \mathbf{(1.2.1.4)}$$

> $map\Big(simplify, subs\Big(RootOf\big((54\, Usubc^2 - 60\, Usubc + 12)\, \_Z^2 - 54\, Usubc^6 + 162\, Usubc^5$

$$- 171\, Usubc^4 + 76\, Usubc^3 - 12\, Usubc^2 \big) = -Usubc \cdot (2 - 3 \cdot Usubc) \cdot \mathrm{sqrt}\left(\frac{(6\, Usubc^2 - 10\, Usubc + 3)}{6\,(9\, Usubc^2 - 10\, Usubc + 2)} \right), \mathbf{(1.2.1.3)} \big) \big);$$

$$Usubc + \frac{1}{6}\, Usubc\,(-2 + 3\, Usubc)\,\sqrt{6}\,\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX + \frac{1}{18}\, \frac{1}{(2\, Usubc - 1)\,(9\, Usubc^2 - 10\, Usubc + 2)^2} \big((1458\, Usubc^6 - 5778\, Usubc^5 + 9045\, Usubc^4 - 7146\, Usubc^3 + 2984\, Usubc^2 - 616\, Usubc + 48)\, Usubc^2 \big)\, XX^2 + \frac{5}{1296}\, \frac{1}{(9\, Usubc^2 - 10\, Usubc + 2)^3\,(2\, Usubc - 1)} \Big(Usubc^3\,(135\, Usubc^2 - 134\, Usubc + 22)\,(-2 + 3\, Usubc)^3\,\sqrt{6}\,\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\,(6\, Usubc^2 - 10\, Usubc + 3) \Big)\, XX^3 + \mathrm{O}(XX^4) \quad \mathbf{(1.2.1.5)}$$

Finally we obtain the following developpement for U around Usubc (recall that XX = (1-w/rhosubc)^{1/2}):

$$> Usubcsing3 := Usubc + \frac{1}{6}\, Usubc\,(-2 + 3\, Usubc)\,\sqrt{6}\,\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX + \frac{1}{18}\, \frac{1}{(9\, Usubc^2 - 10\, Usubc + 2)^2\,(-1 + 2\, Usubc)} \big((1458\, Usubc^6 - 5778\, Usubc^5 + 9045\, Usubc^4 - 7146\, Usubc^3 + 2984\, Usubc^2 - 616\, Usubc + 48)\, Usubc^2 \big)\, XX^2 + \frac{5}{1296}\, \frac{1}{(9\, Usubc^2 - 10\, Usubc + 2)^3\,(-1 + 2\, Usubc)} \Big((135\, Usubc^2 - 134\, Usubc + 22)\,(6\, Usubc^2 - 10\, Usubc + 3)\, Usubc^3\,(-2 + 3\, Usubc)^3\,\sqrt{6}\,\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}} \Big)\, XX^3;$$

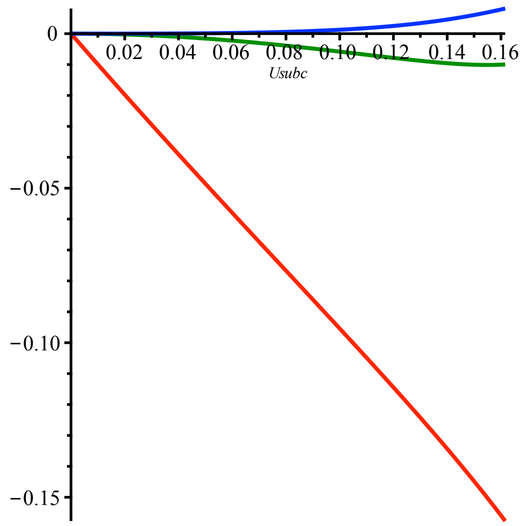
$$Usubcsing3 := Usubc + \frac{Usubc\,(-2 + 3\, Usubc)\,\sqrt{6}\,\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, XX}{6} \quad \mathbf{(1.2.1.6)}$$

$$+ \big((1458\, Usubc^6 - 5778\, Usubc^5 + 9045\, Usubc^4 - 7146\, Usubc^3 + 2984\, Usubc^2 - 616\, Usubc + 48)\, Usubc^2\, XX^2 \big) \big/ \big(18\,(9\, Usubc^2 - 10\, Usubc + 2)^2\,(2\, Usubc$$

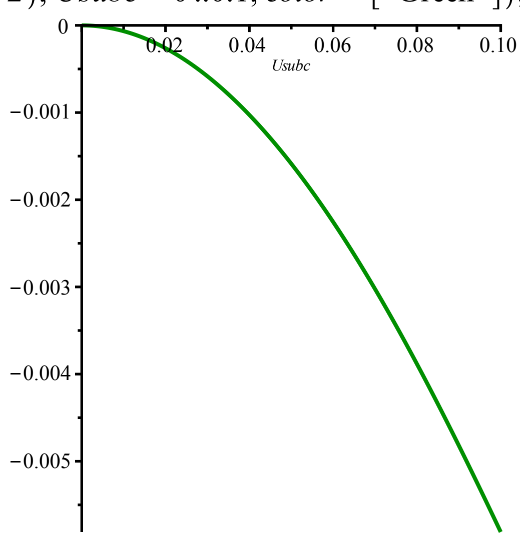
$$- 1)\Big) + \Bigg(5\left(135\,Usubc^2 - 134\,Usubc + 22\right)\left(6\,Usubc^2 - 10\,Usubc + 3\right) Usubc^3 \left(-2 + 3\,Usubc\right)^3 \sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,XX^3\Bigg) \Big/ \Big(1296\left(9\,Usubc^2 - 10\,Usubc + 2\right)^3 \left(2\,Usubc - 1\right)\Big)$$

We check that the coefficients in the development do not cancel for the considered range of values of U.

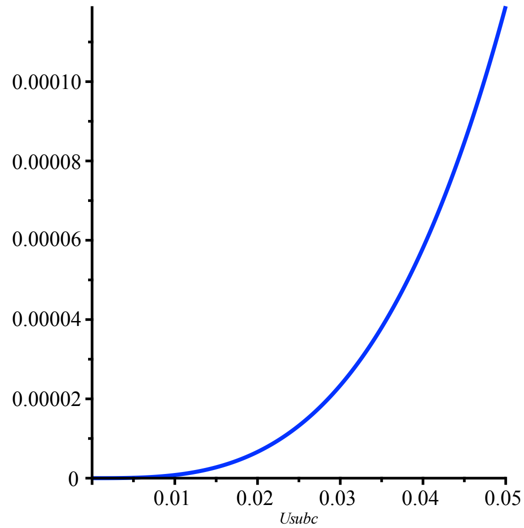
> *plot*([*seq*(*coeff*(*Usubcsing3*, *XX*, *i*), *i* = 1 ..3)], *Usubc* = 0 ..*Uc* − 0.1, *color* = ["Red", "Green", "Blue"]);



> *plot*(*coeff*(*Usubcsing3*, *XX*, 2), *Usubc* = 0 ..0.1, *color* = ["Green"]);



> *plot*(*coeff*(*Usubcsing3*, *XX*, 3), *Usubc* = 0 ..0.05, *color* = ["Blue"]);



Critical regime nu = nuc

We start again from

> $algU;$

$$8\,(\nu+1)^3\,U^5-(11\,\nu+29)\,(\nu+1)^2\,U^4+4\,(\nu+8)\,(\nu+1)^2\,U^3+\left(-128\,w\,\nu^3-12\,\nu^2-32\,\nu-12\right)U^2+8\,\nu\left(16\,\nu^2\,w+1\right)U-32\,w\,\nu^3 \quad \textbf{(1.2.2.1)}$$

We replace w by the value of the radius of convergence rhoc (=t_\nu^3 in the paper) and compute the corresponding singular behavior of U, (with XX=(1-w/rhoc)^1/3)

> $map\left(simplify, algeqtoseries\left(simplify\left(subs\left(w=rhoc\cdot\left(1-XX^3\right), \mathrm{nu}=nuc, algU\right)\right), XX, U, 3\right)\right);$

$$\left[RootOf\left(216\,\_Z^2+\left(-54\sqrt{7}-189\right)\_Z+25\sqrt{7}+55\right)+\mathrm{O}\left(XX^3\right), \frac{5}{9}-\frac{\sqrt{7}}{9}+RootOf\left(39366\,\_Z^3+310\sqrt{7}-425\right)XX+\mathrm{O}\left(XX^{5/3}\right)\right] \quad \textbf{(1.2.2.2)}$$

There are two possible expansions, but since we know that Uc = 5/9 - sqrt(7)/9, it is necessarily the second one.

> $map\left(simplify, op\left(2, algeqtoseries\left(simplify\left(subs\left(w=rhoc\cdot\left(1-XX^3\right), \mathrm{nu}=nuc, algU\right)\right), XX, U, 12\right)\right)\right);$

$$\frac{5}{9}-\frac{\sqrt{7}}{9}+RootOf\left(39366\,\_Z^3+310\sqrt{7}-425\right)XX-\frac{5\,RootOf\left(39366\,\_Z^3+310\sqrt{7}-425\right)^2\left(2\sqrt{7}+1\right)XX^2}{24}+\frac{35\left(-1+2\sqrt{7}\right)XX^3}{10368}-\frac{1645\,RootOf\left(39366\,\_Z^3+310\sqrt{7}-425\right)XX^4}{82944}+\mathrm{O}\left(XX^{14/3}\right) \quad \textbf{(1.2.2.3)}$$

> allvalues(RootOf(39366 _Z^3 + 310 sqrt(7) - 425))

$$\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{108}+\frac{I\sqrt{3}\left(1240\sqrt{7}-1700\right)^{1/3}}{108}, -\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{54}, \frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{108}-\frac{I\sqrt{3}\left(1240\sqrt{7}-1700\right)^{1/3}}{108} \quad \textbf{(1.2.2.4)}$$

There is a unique real root:

> sort(collect(simplify(subs(RootOf(39366 _Z^3 + 310 sqrt(7) - 425) = -(1240 sqrt(7) - 1700)^(1/3)/54, (1.2.2.3))), XX, simplify), XX, ascending);

$$\frac{5}{9}-\frac{\sqrt{7}}{9}+\mathrm{O}\left(XX^{14/3}\right)-\frac{\left(1240\sqrt{7}-1700\right)^{1/3}XX}{54}-\frac{5\left(1240\sqrt{7}-1700\right)^{2/3}\left(2\sqrt{7}+1\right)XX^2}{69984}+\left(-\frac{35}{10368}+\frac{35\sqrt{7}}{5184}\right)XX^3+\frac{1645\left(1240\sqrt{7}-1700\right)^{1/3}XX^4}{4478976} \quad \textbf{(1.2.2.5)}$$

> Ucsing4 := 5/9 - sqrt(7)/9 - (1240 sqrt(7) - 1700)^(1/3) XX/54 - 5 (1240 sqrt(7) - 1700)^(2/3) (2 sqrt(7) + 1) XX^2/69984 + (-35/10368 + 35 sqrt(7)/5184) XX^3 + 1645 (1240 sqrt(7) - 1700)^(1/3) XX^4/4478976;

$$Ucsing4 := \frac{5}{9}-\frac{\sqrt{7}}{9}-\frac{\left(1240\sqrt{7}-1700\right)^{1/3}XX}{54}-\frac{5\left(1240\sqrt{7}-1700\right)^{2/3}\left(2\sqrt{7}+1\right)XX^2}{69984}+\left(-\frac{35}{10368}+\frac{35\sqrt{7}}{5184}\right)XX^3+\frac{1645\left(1240\sqrt{7}-1700\right)^{1/3}XX^4}{4478976} \quad \textbf{(1.2.2.6)}$$

Supercritical regime nu >nuc

We consider the rational parametrization of the critical line in this regime given by K:

> UsupK; nusupK;

$$-\frac{K\sim^2-3}{6K\sim+10}$$

(1.2.3.1)

$$-\frac{K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11}{(K\sim + 3)\,(K\sim^2 - 3)} \qquad \mathbf{(1.2.3.1)}$$

We express the value of (t_nu)^3 = rho_c, in terms of K:

```
> rhosupcK := simplify(subs(nu = nusupK, U = UsupK, wU));
```

$$rhosupcK := -\frac{(K\sim + 1)\,(K\sim^2 + 8\,K\sim + 13)\,(K\sim^2 - 3)^3}{16\,(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11)^3} \qquad \mathbf{(1.2.3.2)}$$

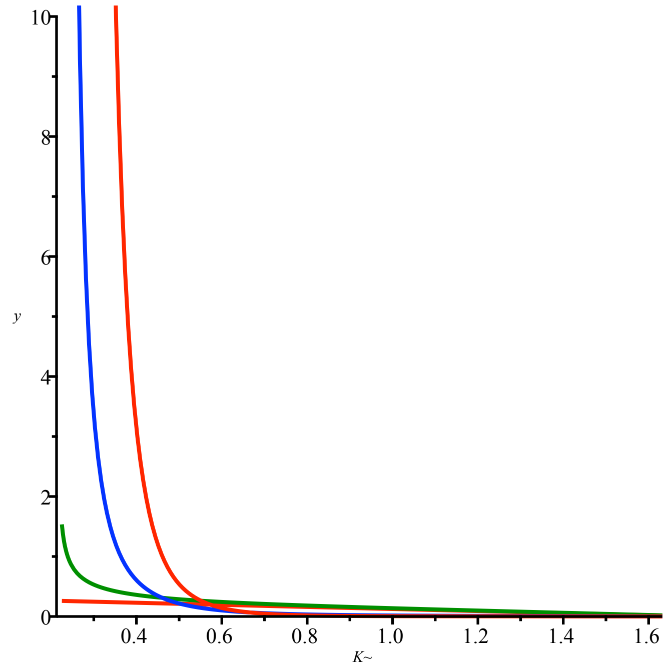
We compute the asymptotic behavior of U around rho (with XX=(1-w/rho)^1/2)

```
> Usupcsing := collect(map(factor, map(simplify, convert(op(2, algeqtoseries(subs(w
      = rhosupcK·(1 - XX^2), subs(nu = nusupK, algU)), XX, U, 6, true)), polynom))), XX,
      factor);
```

$$Usupcsing := -\frac{K\sim^2 - 3}{2\,(3\,K\sim + 5)} + RootOf\big((1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200)\,\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,XX - \big((K\sim^2 - 3)\,(K\sim^2 + 8\,K\sim + 13)\,XX^2\,(9\,K\sim^4 + 14\,K\sim^3 - 18\,K\sim^2 - 10\,K\sim + 29)\,(K\sim + 1)\big)\Big/\big(144\,(3\,K\sim + 5)\,(3\,K\sim^2 + 4\,K\sim - 1)^2\,(2 + K\sim)\big) + \frac{1}{216\,(3\,K\sim^2 + 4\,K\sim - 1)^3\,(2 + K\sim)}\big(5\,(K\sim^2 + 8\,K\sim + 13)\,(9\,K\sim^6 + 40\,K\sim^5 + 43\,K\sim^4 - 48\,K\sim^3 - 97\,K\sim^2 + 24\,K\sim + 77)\,RootOf\big((1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200)\,\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,XX^3\big) \qquad \mathbf{(1.2.3.3)}$$

The coefficients are non vanishing:

```
> plot([seq(coeff(Usupcsing, XX, i), i = 0 ..3)], K = Kc + 0.01 ..Kinfini - 0.1, y = 0 ..10, color
      = ["Red", "Green", "Blue"]);
```



Development in t of the partition function Zplus (Proposition 2.5)

We compute the asymptotic behavior of the generating series of triangulations of the sphere (which corresponds to ((tZ_1)^2+t^2Z_2)/(t^3*nu) by standard manipulations):

Subcritical regime nu < nuc

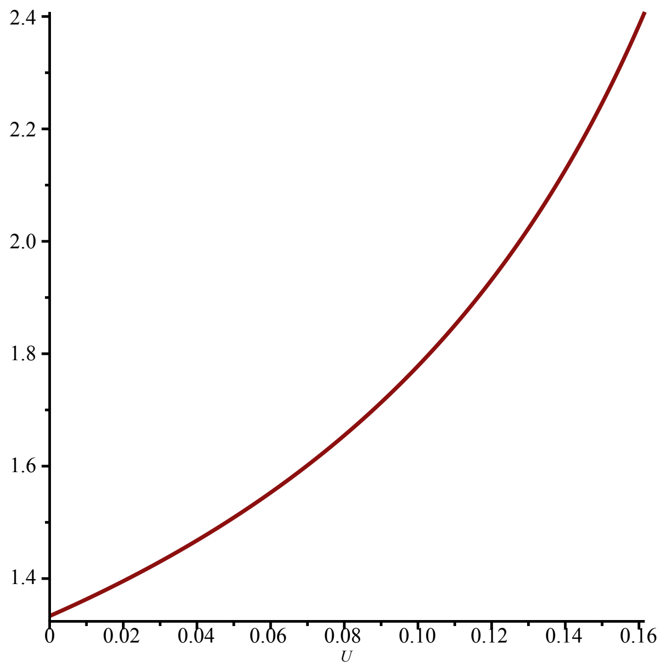
Here U is Unu and XX=(1-w/rho)^1/2

> $Zpsubcdevt := simplify\left(series\left(subs\left(U = Usubcsing3, Usubc = U, \mathrm{nu} = nuUsub, \frac{(tZ1U)^2 + t2Z2U}{wU \cdot \mathrm{nu}}\right), XX, 4\right)\right);$

$$Zpsubcdevt := \frac{810\,U^6 - 3780\,U^5 + 6507\,U^4 - 5805\,U^3 + 2889\,U^2 - 768\,U + 84}{2\,(6\,U^2 - 10\,U + 3)^2\,(-2 + 3\,U)^2} + \frac{-324\,U^6 + 756\,U^5 - 1008\,U^4 + 900\,U^3 - 516\,U^2 + 168\,U - 24}{(-2 + 3\,U)^2\,(6\,U^2 - 10\,U + 3)^2}\,XX^2 + 12\,\frac{\sqrt{\frac{6\,U^2 - 10\,U + 3}{9\,U^2 - 10\,U + 2}}\,\sqrt{6}\,\left(U^2 - U + \frac{1}{3}\right)(U - 2)}{54\,U^3 - 126\,U^2 + 87\,U - 18}\,XX^3 + \mathrm{O}(XX^4) \qquad \textbf{(1.3.1.1)}$$

The singular coefficient does not vanish

> $plot(coeff(Zpsubcdevt, XX, 3), U = 0\,..Uc - 0.1);$



Critical regime nu = nuc

with XX=(1-w/rho)^1/3

> $Zpscritdevt := collect\left(expand\left(rationalize\left(convert\left(series\left(subs\left(U = Ucsing4, \mathrm{nu} = nuc, \frac{(tZ1U)^2 + t2Z2U}{wU \cdot \mathrm{nu}}\right), XX, 5\right), polynom\right)\right)\right), XX, factor\right);$

$$Zpscritdevt := \frac{3\sqrt{7}\left(1240\sqrt{7} - 1700\right)^{1/3} XX^4}{20} + \left(-\frac{476}{25} + \frac{148\sqrt{7}}{25}\right) XX^3 + \frac{263\sqrt{7}}{50} - \frac{308}{25} \qquad \textbf{(1.3.2.1)}$$

Supercritical regime nu >nuc

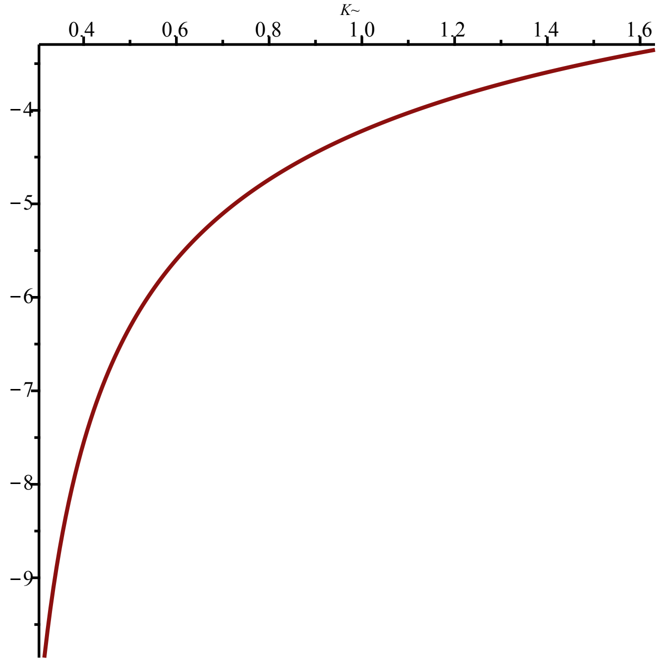
With XX=(1-w/rho)^1/2

> $Zpsupcdevt := simplify\left(series\left(subs\left(U = Usupcsing, \mathrm{nu} = nusupK, \frac{(tZ1U)^2 + t2Z2U}{wU \cdot \mathrm{nu}}\right), XX, 4\right)\right);$

$$Zpsupcdevt := \frac{5K\sim^7 + 95K\sim^6 + 675K\sim^5 + 2617K\sim^4 + 6055K\sim^3 + 7845K\sim^2 + 4809K\sim + 1035}{4\left(K\sim + 1\right)^3 \left(K\sim^2 + 8K\sim + 13\right)^2} \qquad \textbf{(1.3.3.1)}$$

$$- \frac{(K\sim^6 + 12K\sim^5 + 95K\sim^4 + 344K\sim^3 + 651K\sim^2 + 652K\sim + 237)(K\sim + 3)}{(K\sim + 1)^3 (K\sim^2 + 8K\sim + 13)^2} XX^2$$
$$+ \frac{8}{3} \frac{1}{(K\sim^2 + 8K\sim + 13)(K\sim + 1)^4 (K\sim^2 - 3)} ((21K\sim^6 + 242K\sim^5$$
$$+ 1083K\sim^4 + 2388K\sim^3 + 2695K\sim^2 + 1410K\sim + 225)\, RootOf((1296K\sim^4$$
$$+ 6048K\sim^3 + 8928K\sim^2 + 3360K\sim - 1200)\_Z^2 - K\sim^8 - 10K\sim^7 - 24K\sim^6 + 26K\sim^5$$
$$+ 158K\sim^4 + 114K\sim^3 - 192K\sim^2 - 306K\sim - 117))\, XX^3 + O(XX^4)$$

> $plot(coeff(Zpsupcdevt, XX, 3), K = Kc + 0.1 .. Kinfini - 0.1);$



Theorem 3.1: Rational parametrisation for Q^+(t,ty) (denoted Qt here): g.s of trig with monochromatic non simple boundary

We start with the equation satisfied by Q, in terms of Z1 (=Z_1^+) et Z2 = Z_2^+):

> $eqQ := collect\left(simplify\left(\frac{subs(Z = Q - 1, y = y \cdot Q,\ eqZ)}{Q^2}\right), [Q, y], factor\right);$

$$eqQ := Q^3 \nu^2 t^3 y^5 + (-y^4 \nu (\nu - 1) t + \nu (2\nu - 3) t^2 y^3 + \nu^2 t^3 y^2) Q^2 + (-t^2 \nu (2\nu Z1\, t + \nu - 2) y^3 + y^2 (\nu - 1) - y t (\nu + 2)(\nu - 1) + 2\nu (\nu - 1) t^2) Q + (-2 Z1^2 \nu^2 t^2 + 2 Z1^2 \nu t^2 - 2 Z2 \nu^2 t^2 - \nu^2 t^3 + Z1 \nu^2 t + 2 Z2 \nu t^2 + \nu Z1\, t - 2 Z1\, t - \nu + 1) y^2 - (\nu - 1) t (2\nu Z1\, t - \nu - 2) y - 2\nu (\nu - 1) t^2 \quad \textbf{(2.1)}$$

The equation for Qt= Q(nu,t,ty)

> $eqQt := collect\left(subs\left(t = w^{\frac{1}{3}}, simplify\left(\frac{subs\left(Q = Qt, Z1 = \frac{tZ1}{t}, Z2 = \frac{t2Z2}{t^2}, y = y \cdot t, eqQ\right)}{t^2}\right)\right), [Qt, y, w], recursive\right);$

$$eqQt := Qt^3 \nu^2 w^2 y^5 + \left(-w \nu (\nu - 1) y^4 + 2 \nu w \left(\nu - \frac{3}{2}\right) y^3 + \nu^2 w y^2\right) Qt^2 + \left(2 \nu w \left(\left(-tZ1 - \frac{1}{2}\right) \nu + 1\right) y^3 + y^2 (\nu - 1) - y (\nu + 2) (\nu - 1) + 2 \nu (\nu - 1)\right) Qt + \left(-\nu^2 w + \left(-2 tZ1^2 - 2 t2Z2 + tZ1\right) \nu^2 + \left(2 tZ1^2 + 2 t2Z2 + tZ1 - 1\right) \nu - 2 tZ1 + 1\right) y^2 - (\nu - 1) ((2 tZ1 - 1) \nu - 2) y - 2 \nu (\nu - 1) \quad (2.2)$$

> $map(factor, eqQt)$

$$Qt^3 \nu^2 w^2 y^5 - w \nu y^2 (\nu y^2 - 2 \nu y - y^2 - \nu + 3 y) Qt^2 - \left(2 \nu^2 w y^3 tZ1 + \nu^2 w y^3 - 2 \nu w y^3 + \nu^2 y - \nu y^2 - 2 \nu^2 + \nu y + y^2 + 2 \nu - 2 y\right) Qt - \left(2 \nu^2 tZ1^2 + 2 \nu^2 t2Z2 - \nu^2 tZ1 + \nu^2 w - 2 \nu tZ1^2 - 2 \nu t2Z2 - \nu tZ1 + \nu + 2 tZ1 - 1\right) y^2 - (\nu - 1) (2 \nu tZ1 - \nu - 2) y - 2 \nu (\nu - 1) \quad (2.3)$$

We compute the development of the solutions of the equation to identify the right branch (i.e. the one with a formal power series development):

> algeqtoseries(eqQt, y, Qt, 3);

$$\left[-\frac{1}{w} y^{-3} + \frac{1}{\nu w} y^{-2} + O(y^0), -\frac{2 (\nu - 1)}{\nu w} y^{-2} + \frac{\nu - 1}{\nu w} y^{-1} - 1 + O(y), 1 + tZ1\, y + \left(tZ1^2 + t2Z2\right) y^2 + O(y^3)\right] \quad (2.4)$$

> $eqQtU := op(2, factor(numer(subs(w = wU, tZ1 = tZ1U, t2Z2 = t2Z2U, eqQt)))):$
$indets(eqQtU);$
$degree(eqQtU, Qt);$
$degree(eqQtU, \{Qt, y\});$

$$\{Qt, U, \nu, y\}$$
$$3$$
$$8 \quad (2.5)$$

We have a rational parametrisation y(U,V) and Qt(U,V) such that y(U,0)= 0 and Qt(U,0)=1.

> $yUV := \frac{8 \nu \cdot (1 - 2 U)}{U (U \cdot (\nu + 1) - 2)} \cdot (V \cdot (V + 1)) \Big/ \left(V^3 + \frac{9 (\nu + 1) \cdot U^2 - 2 \cdot (3 + 10 \nu) \cdot U + 8 \nu}{U (U \cdot (\nu + 1) - 2)} \cdot V^2\right.$

$$- \frac{9\cdot U\cdot(\nu+1) - 2\cdot(2\nu+3)}{(U\cdot(\nu+1)-2)}\cdot V - 1\Bigg):$$

> $QtUV := \frac{U\,(U\cdot(\nu+1)-2)\cdot(1-\nu)}{P}\cdot\frac{1}{(V+1)^3}\cdot\Bigg(V^3 + \frac{9(\nu+1)\cdot U^2 - 2\cdot(3+10\nu)\cdot U + 8\nu}{U\,(U\cdot(\nu+1)-2)}\cdot V^2 - \frac{9\cdot U\cdot(\nu+1) - 2\cdot(2\nu+3)}{(U\cdot(\nu+1)-2)}\cdot V - 1\Bigg)\cdot\Bigg(V^2 + 2\cdot\frac{(5\cdot(\nu+1)\,U^2 - 2\cdot(3\nu+2)\,U + 2\nu)}{U\,(U\cdot(\nu+1)-2)}\,V - \frac{P}{U\,(U\cdot(\nu+1)-2)\cdot(1-\nu)}\Bigg):$

We check that this parametrizes a solution of eqQt:

> $simplify(subs(y = yUV, Qt = QtUV, eqQtU));$

$$0 \quad \textbf{(2.6)}$$

Lastly, we check that this does correspond to the right branch:

> $collect(eqQtU, Qt, factor);$

$$U^2 y^5 (U\nu + U - 2)^2 (8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu)^3 Qt^3 - 32U\nu^2 y^2 (\nu y^2 - 2\nu y - y^2 - \nu + 3y)(-1 + 2U)^2 (U\nu + U - 2)(8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu)^2 Qt^2 - 64\nu^2(-1+2U)^2(8U^3\nu^2 + 16U^3\nu - 11U^2\nu^2 + 8U^3 - 24U^2\nu + 4U\nu^2 - 13U^2 + 14U\nu + 6U - 4\nu)(3U^6\nu^4y^3 + 12U^6\nu^3y^3 + 18U^6\nu^2y^3 - 18U^5\nu^3y^3 - 4U^4\nu^4y^3 + 12U^6\nu y^3 - 54U^5\nu^2y^3 + 2U^3\nu^4y^3 + 3U^6y^3 - 54U^5\nu y^3 + 51U^4\nu^2y^3 + 12U^3\nu^3y^3 - 18U^5y^3 + 86U^4\nu y^3 - 18U^3\nu^2y^3 - 6U^2\nu^3y^3 + 39U^4y^3 - 64U^3\nu y^3 + 64U^2\nu^4y - 64U^2\nu^3y^2 - 2U^2\nu^2y^3 - 36U^3y^3 - 128U^2\nu^4 + 64U^2\nu^3y + 64U^2\nu^2y^2 + 28U^2\nu y^3 - 64U\nu^4y + 64U\nu^3y^2 + 4U\nu^2y^3 + 128U^2\nu^3 - 128U^2\nu^2y + 12U^2y^3 + 128U\nu^4 - 64U\nu^3y - 64U\nu^2y^2 - 8U\nu y^3 + 16\nu^4y - 16\nu^3y^2 - 128U\nu^3 + 128U\nu^2y - 32\nu^4 + 16\nu^3y + 16\nu^2y^2 + 32\nu^3 - 32\nu^2y)Qt - 512\nu^3(-1+2U)^3(2U^7\nu^5y^2 + 10U^7\nu^4y^2 + 20U^7\nu^3y^2 - 14U^6\nu^4y^2 - 14U^5\nu^5y^2 + 20U^7\nu^2y^2 - 56U^6\nu^3y^2 + 24U^5\nu^5y - 26U^5\nu^4y^2 + 23U^4\nu^5y^2 + 10U^7\nu y^2 - 84U^6\nu^2y^2 + 48U^5\nu^4y + 44U^5\nu^3y^2 - 76U^4\nu^5y + 85U^4\nu^4y^2 - 14U^3\nu^5y^2 + 2U^7y^2 - 56U^6\nu y^2 + 148U^5\nu^2y^2 + 64U^4\nu^5 - 184U^4\nu^4y + 29U^4\nu^3y^2 + 88U^3\nu^5y - 94U^3\nu^4y^2 + 3U^2\nu^5y^2 - 14U^6y^2 - 48U^5\nu^2y + 130U^5\nu y^2 + 64U^4\nu^4 - 32U^4\nu^3y - 155U^4\nu^2y^2 - 120U^3\nu^5 + 248U^3\nu^4y - 80U^3\nu^3y^2 - 44U^2\nu^5y + 49U^2\nu^4y^2 \quad \textbf{(2.7)}$$

$$- 24 U^5 \nu y + 38 U^5 y^2 - 64 U^4 \nu^3 + 184 U^4 \nu^2 y - 172 U^4 \nu y^2 - 136 U^3 \nu^4 + 80 U^3 \nu^3 y + 112 U^3 \nu^2 y^2 + 76 U^2 \nu^5 - 164 U^2 \nu^4 y + 72 U^2 \nu^3 y^2 + 8 U \nu^5 y - 10 U \nu^4 y^2 - 64 U^4 \nu^2 + 108 U^4 \nu y - 50 U^4 y^2 + 120 U^3 \nu^3 - 248 U^3 \nu^2 y + 144 U^3 \nu y^2 + 132 U^2 \nu^4 - 80 U^2 \nu^3 y - 66 U^2 \nu^2 y^2 - 16 U \nu^5 + 56 U \nu^4 y - 36 U \nu^3 y^2 + 136 U^3 \nu^2 - 168 U^3 \nu y + 32 U^3 y^2 - 108 U^2 \nu^3 + 180 U^2 \nu^2 y - 72 U^2 \nu y^2 - 72 U \nu^4 + 40 U \nu^3 y + 32 U \nu^2 y^2 - 8 \nu^4 y + 8 \nu^3 y^2 - 100 U^2 \nu^2 + 108 U^2 \nu y - 8 U^2 y^2 + 64 U \nu^3 - 80 U \nu^2 y + 16 U \nu y^2 + 16 \nu^4 - 8 \nu^3 y - 8 \nu^2 y^2 + 24 U \nu^2 - 24 U \nu y - 16 \nu^3 + 16 \nu^2 y)$$

> *factor*(*subs*($y = 0$, **(2.7)**));

$$2048 \nu^5 (-1 + 2U)^4 (\nu - 1) (8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4 \nu) (Qt - 1) \qquad \textbf{(2.8)}$$

Proposition 3.5

> $yUV := (8 \nu (1 - 2U) V (V + 1)) \Big/ \left(U (U (\nu + 1) - 2) \left(V^3 + \frac{(9 (\nu + 1) U^2 - 2 (3 + 10 \nu) U + 8 \nu) V^2}{U (U (\nu + 1) - 2)} - \frac{(9 U (\nu + 1) - 4 \nu - 6) V}{U (\nu + 1) - 2} - 1\right)\right)$
> ;

$$yUV := (8 \nu (1 - 2U) V (V + 1)) \Big/ \left(U (U (\nu + 1) - 2) \left(V^3 + \frac{(9 (\nu + 1) U^2 - 2 (3 + 10 \nu) U + 8 \nu) V^2}{U (U (\nu + 1) - 2)} - \frac{(9 U (\nu + 1) - 4 \nu - 6) V}{U (\nu + 1) - 2} - 1\right)\right) \qquad \textbf{(3.1)}$$

First we look at the possible poles:

> $factor\left(discrim\left(V^3 + \frac{(9 (\nu + 1) U^2 - 2 (3 + 10 \nu) U + 8 \nu) V^2}{U (U (\nu + 1) - 2)} - \frac{(9 U (\nu + 1) - 4 \nu - 6) V}{U (\nu + 1) - 2} - 1, V\right)\right);$

$$\frac{1}{U^3 (U \nu + U - 2)^4} (64 (-1 + 2U) (108 U^6 \nu^4 + 432 U^6 \nu^3 - 432 U^5 \nu^4 + 648 U^6 \nu^2 - 1620 U^5 \nu^3 + 774 U^4 \nu^4 + 432 U^6 \nu - 2268 U^5 \nu^2 + 2628 U^4 \nu^3 - 693 U^3 \nu^4 + 108 U^6 - 1404 U^5 \nu + 3258 U^4 \nu^2 - 2358 U^3 \nu^3 + 296 U^2 \nu^4 - 324 U^5 + 1728 U^4 \nu - 2529 U^3 \nu^2 + 1308 U^2 \nu^3 - 48 U \nu^4 + 324 U^4 - 972 U^3 \nu + 1044 U^2 \nu^2 - 432 U \nu^3 - 108 U^3 + 216 U^2 \nu - 180 U \nu^2 + 64 \nu^3)) \qquad \textbf{(3.2)}$$

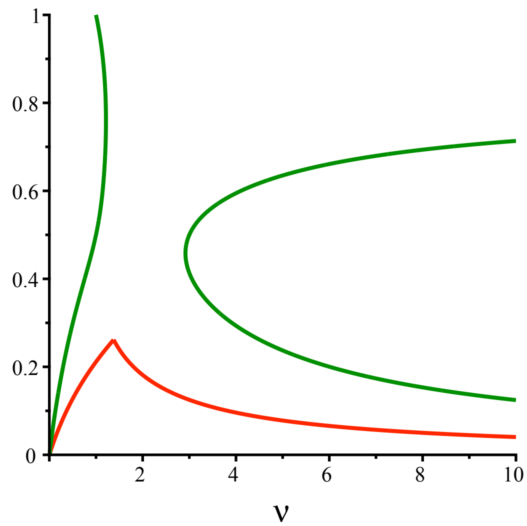
We want to know the sign of this factor:

> $collect\big(108\,U^6\nu^4+432\,U^6\nu^3-432\,U^5\nu^4+648\,U^6\nu^2-1620\,U^5\nu^3+774\,U^4\nu^4+432\,U^6\nu-2268\,U^5\nu^2+2628\,U^4\nu^3-693\,U^3\nu^4+108\,U^6-1404\,U^5\nu+3258\,U^4\nu^2-2358\,U^3\nu^3+296\,U^2\nu^4-324\,U^5+1728\,U^4\nu-2529\,U^3\nu^2+1308\,U^2\nu^3-48\,U\nu^4+324\,U^4-972\,U^3\nu+1044\,U^2\nu^2-432\,U\nu^3-108\,U^3+216\,U^2\nu-180\,U\nu^2+64\,\nu^3, U, factor\big);$

$$108\,(\nu+1)^4U^6-108\,(4\nu+3)\,(\nu+1)^3U^5+18\,(43\nu^2+60\nu+18)\,(\nu+1)^2U^4-9\,(\nu+1)\,(77\nu^3+185\nu^2+96\nu+12)\,U^3+4\nu\,(74\nu^3+327\nu^2+261\nu+54)\,U^2-12\nu^2\,(4\nu^2+36\nu+15)\,U+64\nu^3 \quad \textbf{(3.3)}$$

Can it be 0 for U \in [0,U_c(nu)]. We plot the points corresponding to its zeroes (in green) and the value of U_\nu (in red).

> $pzero := implicitplot(\textbf{(3.3)}, nu = 0..10, U = 0..1, numpoints = 10000, color = \text{"Green"}):$
$display([PUsub, PUsur, pzero]);$



We check when the curves meet. First in the subcritical regime. It is always for U>U_\nu_c

> $factor\big(resultant\big(\textbf{(3.3)}, (3\,U^2\nu+3\,U^2-3\,U\nu-3\,U+\nu), nu\big)\big); fsolve(\%); evalf(Uc);$

$$-81\,U^3\,(23\,U^2-28\,U+8)\,(U-1)^2\,(-1+2\,U)^3$$

0., 0., 0., 0.4580825385, 0.5000000000, 0.5000000000, 0.5000000000, 0.7593087659, 1., 1.

0.2615831877 **(3.4)**

Same in the supercritical regime:

> $factor\big(resultant\big(\textbf{(3.3)}, (4\,U^3\nu^2+8\,U^3\nu-3\,U^2\nu^2+4\,U^3-12\,U^2\nu-9\,U^2+6\,U\nu+6\,U-2), nu\big)\big); fsolve(\%); evalf(Uc);$

$$-4\,U^2\,(200\,U^4-432\,U^3+447\,U^2-232\,U+48)\,(1633\,U^6-5980\,U^5+8856\,U^4-6856\,U^3+2948\,U^2-672\,U+64)\,(-1+2\,U)^4$$

0., 0., 0.5000000000, 0.5000000000, 0.5000000000, 0.5000000000, 0.6745152264, 1.142829079

0.2615831877 **(3.5)**

Hence for U =< U_\nu_c (which is the only values of U of intereset) the discriminant of the denominator does not change sign. To know its sign, we check its value at U=0.

> $subs(U = 0, numer((\mathbf{3.2})));$

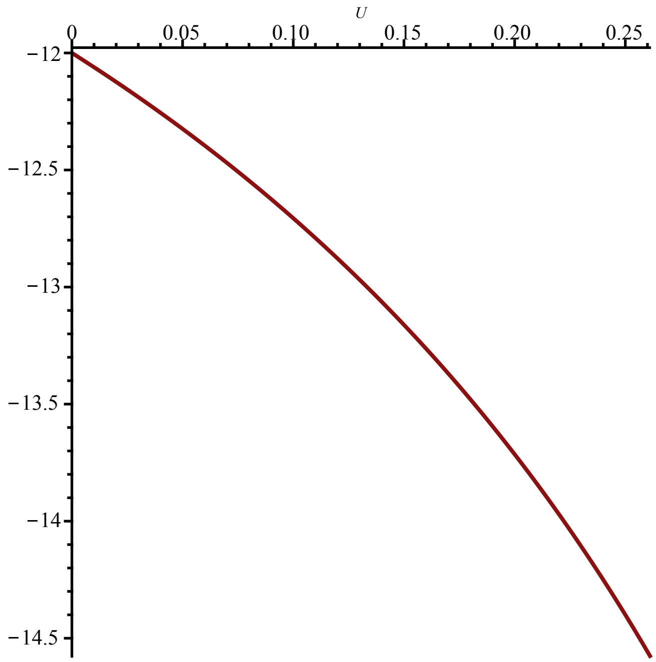
$$-4096\,\nu^3 \qquad \mathbf{(3.6)}$$

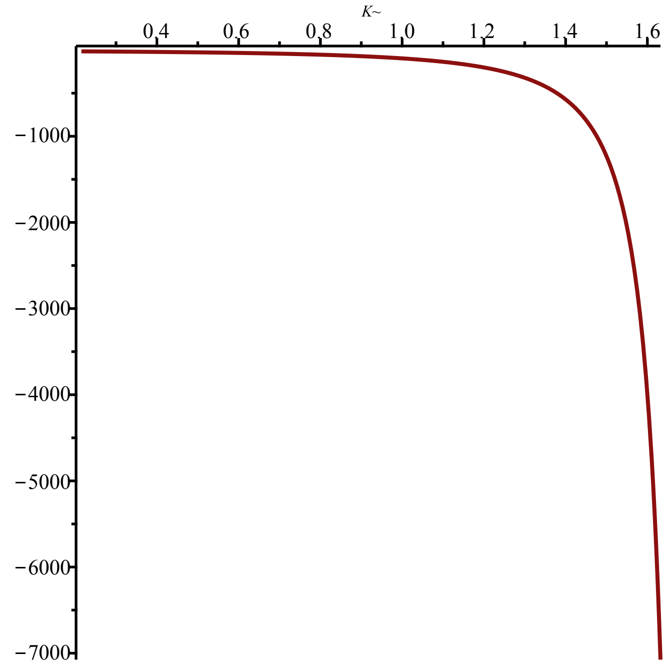
Since it is negative, yUV has a single real pole if U =<U_c:

The leading coefficient of the polynom is positive and the polynom is <0 at V=1 : the pole is after V=1

> $factor\left(subs\left(V = 1, \left(V^3 + \frac{(9\,(\nu+1)\,U^2 - 2\,(3+10\,\nu)\,U + 8\,\nu)\,V^2}{U\,(U\,(\nu+1)-2)} - \frac{(9\,U\,(\nu+1) - 4\,\nu - 6)\,V}{U\,(\nu+1)-2} - 1\right)\right)\right); plot(subs(\mathrm{nu} = nuUsub, \%), U = 0..Uc);$
$plot(subs(\mathrm{nu} = nusupK, U = UsupK, \%\%), K = Kc..Kinfini - 0.1);$

$$-\frac{8\,\nu\,(-1+2\,U)}{U\,(U\,\nu + U - 2)}$$





We now look at the stationary points of y(V):

> *factor*(*diff*(*yUV*, *V*)); *factor*(*subs*(*V* = 0, %));

$$\left(8\left(U^2 V^4 \nu + U^2 V^4 + 2 U^2 V^3 \nu + 2 U^2 V^3 + 18 U^2 V^2 \nu - 2 U V^4 + 18 U^2 V^2 + 2 U^2 V \nu - 4 V^3 U - 24 U V^2 \nu + 2 U^2 V + U^2 \nu - 12 V^2 U + 8 V^2 \nu + U^2 - 4 V U - 2 U\right) \nu \left(-1 + 2 U\right)\right) \Big/ \left(U^2 V^3 \nu + U^2 V^3 + 9 U^2 V^2 \nu + 9 U^2 V^2 - 9 U^2 V \nu - 2 V^3 U - 20 U V^2 \nu - 9 U^2 V - U^2 \nu - 6 V^2 U + 4 U V \nu + 8 V^2 \nu - U^2 + 6 V U + 2 U\right)^2$$

$$\frac{8 \nu \left(-1 + 2 U\right)}{U \left(U \nu + U - 2\right)} \tag{3.7}$$

We have to study the roots of the following polynomial of degree 4:

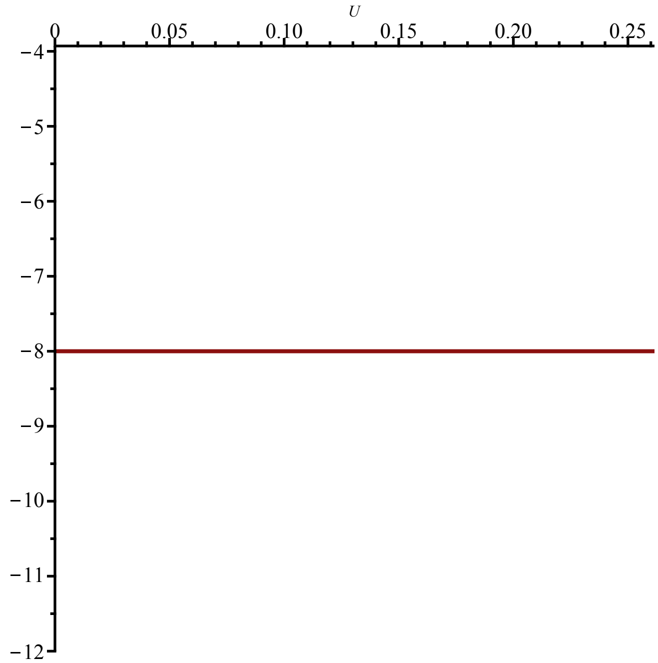
> $eqVcritU := collect\left(\frac{1}{U\left(U\nu + U - 2\right)}\left(\left(U^2 V^4 \nu + U^2 V^4 + 2 U^2 V^3 \nu + 2 U^2 V^3 + 18 U^2 V^2 \nu - 2 U V^4 + 18 U^2 V^2 + 2 U^2 V \nu - 4 V^3 U - 24 U V^2 \nu + 2 U^2 V + U^2 \nu - 12 U V^2 + 8 V^2 \nu + U^2 - 4 V U - 2 U\right)\right), V, factor\right);$

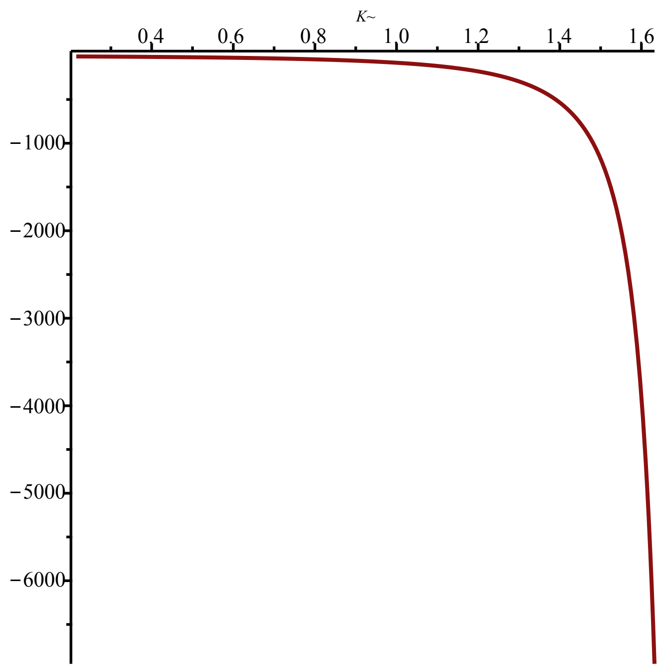
$$eqVcritU := 1 + V^4 + 2 V^3 + \frac{2\left(-2 + 3 U\right)\left(3 U \nu + 3 U - 2 \nu\right) V^2}{U\left(U \nu + U - 2\right)} + 2 V \tag{3.8}$$

We have four stationary points and we can check that they are all real. Indeed the previous polynom is <0 at V=-1, >0 at V=0 and < at V=1 (with a possible double root here in the subcritical case and if U= U_\nu). It is also positive at +infinity and - infinity.

> *factor*(*subs*(*V* = −1, *eqVcritU*)); *plot*(*factor*(*subs*(nu = *nuUsub*, %)), *U* = 0 ..*Uc*);
> *plot*(*subs*(nu = *nusupK*, *U* = *UsupK*, %%), *K* = *Kc* ..*Kinfini* − 0.1);

$$\frac{8\left(-1 + 2 U\right)\left(U \nu + U - \nu\right)}{U\left(U \nu + U - 2\right)}$$





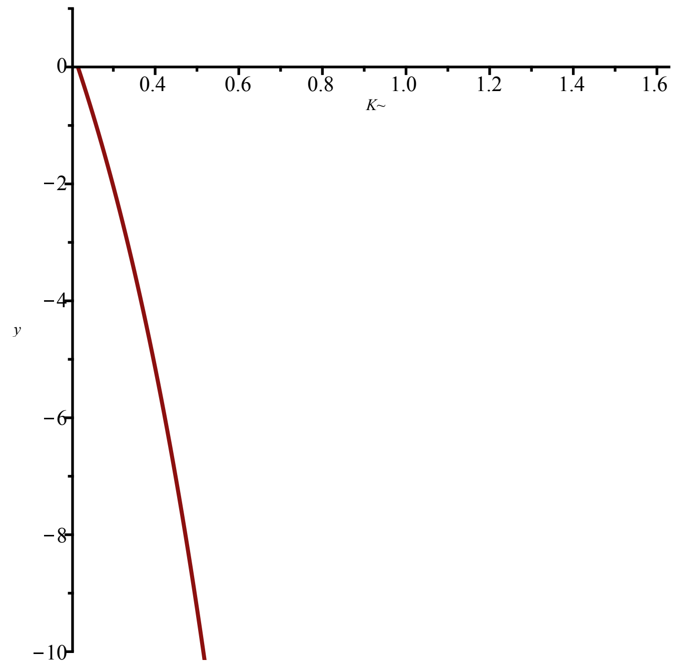
> $factor(subs(V = 0, eqVcritU));$

$$1 \tag{3.9}$$

> $factor(subs(V = 1, eqVcritU)); factor(subs(\mathrm{nu} = nuUsub, \%)); plot(subs(\mathrm{nu} = nusupK, U = UsupK, \%\%), K = Kc..Kinfini - 0.1, y = -10..1);$

$$\frac{8\left(3\,U^2\,\nu + 3\,U^2 - 3\,U\,\nu - 3\,U + \nu\right)}{U\left(U\,\nu + U - 2\right)}$$

$$0$$



The polynom has four real roots : one <-1, one between -1 and 0, one between 0 and +1, and one after 1.

Asymptotic expansion of V in y (on the critical line, i.e when t= t_\nu is fixed and equal to the radius of convergence, Lemma 3.8)

For nu<=nu_c

Recall the values of Unu and nuU in this regime:

> *Unusubc*;

$$\frac{3\nu+3-\sqrt{-3\nu^2+6\nu+9}}{6\nu+6} \quad \textbf{(4.1.1)}$$

> *nuUsub* := *solve*(*algUsubcrit*, nu);

$$nuUsub := -\frac{3U(U-1)}{3U^2-3U+1} \quad \textbf{(4.1.2)}$$

In the parametrization of y in terms of V, we replace nu by its expression in terms of Unusub.

> *yUVsubc* := *factor*(*subs*(nu = *nuUsub*, *yUV*));

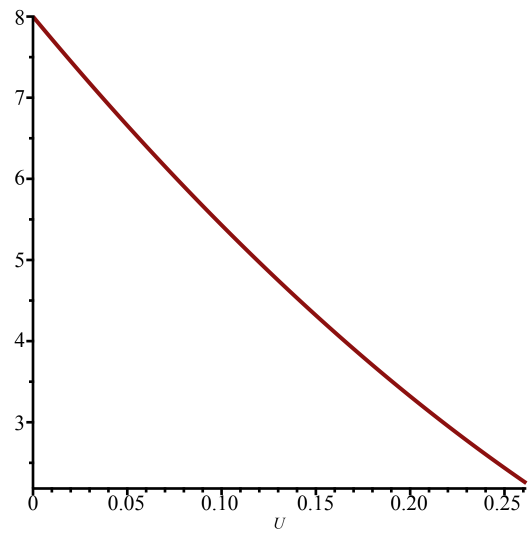
$$yUVsubc := -\frac{24(U-1)V(V+1)}{3UV^3-21V^2U-2V^3-3UV+18V^2-3U+6V+2} \quad \textbf{(4.1.3)}$$

First we look at the poles of y:

> *collect*(*denom*(*yUVsubc*), *V*, *factor*); *factor*(*discrim*(%, *V*));

$$(-2+3U)V^3+(-21U+18)V^2+(-3U+6)V-3U+2$$
$$-5184(23U^2-28U+8)(U-1)^2 \quad \textbf{(4.1.4)}$$

> *plot*((23 U^2 − 28 U + 8), U = 0 ..*Uc*);



Since the discriminant is negative for U\in [0,Uc], yUVsubc has a unique pole. Moreover the leading coefficient of the denominator of yUVsubc is negative, hence by evaluating it at any value of V, we can determine whether it cancels before or after V.

> $factor(subs(V=1, denom(yUVsubc)));$

$$-24\,U+24 \tag{4.1.5}$$

Hence, there is a unique pole, located after V=1.

Now we look at the critical values for (y,V):

> $factor(numer(diff(yUVsubc, V))); solve(\%, V);$

$$24\,(U-1)\,\left(V^2+4\,V+1\right)\,(V-1)^2\,(-2+3\,U)$$

$$1,\,1,\,-2+\sqrt{3},\,-2-\sqrt{3} \tag{4.1.6}$$

> $Vsubl := -2+\sqrt{3} : Vsubc := 1 :$

y(V) is increasing in [Vsubl,Vsubc], y(V) is critical at 1 (corresponding to y=2). We compute the corresponding expansion:

> $simplify(series(yUVsubc, V=1, 4));$

$$2+\frac{-2+3\,U}{12\,U-12}\,(V-1)^3+\mathrm{O}\left((V-1)^4\right) \tag{4.1.7}$$

This gives the development of V around y=2, (YY=(1-y/2)),

> $algeqtoseries(numer(2\cdot(1-YY)-yUVsubc), YY, V, 5);$

$$\left[1+RootOf\left((-2+3\,U)\,\_Z^3+24\,U-24\right)\,YY^{1/3}+\frac{RootOf\left((-2+3\,U)\,\_Z^3+24\,U-24\right)^2\,YY^{2/3}}{2}-\frac{4\,(U-1)\,YY}{-2+3\,U}+\frac{RootOf\left((-2+3\,U)\,\_Z^3+24\,U-24\right)\,YY^{4/3}}{3\,(-2+3\,U)}+\mathrm{O}\left(YY^{5/3}\right)\right] \tag{4.1.8}$$

with Y3Y=YY^1/3=(1-y/2)^1/3:

> $allvalues(RootOf((3\,U-2)\,\_Z^3+24\,U-24));$

(4.1.9)

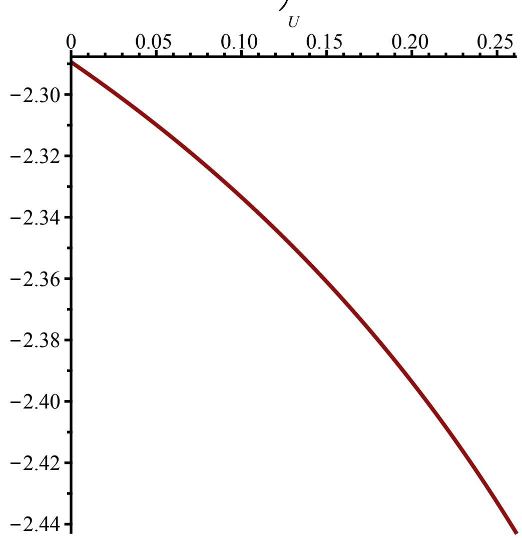
$$\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3},\ \left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3},\ -\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{1/3} \tag{4.1.9}$$

> $Vsubsingy := 1 + RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)\,Y3Y + \frac{RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)^2\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{3\,U-2} + \frac{RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right)\,Y3Y^4}{3\,(3\,U-2)};$

$$Vsubsingy := 1 + RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,Y3Y + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)^2\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{-2+3\,U} + \frac{RootOf\left((-2+3\,U)\,\_Z^3 + 24\,U - 24\right)\,Y3Y^4}{9\,U-6} \tag{4.1.10}$$

We check which solution of the RootOf is real, when U is real:

> $plot\left(\left(-\frac{24\,U-24}{3\,U-2}\right)^{1/3}(-1)^{2/3},\ U = 0\,..Uc\right);$

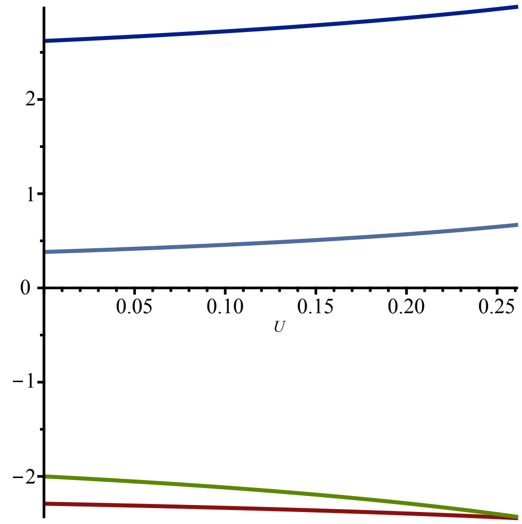


> $Vsubsingy := subs\left(RootOf\left((3\,U-2)\,\_Z^3 + 24\,U - 24\right) = \left(-\frac{24\,U-24}{3\,U-2}\right)^{1/3}(-1)^{2/3},\ Vsubsingy\right);$

$$Vsubsingy := 1 + \left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y - \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3}(-1)^{1/3}\,Y3Y^2}{2} - \frac{4\,(U-1)\,Y3Y^3}{-2+3\,U} + \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y^4}{9\,U-6} \tag{4.1.11}$$

We check that the coefficients do not vanish :

> $plot([seq(coeff(\textbf{(4.1.11)}, Y3Y, i), i = 1..4)], U = 0..Uc)$



For nu>nu_c

For nu>nuc, things get slighly more complicated because we cannot express simply U in terms of nu. Indeed the expression is cubic in U, but we have our parametric expression:

> $nusupK; UsupK;$

$$-\frac{K\sim^3 + 3K\sim^2 + 9K\sim + 11}{(K\sim + 3)(K\sim^2 - 3)}$$

$$-\frac{K\sim^2 - 3}{6K\sim + 10} \quad \textbf{(4.2.1)}$$

Specialisation of y to the critical line for nu > nuc, we first look at the poles of yUVsup

> $yUVsupc := factor(subs(\text{nu} = nusupK, U = UsupK, yUV));$

$$yUVsupc := -\left(8\left(K\sim^3 + 3K\sim^2 + 9K\sim + 11\right)(K\sim + 1)V(V+1)\right)\Big/\left(K\sim^4 V^3 - 7K\sim^4 V^2 - K\sim^4 V - 40K\sim^3 V^2 - 6K\sim^2 V^3 - K\sim^4 + 8K\sim^3 V - 110K\sim^2 V^2 + 14K\sim^2 V - 136K\sim V^2 + 9V^3 + 6K\sim^2 - 24K\sim V - 55V^2 - 33V - 9\right) \quad \textbf{(4.2.2)}$$

We rearrange its denominator :

> $collect(denom(yUVsupc), V, factor);$

$$\left(K\sim^2 - 3\right)^2 V^3 + \left(-7K\sim^4 - 40K\sim^3 - 110K\sim^2 - 136K\sim - 55\right)V^2 - \left(K\sim^2 - 8K\sim - 11\right)\left(K\sim^2 - 3\right)V - \left(K\sim^2 - 3\right)^2 \quad \textbf{(4.2.3)}$$

This is a polynomial of degree 3 and V, we compute its discriminant:

> $factor(discrim(\textbf{(4.2.3)}, V));$

$$-64\left(K\sim^4 + 6K\sim^3 + 30K\sim^2 + 62K\sim + 45\right)\left(23K\sim^6 + 184K\sim^5 + 593K\sim^4 + 1008K\sim^3 + 989K\sim^2 + 568K\sim + 163\right)(K\sim + 1)^2\left(K\sim^2 - 3\right)^2 \quad \textbf{(4.2.4)}$$

This is clearly nonpositive for K between Kc and Kinfini. Hence the denominator of yUVsupc has only one real root. To determine its position with respect to 1, we compute:

> *factor*(*subs*(*V* = 1, *denom*(*yUVsupc*)));

$$-8\,(K\sim+1)\,\left(K\sim^3+3\,K\sim^2+9\,K\sim+11\right) \tag{4.2.5}$$

Since this is negative, and the leading term of denom(yUVsupc) is also negative, we deduct that its unique real root is bigger than 1. We now turn our attention to the possible values for V critical.

> *factor*(*numer*(*diff*(*yUVsupc*, *V*)));

$$8\,(K\sim+1)\,\left(K\sim^3+3\,K\sim^2+9\,K\sim+11\right)\left(K\sim^2\,V^2+4\,K\sim^2\,V+K\sim^2+8\,K\sim\,V-3\,V^2+4\,V-3\right)\left(K\sim^2\,V^2-2\,K\sim^2\,V+K\sim^2-8\,K\sim\,V-3\,V^2-10\,V-3\right) \tag{4.2.6}$$

There are 2 polynomials of degree 2 in V with 4 possible real roots. We first check whether the roots are real:

> $P1 := collect\left(K^2\,V^2+4\,K^2\,V+K^2+8\,K\,V-3\,V^2+4\,V-3,\ V,\ factor\right);\ factor(discrim(\%,\ V));$

$$P1 := \left(K\sim^2-3\right)V^2+4\,(K\sim+1)^2\,V+K\sim^2-3$$
$$4\left(K\sim^2+4\,K\sim+5\right)\left(3\,K\sim^2+4\,K\sim-1\right) \tag{4.2.7}$$

> $simplify\left(subs\left(K=Kc,\ 3\,K^2+4\,K-1\right)\right);$

$$0 \tag{4.2.8}$$

> $P2 := collect\left(K^2\,V^2-2\,K^2\,V+K^2-8\,K\,V-3\,V^2-10\,V-3,\ V,\ factor\right);$
> $factor(discrim(\%,\ V));$

$$P2 := \left(K\sim^2-3\right)V^2+\left(-2\,K\sim^2-8\,K\sim-10\right)V+K\sim^2-3$$
$$32\,(2+K\sim)\,(K\sim+1)^2 \tag{4.2.9}$$

Since the discriminant are nonnegative in the considered range of values for K, there are 4 possible real roots. (The fact that the first term has a double root at Kc is a strong indication that it should give the critical values for V. Let us check it !)

> $V1sol := map\left(factor,\ solve\left(K^2\,V^2+4\,K^2\,V+K^2+8\,K\,V-3\,V^2+4\,V-3,\ [V]\right)\right);$

$$V1sol := \left[\left[V=-\frac{2\,K\sim^2+4\,K\sim-\sqrt{\left(K\sim^2+4\,K\sim+5\right)\left(3\,K\sim^2+4\,K\sim-1\right)}+2}{K\sim^2-3}\right],\left[V=-\frac{2\,K\sim^2+4\,K\sim+\sqrt{\left(K\sim^2+4\,K\sim+5\right)\left(3\,K\sim^2+4\,K\sim-1\right)}+2}{K\sim^2-3}\right]\right] \tag{4.2.10}$$

> $V2sol := map\left(factor,\ solve\left(K^2\,V^2-2\,K^2\,V+K^2-8\,K\,V-3\,V^2-10\,V-3,\ [V]\right)\right)$

$$V2sol := \left[\left[V=\frac{K\sim^2+4\,K\sim+2\sqrt{2}\,\sqrt{(2+K\sim)\,(K\sim+1)^2}+5}{K\sim^2-3}\right],\left[V=\frac{K\sim^2+4\,K\sim-2\sqrt{2}\,\sqrt{(2+K\sim)\,(K\sim+1)^2}+5}{K\sim^2-3}\right]\right] \tag{4.2.11}$$

$$> \ VK11 := -\frac{2K^2 + 4K - \sqrt{(K^2+4K+5)(3K^2+4K-1)} + 2}{K^2-3}:$$

$$VK12 := -\frac{2K^2 + 4K + \sqrt{(K^2+4K+5)(3K^2+4K-1)} + 2}{K^2-3}:$$

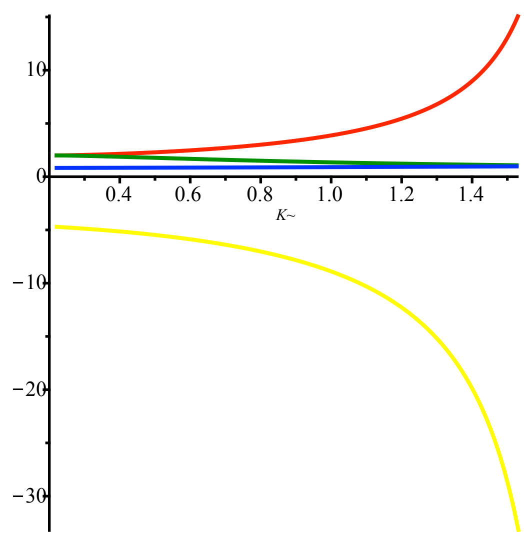
$$VK21 := \frac{K^2 + 4K + 2\sqrt{2}\sqrt{(K+2)(K+1)^2} + 5}{K^2-3}:$$

$$VK22 := \frac{K^2 + 4K - 2\sqrt{2}\sqrt{(K+2)(K+1)^2} + 5}{K^2-3}:$$

We plot the correponding values of y. Since we know that the coefficients of Qt are nonnegative, its radius of convergence must be singular.

> $yK11 := simplify(expand(rationalize(subs(V = VK11, yUVsupc)))):$
$yK12 := simplify(expand(rationalize(subs(V = VK12, yUVsupc)))):$
$yK21 := simplify(expand(rationalize(subs(V = VK21, yUVsupc)))):$
$yK22 := simplify(expand(rationalize(subs(V = VK22, yUVsupc)))):$

> $plot([yK11, yK12, yK21, yK22], K = Kc..Kinfini - 0.2, color = [\text{"Red"}, \text{"Green"}, \text{"Blue"}, \text{"Yellow"}]);$



From the analysis for nu<=nuc, we know that the critical value of y for nu=nuc is equal to 2. The plots indicate that only yK11 and yK12 are possible candidates. Let us check:

> $simplify(subs(K = Kc, yK11)); simplify(subs(K = Kc, yK12));$

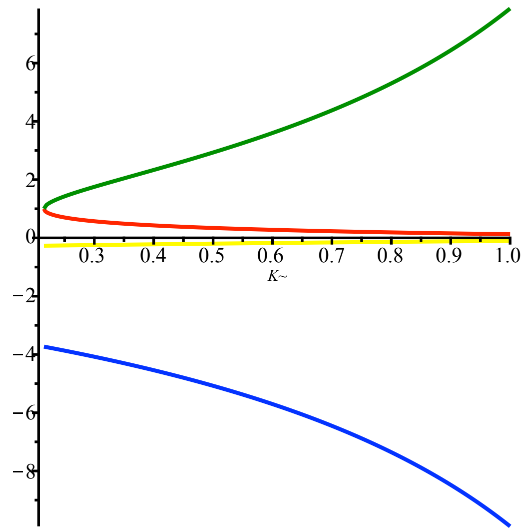
$$2$$

$$2 \qquad \textbf{(4.2.12)}$$

We now have to distinguish between these two values. We come back to the corresponding values of V.

For the range of values of K we consider, VK11 is smaller than VK12. Plots of the roots

> $plot([VK11, VK12, VK21, VK22], K = Kc..1, color = [\text{"Red"}, \text{"Green"}, \text{"Blue"}, \text{"Yellow"}]);$



So that V+ := VK11 and V-:=VK22

We compute the expansion VK11 around yUVsupc.

```
> map(factor, map(expand, simplify(series(yUVsupc, V = VK11, 3), symbolic)));
```

(4.2.13)

$$\Big(4\,(K\sim+1)\,({K\sim}^{3}+3\,{K\sim}^{2}+9\,K\sim+11)\,\big(5\,{K\sim}^{4}+28\,{K\sim}^{3}$$
$$-3\,{K\sim}^{2}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+58\,{K\sim}^{2}$$
$$-8\,K\sim\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+44\,K\sim+5$$
$$-7\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}\,\big)\Big)\Big/\Big(37\,{K\sim}^{8}+348\,{K\sim}^{7}$$
$$-21\,{K\sim}^{6}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+1456\,{K\sim}^{6}$$
$$-144\,{K\sim}^{5}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+3508\,{K\sim}^{5}$$
$$-431\,{K\sim}^{4}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+5314\,{K\sim}^{4}$$
$$-704\,{K\sim}^{3}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+5140\,{K\sim}^{3}$$
$$-687\,{K\sim}^{2}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+3016\,{K\sim}^{2}$$
$$-432\,K\sim\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+956\,K\sim$$
$$-149\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+145\Big)-4\,\Big((K\sim+1)\,\big(3\,{K\sim}^{2}$$
$$+8\,K\sim+7\big)\,({K\sim}^{3}+3\,{K\sim}^{2}+9\,K\sim+11)\,({K\sim}^{2}-3)^{3}\,\big(3\,{K\sim}^{4}$$
$$-2\,{K\sim}^{2}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+16\,{K\sim}^{3}$$
$$-4\,K\sim\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+30\,{K\sim}^{2}$$
$$-2\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+16\,K\sim-5\big)\Big)\Big/\Big(37\,{K\sim}^{8}+348\,{K\sim}^{7}$$
$$-21\,{K\sim}^{6}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+1456\,{K\sim}^{6}$$
$$-144\,{K\sim}^{5}\sqrt{{K\sim}^{2}+4\,K\sim+5}\,\sqrt{3\,{K\sim}^{2}+4\,K\sim-1}+3508\,{K\sim}^{5}$$

$$- 431\, K\sim^4 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 5314\, K\sim^4$$
$$- 704\, K\sim^3 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 5140\, K\sim^3$$
$$- 687\, K\sim^2 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 3016\, K\sim^2$$
$$- 432\, K\sim \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 956\, K\sim$$
$$- 149 \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 145\Big)^2 \Bigg(V$$
$$+ \frac{2\, K\sim^2 + 4\, K\sim - \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} + 2}{K\sim^2 - 3}\Bigg)^2 + \mathrm{O}\Bigg(\Bigg(V$$
$$+ \frac{2\, K\sim^2 + 4\, K\sim - \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} + 2}{K\sim^2 - 3}\Bigg)^3\Bigg)$$

The expansion is quadratic, so we introduce the change of variable YY=(1-y/yK11)^(1/2) and get the following development of V around VK11:

```
> devV11ysupc := sort(map(factor, collect(convert(op(2, simplify(algeqtoseries(numer(yK11
      ·(1 - YY^2) - yUVsupc), YY, V, 6))), polynom), YY)), YY);
```

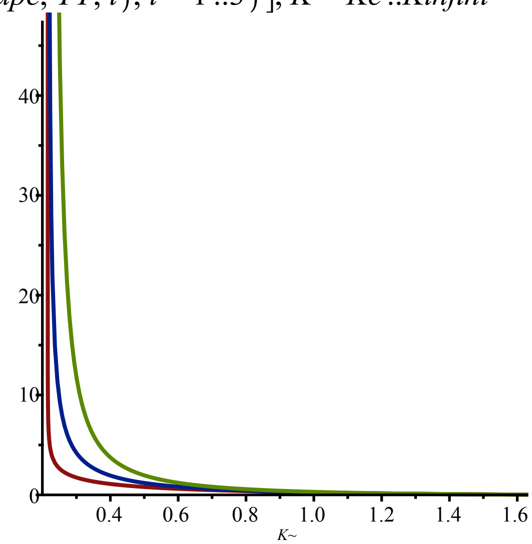
$$devV11ysupc := \Big(RootOf\big(\_Z^2\, (9\, K\sim^{10} + 36\, K\sim^9 - 31\, K\sim^8 - 304\, K\sim^7 - 214\, K\sim^6 \quad \mathbf{(4.2.14)}$$
$$+ 792\, K\sim^5 + 1170\, K\sim^4 - 432\, K\sim^3 - 1539\, K\sim^2 - 540\, K\sim + 189) - 174\, K\sim^{10}$$
$$- 1960\, K\sim^9 + 100\, K\sim^8 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 9950\, K\sim^8$$
$$+ 864\, K\sim^7 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 29664\, K\sim^7$$
$$+ 3304\, K\sim^6 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 56972\, K\sim^6$$
$$+ 7200\, K\sim^5 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 72752\, K\sim^5$$
$$+ 9760\, K\sim^4 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 61372\, K\sim^4$$
$$+ 8480\, K\sim^3 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 32608\, K\sim^3$$
$$+ 4504\, K\sim^2 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} - 9190\, K\sim^2$$
$$+ 1120\, K\sim \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} + 152\, K\sim$$
$$- 4 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)} + 890\big)\, \big(7445 + 78955\, K\sim^{12}$$
$$+ 335800\, K\sim^{11} + 934389\, K\sim^{10} + 729\, K\sim^{14} + 11172\, K\sim^{13} + 1700348\, K\sim^9$$
$$+ 1773231\, K\sim^8 + 182160\, K\sim^7$$
$$- 234096\, K\sim^6 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}$$
$$- 374512\, K\sim^5 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}$$

$$- 462922\, K\sim^{4} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 365048\, K\sim^{3} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 153792\, K\sim^{2} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 19800\, K\sim \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 81796\, K\sim$$
$$- 238\, K\sim^{12} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 2840\, K\sim^{11} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} - 1264840\, K\sim^{3} - 67049\, K\sim^{2}$$
$$- 2565413\, K\sim^{6} - 4211364\, K\sim^{5} - 3327599\, K\sim^{4}$$
$$+ 4118 \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 91230\, K\sim^{8} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 145520\, K\sim^{7} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 14928\, K\sim^{10} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 45368\, K\sim^{9} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}\Big)\, YY^{3}\Big) \Big/ \Big(4\, \left(K\sim^{2} - 3\right) \big(K\sim^{2}$$
$$+ 4\, K\sim + 5\big) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)^{2} \left(3\, K\sim^{2} + 8\, K\sim + 7\right)^{3}\Big) - \Big(\Big(98\, K\sim^{10}$$
$$+ 1048\, K\sim^{9} - 55\, K\sim^{8} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 5078\, K\sim^{8}$$
$$- 444\, K\sim^{7} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 14592\, K\sim^{7}$$
$$- 1636\, K\sim^{6} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 27540\, K\sim^{6}$$
$$- 3556\, K\sim^{5} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 35824\, K\sim^{5}$$
$$- 4866\, K\sim^{4} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 32332\, K\sim^{4}$$
$$- 4180\, K\sim^{3} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 18752\, K\sim^{3}$$
$$- 2228\, K\sim^{2} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} + 4682\, K\sim^{2}$$
$$- 812\, K\sim \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} - 1480\, K\sim$$
$$- 223 \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} - 994\Big)\, YY^{2}\Big) \Big/ \Big(\big(3\, K\sim^{2} + 4\, K\sim$$
$$- 1\big) \left(3\, K\sim^{2} + 8\, K\sim + 7\right)^{2} \left(K\sim^{2} - 3\right)^{2}\Big) + RootOf\Big(\_Z^{2}\, \big(9\, K\sim^{10} + 36\, K\sim^{9} - 31\, K\sim^{8}$$
$$- 304\, K\sim^{7} - 214\, K\sim^{6} + 792\, K\sim^{5} + 1170\, K\sim^{4} - 432\, K\sim^{3} - 1539\, K\sim^{2} - 540\, K\sim$$
$$+ 189\big) - 174\, K\sim^{10} - 1960\, K\sim^{9} + 100\, K\sim^{8} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)}$$
$$- 9950\, K\sim^{8} + 864\, K\sim^{7} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} - 29664\, K\sim^{7}$$
$$+ 3304\, K\sim^{6} \sqrt{\left(K\sim^{2} + 4\, K\sim + 5\right) \left(3\, K\sim^{2} + 4\, K\sim - 1\right)} - 56972\, K\sim^{6}$$

$$+ 7200 K\sim^5 \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} - 72752 K\sim^5$$
$$+ 9760 K\sim^4 \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} - 61372 K\sim^4$$
$$+ 8480 K\sim^3 \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} - 32608 K\sim^3$$
$$+ 4504 K\sim^2 \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} - 9190 K\sim^2$$
$$+ 1120 K\sim \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} + 152 K\sim$$
$$- 4 \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} + 890\Big) YY$$
$$- \frac{2 K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} + 2}{K\sim^2 - 3}$$

We check that the coefficients in the development do not cancel for K in [Kc,Kinfini]

```
> plot([seq(coeff(devV11ysupc, YY, i), i = 1 ..3)], K = Kc ..Kinfini - 0.1);
```



Asymptotic behavior (in t) of V(t,ty) (Lemma 3.9)

For nu<nu_c:

Recall the singular expansion of U in this regime (Usubc is U(nu,tnu^3)):

```
> Usubcsing3;
```

$$Usubc + \frac{Usubc\,(-2 + 3\,Usubc)\sqrt{6}\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,XX}{6} + \Big(\big(1458\,Usubc^6 - 5778\,Usubc^5 + 9045\,Usubc^4 - 7146\,Usubc^3 + 2984\,Usubc^2 - 616\,Usubc + 48\big)\,Usubc^2\,XX^2\Big) \Big/ \Big(18\,(9\,Usubc^2 - 10\,Usubc + 2)^2\,(2\,Usubc - 1)\Big) \quad \textbf{(5.1.1)}$$

$\left(9\, Usubc^2 - 10\, Usubc + 2\right)^3 \left(2\, Usubc - 1\right)\Big)$

We want to compute the development of V around rhosubc (=t_\nu^3 in the paper), for a fixed y. Recall that y is parametrized by U and V:

> *yUV*;

$$\left(8\,\nu\,(1-2\,U)\,V\,(V+1)\right) \Big/ \left(U\,(U\,(\nu+1)-2)\,\left(V^3 + \frac{\left(9\,(\nu+1)\,U^2 - 2\,(3+10\,\nu)\,U + 8\,\nu\right)V^2}{U\,(U\,(\nu+1)-2)} - \frac{\left(9\,U\,(\nu+1) - 4\,\nu - 6\right)V}{U\,(\nu+1)-2} - 1\right)\right) \quad \textbf{(5.1.2)}$$

Since y is fixed, we could write the development of V in terms of this fixed value of y. It turns out that the formulas are simpler when written in terms of V rather than y. Indeed, when U is equal to Usubc (i.e. when w=t^3= rhosubc), nu can be replaced by its value in terms of Usubc, and hence the value of y is fully determined by the value of V in this setting, which we denote by Vsubc:

> *yVsub* := *subs*(*V* = *Vsub*, *U* = *Usubc*, *factor*(*subs*(nu = *nuUsub*, *yUV*)));

$$yVsub := -\left(24\,(Usubc-1)\,Vsub\,(Vsub+1)\right) \Big/ \left(3\,Usubc\,Vsub^3 - 21\,Usubc\,Vsub^2 - 2\,Vsub^3 - 3\,Usubc\,Vsub + 18\,Vsub^2 - 3\,Usubc + 6\,Vsub + 2\right) \quad \textbf{(5.1.3)}$$

When we compute the development of V for w close to rhosubc, we can replace y by the latter value. Here is the new equation we obtain:

> *op*(6, *factor*(*numer*(*yVsub* − *subs*(nu = *subs*(*U* = *Usubc*, *nuUsub*), *yUV*)))); *indets*(%);

$$-6\,U\,Usubc^2\,V^3\,Vsub^2 + 6\,U\,Usubc^2\,V^2\,Vsub^3 + U^2\,V^3\,Vsub^2 - 6\,U\,Usubc^2\,V^3\,Vsub + 6\,U\,Usubc^2\,V\,Vsub^3 + 6\,U\,Usubc\,V^3\,Vsub^2 - 4\,U\,Usubc\,V^2\,Vsub^3 - 3\,Usubc^2\,V^2\,Vsub^3 + U^2\,V^3\,Vsub + 9\,U^2\,V^2\,Vsub^2 + 36\,U\,Usubc^2\,V^2\,Vsub - 36\,U\,Usubc^2\,V\,Vsub^2 + 6\,U\,Usubc\,V^3\,Vsub - 6\,U\,Usubc\,V^2\,Vsub^2 - 4\,U\,Usubc\,V\,Vsub^3 - 2\,U\,V^3\,Vsub^2 - 3\,Usubc^2\,V^2\,Vsub^2 - 3\,Usubc^2\,V\,Vsub^3 + 2\,Usubc\,V^2\,Vsub^3 + 9\,U^2\,V^2\,Vsub - 9\,U^2\,V\,Vsub^2 - 6\,U\,Usubc^2\,V^2 + 6\,U\,Usubc^2\,Vsub^2 - 30\,U\,Usubc\,V^2\,Vsub + 30\,U\,Usubc\,V\,Vsub^2 - 2\,U\,V^3\,Vsub - 6\,U\,V^2\,Vsub^2 - 21\,Usubc^2\,V^2\,Vsub + 21\,Usubc^2\,V\,Vsub^2 + 6\,Usubc\,V^2\,Vsub^2 + 2\,Usubc\,V\,Vsub^3 - 9\,U^2\,V\,Vsub - U^2\,Vsub^2 - 6\,U\,Usubc^2\,V + 6\,U\,Usubc^2\,Vsub + 4\,U\,Usubc\,V^2 + 6\,U\,Usubc\,V\,Vsub - 6\,U\,Usubc\,Vsub^2 - 6\,U\,V^2\,Vsub + 6\,U\,V\,Vsub^2 + 3\,Usubc^2\,V^2 + 3\,Usubc^2\,V\,Vsub + 18\,Usubc\,V^2\,Vsub - 18\,Usubc\,V\,Vsub^2 - U^2\,Vsub + 4\,U\,Usubc\,V - 6\,U\,Usubc\,Vsub + 6\,U\,V\,Vsub + 2\,U\,Vsub^2 + 3\,Usubc^2\,V - 2\,Usubc\,V^2 - 6\,Usubc\,V\,Vsub + 2\,U\,Vsub - 2\,Usubc\,V$$

$$\{U, Usubc, V, Vsub\} \quad \textbf{(5.1.4)}$$

> $eqyUVsub := -6\,U\,Usubc^2\,V^3\,Vsub^2 + 6\,U\,Usubc^2\,V^2\,Vsub^3 + U^2\,V^3\,Vsub^2$

$- 6\,U\,Usubc^2\,V^3\,Vsub + 6\,U\,Usubc^2\,V\,Vsub^3 + 6\,U\,Usubc\,V^3\,Vsub^2$
$- 4\,U\,Usubc\,V^2\,Vsub^3 - 3\,Usubc^2\,V^2\,Vsub^3 + U^2\,V^3\,Vsub + 9\,U^2\,V^2\,Vsub^2$
$+ 36\,U\,Usubc^2\,V^2\,Vsub - 36\,U\,Usubc^2\,V\,Vsub^2 + 6\,U\,Usubc\,V^3\,Vsub$
$- 6\,U\,Usubc\,V^2\,Vsub^2 - 4\,U\,Usubc\,V\,Vsub^3 - 2\,U\,V^3\,Vsub^2 - 3\,Usubc^2\,V^2\,Vsub^2$
$- 3\,Usubc^2\,V\,Vsub^3 + 2\,Usubc\,V^2\,Vsub^3 + 9\,U^2\,V^2\,Vsub - 9\,U^2\,V\,Vsub^2$
$- 6\,U\,Usubc^2\,V^2 + 6\,U\,Usubc^2\,Vsub^2 - 30\,U\,Usubc\,V^2\,Vsub + 30\,U\,Usubc\,V\,Vsub^2$
$- 2\,U\,V^3\,Vsub - 6\,U\,V^2\,Vsub^2 - 21\,Usubc^2\,V^2\,Vsub + 21\,Usubc^2\,V\,Vsub^2$
$+ 6\,Usubc\,V^2\,Vsub^2 + 2\,Usubc\,V\,Vsub^3 - 9\,U^2\,V\,Vsub - U^2\,Vsub^2 - 6\,U\,Usubc^2\,V$
$+ 6\,U\,Usubc^2\,Vsub + 4\,U\,Usubc\,V^2 + 6\,U\,Usubc\,V\,Vsub - 6\,U\,Usubc\,Vsub^2$
$- 6\,U\,V^2\,Vsub + 6\,U\,V\,Vsub^2 + 3\,Usubc^2\,V^2 + 3\,Usubc^2\,V\,Vsub + 18\,Usubc\,V^2\,Vsub$
$- 18\,Usubc\,V\,Vsub^2 - U^2\,Vsub + 4\,U\,Usubc\,V - 6\,U\,Usubc\,Vsub + 6\,U\,V\,Vsub$
$+ 2\,U\,Vsub^2 + 3\,Usubc^2\,V - 2\,Usubc\,V^2 - 6\,Usubc\,V\,Vsub + 2\,U\,Vsub - 2\,V\,Usubc :$

We plug the singular behavior of U in the equation, and deduce from it the asympotic behavior of V (we write V=Vsub + XX*VX, so that we obtain the singular behavior of VX)

> $algeqtoseries(simplify(subs(U = Usubcsing3, V = Vsub + XX \cdot VX, eqyUVsub)), XX, VX, 3, true);$

$$\left[\frac{\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,\sqrt{6}\,Vsub\,(Vsub+1)}{3\,(Vsub-1)} - \frac{1}{18}\Big((Vsub \right. \qquad \textbf{(5.1.5)}$$

$+ 1)\,Vsub\,(6\,Usubc^2 - 10\,Usubc + 3)\,(81\,Usubc^4\,Vsub^3 - 243\,Usubc^4\,Vsub^2$

$- 384\,Usubc^3\,Vsub^3 + 243\,Usubc^4\,Vsub - 288\,Usubc^3\,Vsub^2 + 454\,Vsub^3\,Usubc^2$

$- 81\,Usubc^4 - 792\,Usubc^3\,Vsub + 894\,Usubc^2\,Vsub^2 - 188\,Vsub^3\,Usubc + 168\,Usubc^3$

$+ 846\,Usubc^2\,Vsub - 492\,Usubc\,Vsub^2 + 24\,Vsub^3 - 106\,Usubc^2 - 348\,Usubc\,Vsub$

$+ 72\,Vsub^2 + 20\,Usubc + 48\,Vsub))\Big/\big((2\,Usubc - 1)\,(Vsub^2 + 4\,Vsub$

$$+ 1) \left(9\, Usubc^{2} - 10\, Usubc + 2\right)^{2} \left(Vsub - 1\right)^{2}\Big)\, XX$$

$$- \frac{1}{648} \Bigg(Vsub \sqrt{6} \sqrt{\frac{6\, Usubc^{2} - 10\, Usubc + 3}{9\, Usubc^{2} - 10\, Usubc + 2}} \big(103518\, Usubc^{8}\, Vsub^{7}$$

$$- 281394\, Usubc^{8}\, Vsub^{6} - 647622\, Usubc^{7}\, Vsub^{7} + 118098\, Usubc^{8}\, Vsub^{5}$$

$$+ 1074762\, Usubc^{7}\, Vsub^{6} + 1632015\, Usubc^{6}\, Vsub^{7} + 269730\, Usubc^{8}\, Vsub^{4}$$

$$+ 284310\, Usubc^{7}\, Vsub^{5} - 1507545\, Usubc^{6}\, Vsub^{6} - 2172726\, Usubc^{5}\, Vsub^{7}$$

$$- 255150\, Usubc^{8}\, Vsub^{3} - 143802\, Usubc^{7}\, Vsub^{4} - 2474199\, Usubc^{6}\, Vsub^{5}$$

$$+ 840474\, Usubc^{5}\, Vsub^{6} + 1676186\, Usubc^{4}\, Vsub^{7} + 13122\, Usubc^{8}\, Vsub^{2}$$

$$+ 1609686\, Usubc^{7}\, Vsub^{3} - 2504655\, Usubc^{6}\, Vsub^{4} + 4954950\, Usubc^{5}\, Vsub^{5}$$

$$+ 38330\, Usubc^{4}\, Vsub^{6} - 769072\, Usubc^{3}\, Vsub^{7} + 33534\, Usubc^{8}\, Vsub$$

$$+ 187110\, Usubc^{7}\, Vsub^{2} - 4251447\, Usubc^{6}\, Vsub^{3} + 6349878\, Usubc^{5}\, Vsub^{4}$$

$$- 4764858\, Usubc^{4}\, Vsub^{5} - 266800\, Usubc^{3}\, Vsub^{6} + 205560\, Usubc^{2}\, Vsub^{7}$$

$$- 1458\, Usubc^{8} - 126630\, Usubc^{7}\, Vsub - 907551\, Usubc^{6}\, Vsub^{2}$$

$$+ 6094278\, Usubc^{5}\, Vsub^{3} - 6858106\, Usubc^{4}\, Vsub^{4} + 2511600\, Usubc^{3}\, Vsub^{5}$$

$$+ 125208\, Usubc^{2}\, Vsub^{6} - 29376\, Usubc\, Vsub^{7} + 1674\, Usubc^{7} + 179199\, Usubc^{6}\, Vsub$$

$$+ 1586070\, Usubc^{5}\, Vsub^{2} - 5156362\, Usubc^{4}\, Vsub^{3} + 3948272\, Usubc^{3}\, Vsub^{4}$$

$$- 737016\, Usubc^2\, Vsub^5 - 24192\, Usubc\, Vsub^6 + 1728\, Vsub^7 + 5319\, Usubc^6$$

$$- 112086\, Usubc^5\, Vsub - 1413642\, Usubc^4\, Vsub^2 + 2630576\, Usubc^3\, Vsub^3$$

$$- 1255896\, Vsub^4\, Usubc^2 + 112320\, Usubc\, Vsub^5 + 1728\, Vsub^6 - 12006\, Usubc^5$$

$$+ 20906\, Usubc^4\, Vsub + 703920\, Usubc^3\, Vsub^2 - 790968\, Vsub^3\, Usubc^2$$

$$+ 207360\, Vsub^4\, Usubc - 6912\, Vsub^5 + 9290\, Usubc^4 + 9104\, Usubc^3\, Vsub$$

$$- 196824\, Usubc^2\, Vsub^2 + 128448\, Vsub^3\, Usubc - 13824\, Vsub^4 - 3184\, Usubc^3$$

$$- 4680\, Usubc^2\, Vsub + 28800\, Usubc\, Vsub^2 - 8640\, Vsub^3 + 408\, Usubc^2$$

$$+ 576\, Usubc\, Vsub - 1728\, Vsub^2)) \Big/ \Big(\big(Vsub^2 + 4\, Vsub + 1\big)\big(9\, Usubc^2$$

$$- 10\, Usubc + 2\big)^3\, (2\, Usubc - 1)\, (Vsub - 1)^5\Big)\, XX^2 + O\big(XX^3\big)\Big]$$

> $$Vsubsing3 := sort\Bigg(collect\Bigg(Vsub + XX$$

$$\cdot \Bigg(\frac{Vsub\, (Vsub + 1)\sqrt{6}\, \sqrt{\dfrac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}}{3\, (Vsub - 1)} - \frac{1}{18}\, \Big(\big(6\, Usubc^2$$

$$- 10\, Usubc + 3\big)\, (Vsub + 1)\, Vsub\, \big(81\, Usubc^4\, Vsub^3 - 243\, Usubc^4\, Vsub^2$$

$$- 384\, Usubc^3\, Vsub^3 + 243\, Usubc^4\, Vsub - 288\, Usubc^3\, Vsub^2 + 454\, Usubc^2\, Vsub^3$$

$- 81\, Usubc^4 - 792\, Usubc^3\, Vsub + 894\, Usubc^2\, Vsub^2 - 188\, Usubc\, Vsub^3 + 168\, Usubc^3$

$+ 846\, Usubc^2\, Vsub - 492\, Usubc\, Vsub^2 + 24\, Vsub^3 - 106\, Usubc^2 - 348\, Usubc\, Vsub$

$+ 72\, Vsub^2 + 20\, Usubc + 48\, Vsub)\big)\Big/\Big((-1 + 2\, Usubc)\, \big(Vsub^2 + 4\, Vsub$

$+ 1\big)\, \big(9\, Usubc^2 - 10\, Usubc + 2\big)^2\, (Vsub - 1)^2\Big)\, XX$

$- \frac{1}{648} \left(Vsub \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, \sqrt{6}\, \big(103518\, Usubc^8\, Vsub^7 - 281394\, Usubc^8\, Vsub^6 - \right.$

$+ 284310\, Usubc^7\, Vsub^5 - 1507545\, Usubc^6\, Vsub^6 - 2172726\, Usubc^5\, Vsub^7$

$- 255150\, Usubc^8\, Vsub^3 - 143802\, Usubc^7\, Vsub^4 - 2474199\, Usubc^6\, Vsub^5$

$+ 840474\, Usubc^5\, Vsub^6 + 1676186\, Usubc^4\, Vsub^7 + 13122\, Usubc^8\, Vsub^2$

$+ 1609686\, Usubc^7\, Vsub^3 - 2504655\, Usubc^6\, Vsub^4 + 4954950\, Usubc^5\, Vsub^5$

$+ 38330\, Usubc^4\, Vsub^6 - 769072\, Usubc^3\, Vsub^7 + 33534\, Usubc^8\, Vsub$

$+ 187110\, Usubc^7\, Vsub^2 - 4251447\, Usubc^6\, Vsub^3 + 6349878\, Usubc^5\, Vsub^4$

$- 4764858\, Usubc^4\, Vsub^5 - 266800\, Usubc^3\, Vsub^6 + 205560\, Usubc^2\, Vsub^7$

$- 1458\, Usubc^8 - 126630\, Usubc^7\, Vsub - 907551\, Usubc^6\, Vsub^2 + 6094278\, Usubc^5\, Vsub^3$

$- 6858106\, Usubc^4\, Vsub^4 + 2511600\, Usubc^3\, Vsub^5 + 125208\, Usubc^2\, Vsub^6$

$- 29376\, Usubc\, Vsub^{7} + 1674\, Usubc^{7} + 179199\, Usubc^{6}\, Vsub + 1586070\, Usubc^{5}\, Vsub^{2}$

$- 5156362\, Usubc^{4}\, Vsub^{3} + 3948272\, Usubc^{3}\, Vsub^{4} - 737016\, Usubc^{2}\, Vsub^{5}$

$- 24192\, Usubc\, Vsub^{6} + 1728\, Vsub^{7} + 5319\, Usubc^{6} - 112086\, Usubc^{5}\, Vsub$

$- 1413642\, Usubc^{4}\, Vsub^{2} + 2630576\, Usubc^{3}\, Vsub^{3} - 1255896\, Usubc^{2}\, Vsub^{4}$

$+ 112320\, Usubc\, Vsub^{5} + 1728\, Vsub^{6} - 12006\, Usubc^{5} + 20906\, Usubc^{4}\, Vsub$

$+ 703920\, Usubc^{3}\, Vsub^{2} - 790968\, Usubc^{2}\, Vsub^{3} + 207360\, Usubc\, Vsub^{4} - 6912\, Vsub^{5}$

$+ 9290\, Usubc^{4} + 9104\, Usubc^{3}\, Vsub - 196824\, Usubc^{2}\, Vsub^{2} + 128448\, Usubc\, Vsub^{3}$

$- 13824\, Vsub^{4} - 3184\, Usubc^{3} - 4680\, Usubc^{2}\, Vsub + 28800\, Usubc\, Vsub^{2} - 8640\, Vsub^{3}$

$+ 408\, Usubc^{2} + 576\, Usubc\, Vsub - 1728\, Vsub^{2})) \Big/ \Big((Vsub^{2} + 4\, Vsub$

$+ 1) \left(9\, Usubc^{2} - 10\, Usubc + 2\right)^{3} (-1 + 2\, Usubc)\, (Vsub - 1)^{5}\Big)\, XX^{2} \Big), XX, factor \Big),$

$XX, ascending \Big);$

$$Vsubsing3 := Vsub + \frac{\sqrt{\frac{6\, Usubc^{2} - 10\, Usubc + 3}{9\, Usubc^{2} - 10\, Usubc + 2}}\, \sqrt{6}\, Vsub\, (Vsub + 1)\, XX}{3\, (Vsub - 1)} - \Big((Vsub$$

$$+ 1)\, Vsub \left(6\, Usubc^{2} - 10\, Usubc + 3\right) \left(81\, Usubc^{4}\, Vsub^{3} - 243\, Usubc^{4}\, Vsub^{2}\right.$$

$$- 384\, Usubc^{3}\, Vsub^{3} + 243\, Usubc^{4}\, Vsub - 288\, Usubc^{3}\, Vsub^{2} + 454\, Vsub^{3}\, Usubc^{2}$$

$$- 81\, Usubc^{4} - 792\, Usubc^{3}\, Vsub + 894\, Usubc^{2}\, Vsub^{2} - 188\, Vsub^{3}\, Usubc + 168\, Usubc^{3}$$

$$+ 846\, Usubc^{2}\, Vsub - 492\, Usubc\, Vsub^{2} + 24\, Vsub^{3} - 106\, Usubc^{2} - 348\, Usubc\, Vsub$$

$$\left. + 72\, Vsub^{2} + 20\, Usubc + 48\, Vsub\right) XX^{2} \Big) \Big/ \Big(18\, (2\, Usubc - 1) \left(Vsub^{2} + 4\, Vsub\right.$$

$$\left. + 1\right) \left(9\, Usubc^{2} - 10\, Usubc + 2\right)^{2} (Vsub - 1)^{2} \Big)$$

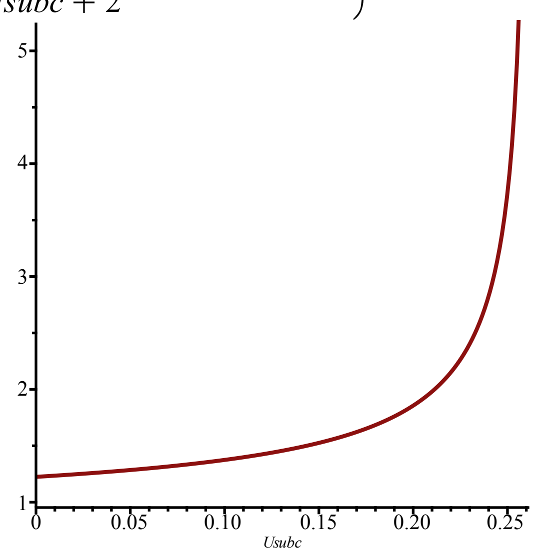
$$- \Bigg(Vsub \sqrt{6} \sqrt{\frac{6\, Usubc^{2} - 10\, Usubc + 3}{9\, Usubc^{2} - 10\, Usubc + 2}}\, (Vsub + 1) \left(6\, Usubc^{2} - 10\, Usubc + 3\right) \left(17253\, Usubc^{6}\, Vs\right.$$

$+ 131406\, Usubc^4\, Vsub^6 - 38880\, Usubc^6\, Vsub^3 - 99954\, Usubc^5\, Vsub^4$
$- 190944\, Usubc^4\, Vsub^5 - 103520\, Usubc^3\, Vsub^6 - 3645\, Usubc^6\, Vsub^2$
$+ 150912\, Usubc^5\, Vsub^3 - 97614\, Usubc^4\, Vsub^4 + 93888\, Usubc^3\, Vsub^5$
$+ 41128\, Usubc^2\, Vsub^6 + 5832\, Usubc^6\, Vsub + 46494\, Usubc^5\, Vsub^2$
$- 257376\, Usubc^4\, Vsub^3 + 210912\, Usubc^3\, Vsub^4 - 21024\, Usubc^2\, Vsub^5$
$- 7872\, Usubc\, Vsub^6 - 243\, Usubc^6 - 11664\, Usubc^5\, Vsub - 100926\, Usubc^4\, Vsub^2$
$+ 230272\, Usubc^3\, Vsub^3 - 120840\, Vsub^4\, Usubc^2 + 1728\, Usubc\, Vsub^5 + 576\, Vsub^6$
$- 126\, Usubc^5 + 6624\, Usubc^4\, Vsub + 89568\, Usubc^3\, Vsub^2 - 109376\, Vsub^3\, Usubc^2$
$+ 28032\, Usubc\, Vsub^4 + 798\, Usubc^4 + 192\, Usubc^3\, Vsub - 37800\, Usubc^2\, Vsub^2$
$+ 25728\, Vsub^3\, Usubc - 2304\, Vsub^4 - 608\, Usubc^3 - 1056\, Usubc^2\, Vsub$
$+ 7488\, Usubc\, Vsub^2 - 2304\, Vsub^3 + 136\, Usubc^2 + 192\, Usubc\, Vsub - 576\, Vsub^2 \big)$
$XX^3 \big) \Big/ \big(648\, (Vsub^2 + 4\, Vsub + 1)\, (9\, Usubc^2 - 10\, Usubc + 2)^3\, (2\, Usubc$
$- 1)\, (Vsub - 1)^5 \big)$

We check that the coefficients in the development do not cancel. Recall that since y\in(0,2), Vsub\in(0,1) (see the proof of Lemma~\ref{lem:weightsclusters}).

For the coefficient of XX:

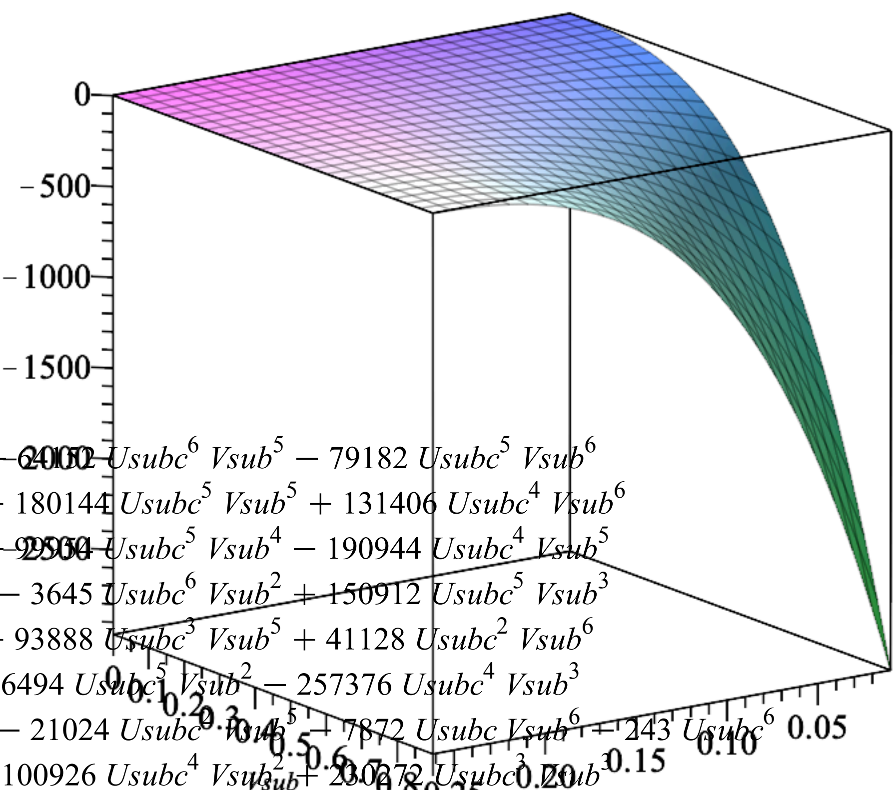
$$> \; plot\left(\sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}},\, Usubc = 0\,..Uc\right);$$



For the coefficient of XX^2:

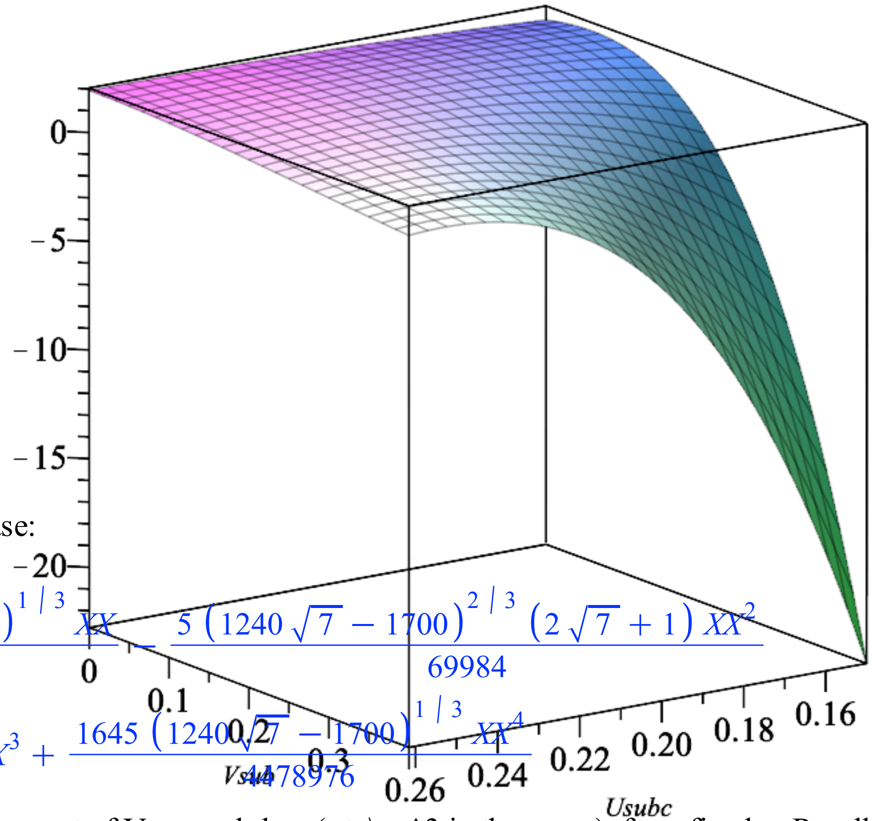
$> \; plot3d\big(\big(17253\, Usubc^6\, Vsub^6 - 64152\, Usubc^6\, Vsub^5 - 79182\, Usubc^5\, Vsub^6$
$+ 83835\, Usubc^6\, Vsub^4 + 180144\, Usubc^5\, Vsub^5 + 131406\, Usubc^4\, Vsub^6$
$- 38880\, Vsub^3\, Usubc^6 - 99954\, Usubc^5\, Vsub^4 - 190944\, Usubc^4\, Vsub^5$
$- 103520\, Usubc^3\, Vsub^6 - 3645\, Vsub^2\, Usubc^6 + 150912\, Vsub^3\, Usubc^5$

$- 97614\, Usubc^4\, Vsub^4 + 93888\, Usubc^3\, Vsub^5 + 41128\, Usubc^2\, Vsub^6$
$+ 5832\, Usubc^6\, Vsub + 46494\, Vsub^2\, Usubc^5 - 257376\, Vsub^3\, Usubc^4$
$+ 210912\, Usubc^3\, Vsub^4 - 21024\, Usubc^2\, Vsub^5 - 7872\, Usubc\, Vsub^6 - 243\, Usubc^6$
$- 11664\, Usubc^5\, Vsub - 100926\, Vsub^2\, Usubc^4 + 230272\, Vsub^3\, Usubc^3$
$- 120840\, Vsub^4\, Usubc^2 + 1728\, Usubc\, Vsub^5 + 576\, Vsub^6 - 126\, Usubc^5$
$+ 6624\, Usubc^4\, Vsub + 89568\, Vsub^2\, Usubc^3 - 109376\, Vsub^3\, Usubc^2$
$+ 28032\, Vsub^4\, Usubc + 798\, Usubc^4 + 192\, Usubc^3\, Vsub - 37800\, Vsub^2\, Usubc^2$
$+ 25728\, Vsub^3\, Usubc - 2304\, Vsub^4 - 608\, Usubc^3 - 1056\, Usubc^2\, Vsub$
$+ 7488\, Vsub^2\, Usubc - 2304\, Vsub^3 + 136\, Usubc^2 + 192\, Usubc\, Vsub - 576\, Vsub^2\big),$
$Usubc = 0.01\,..Uc,\ Vsub = 0\,..0.9\big);$



For the coefficient of XX^3:

> $plot3d\big(17253\, Usubc^6\, Vsub^6 - 64152\, Usubc^6\, Vsub^5 - 79182\, Usubc^5\, Vsub^6$
$+ 83835\, Usubc^6\, Vsub^4 + 180144\, Usubc^5\, Vsub^5 + 131406\, Usubc^4\, Vsub^6$
$- 38880\, Usubc^6\, Vsub^3 - 99954\, Usubc^5\, Vsub^4 - 190944\, Usubc^4\, Vsub^5$
$- 103520\, Usubc^3\, Vsub^6 - 3645\, Usubc^6\, Vsub^2 + 150912\, Usubc^5\, Vsub^3$
$- 97614\, Usubc^4\, Vsub^4 + 93888\, Usubc^3\, Vsub^5 + 41128\, Usubc^2\, Vsub^6$
$+ 5832\, Usubc^6\, Vsub + 46494\, Usubc^5\, Vsub^2 - 257376\, Usubc^4\, Vsub^3$
$+ 210912\, Usubc^3\, Vsub^4 - 21024\, Usubc^2\, Vsub^5 - 7872\, Usubc\, Vsub^6 - 243\, Usubc^6$
$- 11664\, Usubc^5\, Vsub - 100926\, Usubc^4\, Vsub^2 + 230272\, Usubc^3\, Vsub^3$
$- 120840\, Usubc^2\, Vsub^4 + 1728\, Usubc\, Vsub^5 + 576\, Vsub^6 - 126\, Usubc^5$
$+ 6624\, Usubc^4\, Vsub + 89568\, Usubc^3\, Vsub^2 - 109376\, Usubc^2\, Vsub^3$
$+ 28032\, Usubc\, Vsub^4 + 798\, Usubc^4 + 192\, Usubc^3\, Vsub - 37800\, Usubc^2\, Vsub^2$
$+ 25728\, Usubc\, Vsub^3 - 2304\, Vsub^4 - 608\, Usubc^3 - 1056\, Usubc^2\, Vsub$
$+ 7488\, Usubc\, Vsub^2 - 2304\, Vsub^3 + 136\, Usubc^2 + 192\, Usubc\, Vsub - 576\, Vsub^2,\ Usubc$
$= 0.15\,..Uc,\ Vsub = 0\,..0.4\big);$



For nu = nu_c

Recall the expansion of U on this case:

> *Ucsing4*;

$$\frac{5}{9} - \frac{\sqrt{7}}{9} - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7} - 1700\right)^{2/3} \left(2\sqrt{7} + 1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3 + \frac{1645\left(1240\sqrt{7} - 1700\right)^{1/3} XX^4}{4478976}$$

We now want to compute the development of V around rhoc (=t_\nu^3 in the paper), for a fixed y. Recall that y is parametrized by U and V:

> *yUV*;

$$\left(8\nu\left(1 - 2U\right)V\left(V + 1\right)\right) \Bigg/ \left(U\left(U\left(\nu + 1\right) - 2\right)\left(V^3 + \frac{\left(9\left(\nu + 1\right)U^2 - 2\left(3 + 10\nu\right)U + 8\nu\right)V^2}{U\left(U\left(\nu + 1\right) - 2\right)} - \frac{\left(9U\left(\nu + 1\right) - 4\nu - 6\right)V}{U\left(\nu + 1\right) - 2} - 1\right)\right) \qquad \textbf{(5.2.2)}$$

Since y is fixed, we could write the development of V in terms of this fixed value of y. It turns out that the formulas are simpler when written in terms of V rather than y. Indeed, when U is equal to Uc (i.e. when w = rhoc), the value of y is fully determined by the value of V in this setting, which we denote by Vc:

> *yVc* := *factor*(*rationalize*(*subs*(*V* = *Vc*, *U* = *Uc*, *factor*(*subs*(nu = *nuc*, *yUV*))))) ;

$$yVc := \frac{4\left(1 + \sqrt{7}\right) Vc\left(Vc + 1\right)}{2\sqrt{7}\, Vc^2 - Vc^3 + 2\sqrt{7}\, Vc + 5\, Vc^2 - Vc + 1} \qquad \textbf{(5.2.3)}$$

When we compute the development of V for w close to rhoc, we can replace y by the latter value. Here is the new equation we obtain:

> $op(3, factor(numer(yVc - subs(\mathrm{nu} = nuc, yUV)))); indets(\%);$

$$2\,V - 4\sqrt{7}\,V - 27\,U^2\,Vc^2 - 27\,U^2\,Vc + 28\,U\,Vc^2 + 28\,Vc\,U - 4\,U\,V^2 + 2\,V^2 - 2\,V^2\,Vc^3 + 58\,V^2\,Vc^2 - 2\,V\,Vc^3 + 46\,V^2\,Vc - 46\,V\,Vc^2 - 58\,V\,Vc + 30\sqrt{7}\,U\,V\,Vc^2 - 18\sqrt{7}\,U\,V\,Vc + 2\sqrt{7}\,U\,V^3\,Vc^2 - 8\sqrt{7}\,U\,V^2\,Vc^3 + 2\sqrt{7}\,U\,V^3\,Vc + 18\sqrt{7}\,U\,V^2\,Vc^2 - 8\sqrt{7}\,U\,V\,Vc^3 - 30\sqrt{7}\,U\,V^2\,Vc - 4\sqrt{7}\,V^2 - 4\,V\,U + 8\sqrt{7}\,U\,V^2 + 8\sqrt{7}\,U\,V - 2\sqrt{7}\,U\,Vc^2 - 2\sqrt{7}\,U\,Vc + 27\,U^2\,V^3\,Vc^2 + 27\,U^2\,V^3\,Vc + 243\,U^2\,V^2\,Vc^2 - 28\,U\,V^3\,Vc^2 + 4\,U\,V^2\,Vc^3 + 243\,U^2\,V^2\,Vc - 243\,U^2\,V\,Vc^2 - 28\,U\,V^3\,Vc - 252\,U\,V^2\,Vc^2 + 4\,U\,V\,Vc^3 - 243\,U^2\,V\,Vc - 228\,U\,V^2\,Vc + 228\,U\,V\,Vc^2 + 252\,U\,V\,Vc + 4\sqrt{7}\,V^2\,Vc^3 - 8\sqrt{7}\,V^2\,Vc^2 + 4\sqrt{7}\,V\,Vc^3 + 16\sqrt{7}\,V^2\,Vc - 16\sqrt{7}\,V\,Vc^2 + 8\sqrt{7}\,V\,Vc$$

$$\{U, V, Vc\} \qquad \textbf{(5.2.4)}$$

> $eqyUVc := -27\,U^2\,Vc^2 - 27\,U^2\,Vc + 28\,U\,Vc^2 + 28\,Vc\,U + 2\,V - 4\,U\,V^2 - 2\,V^2\,Vc^3 + 58\,V^2\,Vc^2 - 2\,V\,Vc^3 + 46\,V^2\,Vc - 46\,V\,Vc^2 - 58\,V\,Vc - 4\sqrt{7}\,V + 2\,V^2 - 2\sqrt{7}\,U\,Vc^2 - 2\sqrt{7}\,U\,Vc - 243\,U^2\,V\,Vc - 228\,U\,V^2\,Vc + 228\,U\,V\,Vc^2 + 252\,U\,V\,Vc + 4\sqrt{7}\,V^2\,Vc^3 - 8\sqrt{7}\,V^2\,Vc^2 + 4\sqrt{7}\,V\,Vc^3 + 16\sqrt{7}\,V^2\,Vc - 16\sqrt{7}\,V\,Vc^2 + 8\sqrt{7}\,V\,Vc + 27\,U^2\,V^3\,Vc^2 + 27\,U^2\,V^3\,Vc + 243\,U^2\,V^2\,Vc^2 - 28\,U\,V^3\,Vc^2 + 4\,U\,V^2\,Vc^3 + 243\,U^2\,V^2\,Vc - 243\,U^2\,V\,Vc^2 - 28\,U\,V^3\,Vc - 252\,U\,V^2\,Vc^2 + 4\,U\,V\,Vc^3 - 4\sqrt{7}\,V^2 - 4\,V\,U + 8\sqrt{7}\,U\,V^2 + 8\sqrt{7}\,U\,V + 2\sqrt{7}\,U\,V^3\,Vc^2 - 8\sqrt{7}\,U\,V^2\,Vc^3 + 2\sqrt{7}\,U\,V^3\,Vc + 18\sqrt{7}\,U\,V^2\,Vc^2 - 8\sqrt{7}\,U\,V\,Vc^3 - 30\sqrt{7}\,U\,V^2\,Vc + 30\sqrt{7}\,U\,V\,Vc^2 - 18\sqrt{7}\,U\,V\,Vc:$

We plug the singular behavior of U in the equation, and deduce from it the asympotic behavior of V (we write V=Vc + XX*VX, so that we obtain the singular behavior of VX, recall that XX = (1- w/rhoc)^{1/3})

> $simplify(map(simplify, map(expand, map(rationalize, op(1, algeqtoseries(simplify(subs(U = Ucsing4, V = Vc + XX\cdot VX, eqyUVc)), XX, VX, 4, true))))));$

$$\frac{(Vc+1)\,Vc\,\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)}{54\,Vc-54} + \frac{1}{69984}\,\frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\,Vc\,(17\,Vc+7)}{(Vc-1)^2}\,XX + \frac{5}{384}\,\frac{1}{(Vc-1)^5\,(Vc^2+4\,Vc+1)}\left(Vc\,(97\,Vc^7+241\,Vc^6-521\,Vc^5 - 1673\,Vc^4-1113\,Vc^3-121\,Vc^2+Vc+17)\right)XX^2 + \frac{5}{4478976}\,\frac{1}{(Vc-1)^7\,(Vc^2+4\,Vc+1)}\left((Vc+1)\,Vc\,(15763\,Vc^8-12590\,Vc^7 - 121424\,Vc^6+6094\,Vc^5+392002\,Vc^4+315982\,Vc^3+66784\,Vc^2+1234\,Vc-293)\right. \qquad \textbf{(5.2.5)}$$

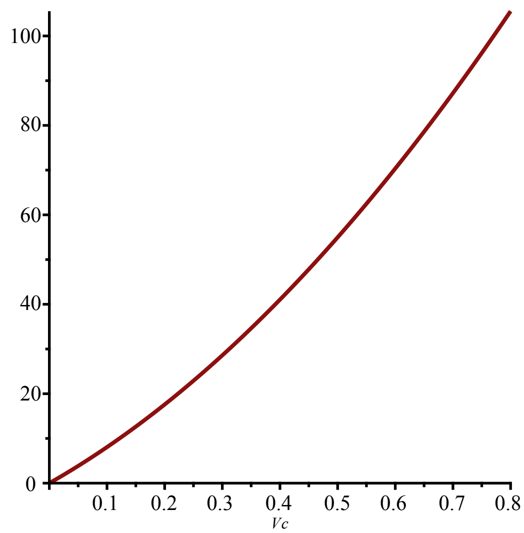
$$\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)\Big)\, XX^3 + O\left(XX^4\right)$$

> $Vcsing4 := sort\Bigg(collect\Bigg(Vc + XX\cdot\Bigg(\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)Vc\,(Vc+1)}{54\,Vc-54} + \frac{1}{69984}\,\frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\,Vc\,(17\,Vc+7)}{(Vc-1)^2}\,XX + \frac{5}{384}\,\frac{1}{(Vc-1)^5\left(Vc^2+4\,Vc+1\right)}\Big(Vc\left(97\,Vc^7+241\,Vc^6-521\,Vc^5-1673\,Vc^4 - 1113\,Vc^3-121\,Vc^2+Vc+17\right)\Big)XX^2 + \frac{5}{4478976}\,\frac{1}{\left(Vc^2+4\,Vc+1\right)(Vc-1)^7}\Big((Vc+1)\left(2\sqrt{7}+1\right)\big(15763\,Vc^8 - 12590\,Vc^7-121424\,Vc^6+6094\,Vc^5+392002\,Vc^4+315982\,Vc^3+66784\,Vc^2 + 1234\,Vc-293\big)\left(1240\sqrt{7}-1700\right)^{1/3}Vc\Big)XX^3\Bigg), XX, factor\Bigg), XX, ascending\Bigg);$

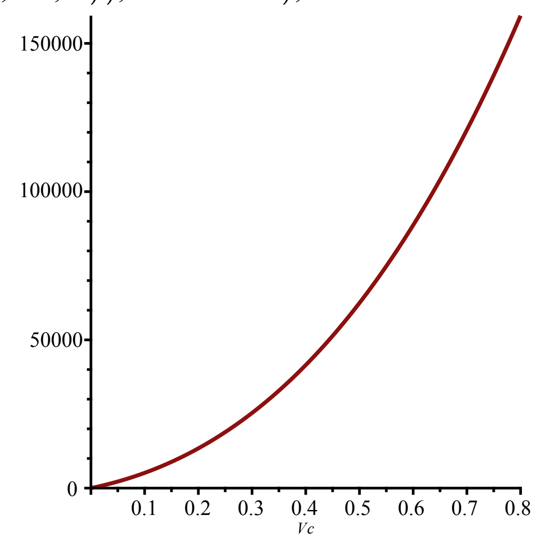
$$Vcsing4 := Vc + \frac{\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)Vc\,(Vc+1)\,XX}{54\,(Vc-1)} + \frac{\left(1240\sqrt{7}-1700\right)^{2/3}\left(29+4\sqrt{7}\right)(Vc+1)\,Vc\,(17\,Vc+7)\,XX^2}{69984\,(Vc-1)^2} + \frac{1}{384\,(Vc-1)^5\left(Vc^2+4\,Vc+1\right)}\Big(5\,Vc\,(Vc+1)\left(97\,Vc^6+144\,Vc^5-665\,Vc^4 - 1008\,Vc^3-105\,Vc^2-16\,Vc+17\right)XX^3\Big) + \frac{1}{4478976\,(Vc-1)^7\left(Vc^2+4\,Vc+1\right)}\Big(5\,(Vc+1)\,Vc\left(15763\,Vc^8-12590\,Vc^7 - 121424\,Vc^6+6094\,Vc^5+392002\,Vc^4+315982\,Vc^3+66784\,Vc^2+1234\,Vc-293\right)\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)XX^4\Big) \qquad \textbf{(5.2.6)}$$

We check that the coefficient does not vanish for Vc\in (0,1)

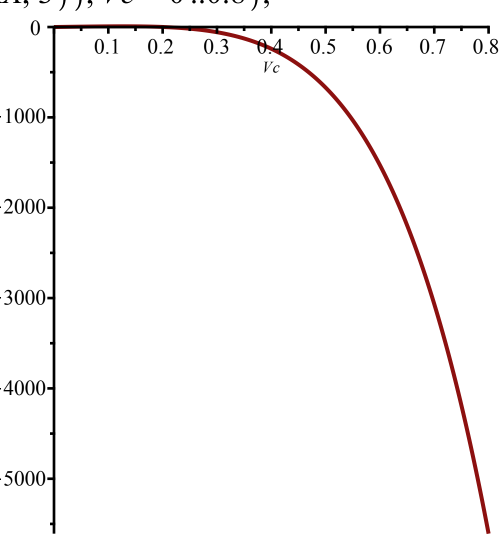
> $plot(numer(coeff(Vcsing4, XX, 1)), Vc = 0..0.8);$



> $plot(numer(coeff(Vcsing4, XX, 2)), Vc = 0..0.8);$



> $plot(numer(coeff(Vcsing4, XX, 3)), Vc = 0..0.8);$



For nu > nu_c

We consider again the rational parametrization of the critical line in this regime given by K:

```
> UsupK; nusupK;
```

$$-\frac{K\sim^2-3}{6K\sim+10}$$

$$-\frac{K\sim^3+3K\sim^2+9K\sim+11}{(K\sim+3)\left(K\sim^2-3\right)} \qquad \textbf{(5.3.1)}$$

And the expansion of U in this regime

```
> Usupcsing;
```

$$-\frac{K\sim^2-3}{2\,(3K\sim+5)}+RootOf\big((1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200)\,\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\big)\,XX-\big((K\sim^2-3)(K\sim^2+8K\sim+13)\,XX^2\,(9K\sim^4+14K\sim^3-18K\sim^2-10K\sim+29)(K\sim+1)\big)\Big/\big(144\,(3K\sim+5)\,(3K\sim^2+4K\sim-1)^2\,(2+K\sim)\big)+\frac{1}{216\,(3K\sim^2+4K\sim-1)^3\,(2+K\sim)}\big(5\,(K\sim^2+8K\sim+13)\,(9K\sim^6+40K\sim^5+43K\sim^4-48K\sim^3-97K\sim^2+24K\sim+77)\,RootOf\big((1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200)\,\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\big)\,XX^3\big) \qquad \textbf{(5.3.2)}$$

As before, we turn our attention to V, and replace U and nu by their expression in terms of K, in the rational parametrization of y. Hence we obtain an expression between y (which is fixed) and K and Vsup

```
> yKsup := simplify(subs(V = Vsup, U = UsupK, nu = nusupK, yUV));
```

$$yKsup := -\big(8\,(K\sim^3+3K\sim^2+9K\sim+11)\,(K\sim+1)\,Vsup\,(Vsup+1)\big)\Big/\big((K\sim^2-3)^2\,Vsup^3+(-7K\sim^4-40K\sim^3-110K\sim^2-136K\sim-55)\,Vsup^2-(K\sim^2-8K\sim-11)\,(K\sim^2-3)\,Vsup-(K\sim^2-3)^2\big) \qquad \textbf{(5.3.3)}$$

```
> op(2, numer(factor(yKsup - subs(nu = nusupK, yUV))));
```

$$2K\sim^4\,U\,V^3\,Vsup^2-2K\sim^4\,U\,V^2\,Vsup^3+2K\sim^4\,U\,V^3\,Vsup-2K\sim^4\,U\,V\,Vsup^3+K\sim^4\,V^2\,Vsup^3+8K\sim^3\,U\,V^3\,Vsup^2+12K\sim^2\,U^2\,V^3\,Vsup^2-12K\sim^4\,U\,V^2\,Vsup+12K\sim^4\,U\,V\,Vsup^2+K\sim^4\,V^2\,Vsup^2+K\sim^4\,V\,Vsup^3+8K\sim^3\,U\,V^3\,Vsup+24K\sim^3\,U\,V^2\,Vsup^2+12K\sim^2\,U^2\,V^3\,Vsup+108K\sim^2\,U^2\,V^2\,Vsup^2+12K\sim^2\,U\,V^2\,Vsup^3+32K\sim\,U^2\,V^3\,Vsup^2+2K\sim^4\,U\,V^2-2K\sim^4\,U\,Vsup^2+7K\sim^4\,V^2\,Vsup-7K\sim^4\,V\,Vsup^2-72K\sim^3\,U\,V^2\,Vsup+72K\sim^3\,U\,V\,Vsup^2-8K\sim^3\,V^2\,Vsup^2+108K\sim^2\,U^2\,V^2\,Vsup-108K\sim^2\,U^2\,V\,Vsup^2-20K\sim^2\,U\,V^2\,Vsup^2+12K\sim^2\,U\,V\,Vsup^3-6K\sim^2\,V^2\,Vsup^3+32K\sim\,U^2\,V^3\,Vsup+288K\sim\,U^2\,V^2\,Vsup^2 \qquad \textbf{(5.3.4)}$$

$$
\begin{aligned}
&- 24\, K{\sim}\, U\, V^{3}\, \mathit{Vsup}^{2} + 20\, U^{2}\, V^{3}\, \mathit{Vsup}^{2} + 2\, K{\sim}^{4}\, U\, V - 2\, K{\sim}^{4}\, U\, \mathit{Vsup} - K{\sim}^{4}\, V^{2} \\
&- K{\sim}^{4}\, V\, \mathit{Vsup} - 24\, K{\sim}^{3}\, U\, V\, \mathit{Vsup} - 8\, K{\sim}^{3}\, U\, \mathit{Vsup}^{2} + 40\, K{\sim}^{3}\, V^{2}\, \mathit{Vsup} - 40\, K{\sim}^{3}\, V\, \mathit{Vsup}^{2} \\
&- 108\, K{\sim}^{2}\, U^{2}\, V\, \mathit{Vsup} - 12\, K{\sim}^{2}\, U^{2}\, \mathit{Vsup}^{2} - 268\, K{\sim}^{2}\, U\, V^{2}\, \mathit{Vsup} + 268\, K{\sim}^{2}\, U\, V\, \mathit{Vsup}^{2} \\
&- 14\, K{\sim}^{2}\, V^{2}\, \mathit{Vsup}^{2} - 6\, K{\sim}^{2}\, V\, \mathit{Vsup}^{3} + 288\, K{\sim}\, U^{2}\, V^{2}\, \mathit{Vsup} - 288\, K{\sim}\, U^{2}\, V\, \mathit{Vsup}^{2} \\
&- 24\, K{\sim}\, U\, V^{3}\, \mathit{Vsup} - 200\, K{\sim}\, U\, V^{2}\, \mathit{Vsup}^{2} + 20\, U^{2}\, V^{3}\, \mathit{Vsup} + 180\, U^{2}\, V^{2}\, \mathit{Vsup}^{2} \\
&- 18\, U\, V^{3}\, \mathit{Vsup}^{2} - 18\, U\, V^{2}\, \mathit{Vsup}^{3} - K{\sim}^{4}\, V - 8\, K{\sim}^{3}\, U\, \mathit{Vsup} + 8\, K{\sim}^{3}\, V\, \mathit{Vsup} \\
&- 12\, K{\sim}^{2}\, U^{2}\, \mathit{Vsup} - 12\, K{\sim}^{2}\, U\, V^{2} + 20\, K{\sim}^{2}\, U\, V\, \mathit{Vsup} + 110\, K{\sim}^{2}\, V^{2}\, \mathit{Vsup} \\
&- 110\, K{\sim}^{2}\, V\, \mathit{Vsup}^{2} - 288\, K{\sim}\, U^{2}\, V\, \mathit{Vsup} - 32\, K{\sim}\, U^{2}\, \mathit{Vsup}^{2} - 424\, K{\sim}\, U\, V^{2}\, \mathit{Vsup} \\
&+ 424\, K{\sim}\, U\, V\, \mathit{Vsup}^{2} + 24\, K{\sim}\, V^{2}\, \mathit{Vsup}^{2} + 180\, U^{2}\, V^{2}\, \mathit{Vsup} - 180\, U^{2}\, V\, \mathit{Vsup}^{2} \\
&- 18\, U\, V^{3}\, \mathit{Vsup} - 164\, U\, V^{2}\, \mathit{Vsup}^{2} - 18\, U\, V\, \mathit{Vsup}^{3} + 9\, V^{2}\, \mathit{Vsup}^{3} - 12\, K{\sim}^{2}\, U\, V \\
&+ 6\, K{\sim}^{2}\, V^{2} + 14\, K{\sim}^{2}\, V\, \mathit{Vsup} - 32\, K{\sim}\, U^{2}\, \mathit{Vsup} + 200\, K{\sim}\, U\, V\, \mathit{Vsup} + 24\, K{\sim}\, U\, \mathit{Vsup}^{2} \\
&+ 136\, K{\sim}\, V^{2}\, \mathit{Vsup} - 136\, K{\sim}\, V\, \mathit{Vsup}^{2} - 180\, U^{2}\, V\, \mathit{Vsup} - 20\, U^{2}\, \mathit{Vsup}^{2} - 208\, U\, V^{2}\, \mathit{Vsup} \\
&+ 208\, U\, V\, \mathit{Vsup}^{2} + 33\, V^{2}\, \mathit{Vsup}^{2} + 9\, V\, \mathit{Vsup}^{3} + 6\, K{\sim}^{2}\, V + 24\, K{\sim}\, U\, \mathit{Vsup} \\
&- 24\, K{\sim}\, V\, \mathit{Vsup} - 20\, U^{2}\, \mathit{Vsup} + 18\, U\, V^{2} + 164\, U\, V\, \mathit{Vsup} + 18\, U\, \mathit{Vsup}^{2} + 55\, V^{2}\, \mathit{Vsup} \\
&- 55\, V\, \mathit{Vsup}^{2} + 18\, V\, U + 18\, U\, \mathit{Vsup} - 9\, V^{2} - 33\, V\, \mathit{Vsup} - 9\, V
\end{aligned}
$$

> $\mathit{eqyKsup} := 2\, K^{4}\, U\, V^{3}\, \mathit{Vsup}^{2} - 2\, K^{4}\, U\, V^{2}\, \mathit{Vsup}^{3} + 2\, K^{4}\, U\, V^{3}\, \mathit{Vsup} - 2\, K^{4}\, U\, V\, \mathit{Vsup}^{3}$

$$
\begin{aligned}
&+ K^{4}\, V^{2}\, \mathit{Vsup}^{3} + 8\, K^{3}\, U\, V^{3}\, \mathit{Vsup}^{2} + 12\, K^{2}\, U^{2}\, V^{3}\, \mathit{Vsup}^{2} - 12\, K^{4}\, U\, V^{2}\, \mathit{Vsup} \\
&+ 12\, K^{4}\, U\, V\, \mathit{Vsup}^{2} + K^{4}\, V^{2}\, \mathit{Vsup}^{2} + K^{4}\, V\, \mathit{Vsup}^{3} + 8\, K^{3}\, U\, V^{3}\, \mathit{Vsup} + 24\, K^{3}\, U\, V^{2}\, \mathit{Vsup}^{2} \\
&+ 12\, K^{2}\, U^{2}\, V^{3}\, \mathit{Vsup} + 108\, K^{2}\, U^{2}\, V^{2}\, \mathit{Vsup}^{2} + 12\, K^{2}\, U\, V^{2}\, \mathit{Vsup}^{3} + 32\, K\, U^{2}\, V^{3}\, \mathit{Vsup}^{2} \\
&+ 2\, K^{4}\, U\, V^{2} - 2\, K^{4}\, U\, \mathit{Vsup}^{2} + 7\, K^{4}\, V^{2}\, \mathit{Vsup} - 7\, K^{4}\, V\, \mathit{Vsup}^{2} - 72\, K^{3}\, U\, V^{2}\, \mathit{Vsup} \\
&+ 72\, K^{3}\, U\, V\, \mathit{Vsup}^{2} - 8\, K^{3}\, V^{2}\, \mathit{Vsup}^{2} + 108\, K^{2}\, U^{2}\, V^{2}\, \mathit{Vsup} - 108\, K^{2}\, U^{2}\, V\, \mathit{Vsup}^{2} \\
&- 20\, K^{2}\, U\, V^{2}\, \mathit{Vsup}^{2} + 12\, K^{2}\, U\, V\, \mathit{Vsup}^{3} - 6\, K^{2}\, V^{2}\, \mathit{Vsup}^{3} + 32\, K\, U^{2}\, V^{3}\, \mathit{Vsup} \\
&+ 288\, K\, U^{2}\, V^{2}\, \mathit{Vsup}^{2} - 24\, K\, U\, V^{3}\, \mathit{Vsup}^{2} + 20\, U^{2}\, V^{3}\, \mathit{Vsup}^{2} + 2\, K^{4}\, U\, V - 2\, K^{4}\, U\, \mathit{Vsup} \\
&- K^{4}\, V^{2} - K^{4}\, V\, \mathit{Vsup} - 24\, K^{3}\, U\, V\, \mathit{Vsup} - 8\, K^{3}\, U\, \mathit{Vsup}^{2} + 40\, K^{3}\, V^{2}\, \mathit{Vsup} \\
&- 40\, K^{3}\, V\, \mathit{Vsup}^{2} - 108\, K^{2}\, U^{2}\, V\, \mathit{Vsup} - 12\, K^{2}\, U^{2}\, \mathit{Vsup}^{2} - 268\, K^{2}\, U\, V^{2}\, \mathit{Vsup} \\
&+ 268\, K^{2}\, U\, V\, \mathit{Vsup}^{2} - 14\, K^{2}\, V^{2}\, \mathit{Vsup}^{2} - 6\, K^{2}\, V\, \mathit{Vsup}^{3} + 288\, K\, U^{2}\, V^{2}\, \mathit{Vsup} \\
&- 288\, K\, U^{2}\, V\, \mathit{Vsup}^{2} - 24\, K\, U\, V^{3}\, \mathit{Vsup} - 200\, K\, U\, V^{2}\, \mathit{Vsup}^{2} + 20\, U^{2}\, V^{3}\, \mathit{Vsup} \\
&+ 180\, U^{2}\, V^{2}\, \mathit{Vsup}^{2} - 18\, U\, V^{3}\, \mathit{Vsup}^{2} - 18\, U\, V^{2}\, \mathit{Vsup}^{3} - K^{4}\, V - 8\, K^{3}\, U\, \mathit{Vsup} \\
&+ 8\, K^{3}\, V\, \mathit{Vsup} - 12\, K^{2}\, U^{2}\, \mathit{Vsup} - 12\, K^{2}\, U\, V^{2} + 20\, K^{2}\, U\, V\, \mathit{Vsup} + 110\, K^{2}\, V^{2}\, \mathit{Vsup} \\
&- 110\, K^{2}\, V\, \mathit{Vsup}^{2} - 288\, K\, U^{2}\, V\, \mathit{Vsup} - 32\, K\, U^{2}\, \mathit{Vsup}^{2} - 424\, K\, U\, V^{2}\, \mathit{Vsup} \\
&+ 424\, K\, U\, V\, \mathit{Vsup}^{2} + 24\, K\, V^{2}\, \mathit{Vsup}^{2} + 180\, U^{2}\, V^{2}\, \mathit{Vsup} - 180\, U^{2}\, V\, \mathit{Vsup}^{2} \\
&- 18\, U\, V^{3}\, \mathit{Vsup} - 164\, U\, V^{2}\, \mathit{Vsup}^{2} - 18\, U\, V\, \mathit{Vsup}^{3} + 9\, V^{2}\, \mathit{Vsup}^{3} - 12\, K^{2}\, U\, V + 6\, K^{2}\, V^{2} \\
&+ 14\, K^{2}\, V\, \mathit{Vsup} - 32\, K\, U^{2}\, \mathit{Vsup} + 200\, K\, U\, V\, \mathit{Vsup} + 24\, K\, U\, \mathit{Vsup}^{2} + 136\, K\, V^{2}\, \mathit{Vsup} \\
&- 136\, K\, V\, \mathit{Vsup}^{2} - 180\, U^{2}\, V\, \mathit{Vsup} - 20\, U^{2}\, \mathit{Vsup}^{2} - 208\, U\, V^{2}\, \mathit{Vsup} + 208\, U\, V\, \mathit{Vsup}^{2} \\
&+ 33\, V^{2}\, \mathit{Vsup}^{2} + 9\, V\, \mathit{Vsup}^{3} + 6\, K^{2}\, V + 24\, K\, U\, \mathit{Vsup} - 24\, K\, V\, \mathit{Vsup} - 20\, U^{2}\, \mathit{Vsup} \\
&+ 18\, U\, V^{2} + 164\, U\, V\, \mathit{Vsup} + 18\, U\, \mathit{Vsup}^{2} + 55\, V^{2}\, \mathit{Vsup} - 55\, V\, \mathit{Vsup}^{2} + 18\, V\, U
\end{aligned}
$$

$$+ 18\,U\,Vsup - 9\,V^2 - 33\,V\,Vsup - 9\,V:$$

We can replace U by its singular behavior in terms of K, and compute the corresponding expansion for V

```
> Vsupsing := sort(collect(Vsup + convert(simplify(op(2, algeqtoseries(subs(V = Vsup + VV,
    subs(U = Usupcsing, eqyKsup)), XX, VV, 3 , true))), polynom), XX, factor), XX,
    ascending);
```

$$
\begin{aligned}
Vsupsing :=\ & Vsup + \Big(4\, RootOf\big(\big(1296\, K\sim^4 + 6048\, K\sim^3 + 8928\, K\sim^2 + 3360\, K\sim \\
& - 1200\big)\, \_Z^2 - K\sim^8 - 10\, K\sim^7 - 24\, K\sim^6 + 26\, K\sim^5 + 158\, K\sim^4 + 114\, K\sim^3 - 192\, K\sim^2 \\
& - 306\, K\sim - 117\big)\, Vsup\, (Vsup - 1)\, (Vsup + 1)\, (3\, K\sim + 5)\, XX\Big) \Big/ \Big((K\sim \\
& + 1)\, \big(K\sim^2\, Vsup^2 + 4\, K\sim^2\, Vsup + K\sim^2 + 8\, K\sim\, Vsup - 3\, Vsup^2 + 4\, Vsup - 3\big)\Big) \\
& + \Big(\big(K\sim^2 - 3\big)\, (Vsup - 1)\, Vsup\, \big(K\sim^2 + 8\, K\sim + 13\big)\, (Vsup + 1)\, \big(9\, K\sim^{10}\, Vsup^6 + 108\, Vsup^5\, K\sim^{10} + 38 \\
& + 459\, Vsup^2\, K\sim^{10} + 4604\, K\sim^9\, Vsup^3 + 1115\, K\sim^8\, Vsup^4 - 2800\, Vsup^5\, K\sim^7 \\
& - 262\, K\sim^6\, Vsup^6 + 108\, Vsup\, K\sim^{10} + 2418\, Vsup^2\, K\sim^9 + 7248\, K\sim^8\, Vsup^3 \\
& - 12688\, K\sim^7\, Vsup^4 - 4352\, Vsup^5\, K\sim^6 + 900\, K\sim^5\, Vsup^6 + 9\, K\sim^{10} + 480\, Vsup\, K\sim^9 \\
& + 2399\, Vsup^2\, K\sim^8 - 5136\, K\sim^7\, Vsup^3 - 24378\, K\sim^6\, Vsup^4 + 2616\, Vsup^5\, K\sim^5 \\
& + 1386\, K\sim^4\, Vsup^6 + 20\, K\sim^9 + 52\, Vsup\, K\sim^8 - 7768\, Vsup^2\, K\sim^7 - 20080\, K\sim^6\, Vsup^3 \\
& + 3608\, K\sim^5\, Vsup^4 + 9872\, K\sim^4\, Vsup^5 - 648\, K\sim^3\, Vsup^6 - 87\, K\sim^8 - 2560\, Vsup\, K\sim^7 \\
& - 17202\, Vsup^2\, K\sim^6 + 5288\, K\sim^5\, Vsup^3 + 49854\, K\sim^4\, Vsup^4 + 4176\, Vsup^5\, K\sim^3 \\
& - 1971\, K\sim^2\, Vsup^6 - 208\, K\sim^7 - 3248\, Vsup\, K\sim^6 - 628\, Vsup^2\, K\sim^5 + 51520\, K\sim^4\, Vsup^3 \\
& + 44720\, K\sim^3\, Vsup^4 - 4044\, K\sim^2\, Vsup^5 - 378\, K\sim\, Vsup^6 + 290\, K\sim^6 + 2400\, Vsup\, K\sim^5 \\
& + 21918\, K\sim^4\, Vsup^2 + 47984\, K\sim^3\, Vsup^3 + 3103\, K\sim^2\, Vsup^4 - 2268\, Vsup^5\, K\sim \\
& + 513\, Vsup^6 + 792\, K\sim^5 + 6416\, K\sim^4\, Vsup + 9704\, K\sim^3\, Vsup^2 + 12184\, K\sim^2\, Vsup^3 \\
& - 9708\, K\sim\, Vsup^4 + 612\, Vsup^5 - 342\, K\sim^4 + 2880\, K\sim^3\, Vsup - 12713\, K\sim^2\, Vsup^2 \\
& + 2236\, K\sim\, Vsup^3 - 1641\, Vsup^4 - 1296\, K\sim^3 - 156\, K\sim^2\, Vsup - 9582\, K\sim\, Vsup^2 \\
& + 3760\, Vsup^3 - 27\, K\sim^2 - 525\, Vsup^2 + 756\, K\sim - 36\, Vsup + 189\big)\, XX^2\Big) \Big/ \Big(18\, (2 \\
& + K\sim)\, \big(K\sim^2\, Vsup^2 - 2\, K\sim^2\, Vsup + K\sim^2 - 8\, K\sim\, Vsup - 3\, Vsup^2 - 10\, Vsup \\
& - 3\big)\, \big(3\, K\sim^2 + 4\, K\sim - 1\big)^2\, \big(K\sim^2\, Vsup^2 + 4\, K\sim^2\, Vsup + K\sim^2 + 8\, K\sim\, Vsup \\
& - 3\, Vsup^2 + 4\, Vsup - 3\big)^3\Big) + \Big(\big(639\, K\sim^{16}\, Vsup^{10} + 10674\, K\sim^{16}\, Vsup^9 \\
& + 2828\, K\sim^{15}\, Vsup^{10} + 71091\, K\sim^{16}\, Vsup^8 + 63568\, K\sim^{15}\, Vsup^9 - 5716\, K\sim^{14}\, Vsup^{10} \\
& + 239112\, K\sim^{16}\, Vsup^7 + 501060\, K\sim^{15}\, Vsup^8 + 9688\, K\sim^{14}\, Vsup^9 - 41724\, K\sim^{13}\, Vsup^{10} \\
& + 426006\, Vsup^6\, K\sim^{16} + 1800240\, K\sim^{15}\, Vsup^7 + 679404\, K\sim^{14}\, Vsup^8 \\
& - 648048\, K\sim^{13}\, Vsup^9 - 2396\, K\sim^{12}\, Vsup^{10} + 395820\, K\sim^{16}\, Vsup^5 + 3218472\, Vsup^6\, K\sim^{15} \\
& + 3217728\, K\sim^{14}\, Vsup^7 - 3113492\, K\sim^{13}\, Vsup^8 - 1083928\, K\sim^{12}\, Vsup^9 \\
& + 243660\, K\sim^{11}\, Vsup^{10} + 196614\, K\sim^{16}\, Vsup^4 + 2711568\, K\sim^{15}\, Vsup^5
\end{aligned}
$$

$+ 5754840\, Vsup^{6}\, K\sim^{14} - 8503920\, K\sim^{13}\, Vsup^{7} - 9997948\, K\sim^{12}\, Vsup^{8}$

$+ 2001744\, K\sim^{11}\, Vsup^{9} + 204292\, K\sim^{10}\, Vsup^{10} + 52488\, Vsup^{3}\, K\sim^{16}$

$+ 1168632\, K\sim^{15}\, Vsup^{4} + 2279280\, K\sim^{14}\, Vsup^{5} - 15721032\, Vsup^{6}\, K\sim^{13}$

$- 37331648\, K\sim^{12}\, Vsup^{7} - 197788\, K\sim^{11}\, Vsup^{8} + 6365672\, K\sim^{10}\, Vsup^{9}$

$- 694620\, K\sim^{9}\, Vsup^{10} + 6939\, Vsup^{2}\, K\sim^{16} + 290832\, Vsup^{3}\, K\sim^{15} - 456456\, K\sim^{14}\, Vsup^{4}$

$- 28678544\, K\sim^{13}\, Vsup^{5} - 70210936\, K\sim^{12}\, Vsup^{6} - 21406352\, K\sim^{11}\, Vsup^{7}$

$+ 33635172\, K\sim^{10}\, Vsup^{8} - 9456\, K\sim^{9}\, Vsup^{9} - 962430\, K\sim^{8}\, Vsup^{10} + 306\, K\sim^{16}\, Vsup$

$+ 40332\, Vsup^{2}\, K\sim^{15} - 183552\, Vsup^{3}\, K\sim^{14} - 18557016\, K\sim^{13}\, Vsup^{4}$

$- 100072048\, Vsup^{5}\, K\sim^{12} - 46519576\, K\sim^{11}\, Vsup^{6} + 96135168\, K\sim^{10}\, Vsup^{7}$

$+ 34999724\, K\sim^{9}\, Vsup^{8} - 14324532\, K\sim^{8}\, Vsup^{9} + 919620\, K\sim^{7}\, Vsup^{10} - 9\, K\sim^{16}$

$+ 352\, K\sim^{15}\, Vsup + 15420\, Vsup^{2}\, K\sim^{14} - 4160976\, K\sim^{13}\, Vsup^{3} - 46512472\, K\sim^{12}\, Vsup^{4}$

$- 95900080\, Vsup^{5}\, K\sim^{11} + 169920072\, K\sim^{10}\, Vsup^{6} + 166843984\, K\sim^{9}\, Vsup^{7}$

$- 32436278\, K\sim^{8}\, Vsup^{8} - 10312848\, K\sim^{7}\, Vsup^{9} + 1966068\, K\sim^{6}\, Vsup^{10} - 124\, K\sim^{15}$

$- 9224\, K\sim^{14}\, Vsup - 300668\, Vsup^{2}\, K\sim^{13} - 7092608\, K\sim^{12}\, Vsup^{3} - 3437704\, K\sim^{11}\, Vsup^{4}$

$+ 95370320\, Vsup^{5}\, K\sim^{10} + 328474808\, K\sim^{9}\, Vsup^{6} - 5649712\, K\sim^{8}\, Vsup^{7}$

$- 72502036\, K\sim^{7}\, Vsup^{8} + 11587752\, K\sim^{6}\, Vsup^{9} - 291924\, K\sim^{5}\, Vsup^{10} - 292\, K\sim^{14}$

$- 23232\, Vsup\, K\sim^{13} - 408556\, Vsup^{2}\, K\sim^{12} + 10442704\, K\sim^{11}\, Vsup^{3}$

$+ 151944168\, K\sim^{10}\, Vsup^{4} + 222863664\, Vsup^{5}\, K\sim^{9} + 53109364\, K\sim^{8}\, Vsup^{6}$

$- 232332400\, K\sim^{7}\, Vsup^{7} - 13193356\, K\sim^{6}\, Vsup^{8} + 15846192\, K\sim^{5}\, Vsup^{9}$

$- 1845612\, K\sim^{4}\, Vsup^{10} + 1452\, K\sim^{13} + 62792\, Vsup\, K\sim^{12} + 805388\, Vsup^{2}\, K\sim^{11}$

$+ 41286336\, K\sim^{10}\, Vsup^{3} + 215331560\, K\sim^{9}\, Vsup^{4} - 30420376\, Vsup^{5}\, K\sim^{8}$

$- 378570056\, K\sim^{7}\, Vsup^{6} - 183172544\, K\sim^{6}\, Vsup^{7} + 41919588\, K\sim^{5}\, Vsup^{8}$

$+ 160488\, K\sim^{4}\, Vsup^{9} - 339228\, K\sim^{3}\, Vsup^{10} + 5908\, K\sim^{12} + 247584\, Vsup\, K\sim^{11}$

$+ 1206516\, Vsup^{2}\, K\sim^{10} + 20772976\, K\sim^{9}\, Vsup^{3} - 2753420\, K\sim^{8}\, Vsup^{4}$

$- 322010064\, Vsup^{5}\, K\sim^{7} - 381755416\, K\sim^{6}\, Vsup^{6} + 19549104\, K\sim^{5}\, Vsup^{7}$

$+ 18201492\, K\sim^{4}\, Vsup^{8} - 5848848\, K\sim^{3}\, Vsup^{9} + 676188\, K\sim^{2}\, Vsup^{10} - 4380\, K\sim^{11}$

$- 73336\, Vsup\, K\sim^{10} - 1953052\, Vsup^{2}\, K\sim^{9} - 58484464\, K\sim^{8}\, Vsup^{3}$

$- 231714776\, K\sim^{7}\, Vsup^{4} - 174497008\, Vsup^{5}\, K\sim^{6} - 87082392\, K\sim^{5}\, Vsup^{6}$

$+ 76408960\, K\sim^{4}\, Vsup^{7} - 11266452\, K\sim^{3}\, Vsup^{8} - 703080\, K\sim^{2}\, Vsup^{9}$

$+ 102060\, K\sim\, Vsup^{10} - 40460\, K\sim^{10} - 1060608\, Vsup\, K\sim^{9} - 1505366\, Vsup^{2}\, K\sim^{8}$

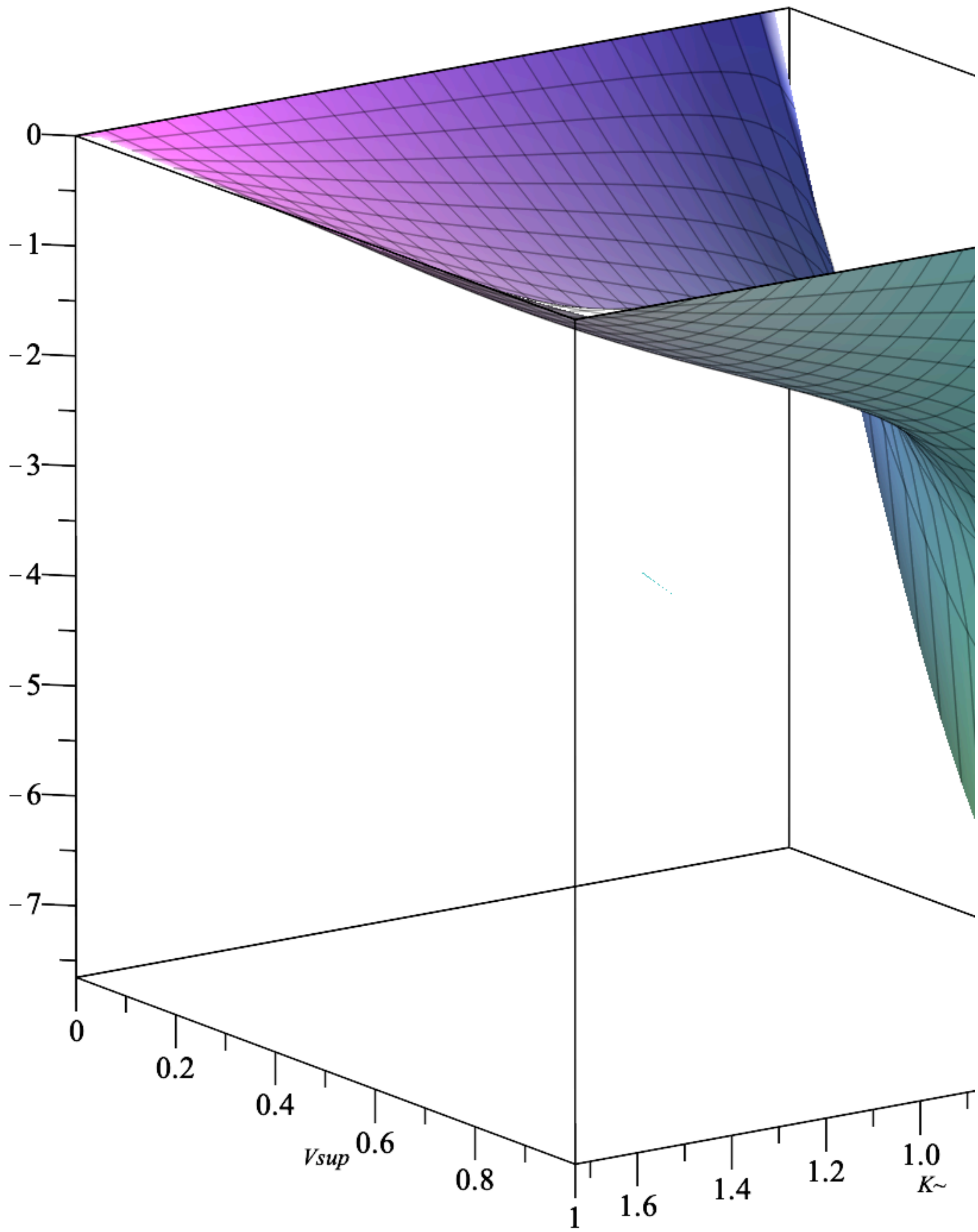
$- 78414544\, K\sim^{7}\, Vsup^{3} - 149999800\, K\sim^{6}\, Vsup^{4} + 84505936\, Vsup^{5}\, K\sim^{5}$

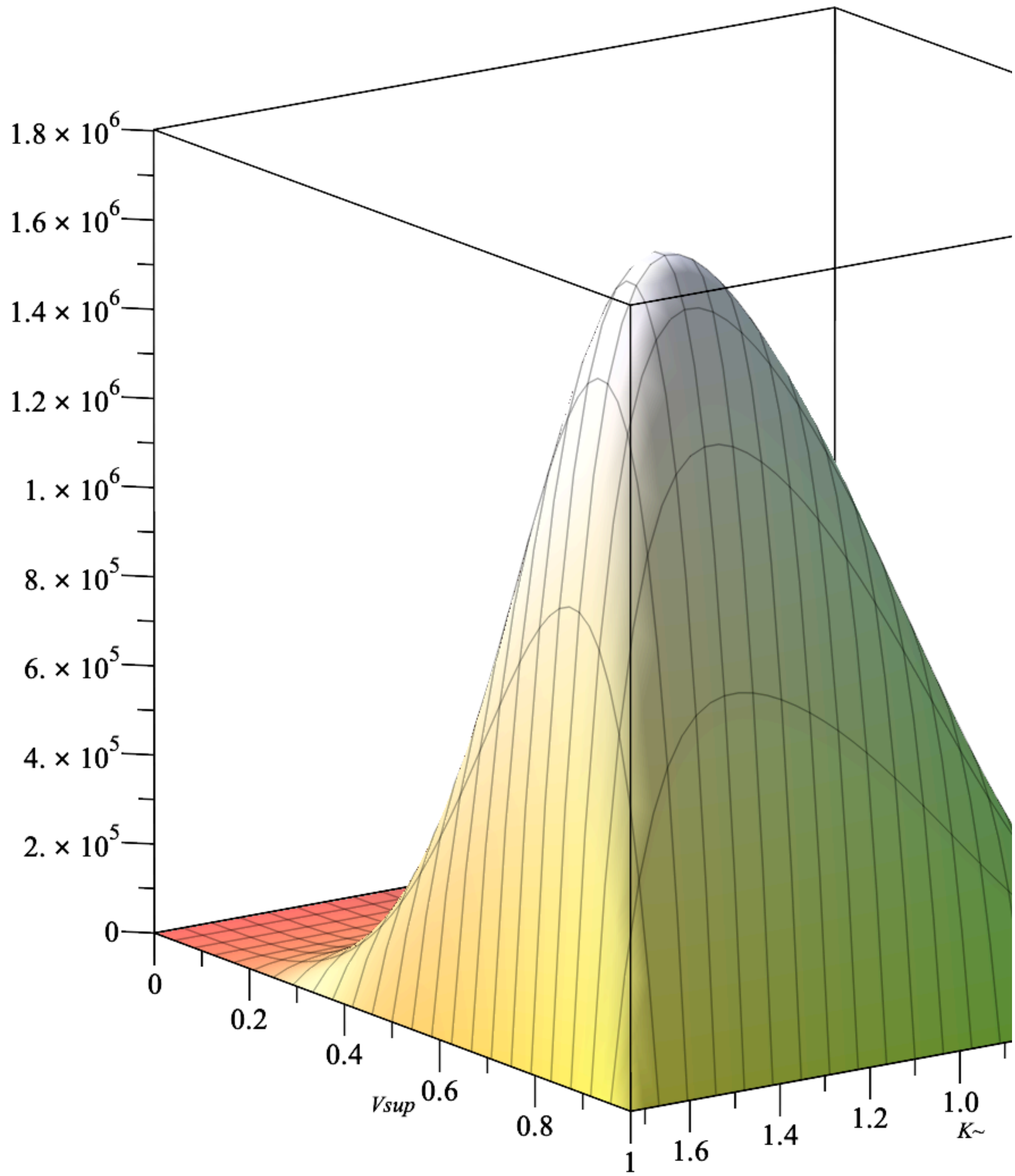
$+ 46216232\, K\sim^{4}\, Vsup^{6} + 18193104\, K\sim^{3}\, Vsup^{7} - 5913540\, K\sim^{2}\, Vsup^{8}$

$$+ 1123632\, K\sim Vsup^{9} - 132921\, Vsup^{10} - 13140\, K\sim^{9} - 655956\, Vsup\, K\sim^{8}$$
$$+ 6247748\, Vsup^{2} K\sim^{7} - 2589824\, K\sim^{6} Vsup^{3} + 44865336\, K\sim^{5} Vsup^{4}$$
$$+ 31232464\, K\sim^{4} Vsup^{5} + 34111032\, K\sim^{3} Vsup^{6} - 11395968\, K\sim^{2} Vsup^{7}$$
$$+ 1080324\, K\sim Vsup^{8} + 188082\, Vsup^{9} + 136290\, K\sim^{8} + 1992096\, Vsup\, K\sim^{7}$$
$$+ 4559972\, Vsup^{2} K\sim^{6} + 42001296\, K\sim^{5} Vsup^{3} + 90548744\, K\sim^{4} Vsup^{4}$$
$$- 97894032\, Vsup^{5} K\sim^{3} + 32556024\, K\sim^{2} Vsup^{6} - 7932816\, K\sim Vsup^{7} + 359019\, Vsup^{8}$$
$$+ 119340\, K\sim^{7} + 2550024\, Vsup\, K\sim^{6} - 8722260\, Vsup^{2} K\sim^{5} + 7858240\, K\sim^{4} Vsup^{3}$$
$$+ 65235816\, K\sim^{3} Vsup^{4} - 77490896\, K\sim^{2} Vsup^{5} + 22431528\, K\sim Vsup^{6} - 2266776\, Vsup^{7}$$
$$- 234108\, K\sim^{6} - 1073088\, Vsup\, K\sim^{5} - 9440892\, K\sim^{4} Vsup^{2} - 25004304\, K\sim^{3} Vsup^{3}$$
$$+ 49526232\, K\sim^{2} Vsup^{4} - 23701744\, Vsup^{5} K\sim + 4257990\, Vsup^{6} - 313308\, K\sim^{5}$$
$$- 3196152\, K\sim^{4} Vsup + 951588\, K\sim^{3} Vsup^{2} - 22509888\, K\sim^{2} Vsup^{3} + 22575672\, K\sim Vsup^{4}$$
$$- 5011444\, Vsup^{5} + 167076\, K\sim^{4} - 997920\, K\sim^{3} Vsup + 3048300\, K\sim^{2} Vsup^{2}$$
$$- 8692272\, K\sim Vsup^{3} + 2948310\, Vsup^{4} + 366444\, K\sim^{3} + 1036152\, K\sim^{2} Vsup$$
$$+ 425196\, K\sim Vsup^{2} - 1041624\, Vsup^{3} + 19116\, K\sim^{2} + 777600\, K\sim Vsup + 42147\, Vsup^{2}$$
$$- 160380\, K\sim + 149202\, Vsup - 57105\big)\, RootOf\Big(\big(1296\, K\sim^{4} + 6048\, K\sim^{3} + 8928\, K\sim^{2}$$
$$+ 3360\, K\sim - 1200\big)\, \_Z^{2} - K\sim^{8} - 10\, K\sim^{7} - 24\, K\sim^{6} + 26\, K\sim^{5} + 158\, K\sim^{4} + 114\, K\sim^{3}$$
$$- 192\, K\sim^{2} - 306\, K\sim - 117\Big)\, (Vsup - 1)\, Vsup\, \big(K\sim^{2} + 8\, K\sim + 13\big)\, (Vsup + 1)\, (3\, K\sim$$
$$+ 5)\, XX^{3}\Big) \Big/ \Big(54\, (2 + K\sim)\, (K\sim + 1)\, \big(K\sim^{2} Vsup^{2} - 2\, K\sim^{2} Vsup + K\sim^{2} - 8\, K\sim Vsup$$
$$- 3\, Vsup^{2} - 10\, Vsup - 3\big)\, \big(3\, K\sim^{2} + 4\, K\sim - 1\big)^{3}\, \big(K\sim^{2} Vsup^{2} + 4\, K\sim^{2} Vsup + K\sim^{2}$$
$$+ 8\, K\sim Vsup - 3\, Vsup^{2} + 4\, Vsup - 3\big)^{5}\Big)$$

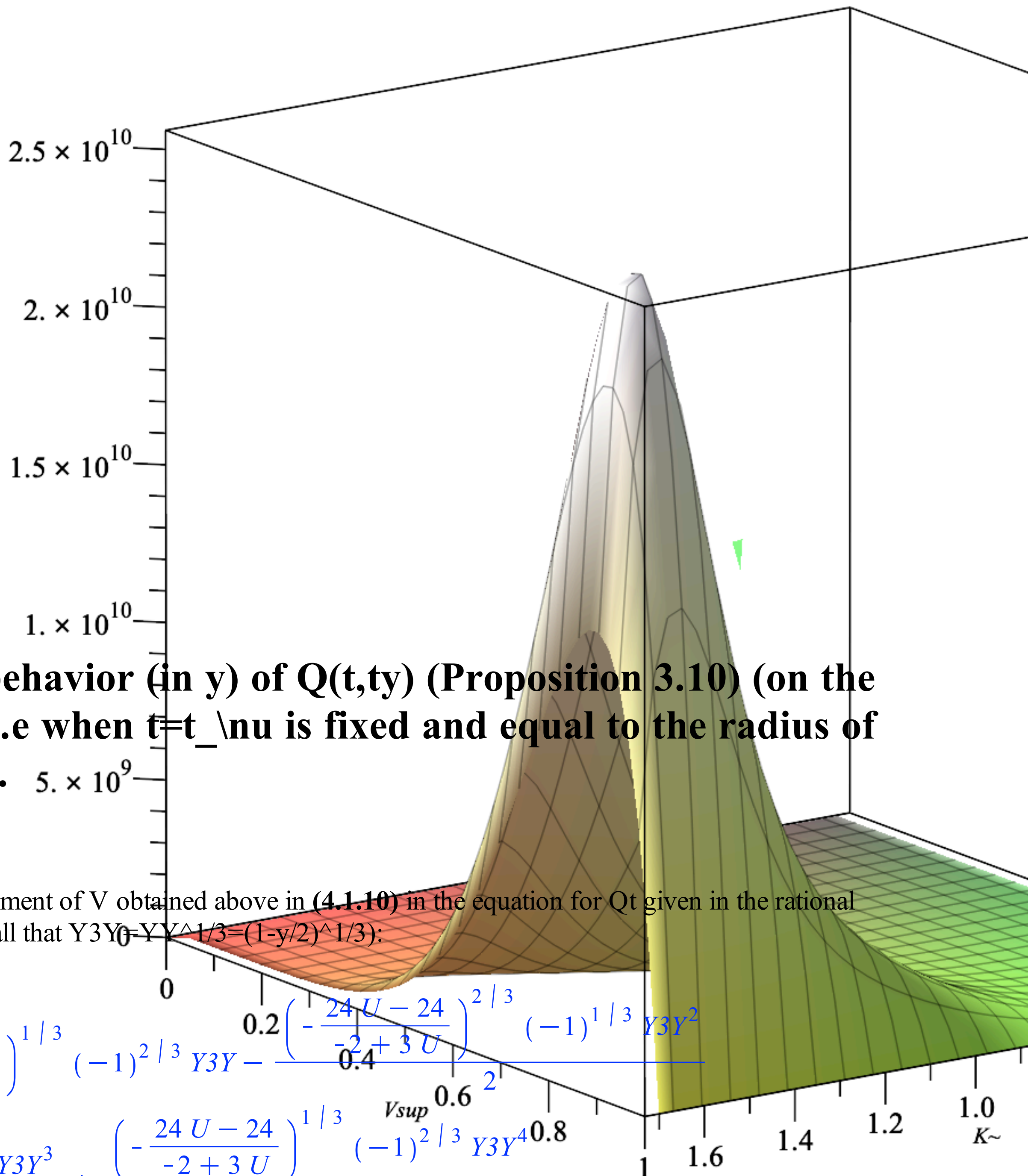
plots of the coefficients

```
> plot3d(numer(coeff(Vsupsing, XX, 1)), K = Kc..Kinfini, Vsup = 0..1);
    plot3d(numer(coeff(Vsupsing, XX, 2)), K = Kc..Kinfini, Vsup = 0..1);
    plot3d(numer(coeff(Vsupsing, XX, 3)), K = Kc..Kinfini, Vsup = 0..1);
```


0
−1
−2
−3
−4
−5
−6
−7
0
0.2
0.4
Vsup
0.6
0.8
1
1.6
1.4
1.2
1.0
K~



1.8 × 10^6
1.6 × 10^6
1.4 × 10^6
1.2 × 10^6
1. × 10^6
8. × 10^5
6. × 10^5
4. × 10^5
2. × 10^5
0
0
0.2
0.4
Vsup
0.6
0.8
1
1.6
1.4
1.2
1.0
K~



Asymptotic behavior (in y) of Q(t,ty) (Proposition 3.10) (on the critical line, i.e when t=t_\nu is fixed and equal to the radius of convergence).

For nu ≤ nu_c:

We plug the development of V obtained above in **(4.1.10)** in the equation for Qt given in the rational parametrization (recall that Y3Y=YY^1/3=(1-y/2)^1/3):

> *Vsubsingy*;

$$1+\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y-\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3}(-1)^{1/3}\,Y3Y^2}{2}-\frac{4\,(U-1)\,Y3Y^3}{-2+3\,U}+\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y^4}{9\,U-6}$$

In the rational parametrization for Qt (given in QtUV), we replace nu by its value in terms of U:

> *QtUVsubc* := *factor*(*subs*(nu = *nuUsub*, *QtUV*));

$$QtUVsubc := \frac{1}{(-2+3U)(V+1)^3(6U^2-10U+3)}\big((3UV^3-21UV^2-2V^3 - 3VU+18V^2-3U+6V+2)(6U^2V^2-12U^2V-6UV^2-6U^2+12VU + V^2+10U-2V-3)\big) \quad \textbf{(6.1.2)}$$

> map(factor, collect(simplify(series(subs(V = Vsubsingy, QtUVsubc), Y3Y, 4)), Y3Y));

$$\frac{12(3U-1)(U-1)^2}{(-2+3U)(6U^2-10U+3)} + \frac{-\frac{3I}{2}(-\sqrt{3}+I)(9U^2-10U+2)(U-1)\left(-\frac{24(U-1)}{-2+3U}\right)^{2/3}}{(-2+3U)(6U^2-10U+3)}Y3Y^2 + 12\frac{(45U^2-45U+8)(U-1)^2}{(-2+3U)^2(6U^2-10U+3)}Y3Y^3 + O(Y3Y^4) \quad \textbf{(6.1.3)}$$

We check if/when the leading term in the development of Qt cancels out. There are two roots either U=1 (which is not possible in this range of nu) or

> solve$(9U^2-10U+2)$; Uc

$$\frac{5}{9}+\frac{\sqrt{7}}{9}, \frac{5}{9}-\frac{\sqrt{7}}{9}$$

$$\frac{5}{9}-\frac{\sqrt{7}}{9} \quad \textbf{(6.1.4)}$$

The leading term cancels for U=Uc, we compute the corresponding development:

> map(factor, collect(simplify(series(subs(V = Vsubsingy, U = Uc, QtUVsubc), Y3Y, 5)), Y3Y));

$$\frac{2}{5}+\frac{2\sqrt{7}}{5}+\left(-\frac{14}{5}-\frac{2\sqrt{7}}{5}\right)Y3Y^3+\left(\frac{2\left(46+16\sqrt{7}\right)^{1/3}\sqrt{7}}{5} + \frac{2\left(46+16\sqrt{7}\right)^{1/3}}{5}\right)Y3Y^4+O(Y3Y^5) \quad \textbf{(6.1.5)}$$

We obtain a singularity in (1-y/2)^(4/3).

For nu>nu_c

We use the same rational parametrization of U and nu in terms of K, and replace in \hat Q, their expression in terms of K. Then we use the development of V obtained in **(4.2.14)** (with YY=(1-y/yK11)^(1/2)) and substitute it in the expression of Qt:

> QtUVsupc := factor(subs(nu = nusupK, U = UsupK, QtUV)) :

> devQtsur := map(factor, collect(simplify(series(subs(V = devV11ysupc, QtUVsupc), YY, 4)), YY));

$$devQtsur := \Big(4\big(37K\sim^8+348K\sim^7-21K\sim^6\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1} \quad \textbf{(6.2.1)}$$

$$+ 1456 K\sim^{6} - 144 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 3508 K\sim^{5}$$

$$- 431 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 5314 K\sim^{4}$$

$$- 704 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 5140 K\sim^{3}$$

$$- 687 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 3016 K\sim^{2}$$

$$- 432 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 956 K\sim$$

$$- 149 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 145) (K\sim^{2} + 4 K\sim + 5) (5 K\sim^{4}$$

$$- (3 K\sim^{2} + 4 K\sim - 1)^{3/2} \sqrt{K\sim^{2} + 4 K\sim + 5} + 20 K\sim^{3} + 26 K\sim^{2} + 4 K\sim - 11)) \Big/$$

$$((K\sim^{2} + 8 K\sim + 13) (K\sim^{2} + 4 K\sim - \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 5)^{3} (K\sim^{2} - 3)^{3}) + 8 ((-106035 + 34322652 K\sim^{12} + 94811152 K\sim^{11}$$

$$+ 198675880 K\sim^{10} - 52888765 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 9232696 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 2484785 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 380604 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 10557 K\sim^{16} + 194672 K\sim^{15}$$

$$+ 1689784 K\sim^{14} + 9134528 K\sim^{13} + 321307488 K\sim^{9} + 404013790 K\sim^{8}$$

$$+ 394520720 K\sim^{7} - 6095 K\sim^{14} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 40391073 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 52872400 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 23482148 K\sim^{9} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 3247736 K\sim^{11} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 10235523 K\sim^{10} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 96140 K\sim^{13} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 710421 K\sim^{12} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 326688 K\sim$$

$$- 40181524 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 22752079 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$- 24891 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 16980208 K\sim^{3} + 1771480 K\sim^{2}$$

$$+ 295981128 K\sim^{6} + 166429952 K\sim^{5} + 66693052 K\sim^{4}) (K\sim^{2} + 4 K\sim + 5)^{2}) \Big/$$

$$\Big((K\sim^2-3)^3 \left(K\sim^2+4K\sim-\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+5\right)^5 (3K\sim^2 + 8K\sim+7)(K\sim^2+8K\sim+13)\Big) YY^2 + 32\Big((K\sim^2+4K\sim+5)^4 \big(361K\sim^{12} - 208K\sim^{10}\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 4552K\sim^{11} - 2072K\sim^9\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 26718K\sim^{10} - 9608K\sim^8\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 96200K\sim^9 - 27200K\sim^7\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 236399K\sim^8 - 52192K\sim^6\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 417744K\sim^7 - 71600K\sim^5\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 544964K\sim^6 - 71696K\sim^4\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 530576K\sim^5 - 51968K\sim^3\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 383799K\sim^4 - 26768K\sim^2\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 198632K\sim^3 - 9592K\sim\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 65662K\sim^2 - 1960\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 10536K\sim+401\big) \mathit{RootOf}\big(\_Z^2(9K\sim^{10}+36K\sim^9-31K\sim^8-304K\sim^7-214K\sim^6+792K\sim^5+1170K\sim^4 - 432K\sim^3-1539K\sim^2-540K\sim+189) - 174K\sim^{10} + 100K\sim^8\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 1960K\sim^9 + 864K\sim^7\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 9950K\sim^8 + 3304K\sim^6\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 29664K\sim^7 + 7200K\sim^5\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 56972K\sim^6 + 9760K\sim^4\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 72752K\sim^5 + 8480K\sim^3\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 61372K\sim^4 + 4504K\sim^2\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 32608K\sim^3 + 1120K\sim\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} - 9190K\sim^2 - 4\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1} + 152K\sim+890\big)\Big) \Big/ \Big((K\sim^2 - 3)\left(K\sim^2+4K\sim-\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+5\right)^6 (3K\sim^2 + 8K\sim+7)^2 (K\sim^2+8K\sim+13)\Big) YY^3 + \mathrm{O}(YY^4)$$

We get an expansion in (1-y/y+)^3/2

Asymptotic behavior (in t) of Q(t,ty) (Proposition 3.12)

For nu<nu_c:

We plug the developments of U and V obtained above (in **(5.1.1)** and **(5.1.6)**, with (1-w/rhosubc)^{1/2}) in the rational parametrization of Q, and check that the coefficient of XX cancels out. Hence it gives the expected singular behavior.

> Qtsubcsing3 := convert(simplify(series(subs(U = Usubcsing3, V = Vsubsing3, subs(nu = subs(U = Usubc, nuUsub), QtUV)), XX, 4)), polynom);

$$Qtsubcsing3 := \left(18\left(\left(Vsub^3 - 7\,Vsub^2 - Vsub - 1\right)Usubc - \frac{2\,Vsub^3}{3} + 6\,Vsub^2 + 2\,Vsub + \frac{2}{3}\right)\left(\left(Vsub^2 - 2\,Vsub - 1\right)Usubc^2 + \left(-Vsub^2 + 2\,Vsub + \frac{5}{3}\right)Usubc + \frac{Vsub^2}{6} - \frac{Vsub}{3} - \frac{1}{2}\right)\right) \Big/ \left((-2 + 3\,Usubc)\,(Vsub+1)^3\left(6\,Usubc^2 - 10\,Usubc + 3\right)\right) + \left(18\,Vsub\left(\left(\left(Usubc^2 - Usubc + \frac{1}{6}\right)Vsub^5 + \left(-10\,Usubc^2 + 10\,Usubc - \frac{5}{3}\right)Vsub^3 + \left(16\,Usubc^2 - \frac{52}{3}\,Usubc + \frac{10}{3}\right)Vsub^2 + \left(9\,Usubc^2 - 17\,Usubc + \frac{11}{2}\right)Vsub - \frac{4\,Usubc}{3} + \frac{2}{3}\right)\left(\left(-\frac{2}{3} + Usubc\right)Vsub^3 + (-7\,Usubc + 6)\,Vsub^2 + (-Usubc + 2)\,Vsub - Usubc + \frac{2}{3}\right)XX^2\right) \Big/ \left(\left(Vsub^2 + 4\,Vsub + 1\right)(Vsub - 1)^2\,(-2 + 3\,Usubc)\,(Vsub+1)^3\left(6\,Usubc^2 - 10\,Usubc + 3\right)\right) - \left(4\,(Vsub + 1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\left(\left(Vsub^3 - 7\,Vsub^2 - Vsub - 1\right)Usubc - \frac{2\,Vsub^3}{3} + 6\,Vsub^2 + 2\,Vsub + \frac{2}{3}\right)XX^3\right) \Big/ \left(9\left(-\frac{2}{3} + Usubc\right)\left(Vsub^2 + 4\,Vsub + 1\right)(Vsub - 1)^4\right) \quad \textbf{(7.1.1)}$$

> coeff(Qtsubcsing3, XX, 1);

$$0 \quad \textbf{(7.1.2)}$$

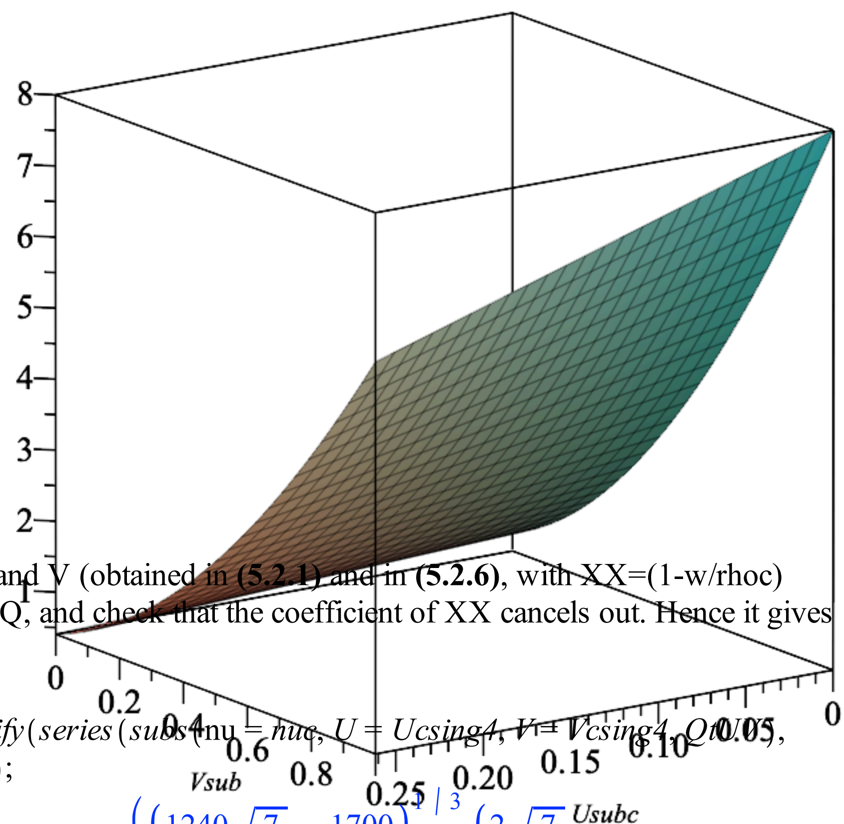
> coeff(Qtsubcsing3, XX, 3);

(7.1.3)

$$-\left(4\,(Vsub+1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)\right)\Big/\left(9\left(-\frac{2}{3}+Usubc\right)\left(Vsub^2+4\,Vsub+1\right)(Vsub-1)^4\right) \qquad \textbf{(7.1.3)}$$

We check that the coefficient does not vanish (the denominator is clearly not zero in the range of values of interest, V in (0,1) and U in (0,1/2)

$> plot3d\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3},\,Usubc=0\,..Uc,\,Vsub=0\,..1\right)$



For nu = nu_c

We plug again the developments of U and V (obtained in **(5.2.1)** and in **(5.2.6)**, with XX=(1-w/rhoc)^1/3) in the rational parametrization of Q, and check that the coefficient of XX cancels out. Hence it gives the expected singular behavior.

$> Qtcsing4 := collect(convert(simplify(series(subs(nu=nuc, U=Ucsing4, V=Vcsing4, Qt0UV), XX, 5)), polynom), XX, factor);$

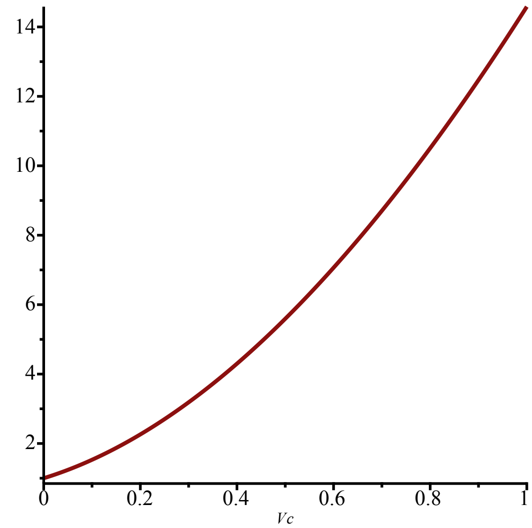
$$Qtcsing4 := \frac{1}{36\,(Vc-1)^4\left(Vc^2+4\,Vc+1\right)}\left(\left(1240\sqrt{7}-1700\right)^{1/3}\left(2\sqrt{7}+1\right)\left(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\right)Vc\,(Vc+1)\,XX^4\right)$$
$$+\frac{1}{5\,(Vc-1)^2\left(Vc^2+4\,Vc+1\right)(Vc+1)^3}\left(\left(Vc^5-10\,Vc^3+4\,Vc^2-63\,Vc\right.\right.$$

$$- 12)\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right) Vc\,XX^3\Big)$$

$$+ \frac{\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1\right)\left(Vc^2 - 2\,Vc + 5\right)}{5\,(Vc+1)^3}$$

We check that the coefficient does not vanish for Vc\in (0,1)

> $plot\left(2\sqrt{7}\,Vc^2 - Vc^3 + 2\sqrt{7}\,Vc + 5\,Vc^2 - Vc + 1,\ Vc = 0\,..1\right);$



For nu > nu_c

We replace U and V by their singular expansion (obtained in **(5.3.2)** and **(5.3.5)**, with XX=(1-w/rho)^1/2) in the expression of Q given by the rational parametrizations :

> $Qtsupsing := simplify(series(subs(U = Usupcsing, V = Vsupsing, subs(\mathrm{nu} = nusupK, QtUV)), XX, 4));$

$$Qtsupsing := \frac{1}{(Vsup+1)^3\left(K\sim^2 + 8\,K\sim + 13\right)\left(K\sim^2 - 3\right)^3}\Big(\Big(\Big(\left(Vsup^2 - 2\,Vsup - 1\right)K\sim^4 \qquad \textbf{(7.3.1)}$$

$$+ (-24\,Vsup - 8)\,K\sim^3 + \left(-6\,Vsup^2 - 68\,Vsup - 10\right)K\sim^2 + (-56\,Vsup + 24)\,K\sim$$

$$+ 9\,Vsup^2 - 2\,Vsup + 39\Big)\Big(\left(Vsup^3 - 7\,Vsup^2 - Vsup - 1\right)K\sim^4 + \big(-40\,Vsup^2$$

$$+ 8\,Vsup\big)K\sim^3 + \left(-6\,Vsup^3 - 110\,Vsup^2 + 14\,Vsup + 6\right)K\sim^2 + \big(-136\,Vsup^2$$

$$- 24\,Vsup\big)K\sim + 9\,Vsup^3 - 55\,Vsup^2 - 33\,Vsup - 9\Big)\Big) + \Bigg(\Bigg(\left(K\sim^2 - 3\right)^4 Vsup^5$$

$$- 24\,(2 + K\sim)\left(K\sim^2 - 3\right)^2\left(K\sim^2 + \frac{4}{3}\,K\sim - \frac{1}{3}\right) Vsup^4 - 10\left(K\sim^2 - 3\right)^2\Bigg(K\sim^2$$

$$+ \frac{8}{5}\,K\sim + \frac{1}{5}\Bigg)\left(K\sim^2 + 8\,K\sim + 13\right) Vsup^3 + 16\,\big(K\sim^6 + 14\,K\sim^5 + 83\,K\sim^4 + 236\,K\sim^3$$

$$+ 307\,K\sim^2 + 126\,K\sim - 31\big)\,(K\sim + 1)^2\, Vsup^2 + \big(9\,K\sim^8 + 96\,K\sim^7 + 468\,K\sim^6$$

$$+ 1184\,K\sim^5 + 1062\,K\sim^4 - 2016\,K\sim^3 - 6668\,K\sim^2 - 7200\,K\sim - 2919\big)\,Vsup - 8\,(2$$

$$+ K\sim)\left(K\sim^2 + 4\,K\sim + 5\right)\left(K\sim^2 - 3\right)^2\Big)\,Vsup\,\Big(\left(K\sim^2 - 3\right)^2 Vsup^3 + \big(-7\,K\sim^4 - 40\,K\sim^3$$

$- 110\,K\sim^2 - 136\,K\sim - 55\big)\,Vsup^2 - \big(K\sim^2 - 8\,K\sim - 11\big)\,\big(K\sim^2 - 3\big)\,Vsup - \big(K\sim^2$
$- 3\big)^2\big)\big)\Big/\Big(\big(\big(\big(K\sim^2 - 3\big)\,Vsup^2 + \big(-2\,K\sim^2 - 8\,K\sim - 10\big)\,Vsup + K\sim^2 - 3\big)\,\big(K\sim^2$
$- 3\big)^3\,\big(\big(K\sim^2 - 3\big)\,Vsup^2 + 4\,(K\sim + 1)^2\,Vsup + K\sim^2 - 3\big)\,\big(K\sim^2 + 8\,K\sim$
$+ 13\big)\,(Vsup + 1)^3\Big)\,XX^2 + 32\,\Big(RootOf\big(\big(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2$
$+ 3360\,K\sim - 1200\big)\,\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3$
$- 192\,K\sim^2 - 306\,K\sim - 117\big)\,Vsup\,\big(\big(K\sim^2 - 3\big)^2\,Vsup^3 + \big(-7\,K\sim^4 - 40\,K\sim^3 - 110\,K\sim^2$
$- 136\,K\sim - 55\big)\,Vsup^2 - \big(K\sim^2 - 8\,K\sim - 11\big)\,\big(K\sim^2 - 3\big)\,Vsup - \big(K\sim^2 - 3\big)^2\big)$
$\big((K\sim + 1)^2 Vsup^2 + \big(K\sim^2 - 3\big)\,Vsup + (K\sim + 1)^2\big)\,\left(K\sim + \frac{5}{3}\right)\,(Vsup + 1)\Big)\Big/\Big(\big(K\sim$
$+ 1)\,\big(\big(K\sim^2 - 3\big)\,Vsup^2 + \big(-2\,K\sim^2 - 8\,K\sim - 10\big)\,Vsup + K\sim^2 - 3\big)\,\big(\big(K\sim^2$
$- 3\big)\,Vsup^2 + 4\,(K\sim + 1)^2\,Vsup + K\sim^2 - 3\big)^3\Big)\,XX^3 + \mathrm{O}\big(XX^4\big)$

> *coeff*(*Qtsupsing*, *XX*, 1);

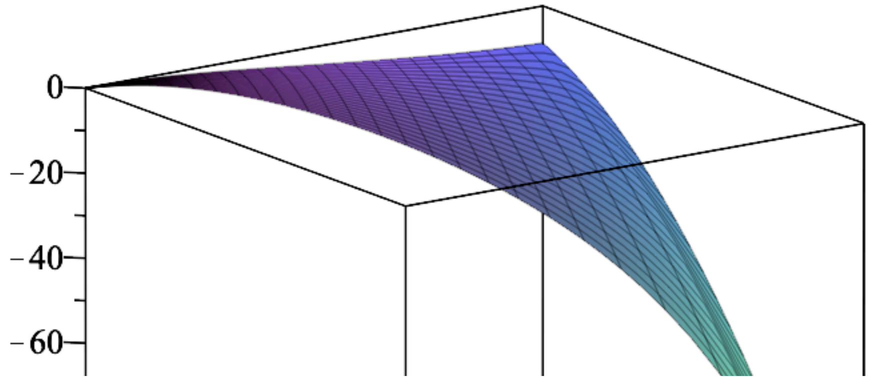
$$0 \qquad \textbf{(7.3.2)}$$

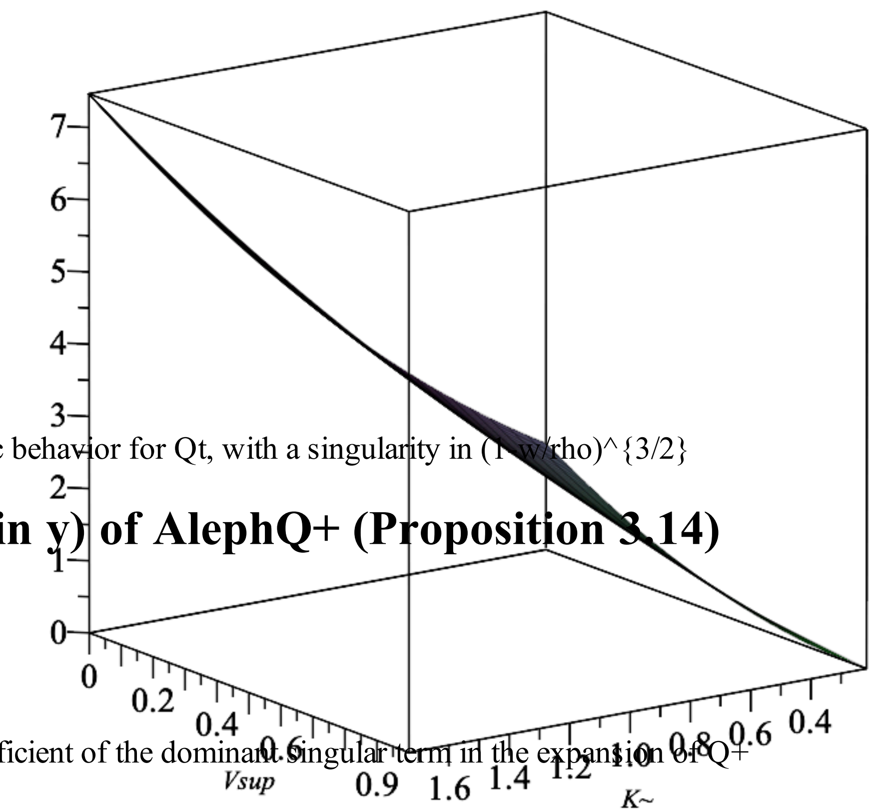
We check that the coefficient of XX^3 does not cancel :

> *factor*(*numer*(*coeff*(*Qtsupsing*, *XX*, 3)));

$32\,RootOf\big(\big(1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200\big)\,\_Z^2 - K\sim^8 - 10\,K\sim^7$ **(7.3.3)**
$- 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)$
$Vsup\,\big(K\sim^4\,Vsup^3 - 7\,K\sim^4\,Vsup^2 - K\sim^4\,Vsup - 40\,K\sim^3\,Vsup^2 - 6\,K\sim^2\,Vsup^3 - K\sim^4$
$+ 8\,K\sim^3\,Vsup - 110\,K\sim^2\,Vsup^2 + 14\,K\sim^2\,Vsup - 136\,K\sim\,Vsup^2 + 9\,Vsup^3 + 6\,K\sim^2$
$- 24\,K\sim\,Vsup - 55\,Vsup^2 - 33\,Vsup - 9\big)\,\big(K\sim^2\,Vsup^2 + K\sim^2\,Vsup + 2\,K\sim\,Vsup^2 + K\sim^2$
$+ Vsup^2 + 2\,K\sim - 3\,Vsup + 1\big)\,(3\,K\sim + 5)\,(Vsup + 1)$

> $plot3d\big(\big(K^4\,Vsup^3 - 7\,K^4\,Vsup^2 - K^4\,Vsup - 40\,K^3\,Vsup^2 - 6\,K^2\,Vsup^3 - K^4 + 8\,K^3\,Vsup - 110\,K^2\,Vsup^2 + 14\,K^2\,Vsup - 136\,K\,Vsup^2 + 9\,Vsup^3 + 6\,K^2 - 24\,K\,Vsup - 55\,Vsup^2 - 33\,Vsup - 9\big), K = Kc\,..Kinfini, Vsup = 0\,..VK11\big);$
$plot3d\big(K^2\,Vsup^2 + K^2\,Vsup + 2\,K\,Vsup^2 + K^2 + Vsup^2 + 2\,K - 3\,Vsup + 1, K = Kc\,..Kinfini, Vsup = 0\,..VK11\big);$





>

We hence get the desired asymptotic behavior for Qt, with a singularity in (1-w/rho)^{3/2}

Asymptotic behavior (in y) of AlephQ+ (Proposition 3.14)

nu <nuc

The function alephQplus is the coefficient of the dominant singular term in the expansion of Q+

> *Qtsubcsing3*;
alephQplussubc := *coeff*(*Qtsubcsing3*, *XX*, 3);

$$\left(18\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)\left(\left(Vsub^2-2\,Vsub-1\right)Usubc^2+\left(-Vsub^2+2\,Vsub+\frac{5}{3}\right)Usubc+\frac{Vsub^2}{6}-\frac{Vsub}{3}-\frac{1}{2}\right)\right)\Big/\left(\left(-2+3\,Usubc\right)\left(Vsub+1\right)^3\left(6\,Usubc^2-10\,Usubc+3\right)\right)$$

$$+\left(18\,Vsub\left(\left(Usubc^2-Usubc+\frac{1}{6}\right)Vsub^5+\left(-10\,Usubc^2+10\,Usubc-\frac{5}{3}\right)Vsub^3+\left(16\,Usubc^2-\frac{52}{3}\,Usubc+\frac{10}{3}\right)Vsub^2+\left(9\,Usubc^2-17\,Usubc+\frac{11}{2}\right)Vsub-\frac{4\,Usubc}{3}+\frac{2}{3}\right)\left(\left(-\frac{2}{3}+Usubc\right)Vsub^3+(-7\,Usubc+6)\,Vsub^2+(-Usubc+2)\,Vsub-Usubc+\frac{2}{3}\right)XX^2\right)\Big/\left(\left(Vsub^2+4\,Vsub+1\right)(Vsub-1)^2\,(-2+3\,Usubc)\,(Vsub+1)^3\left(6\,Usubc^2-10\,Usubc+3\right)\right)-\left(4\,(Vsub+1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)XX^3\right)\Big/\left(9\left(-\frac{2}{3}+Usubc\right)\left(Vsub^2+4\,Vsub+1\right)(Vsub-1)^4\right)$$

$$alephQplussubc:=-\left(4\,(Vsub+1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)Usubc-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)\right)\Big/\left(9\left(-\frac{2}{3}+Usubc\right)\left(Vsub^2+4\,Vsub+1\right)(Vsub-1)^4\right) \quad \textbf{(8.1.1)}$$

We use the development of V that we already computed above

> *Vsubsingy*;

$$1+\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y-\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3}(-1)^{1/3}\,Y3Y^2}{2}-\frac{4\,(U-1)\,Y3Y^3}{-2+3\,U}+\frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3}(-1)^{2/3}\,Y3Y^4}{9\,U-6} \quad \textbf{(8.1.2)}$$

And plug it into the expression of alephQplussubc

> *simplify*(*series*(*subs*(*Vsub* = *Vsubsingy*, *Usubc* = *U*, *alephQplussubc*), *Y3Y*, 8));

$$\frac{2\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{27\left(\frac{-24\,U+24}{-2+3\,U}\right)^{1/3}}\,Y3Y^{-4} \quad \textbf{(8.1.3)}$$

$$- 2 \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}\,(U - 1)\,\left(\mathrm{I}\sqrt{3} + 1\right)^2}{\left(\frac{-24 U + 24}{-2 + 3 U}\right)^{2/3} (-54 + 81 U)} Y3Y^{-2}$$

$$- \frac{2}{81} \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}\,\left(\mathrm{I}\sqrt{3} + 1\right)}{\left(\frac{-24 U + 24}{-2 + 3 U}\right)^{1/3}} Y3Y^{-1}$$

$$+ \frac{(-40 U + 24)\sqrt{6}\,\sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}}{-162 + 243 U}$$

$$- \frac{2}{81} \frac{\left(51 U^2 - 77 U + 26\right)\left(\mathrm{I}\sqrt{3} + 1\right)^2 \sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}\,\sqrt{3}\,\sqrt{2}}{\left(\frac{-24 U + 24}{-2 + 3 U}\right)^{2/3} (-2 + 3 U)^2} Y3Y$$

$$+ \frac{2}{243} \frac{\left(18 U^2 - 30 U + 19\right)\sqrt{2}\,\sqrt{3}\,\sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}\,\left(\mathrm{I}\sqrt{3} + 1\right)}{\left(\frac{-24 U + 24}{-2 + 3 U}\right)^{1/3} (-2 + 3 U)^2} Y3Y^2$$

$$+ \frac{2}{81} \frac{\sqrt{6}\,\sqrt{\frac{6 U^2 - 10 U + 3}{9 U^2 - 10 U + 2}}\,\left(65 U^2 - 81 U + 26\right)}{(-2 + 3 U)^2} Y3Y^3 + \mathrm{O}\left(Y3Y^4\right)$$

> *algeqtoseries*(*numer*(2·(1 − *YY*) − *yUVsubc*), *YY*, *V*, 10);

$$\left[1 + RootOf\left((-2 + 3 U)\,\_Z^3 + 24 U - 24\right) YY^{1/3}\right. \quad \textbf{(8.1.4)}$$

$$+ \frac{RootOf\left((-2 + 3 U)\,\_Z^3 + 24 U - 24\right)^2 YY^{2/3}}{2} - \frac{4 (U - 1)\,YY}{-2 + 3 U}$$

$$+ \frac{RootOf\left((-2 + 3 U)\,\_Z^3 + 24 U - 24\right) YY^{4/3}}{3 (-2 + 3 U)}$$

$$+ \frac{(5 U - 3)\,RootOf\left((-2 + 3 U)\,\_Z^3 + 24 U - 24\right)^2 YY^{5/3}}{6 (-2 + 3 U)} - \frac{4 (U - 1)\,YY^2}{-2 + 3 U}$$

$$- \frac{2 \left(10 U^2 - 20 U + 9\right) RootOf\left((-2 + 3 U)\,\_Z^3 + 24 U - 24\right) YY^{7/3}}{9 (-2 + 3 U)^2}$$

$$+\frac{\left(29U^2-33U+9\right)\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)^2 YY^{8/3}}{18(-2+3U)^2}$$

$$-\frac{4(U-1)YY^3}{-2+3U}+\mathrm{O}\left(YY^{10/3}\right)\Big]$$

with Y3Y=YY^1/3=(1-y/2)^1/3:

> $allvalues\left(RootOf\left((3U-2)\_Z^3+24U-24\right)\right);$

$$\left(-\frac{24U-24}{-2+3U}\right)^{1/3},\ \left(-\frac{24U-24}{-2+3U}\right)^{1/3}(-1)^{2/3},\ -\left(-\frac{24U-24}{-2+3U}\right)^{1/3}(-1)^{1/3} \quad \textbf{(8.1.5)}$$

> $VsubsingyPrecis := 1+RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y$
$+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^2}{2}-\frac{4(U-1)Y3Y^3}{-2+3U}$
$+\frac{RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y^4}{3(-2+3U)}$
$+\frac{(5U-3)RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^5}{6(-2+3U)}-\frac{4(U-1)Y3Y^6}{-2+3U}$
$-\frac{2\left(10U^2-20U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)Y3Y^7}{9(-2+3U)^2}$
$+\frac{\left(29U^2-33U+9\right)RootOf\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^8}{18(-2+3U)^2}$
$-\frac{4(U-1)Y3Y^9}{-2+3U};$

$$VsubsingyPrecis := 1+\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)Y3Y \quad \textbf{(8.1.6)}$$

$$+\frac{\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^2}{2}-\frac{4(U-1)Y3Y^3}{-2+3U}$$

$$+\frac{\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)Y3Y^4}{9U-6}$$

$$+\frac{(5U-3)\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^5}{-12+18U}-\frac{4(U-1)Y3Y^6}{-2+3U}$$

$$-\frac{2\left(10U^2-20U+9\right)\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)Y3Y^7}{9(-2+3U)^2}$$

$$+\frac{\left(29U^2-33U+9\right)\mathrm{RootOf}\left((-2+3U)\_Z^3+24U-24\right)^2 Y3Y^8}{18(-2+3U)^2}$$

$$-\frac{4(U-1)Y3Y^9}{-2+3U}$$

> $VsubsingyPrecis := subs\Bigg(RootOf\left((3U-2)\_Z^3+24U-24\right)=\left(-\frac{24U-24}{3U-2}\right)^{1/}$

$$\sqrt[3]{}\;(-1)^{2/3},\ VsubsingyPrecis\Big);$$

$$VsubsingyPrecis := 1 + \left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3} (-1)^{2/3}\, Y3Y - \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3} (-1)^{1/3}\, Y3Y^2}{2} - \frac{4\,(U-1)\, Y3Y^3}{-2+3\,U} + \frac{\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3} (-1)^{2/3}\, Y3Y^4}{9\,U-6} - \frac{(5\,U-3)\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3} (-1)^{1/3}\, Y3Y^5}{-12+18\,U} - \frac{4\,(U-1)\, Y3Y^6}{-2+3\,U} - \frac{2\left(10\,U^2-20\,U+9\right)\left(-\frac{24\,U-24}{-2+3\,U}\right)^{1/3} (-1)^{2/3}\, Y3Y^7}{9\,(-2+3\,U)^2} - \frac{\left(29\,U^2-33\,U+9\right)\left(-\frac{24\,U-24}{-2+3\,U}\right)^{2/3} (-1)^{1/3}\, Y3Y^8}{18\,(-2+3\,U)^2} - \frac{4\,(U-1)\, Y3Y^9}{-2+3\,U} \quad \textbf{(8.1.7)}$$

> *simplify*(*series*(*subs*(*Vsub* = *VsubsingyPrecis*, *Usubc* = *U*, *alephQplussubc*), *Y3Y*, 5));

$$\frac{2\sqrt{3}\sqrt{2}\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{27\left(\frac{-24\,U+24}{-2+3\,U}\right)^{1/3}}\, Y3Y^{-4} - 2\,\frac{\sqrt{3}\sqrt{2}\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\,(U-1)\left(\mathrm{I}\sqrt{3}+1\right)^2}{\left(\frac{-24\,U+24}{-2+3\,U}\right)^{2/3}(-54+81\,U)}\, Y3Y^{-2} - \frac{2}{81}\,\frac{\sqrt{3}\sqrt{2}\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\left(\mathrm{I}\sqrt{3}+1\right)}{\left(\frac{-24\,U+24}{-2+3\,U}\right)^{1/3}}\, Y3Y^{-1} + \mathrm{O}(Y3Y) \quad \textbf{(8.1.8)}$$

>

>

>

nu=nuc

The function alephQplus is the coefficient of the dominant singular term in the expansion of Q+

> Qtcsing4; alephQplusc := coeff(Qtcsing4, XX, 4);

$$\frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\Big(\big(1240\sqrt{7}-1700\big)^{1/3}\,\big(2\sqrt{7}+1\big)\,\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,Vc\,(Vc+1)\,XX^4\Big)+\frac{1}{5\,(Vc-1)^2\,(Vc^2+4\,Vc+1)\,(Vc+1)^3}\Big(\big(Vc^5-10\,Vc^3+4\,Vc^2-63\,Vc-12\big)\,\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,Vc\,XX^3\Big)+\frac{\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,\big(Vc^2-2\,Vc+5\big)}{5\,(Vc+1)^3}$$

$$alephQplusc := \frac{1}{36\,(Vc-1)^4\,(Vc^2+4\,Vc+1)}\Big(\big(1240\sqrt{7}-1700\big)^{1/3}\,\big(2\sqrt{7}+1\big)\,\big(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\big)\,Vc\,(Vc+1)\Big) \qquad \textbf{(8.2.1)}$$

> simplify(subs(U = Uc, Vsubsingy));

$$\frac{1}{3\,\big(1+\sqrt{7}\big)^2}\Big(6\,Y3Y\,\big(4+\sqrt{7}\big)^{1/3}\,\big(Y3Y^3-\sqrt{7}-1\big)\,\big(1+\sqrt{7}\big)^{2/3}+6\,Y3Y^2\,\big(4+\sqrt{7}\big)^{2/3}\,\big(1+\sqrt{7}\big)^{4/3}-20\sqrt{7}\,Y3Y^3-44\,Y3Y^3+6\sqrt{7}+24\Big) \qquad \textbf{(8.2.2)}$$

> collect(expand(rationalize(simplify(series(subs(Vc = **(8.2.2)**, alephQplusc), Y3Y, 4)))), Y3Y, factor);

$$\left(-\frac{5\,20^{1/3}\,\big(6739+2263\sqrt{7}\big)^{1/3}}{972}+\frac{7\,20^{1/3}\,\big(6739+2263\sqrt{7}\big)^{1/3}\sqrt{7}}{1944}\right)Y3Y^{-4}+\left(-\frac{13\,\big(7585+3730\sqrt{7}\big)^{1/3}}{1944}+\frac{\big(7585+3730\sqrt{7}\big)^{1/3}\sqrt{7}}{1944}\right)Y3Y^{-2}+\left(\frac{5\,\big(134780+45260\sqrt{7}\big)^{1/3}}{2916}-\frac{7\,\big(134780+45260\sqrt{7}\big)^{1/3}\sqrt{7}}{5832}\right)Y3Y^{-1}+O\big(Y3Y^0\big) \qquad \textbf{(8.2.3)}$$

>

Variante avec developpement poussé un cran plus loin :

> map(simplify, series(simplify(subs(U = Uc, VsubsingyPrecis)), Y3Y, 10));

$$1+\frac{\big(-20\sqrt{7}-44\big)\,\big(4+\sqrt{7}\big)^{1/3}}{\big(1+\sqrt{7}\big)^{10/3}}\,Y3Y+\frac{\big(20\sqrt{7}+44\big)\,\big(4+\sqrt{7}\big)^{2/3}}{\big(1+\sqrt{7}\big)^{11/3}}\,Y3Y^2 \qquad \textbf{(8.2.4)}$$

$$- \frac{8}{3} \frac{79+31\sqrt{7}}{(1+\sqrt{7})^4} Y3Y^3 + 4 \frac{(4+\sqrt{7})^{4/3}}{(1+\sqrt{7})^{10/3}} Y3Y^4$$
$$+ \frac{4}{9} \frac{(4+\sqrt{7})^{2/3} (22\sqrt{7}+43)}{(1+\sqrt{7})^{11/3}} Y3Y^5 - \frac{8}{3} \frac{79+31\sqrt{7}}{(1+\sqrt{7})^4} Y3Y^6$$
$$+ \frac{4}{81} \frac{(4+\sqrt{7})^{1/3} (229\sqrt{7}+709)}{(1+\sqrt{7})^{10/3}} Y3Y^7$$
$$+ \frac{2}{81} \frac{(4+\sqrt{7})^{2/3} (179\sqrt{7}+221)}{(1+\sqrt{7})^{11/3}} Y3Y^8 - \frac{8}{3} \frac{79+31\sqrt{7}}{(1+\sqrt{7})^4} Y3Y^9$$

> *map*(*simplify*, *map*(*expand*, *map*(*rationalize*, *series*(*subs*(*Vc* = **(8.2.4)**, *alephQplusc*), *Y3Y*, 5))));

$$\frac{(4+\sqrt{7})^{2/3} (1240\sqrt{7}-1700)^{1/3} (1+\sqrt{7})^{1/3} (-10+7\sqrt{7})}{1944} Y3Y^{-4}$$
$$+ \frac{(-130120450-49180909\sqrt{7}) (1240\sqrt{7}-1700)^{1/3}}{(4+\sqrt{7})^{5/3} (1+\sqrt{7})^{1/3} (303443808+114690840\sqrt{7})} Y3Y^{-2}$$
$$- \frac{1}{5832} (4+\sqrt{7})^{2/3} (1240\sqrt{7}-1700)^{1/3} (1+\sqrt{7})^{1/3} (-10+7\sqrt{7}) Y3Y^{-1}$$
$$+ \mathrm{O}(Y3Y) \quad \textbf{(8.2.5)}$$

>

>

>

Proof of proposition 3.15: expansion of the radius of convergence (in y) of Q(t,ty).

To identify the singularities (in y) of the series Q(t,ty) for a fixed t\in (0,t_nu), we start from the parametrization of y by U and V given by:

> *yUV*;

$$(8\nu(1-2U)V(V+1)) \Big/ \left(U(U(\nu+1)-2)\left(V^3 + \frac{(9(\nu+1)U^2-2(3+10\nu)U+8\nu)V^2}{U(U(\nu+1)-2)} - \frac{(9U(\nu+1)-4\nu-6)V}{U(\nu+1)-2} - 1\right)\right) \quad \textbf{(9.1)}$$

Since t is fixed, U is fixed and the possible values for a singularity in y corresponds to the roots of the quantity:

> *eqVcritU*;

(9.2)

$$1+V^4+2V^3+\frac{2(-2+3U)(3U\nu+3U-2\nu)V^2}{U(U\nu+U-2)}+2V \quad \textbf{(9.2)}$$

To see the singularities in U of the roots, we look at the discriminant

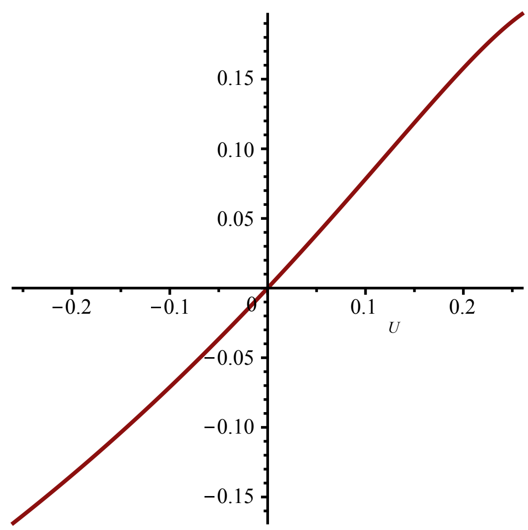
> *factor*(*discrim*(*eqVcritU*, *V*));

$$\frac{1}{U^4(U\nu+U-2)^4}\left(1024(-1+2U)(U\nu+U-\nu)(3U^2\nu+3U^2-3U\nu-3U+\nu)(15U^2\nu+15U^2-24U\nu-6U+8\nu)^2\right) \quad \textbf{(9.3)}$$

Only the last term may pose problems (the one before is the equation for Usubc). We look at wich values of nu may cancel it:

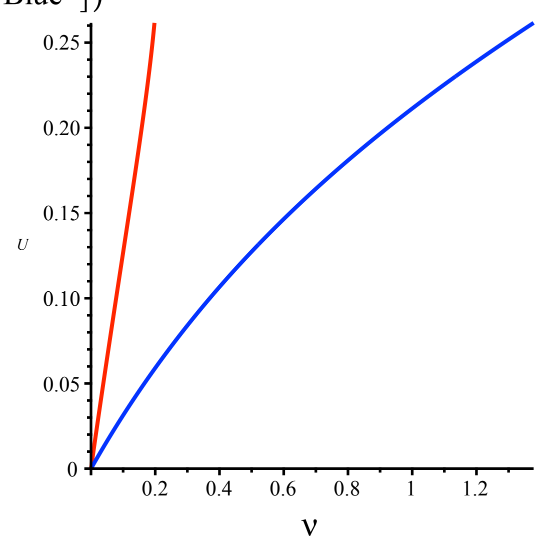
> *solve*((15 U^2 ν + 15 U^2 − 24 U ν − 6 U + 8 ν), nu); *plot*(%, *U* = -*Uc* ..*Uc*);

$$-\frac{3U(5U-2)}{15U^2-24U+8}$$



The discriminant can only be 0 in the subcritical regime. When the roots are real there is no problem:

> *implicitplot*([(15 U^2 ν + 15 U^2 − 24 U ν − 6 U + 8 ν), *algUsubcrit*], nu = 0 ..*nuc*, *U* = -*Uc* ..*Uc*, *color* = ["Red", "Blue"])

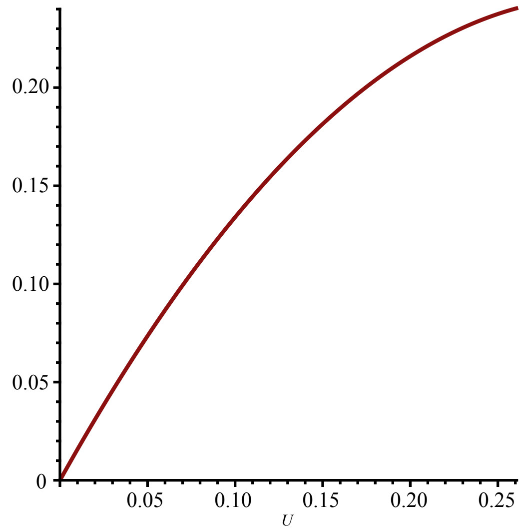


If the roots are complex no problem either as they have modulus larger than Usubc:

> $factor\left(subs\left(\text{nu} = nuUsub, \frac{8\,\text{nu}}{15\,(\text{nu}+1)}\right) - U^2\right);$

$$-\frac{(13\,U - 8)\,U}{5} \tag{9.4}$$

> $plot\left(subs\left(\text{nu} = nuUsub, \frac{8\,\text{nu}}{15\,(\text{nu}+1)}\right) - U^2, U = 0\,..Uc\right);$



For nu < nu_c

The strategy of the proof consists in replacing U by its singular behavior around rho obtained above. Recall indeed that, we have the following development for U (with XX=(1-w/rho)^1/2 and Usubc = value of U for t=t_\nu):

> $Usubcsing3;$

$$Usubc + \frac{Usubc\,(-2 + 3\,Usubc)\,\sqrt{6}\,\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\;XX}{6} + \big((1458\,Usubc^6 \tag{9.1.1}$$

$$- 5778\,Usubc^5 + 9045\,Usubc^4 - 7146\,Usubc^3 + 2984\,Usubc^2 - 616\,Usubc + 48)$$

$$Usubc^2\,XX^2\big) \big/ \big(18\,(9\,Usubc^2 - 10\,Usubc + 2)^2\,(2\,Usubc - 1)\big)$$

$$(9\,Usubc^2 - 10\,Usubc + 2)^3\,(2\,Usubc - 1)\big)$$

In eqUVcritc, we replace U by its development and nu by its expression in termes of Usubc. We also know that for U=Usubc, the radius of convergence in y is 2, corresponding to V=1, so that we set V=1+VV.

> $eqVVpsubsing := convert(map(factor, series(numer(simplify(subs(U = Usubcsing3, \text{nu} = subs(U = Usubc, nuUsub), V = 1 + VV, eqVcritU))), XX, 4)), polynom);$

$$eqVVpsubsing := -1679616\,VV^2\,(VV^2 + 6\,VV + 6)\,(2\,Usubc - 1)^3\,(9\,Usubc^2 - 10\,Usubc \tag{9.1.2}$$

$$+ 2)^7 - 559872\sqrt{6}\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,(2\,Usubc - 1)^2\,(9\,Usubc^2$$
$$- 10\,Usubc + 2)^7\,(3\,Usubc^2\,VV^4 + 18\,Usubc^2\,VV^3 - 4\,Usubc\,VV^4 + 18\,Usubc^2\,VV^2$$
$$- 24\,Usubc\,VV^3 + VV^4 - 48\,Usubc\,VV^2 + 6\,VV^3 - 48\,Usubc\,VV + 18\,VV^2 - 24\,Usubc$$
$$+ 24\,VV + 12)\,XX - 93312\,Usubc\,(2\,Usubc - 1)\,(-2 + 3\,Usubc)\,(6\,Usubc^2$$
$$- 10\,Usubc + 3)\,(9\,Usubc^2 - 10\,Usubc + 2)^5\,(162\,Usubc^4\,VV^4 + 972\,Usubc^4\,VV^3$$
$$- 426\,Usubc^3\,VV^4 + 972\,Usubc^4\,VV^2 - 2556\,Usubc^3\,VV^3 + 373\,Usubc^2\,VV^4$$
$$- 5148\,Usubc^3\,VV^2 + 2238\,Usubc^2\,VV^3 - 126\,Usubc\,VV^4 - 5184\,Usubc^3\,VV$$
$$+ 6222\,Usubc^2\,VV^2 - 756\,Usubc\,VV^3 + 14\,VV^4 - 2592\,Usubc^3 + 7968\,Usubc^2\,VV$$
$$- 2580\,Usubc\,VV^2 + 84\,VV^3 + 3984\,Usubc^2 - 3648\,Usubc\,VV + 324\,VV^2$$
$$- 1824\,Usubc + 480\,VV + 240)\,XX^2$$
$$+ 2592\sqrt{6}\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,Usubc^2\,(2\,Usubc - 1)\,(6\,Usubc^2$$
$$- 10\,Usubc + 3)\,(-2 + 3\,Usubc)^2\,(9\,Usubc^2 - 10\,Usubc + 2)^4\,(891\,Usubc^4\,VV^4$$
$$+ 5346\,Usubc^4\,VV^3 - 1338\,Usubc^3\,VV^4 + 75330\,Usubc^4\,VV^2 - 8028\,Usubc^3\,VV^3$$
$$+ 515\,Usubc^2\,VV^4 + 139968\,Usubc^4\,VV - 136980\,Usubc^3\,VV^2 + 3090\,Usubc^2\,VV^3$$
$$+ 6\,Usubc\,VV^4 + 69984\,Usubc^4 - 257904\,Usubc^3\,VV + 79710\,Usubc^2\,VV^2$$
$$+ 36\,Usubc\,VV^3 - 14\,VV^4 - 128952\,Usubc^3 + 153240\,Usubc^2\,VV - 15780\,Usubc\,VV^2$$
$$- 84\,VV^3 + 76620\,Usubc^2 - 31632\,Usubc\,VV + 900\,VV^2 - 15816\,Usubc + 1968\,VV$$
$$+ 984)\,XX^3$$

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

```
> algeqtoseries(eqVVpsubsing, XX, VV, 1);
```

$$\left[RootOf(\_Z^2 + 6\,\_Z + 6) + O(XX),\ RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}} + 3\,\_Z^2\right)\sqrt{XX} + O(XX)\right] \quad \textbf{(9.1.3)}$$

The right branch is the second one, and we can compute a full expansion

```
> op(2, algeqtoseries(eqVVpsubsing, XX, VV, 3));
```

$$RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}} + 3\,\_Z^2\right)\sqrt{XX} \quad \textbf{(9.1.4)}$$

$$+\frac{\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX}{3}+\frac{1}{36\left(9\,Usubc^2-10\,Usubc+2\right)}\Bigg(\left(69\,Usubc^2-74\,Usubc+14\right)\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)XX^{3/2}\Bigg)+O\left(XX^2\right)$$

> *Vpsubsing* := 1 + *RootOf*(-2 sqrt(6) sqrt((6 Usubc^2 - 10 Usubc + 3)/(9 Usubc^2 - 10 Usubc + 2)) + 3 _Z^2) sqrt(XX) + sqrt(6) sqrt((6 Usubc^2 - 10 Usubc + 3)/(9 Usubc^2 - 10 Usubc + 2)) XX/3 + 1/(36 (9 Usubc^2 - 10 Usubc + 2)) (RootOf(-2 sqrt(6) sqrt((6 Usubc^2 - 10 Usubc + 3)/(9 Usubc^2 - 10 Usubc + 2)) + 3 _Z^2) sqrt((6 Usubc^2 - 10 Usubc + 3)/(9 Usubc^2 - 10 Usubc + 2)) sqrt(6) (69 Usubc^2 - 74 Usubc + 14) XX^(3/2)) :

$$Vpsubsing := 1+RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{XX}+\frac{\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,XX}{3}+\frac{1}{36\left(9\,Usubc^2-10\,Usubc+2\right)}\left(RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\sqrt{6}\left(69\,Usubc^2-74\,Usubc+14\right)XX^{3/2}\right):$$

We can now plug this expansion in the expression of y in terms of U and V, we get:

> *ypsubsing* := *map*(*simplify*, *series*(*subs*(*V* = *Vpsubsing*, *U* = *Usubcsing3*, nu = *subs*(*U* = *Usubc*, *nuUsub*), *yUV*), *XX*, 2));

$$ypsubsing := 2-\frac{1}{9\,Usubc-9}\left(3\left(-\frac{2}{3}+Usubc\right)\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)XX^{3/2}\right)+O\left(XX^2\right) \qquad \textbf{(9.1.5)}$$

We now compute the expansion for the negative value of y which is singular. We first compute the corresponding value of V at t_\nu. We already know from **(4.1.6)** that it is -2+sqrt(3).

> *eqVVmsubsing* := *convert*(*map*(*factor*, *series*(*numer*(*simplify*(*subs*(*U* = *Usubcsing3*, nu = *subs*(*U* = *Usubc*, *nuUsub*), *V* = -2 + sqrt(3) + *VV*, *eqVcritU*))), *XX*, 4)), *polynom*) :

> *algeqtoseries*(*eqVVmsubsing*, *XX*, *VV*, 1);

$$\left[-2\sqrt{3}+O(XX),\ 3-\sqrt{3}+O\left(\sqrt{XX}\right),\right. \qquad \textbf{(9.1.6)}$$

$$\frac{\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,\sqrt{2}\,\left(7\sqrt{3}-12\right)}{3\left(2\sqrt{3}-3\right)}\,XX+\mathrm{O}\left(XX^2\right)\Bigg]$$

The right branch is the third one without the constant term and we can compute a full expansion :

```
> map(simplify, op(3, algeqtoseries(eqVVmsubsing, XX, VV, 3)));
```

$$\frac{\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,\sqrt{2}\,\left(7\sqrt{3}-12\right)}{6\sqrt{3}-9}\,XX - \frac{1}{3}\,\frac{1}{\left(2\sqrt{3}-3\right)^3\left(9\,Usubc^2-10\,Usubc+2\right)^2}\Big(\left(2217\sqrt{3}\,Usubc^2 - 2314\sqrt{3}\,Usubc-3840\,Usubc^2+418\sqrt{3}+4008\,Usubc-724\right)\left(6\,Usubc^2 - 10\,Usubc+3\right)\Big)\,XX^2 - \frac{1}{72}\Bigg(\Big(8863911\sqrt{3}\,Usubc^6-15352740\,Usubc^6 - 48819690\sqrt{3}\,Usubc^5+84558168\,Usubc^5+87820434\sqrt{3}\,Usubc^4 - 152109432\,Usubc^4-72112960\sqrt{3}\,Usubc^3+124903296\,Usubc^3 + 29235912\sqrt{3}\,Usubc^2-50638080\,Usubc^2-5621600\sqrt{3}\,Usubc+9736896\,Usubc + 409152\sqrt{3}-708672\Big)\left(6\,Usubc^2-10\,Usubc + 3\right)\sqrt{2}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\Bigg)\Bigg/\Big(\left(9\,Usubc^2-10\,Usubc+2\right)^3\left(2\,Usubc - 1\right)\left(2\sqrt{3}-3\right)^5\Big)\,XX^3+\mathrm{O}\left(XX^4\right) \qquad \mathbf{(9.1.7)}$$

$$> Vmsubsing := \text{-}2+\operatorname{sqrt}(3)+\frac{\sqrt{2}\,\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,\left(7\sqrt{3}-12\right)}{6\sqrt{3}-9}\,XX - \frac{1}{3}\,\frac{1}{\left(2\sqrt{3}-3\right)^3\left(9\,Usubc^2-10\,Usubc+2\right)^2}\Big(\left(6\,Usubc^2-10\,Usubc + 3\right)\left(2217\sqrt{3}\,Usubc^2-2314\sqrt{3}\,Usubc-3840\,Usubc^2+418\sqrt{3}+4008\,Usubc - 724\right)\Big)\,XX^2-\frac{1}{72}\Bigg(\Big(8863911\sqrt{3}\,Usubc^6-15352740\,Usubc^6$$

$$- 48819690\sqrt{3}\,Usubc^5 + 84558168\,Usubc^5 + 87820434\sqrt{3}\,Usubc^4$$

$$- 152109432\,Usubc^4 - 72112960\sqrt{3}\,Usubc^3 + 124903296\,Usubc^3$$

$$+ 29235912\sqrt{3}\,Usubc^2 - 50638080\,Usubc^2 - 5621600\sqrt{3}\,Usubc + 9736896\,Usubc$$

$$+ 409152\sqrt{3} - 708672\Big)\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,\big(6\,Usubc^2 - 10\,Usubc$$

$$+ 3\big)\sqrt{2}\Bigg) \Bigg/ \Big(\big(2\sqrt{3} - 3\big)^5 (2\,Usubc - 1)\,\big(9\,Usubc^2 - 10\,Usubc + 2\big)^3\Big)\,XX^3 :$$

We can now plug this expansion in the expression of y in terms of U and V, we get:

> *ymsubsing* := *map*(*simplify*, *series*(*subs*(*V* = *Vmsubsing*, *U* = *Usubcsing3*, nu = *subs*(*U* = *Usubc*, *nuUsub*), *yUV*), *XX*, 4));

$$ymsubsing := -\frac{4\,(Usubc - 1)\,\big(-2 + \sqrt{3}\big)\big(\sqrt{3} - 1\big)}{(21\,Usubc - 16)\sqrt{3} - 37\,Usubc + 28} \tag{9.1.8}$$

$$- 4\,\frac{\big(6\,Usubc^2 - 10\,Usubc + 3\big)\,(-2 + 3\,Usubc)\,\big(780\sqrt{3} - 1351\big)\,(Usubc - 1)}{\big(21\sqrt{3}\,Usubc - 16\sqrt{3} - 37\,Usubc + 28\big)^2\,(2\,Usubc - 1)\,\big(2\sqrt{3} - 3\big)^3}$$

$$XX^2 - \frac{8}{9}\Bigg(\big(6\,Usubc^2 - 10\,Usubc + 3\big)\sqrt{2}\,(-2$$

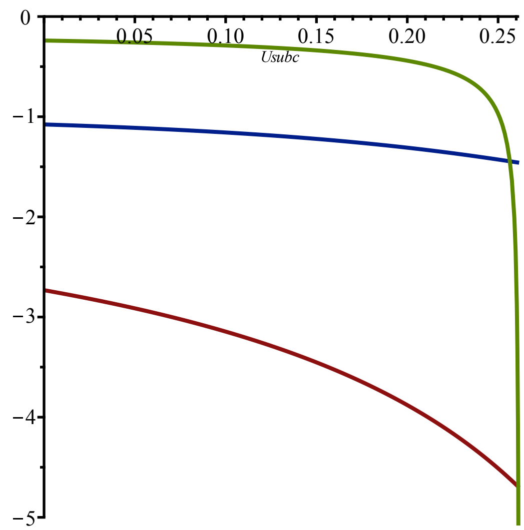
$$+ 3\,Usubc)\sqrt{\frac{6\,Usubc^2 - 10\,Usubc + 3}{9\,Usubc^2 - 10\,Usubc + 2}}\,(Usubc - 1)\,\big(1380661\sqrt{3}\,Usubc$$

$$- 2391375\,Usubc - 1048348\sqrt{3} + 1815792\big)\Bigg) \Bigg/ \Big((2\,Usubc$$

$$- 1)\,\big(21\sqrt{3}\,Usubc - 16\sqrt{3} - 37\,Usubc + 28\big)^3\,\big(2\sqrt{3} - 3\big)^5\Big)\,XX^3 + O\big(XX^4\big)$$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by plotting the values of the coefficients:

> *plot*([*coeff*(*ymsubsing*, *XX*, 0), *coeff*(*ymsubsing*, *XX*, 2), *coeff*(*ymsubsing*, *XX*, 3)], *Usubc* = 0..*Uc*);



For nu = nu_c

The proof is similar as the subcritical case, except that the singular expansion of U around rho is different. With XX=(1-w/rhoc)^1/3, we have:

> *Ucsing4;*

$$\frac{5}{9} - \frac{\sqrt{7}}{9} - \frac{\left(1240\sqrt{7}-1700\right)^{1/3} XX}{54} - \frac{5\left(1240\sqrt{7}-1700\right)^{2/3}\left(2\sqrt{7}+1\right) XX^2}{69984} + \left(-\frac{35}{10368} + \frac{35\sqrt{7}}{5184}\right) XX^3 + \frac{1645\left(1240\sqrt{7}-1700\right)^{1/3} XX^4}{4478976} \quad \textbf{(9.2.1)}$$

In eqVcritU, we replace U by its development and nu by its expression in termes of Usubc. We also know that for U=Uc, the radius of convergence in y is 2, corresponding to V=1, so that we set V=1+VV.

> *eqVVpcsing* := *convert*(*map*(*factor*, *series*(*numer*(*simplify*(*subs*(*U* = *Ucsing4*, nu = *nuc*, *V* = 1 + *VV*, *eqVcritU*))), *XX*, 4)), *polynom*);

$$eqVVpcsing := 4458050224128\left(-14+\sqrt{7}\right) VV^2 \left(VV^2+6\,VV+6\right) + 743008370688\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}\left(VV^4+6\,VV^3+42\,VV^2+72\,VV+36\right) XX - 30958682112\,50^{1/3}\left(-14+\sqrt{7}\right)\left(17\,VV^4+102\,VV^3+570\,VV^2+936\,VV+468\right) XX^2 - 58047528960\left(-14+\sqrt{7}\right)\left(11\,VV^4+66\,VV^3+302\,VV^2+472\,VV+236\right) XX^3 \quad \textbf{(9.2.2)}$$

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

> *algeqtoseries*(*eqVVpcsing*, *XX*, *VV*, 1);

$$\left[RootOf\left(\_Z^2+6\,\_Z+6\right)+O(XX),\ RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\,\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\sqrt{XX}+O(XX)\right] \quad \textbf{(9.2.3)}$$

The right branch is the second one, and we can compute a full expansion

> op(2, algeqtoseries(eqVVpcsing, XX, VV, 3));

$$RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\sqrt{XX}+\left(\frac{\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}}{27}+\frac{\left(1240\sqrt{7}-1700\right)^{1/3}}{54}\right)XX+\frac{1}{96}\left(5\,50^{2/3}\,RootOf\left(-2\left(\frac{2\,50^{2/3}\sqrt{7}}{5}-\frac{50^{2/3}}{5}\right)\sqrt{7}+27\_Z^2-\frac{2\,50^{2/3}\sqrt{7}}{5}+\frac{50^{2/3}}{5}\right)XX^{3/2}\right)+O(XX^2) \quad \mathbf{(9.2.4)}$$

> Vpcsing := 1 + RootOf(-2 (1240 √7 − 1700)^(1/3) √7 + 27 _Z^2 − (1240 √7 − 1700)^(1/3)) √XX + ((1240 √7 − 1700)^(1/3)/54 + (1240 √7 − 1700)^(1/3) √7/27) XX + (1/2592 (25 RootOf(-2 (1240 √7 − 1700)^(1/3) √7 + 27 _Z^2 − (1240 √7 − 1700)^(1/3)) (1240 √7 − 1700)^(1/3)) + 1/1296 (25 RootOf(-2 (1240 √7 − 1700)^(1/3) √7 + 27 _Z^2 − (1240 √7 − 1700)^(1/3)) (1240 √7 − 1700)^(1/3) √7)) XX^(3/2) :

We can now plug this expansion in the expression of y in terms of U and V, we get the desired asymptotics (recall that XX=(1-w/rhoc)^1/3)

> ypcsing := map(simplify, series(subs(V = Vpcsing, U = Ucsing4, nu = nuc, yUV), XX, 2));

$$ypcsing := 2+\frac{1}{\left(-5+\sqrt{7}\right)^2\left(7+\sqrt{7}\right)^2\left(7+13\sqrt{7}\right)}\left(\left(-812\sqrt{7}+784\right)XX^{3/2}\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\right)+O(XX^2) \quad \mathbf{(9.2.5)}$$

We only need to check that the leading coefficient is not zero :

> evalf(allvalues(1/((-5 + √7)^2 (7 + √7)^2 (7 + 13 √7)) ((-812 √7 + 784) (1240 √7 − 1700)^(1/3) RootOf(-2 (1240 √7 − 1700)^(1/3) √7 + 27 _Z^2 − (1240 √7 − 1700)^(1/3)))))

$$-1.226670602,\ 1.226670602 \quad \mathbf{(9.2.6)}$$

We now compute the expansion for the negative singular value of y. We already know from **??** that y(V) is increasing in V for V between -2+sqrt(3) and 1 so that the corresponding value of V is -2+sqrt(3). We start from the same expression as above, only replacing V by -2+sqrt(3) + VV.

> eqVVmcsing := convert(map(factor, series(numer(simplify(subs(U = Ucsing4, nu = nuc, V = -2 + sqrt(3) + VV, eqVcritU))), XX, 4)), polynom);

$$eqVVmcsing := 4458050224128\left(-14+\sqrt{7}\right)\left(2\sqrt{3}+VV\right)VV\left(VV-3+\sqrt{3}\right)^2 \quad \mathbf{(9.2.7)}$$

$$+ 743008370688 \left(1240\sqrt{7} - 1700\right)^{1/3} \sqrt{7} \left(4\sqrt{3}\, VV^3 + VV^4 - 18\sqrt{3}\, VV^2 - 6\, VV^3 + 96\sqrt{3}\, VV + 60\, VV^2 - 144\sqrt{3} - 180\, VV + 252\right) XX - 30958682112\, 50^{1/3} \left(-14 + \sqrt{7}\right) \left(68\sqrt{3}\, VV^3 + 17\, VV^4 - 306\sqrt{3}\, VV^2 - 102\, VV^3 + 1344\sqrt{3}\, VV + 876\, VV^2 - 1872\sqrt{3} - 2484\, VV + 3276\right) XX^2 - 58047528960 \left(-14 + \sqrt{7}\right) \left(44\sqrt{3}\, VV^3 + 11\, VV^4 - 198\sqrt{3}\, VV^2 - 66\, VV^3 + 736\sqrt{3}\, VV + 500\, VV^2 - 944\sqrt{3} - 1340\, VV + 1652\right) XX^3$$

> *algeqtoseries*(*eqVVmcsing*, *XX*, *VV*, 1);

$$\left[-2\sqrt{3} + O(XX),\ 3 - \sqrt{3} + O\left(\sqrt{XX}\right),\ \left(-\frac{\left(1240\sqrt{7} - 1700\right)^{1/3}}{54} + \frac{\left(1240\sqrt{7} - 1700\right)^{1/3}\sqrt{3}}{81} - \frac{\left(1240\sqrt{7} - 1700\right)^{1/3}\sqrt{7}}{27} + \frac{2\sqrt{7}\left(1240\sqrt{7} - 1700\right)^{1/3}\sqrt{3}}{81}\right) XX + O\left(XX^2\right)\right] \quad \textbf{(9.2.8)}$$

The right branch is the third one without the constant term and we can compute a full expansion :

> *map*(*simplify*, *op*(3, *algeqtoseries*(*eqVVmcsing*, *XX*, *VV*, 4)));

$$\frac{\left(1240\sqrt{7} - 1700\right)^{1/3}\left(2\sqrt{3} - 3\right)\left(2\sqrt{7} + 1\right)}{162} XX + \frac{1}{1296}\frac{1}{\left(2\sqrt{3} - 3\right)\left(-14 + \sqrt{7}\right)}\left(-2\left(14 + \sqrt{7}\right)\left(-2 + \sqrt{3}\right)\left(1240\sqrt{7} - 1700\right)^{2/3} - 1404\left(\sqrt{3} - \frac{7}{4}\right)\left(-14 + \sqrt{7}\right) 50^{1/3}\right) XX^2 + \frac{1}{1152}\frac{1}{\left(2\sqrt{3} - 3\right)^2\left(-14 + \sqrt{7}\right)^2}\left(728\left(-\frac{1}{2} + \sqrt{7}\right) 50^{1/3}\left(\sqrt{3} - \frac{24}{13}\right)\left(1240\sqrt{7} - 1700\right)^{1/3} - 510440\left(\sqrt{7} - \frac{29}{4}\right)\left(\sqrt{3} - \frac{6303}{3646}\right)\right) XX^3 + \frac{1}{4608}\frac{1}{\left(2\sqrt{3} - 3\right)^3\left(-14 + \sqrt{7}\right)^3}\left(-10614870\left(\sqrt{3} - \frac{9062}{5229}\right)\left(\sqrt{7} - \frac{28}{29}\right)\left(1240\sqrt{7} - 1700\right)^{1/3} - 151704\left(\sqrt{3} - \frac{4693}{2709}\right)\left(-14 + \sqrt{7}\right) 50^{1/3}\left(1240\sqrt{7} - 1700\right)^{2/3} - 1956955\, 50^{2/3}\left(\sqrt{3} - \frac{438}{253}\right)\left(\sqrt{7} - \frac{434}{85}\right)\right) XX^4 + O\left(XX^5\right) \quad \textbf{(9.2.9)}$$

> $Vmcsing := -2 + \mathrm{sqrt}(3) + \frac{\left(2\sqrt{7} + 1\right)\left(1240\sqrt{7} - 1700\right)^{1/3}\left(2\sqrt{3} - 3\right)}{162} XX$

$$+\frac{1}{2592}\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)\left(-14+\sqrt{7}\right)}\left(-2\left(14+\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(1240\sqrt{7}-1700\right)^{2/3}-1404\,50^{1/3}\left(\sqrt{3}-\frac{7}{4}\right)\left(-14+\sqrt{7}\right)\right)XX^2$$

$$+\frac{1}{4608}\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)^2\left(-14+\sqrt{7}\right)^2}\left(728\left(\sqrt{7}-\frac{1}{2}\right)50^{1/3}\left(\sqrt{3}-\frac{24}{13}\right)\left(1240\sqrt{7}-1700\right)^{1/3}-510440\left(\sqrt{3}-\frac{6303}{3646}\right)\left(\sqrt{7}-\frac{29}{4}\right)\right)XX^3$$

$$+\frac{1}{36864}\frac{1}{\left(\sqrt{3}-\frac{3}{2}\right)^3\left(-14+\sqrt{7}\right)^3}\left(-10614870\left(\sqrt{7}-\frac{28}{29}\right)\left(\sqrt{3}-\frac{9062}{5229}\right)\left(1240\sqrt{7}-1700\right)^{1/3}-151704\,50^{1/3}\left(\sqrt{3}-\frac{4693}{2709}\right)\left(-14+\sqrt{7}\right)\left(1240\sqrt{7}-1700\right)^{2/3}-1956955\left(\sqrt{7}-\frac{434}{85}\right)\left(\sqrt{3}-\frac{438}{253}\right)50^{2/3}\right)XX^4:$$

We can now plug this expansion in the expression of y in terms of U and V, we get:

> ymcsing := map(simplify, series(subs(V = Vmcsing, U = Ucsing4, nu = nuc, yUV), XX, 5));

$$ymcsing := -\frac{4\left(-\frac{1}{2}+\sqrt{7}\right)\left(7+\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)}{\left(24\sqrt{7}+231\right)\sqrt{3}-38\sqrt{7}-413}+\frac{27440}{3}\left(\left(362\sqrt{3}-627\right)\left(78806\sqrt{7}-181693\right)\right)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^2\left(7+\sqrt{7}\right)^2\left(-5+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^2\left(2\sqrt{3}-3\right)^2\right)XX^3$$
$$+\frac{1}{8}\left(\left(\left(330082753200510240\sqrt{3}-571720105508325550\right)\sqrt{7}-1024391999457256185\sqrt{3}+1774299006738717515\right)\left(1240\sqrt{7}-1700\right)^{1/3}+15704812680490206\left(\left(\sqrt{3}-\frac{33518496652}{19351912887}\right)\sqrt{7}-\frac{8999600785\sqrt{3}}{4300425086}+\frac{140289893863}{38703825774}\right)50^{1/3}\left(1240\sqrt{7}-1700\right)^{2/3}+\left(-586902892127647914\,50^{2/3}\sqrt{3}+1016545638114735092\,50^{2/3}\right)\sqrt{7}+1392843856906107333\,50^{2/3}\sqrt{3}-2412476352951155491\,50^{2/3}\right)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^3\left(7+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^3\left(2\sqrt{3}\right.\right.$$ **(9.2.10)**

$- 3)^3 \left(-5+\sqrt{7}\right)^4\right) XX^4 + O\left(XX^5\right)$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by evaluating the values of the coefficients:

```
> evalf(coeff(ymcsing, XX, 0));
  evalf(coeff(ymcsing, XX, 3));
  evalf(coeff(ymcsing, XX, 4));
```

$$-4.702978452$$
$$-1.459540327$$
$$22.57932079 \qquad \textbf{(9.2.11)}$$

For nu>nu_c

For nu>nu_c, we cannot replace nu by its value in terms of Usupc and have to rely on the rational parametrization by K. In this case the parametrization of y by K and V *

```
> yUVsupc;
```

$$-\left(8\left(K\sim^3+3K\sim^2+9K\sim+11\right)\left(K\sim+1\right)V(V+1)\right)\Big/\left(K\sim^4V^3-7K\sim^4V^2-K\sim^4V-40K\sim^3V^2-6K\sim^2V^3-K\sim^4+8K\sim^3V-110K\sim^2V^2+14K\sim^2V-136K\sim V^2+9V^3+6K\sim^2-24K\sim V-55V^2-33V-9\right) \qquad \textbf{(9.3.1)}$$

```
> Usupcsing;
```

$$-\frac{K\sim^2-3}{2\left(3K\sim+5\right)}+\mathit{RootOf}\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\right)XX-\left(\left(K\sim^2-3\right)\left(K\sim^2+8K\sim+13\right)XX^2\left(9K\sim^4+14K\sim^3-18K\sim^2-10K\sim+29\right)\left(K\sim+1\right)\right)\Big/\left(144\left(3K\sim+5\right)\left(3K\sim^2+4K\sim-1\right)^2\left(2+K\sim\right)\right)+\frac{1}{216\left(3K\sim^2+4K\sim-1\right)^3\left(2+K\sim\right)}\left(5\left(K\sim^2+8K\sim+13\right)\left(9K\sim^6+40K\sim^5+43K\sim^4-48K\sim^3-97K\sim^2+24K\sim+77\right)\mathit{RootOf}\left(\left(1296K\sim^4+6048K\sim^3+8928K\sim^2+3360K\sim-1200\right)\_Z^2-K\sim^8-10K\sim^7-24K\sim^6+26K\sim^5+158K\sim^4+114K\sim^3-192K\sim^2-306K\sim-117\right)XX^3\right) \qquad \textbf{(9.3.2)}$$

In eqVcritU, we replace U by its development and nu by its expression in termes of K. We also know that for U=Uc, the radius of convergence coresponds to V=VK11, so that we set V=VK11+VV.

```
> eqVVpsupsing := convert(map(factor, series(numer(simplify(subs(U = Usupcsing, nu
      = nusupK, V = VK11 + VV, eqVcritU))), XX, 4)), polynom) :
```

Since VV is a root of this expression, we can compute its expansion in terms of XX via the command algeqtoseries. We first compute the first terms to identify the right branch (by definifition, VV has no constant term)

> *simplify*(*algeqtoseries*(*eqVVpsupsing*, *XX*, *VV*, 1));

$$\Big[RootOf\Big(\big(K\sim^6 - 9\,K\sim^4 + 27\,K\sim^2 - 27\big)\,\_Z^3 + \big(-6\,K\sim^6 \qquad \textbf{(9.3.3)}$$

$$+ 4\,K\sim^4\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 16\,K\sim^5 + 22\,K\sim^4$$

$$- 24\,K\sim^2\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 96\,K\sim^3 + 30\,K\sim^2$$

$$+ 36\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 144\,K\sim - 126\big)\,\_Z^2 + \big(24\,K\sim^6$$

$$- 18\,K\sim^4\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 120\,K\sim^5$$

$$- 48\,K\sim^3\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 152\,K\sim^4$$

$$+ 12\,K\sim^2\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 208\,K\sim^3$$

$$+ 144\,K\sim\,\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 664\,K\sim^2$$

$$+ 126 \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 456 K\sim - 24\Big) \_Z - 36 K\sim^6$$

$$+ 24 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 288 K\sim^5$$

$$+ 112 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 956 K\sim^4$$

$$+ 208 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 1600 K\sim^3$$

$$+ 176 K\sim \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 1292 K\sim^2$$

$$+ 56 \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1} - 288 K\sim + 140\Big) + \mathrm{O}(XX),$$

$$-\Bigg(64 \left(K\sim + \frac{5}{3}\right) \Bigg(\sqrt{K\sim^2 + 4 K\sim + 5}\ (K\sim + 1)^2 \sqrt{3 K\sim^2 + 4 K\sim - 1} - \frac{7 K\sim^4}{4}$$
$$- 8 K\sim^3 - \frac{27 K\sim^2}{2} - 8 K\sim + \frac{1}{4}\Bigg) (K\sim + 1)\ (2 + K\sim)\ \mathit{RootOf}\Big(\big(1296 K\sim^4 + 6048 K\sim^3$$
$$+ 8928 K\sim^2 + 3360 K\sim - 1200\big) \_Z^2 - K\sim^8 - 10 K\sim^7 - 24 K\sim^6 + 26 K\sim^5 + 158 K\sim^4$$
$$+ 114 K\sim^3 - 192 K\sim^2 - 306 K\sim - 117\Big)\Bigg) \Bigg/ \Bigg(\Bigg(\Bigg(\sqrt{K\sim^2 + 4 K\sim + 5}\ (K\sim$$
$$+ 1)^2 \sqrt{3 K\sim^2 + 4 K\sim - 1} - \frac{3 K\sim^4}{2} - 8 K\sim^3 - 15 K\sim^2 - 8 K\sim + \frac{5}{2}\Bigg) \left(K\sim^2 + \frac{8}{3} K\sim\right.$$
$$\left. + \frac{7}{3}\right) \left(K\sim^2 - 3\right)^2\Bigg) XX + \mathrm{O}(XX^2)\Bigg]$$

The right branch is the second one, and we can compute a full expansion

> $Vpsupsing := VK11 + collect(convert(op(2, algeqtoseries(eqVVpsupsing, XX, VV, 3)), polynom), XX, factor)$:

We can now plug this expansion in the expression of y in terms of U and V, we get:

> $ypsupsing := collect(convert(map(expand, map(rationalize, map(simplify, series(subs(V = Vpsupsing, U = Usupcsing, \mathrm{nu} = nusupK, yUV), XX, 2)))), polynom), XX, factor)$;

$$ypsupsing := -\Big(16\,(3\,K\sim+5)\,(3\,K\sim^2+8\,K\sim+7)\,(K\sim^3+3\,K\sim^2+9\,K\sim+11)\,\Big(36\,K\sim^{10} + 31\,K\sim^8\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 384\,K\sim^9 + 248\,K\sim^7\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 1996\,K\sim^8 + 844\,K\sim^6\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 6624\,K\sim^7 + 1544\,K\sim^5\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 14952\,K\sim^6 + 1818\,K\sim^4\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 22176\,K\sim^5 + 2088\,K\sim^3\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 19160\,K\sim^4 + 2508\,K\sim^2\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 6560\,K\sim^3 + 1816\,K\sim\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} - 1804\,K\sim^2 + 479\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} - 1184\,K\sim+220\Big)\,RootOf\big((1296\,K\sim^4 + 6048\,K\sim^3 + 8928\,K\sim^2 + 3360\,K\sim - 1200)\,\_Z^2 - K\sim^8 - 10\,K\sim^7 - 24\,K\sim^6 + 26\,K\sim^5 + 158\,K\sim^4 + 114\,K\sim^3 - 192\,K\sim^2 - 306\,K\sim - 117\big)\,XX\Big)\Big/\Big((K\sim^2+4\,K\sim+5)\,(23\,K\sim^6 + 184\,K\sim^5 + 593\,K\sim^4 + 1008\,K\sim^3 + 989\,K\sim^2 + 568\,K\sim + 163)^2\,(K\sim^2-3)^2\Big) - \Big(4\,(K\sim+1)\,(K\sim^3+3\,K\sim^2+9\,K\sim+11)\,\Big(2\,K\sim^4 + 3\,K\sim^2\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 8\,K\sim^3 + 4\,K\sim\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 24\,K\sim^2 - \sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 40\,K\sim + 22\Big)\Big)\Big/\Big((23\,K\sim^6 + 184\,K\sim^5 + 593\,K\sim^4 + 1008\,K\sim^3 + 989\,K\sim^2 + 568\,K\sim + 163)\,(K\sim^2-3)\Big) \quad \textbf{(9.3.4)}$$

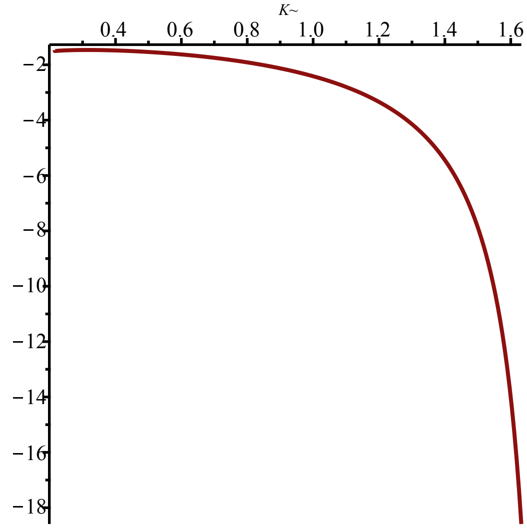
The coefficient in XX does not vanish:

> $fsolve\Big(36\,K\sim^{10} + 31\,K\sim^8\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 384\,K\sim^9 + 248\,K\sim^7\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 1996\,K\sim^8 + 844\,K\sim^6\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 6624\,K\sim^7 + 1544\,K\sim^5\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 14952\,K\sim^6 + 1818\,K\sim^4\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 22176\,K\sim^5 + 2088\,K\sim^3\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 19160\,K\sim^4 + 2508\,K\sim^2\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 6560\,K\sim^3$

$$+ 1816\,K\sim\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1804\,K\sim^2$$
$$+ 479\sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} - 1184\,K\sim + 220\Big)$$

$$-1.548583770 \quad \textbf{(9.3.5)}$$

> $plot(coeff(ypsupsing, XX, 1), K = Kc..Kinfini - 0.1);$



>

>

We now compute the expansion for the negative singular value of y. At t_\nu, the negative singularity is given by VK22:

> $eqVVmsupsing := convert(map(factor, series(numer(simplify(subs(U = Usupcsing, \mathrm{nu} = nusupK, V = VK22 + VV, eqVcritU))), XX, 4)), polynom) : degree(\%, XX);$

$$3 \quad \textbf{(9.3.6)}$$

> $algeqtoseries(eqVVmsupsing, XX, VV, 1);$

$$\Big[RootOf\Big(\big(K\sim^6 - 9\,K\sim^4 + 27\,K\sim^2 - 27\big)\,\_Z^3 + \big(-8\,K\sim^5\sqrt{2}\,\sqrt{2 + K\sim} + 6\,K\sim^6 \quad \textbf{(9.3.7)}$$

$$- 8\,K\sim^4\sqrt{2}\,\sqrt{2 + K\sim} + 16\,K\sim^5 + 48\,K\sim^3\sqrt{2}\,\sqrt{2 + K\sim} - 22\,K\sim^4$$

$$+ 48\,K\sim^2\sqrt{2}\,\sqrt{2 + K\sim} - 96\,K\sim^3 - 72\sqrt{2}\,\sqrt{2 + K\sim}\,K\sim - 30\,K\sim^2 - 72\sqrt{2}\,\sqrt{2 + K\sim}$$

$$+ 144\,K\sim + 126\big)\,\_Z^2 + \big(-36\,K\sim^5\sqrt{2}\,\sqrt{2 + K\sim} + 6\,K\sim^6 - 132\,K\sim^4\sqrt{2}\,\sqrt{2 + K\sim}$$

$$+ 72\,K\sim^5 - 72\,K\sim^3\sqrt{2}\,\sqrt{2 + K\sim} + 218\,K\sim^4 + 312\,K\sim^2\sqrt{2}\,\sqrt{2 + K\sim} + 80\,K\sim^3$$

$$+ 540\sqrt{2}\,\sqrt{2 + K\sim}\,K\sim - 574\,K\sim^2 + 252\sqrt{2}\,\sqrt{2 + K\sim} - 888\,K\sim - 402\big)\,\_Z$$

$- 24 K\sim^5 \sqrt{2} \sqrt{2+K\sim} - 184 K\sim^4 \sqrt{2} \sqrt{2+K\sim} + 96 K\sim^5 - 592 K\sim^3 \sqrt{2} \sqrt{2+K\sim}$

$+ 640 K\sim^4 - 976 K\sim^2 \sqrt{2} \sqrt{2+K\sim} + 1728 K\sim^3 - 824 \sqrt{2} \sqrt{2+K\sim}\, K\sim + 2368 K\sim^2$

$- 280 \sqrt{2} \sqrt{2+K\sim} + 1632 K\sim + 448\big) + O(XX), -\big(16\, RootOf\big(\big(1296 K\sim^4$
$+ 6048 K\sim^3 + 8928 K\sim^2 + 3360 K\sim - 1200\big)\, \_Z^2 - K\sim^8 - 10 K\sim^7 - 24 K\sim^6 + 26 K\sim^5$
$+ 158 K\sim^4 + 114 K\sim^3 - 192 K\sim^2 - 306 K\sim - 117\big)\, \big(-12 K\sim^5 \sqrt{2} \sqrt{2+K\sim} + 3 K\sim^6$
$- 104 K\sim^4 \sqrt{2} \sqrt{2+K\sim} + 59 K\sim^5 - 368 K\sim^3 \sqrt{2} \sqrt{2+K\sim} + 360 K\sim^4$
$- 656 K\sim^2 \sqrt{2} \sqrt{2+K\sim} + 1038 K\sim^3 - 580 \sqrt{2} \sqrt{2+K\sim}\, K\sim + 1583 K\sim^2$
$- 200 \sqrt{2} \sqrt{2+K\sim} + 1251 K\sim + 410\big)\big) \Big/ \big(3 K\sim^8 \sqrt{2} \sqrt{2+K\sim}$
$+ 20 K\sim^7 \sqrt{2} \sqrt{2+K\sim} - 12 K\sim^8 + 36 K\sim^6 \sqrt{2} \sqrt{2+K\sim} - 68 K\sim^7$
$- 52 K\sim^5 \sqrt{2} \sqrt{2+K\sim} - 76 K\sim^6 - 262 K\sim^4 \sqrt{2} \sqrt{2+K\sim} + 260 K\sim^5$
$- 228 K\sim^3 \sqrt{2} \sqrt{2+K\sim} + 724 K\sim^4 + 276 K\sim^2 \sqrt{2} \sqrt{2+K\sim} + 276 K\sim^3$
$+ 612 \sqrt{2} \sqrt{2+K\sim}\, K\sim - 996 K\sim^2 + 315 \sqrt{2} \sqrt{2+K\sim} - 1332 K\sim - 504\big)\, XX +$
$O(XX^2)\big]$

The right branch is the second one without the constant term and we can compute a full expansion :

```
> Vmsupsing := VK22 + collect(convert(map(expand, map(rationalize, map(simplify, op(2,
      algeqtoseries(eqVVmsupsing, XX, VV, 4))))), polynom), XX, factor) :
```

We can now plug this expansion in the expression of y in terms of U and V, we get:

```
> ymsupsing := collect(convert(map(expand, map(rationalize, map(simplify, series(subs(V
      = Vmsupsing, U = Usupcsing, nu = nusupK, yUV), XX, 4)))), polynom), XX, factor);
```

$ymsupsing := \big(4\, RootOf\big(\big(1296 K\sim^4 + 6048 K\sim^3 + 8928 K\sim^2 + 3360 K\sim - 1200\big)\, \_Z^2$
$- K\sim^8 - 10 K\sim^7 - 24 K\sim^6 + 26 K\sim^5 + 158 K\sim^4 + 114 K\sim^3 - 192 K\sim^2 - 306 K\sim$
$- 117\big)\, (3 K\sim + 5)\, (K\sim + 1)\, \big(K\sim^2 + 8 K\sim + 13\big)\, \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big)\, \big(3 K\sim^4$
$+ 12 K\sim^3 + 22 K\sim^2 + 28 K\sim + 19\big)\, \big(K\sim^6 \sqrt{2} + 6 K\sim^5 \sqrt{2} + 27 K\sim^4 \sqrt{2}$
$+ 16 K\sim^4 \sqrt{2+K\sim} + 108 K\sim^3 \sqrt{2} + 80 K\sim^3 \sqrt{2+K\sim} + 339 K\sim^2 \sqrt{2}$
$+ 240 K\sim^2 \sqrt{2+K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2+K\sim}\, K\sim + 377 \sqrt{2} + 352 \sqrt{2+K\sim}\big)$
$XX^3\big) \Big/ \big(3 \sqrt{2+K\sim}\, \big(3 K\sim^2 + 8 K\sim + 7\big)^3 \big(K\sim^2 - 3\big)^2 \big(K\sim^4 + 6 K\sim^3 + 30 K\sim^2$
$+ 62 K\sim + 45\big)^2\big) + \big(\big(K\sim^2 + 8 K\sim + 13\big)\, \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big)\, (K\sim + 1)^2\, \big(K\sim^6 \sqrt{2} + 6 K\sim^5$
$+ 339 K\sim^2 \sqrt{2} + 240 K\sim^2 \sqrt{2+K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2+K\sim}\, K\sim + 377 \sqrt{2}$
$+ 352 \sqrt{2+K\sim}\big)\, XX^2\big) \Big/ \big(2 \sqrt{2+K\sim}\, \big(K\sim^2 - 3\big)\, \big(K\sim^4 + 6 K\sim^3 + 30 K\sim^2$
$+ 62 K\sim + 45\big)^2 \big(3 K\sim^2 + 8 K\sim + 7\big)\big) + \big(2\, \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big)\, \big(K\sim^3$

$+ 4\sqrt{2}\sqrt{2+K\sim}\,K\sim + 3K\sim^2 + 8\sqrt{2}\sqrt{2+K\sim} + 9K\sim + 11))\big/((K\sim^2-3)\,(K\sim^4 + 6K\sim^3 + 30K\sim^2 + 62K\sim + 45))$

> *coeff*(*ymsupsing*, *XX*, 1);

$$0 \qquad \textbf{(9.3.9)}$$

which gives the desired asymptotics, provided that all coefficients are not 0. We check the latter fact by plotting the values of the coefficients:

> *coeff*(*ymsupsing*, *XX*, 3) ;

$$(4\,RootOf((1296K\sim^4 + 6048K\sim^3 + 8928K\sim^2 + 3360K\sim - 1200)\,\_Z^2 - K\sim^8 - 10K\sim^7 - 24K\sim^6 + 26K\sim^5 + 158K\sim^4 + 114K\sim^3 - 192K\sim^2 - 306K\sim - 117)\,(3K\sim + 5)\,(K\sim+1)\,(K\sim^2 + 8K\sim + 13)\,(K\sim^3 + 3K\sim^2 + 9K\sim + 11)\,(3K\sim^4 + 12K\sim^3 + 22K\sim^2 + 28K\sim + 19)\,(K\sim^6\sqrt{2} + 6K\sim^5\sqrt{2} + 27K\sim^4\sqrt{2} + 16K\sim^4\sqrt{2+K\sim} + 108K\sim^3\sqrt{2} + 80K\sim^3\sqrt{2+K\sim} + 339K\sim^2\sqrt{2} + 240K\sim^2\sqrt{2+K\sim} + 582K\sim\sqrt{2} + 464\sqrt{2+K\sim}\,K\sim + 377\sqrt{2} + 352\sqrt{2+K\sim}))\big/(3\sqrt{2+K\sim}\,(3K\sim^2 + 8K\sim + 7)^3\,(K\sim^2-3)^2\,(K\sim^4 + 6K\sim^3 + 30K\sim^2 + 62K\sim + 45)^2) \qquad \textbf{(9.3.10)}$$

> $fsolve((6561K\sim^{12} + 78732K\sim^{11} + 409698K\sim^{10} + 1193292K\sim^9 - 81K\sim^8\sqrt{3} + 2091447K\sim^8 - 558K\sim^7\sqrt{3} + 2242008K\sim^7 - 1464K\sim^6\sqrt{3} + 1680540K\sim^6 - 1538K\sim^5\sqrt{3} + 1774776K\sim^5 + 118K\sim^4\sqrt{3} + 2700207K\sim^4 + 998K\sim^3\sqrt{3} + 2637660K\sim^3 - 1072K\sim^2\sqrt{3} + 1168962K\sim^2 - 2758K\sim\sqrt{3} + 36540K\sim - 1421\sqrt{3} - 95175))$

$$-1.843693340,\ 0.2186477174 \qquad \textbf{(9.3.11)}$$

> *evalf*(*Kc*);

$$0.2152504369 \qquad \textbf{(9.3.12)}$$

Monochromatic simple boundary (Section 3.5)

Rational parametrization (Theorem 3.2)

The algebraic equation for Z(nu,t,t x) in terms of U

> $eqZtUx := numer\Bigg(factor\Bigg(subs\Bigg(w = wU,\ numer\Bigg(factor\Bigg(\frac{1}{w^{\frac{2}{3}}}\Bigg(simplify\Bigg(subs\Bigg(y = t\cdot x,\ Z1 = \frac{tZ1U}{t},\ Z2 = \frac{t2Z2U}{t^2},\ t = w^{\frac{1}{3}},\ Z$

$$= Zt, eqZ\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big) : indets(\%);$$

$$\{U, Zt, \nu, x\} \tag{10.1.1}$$

The parametrization for x and Z:

> XUW := subs(V = W, yUV·QtUV);

$$XUW := \left(8 \nu (1-2U) W (-\nu+1) \left(W^2 + \frac{2\left(5(\nu+1)U^2 - 2(3\nu+2)U + 2\nu\right)W}{U(U(\nu+1)-2)} - \frac{8(\nu+1)^2 U^3 - (11\nu+13)(\nu+1)U^2 + 2(\nu+3)(2\nu+1)U - 4\nu}{U(U(\nu+1)-2)(-\nu+1)}\right)\right) \Big/ \left((W+1)^2 \left(8(\nu+1)^2 U^3 - (11\nu+13)(\nu+1)U^2 + 2(\nu+3)(2\nu+1)U - 4\nu\right)\right) \tag{10.1.2}$$

> ZtUW := factor(subs(V = W, QtUV − 1));

$$ZtUW := -\Big(W\big(U^4 W^4 \nu^3 + 16 U^5 W^2 \nu^3 + U^4 W^4 \nu^2 + 19 U^4 W^3 \nu^3 + 48 U^5 W^2 \nu^2 + 96 U^5 W \nu^3 - U^4 W^4 \nu + 19 U^4 W^3 \nu^2 + 59 U^4 W^2 \nu^3 - 4 U^3 W^4 \nu^2 - 32 U^3 W^3 \nu^3 + 48 U^5 W^2 \nu + 288 U^5 W \nu^2 - 48 U^5 \nu^3 - U^4 W^4 - 19 U^4 W^3 \nu - 21 U^4 W^2 \nu^2 - 383 U^4 W \nu^3 - 52 U^3 W^3 \nu^2 - 296 U^3 W^2 \nu^3 + 12 U^2 W^3 \nu^3 + 16 U^5 W^2 + 288 U^5 W \nu - 144 U^5 \nu^2 - 19 U^4 W^3 - 219 U^4 W^2 \nu - 927 U^4 W \nu^2 + 88 U^4 \nu^3 + 4 U^3 W^4 + 32 U^3 W^3 \nu - 28 U^3 W^2 \nu^2 + 480 U^3 W \nu^3 + 4 U^2 W^4 \nu + 64 U^2 W^3 \nu^2 + 356 U^2 W^2 \nu^3 + 96 U^5 W - 144 U^5 \nu - 139 U^4 W^2 - 705 U^4 W \nu + 264 U^4 \nu^2 + 52 U^3 W^3 + 440 U^3 W^2 \nu + 1092 U^3 W \nu^2 - 56 U^3 \nu^3 - 4 U^2 W^4 - 48 U^2 W^3 \nu - 40 U^2 W^2 \nu^2 - 252 U^2 W \nu^3 - 24 U W^3 \nu^2 - 176 U W^2 \nu^3 - 48 U^5 - 161 U^4 W + 264 U^4 \nu + 172 U^3 W^2 + 704 U^3 W \nu - 176 U^3 \nu^2 - 28 U^2 W^3 - 368 U^2 W^2 \nu - 624 U^2 W \nu^2 + 12 U^2 \nu^3 + 24 U W^3 \nu + 88 U W^2 \nu^2 + 48 U W \nu^3 + 32 W^2 \nu^3 + 88 U^4 + 92 U^3 W - 184 U^3 \nu - 60 U^2 W^2 - 352 U^2 W \nu + 56 U^2 \nu^2 + 104 U W^2 \nu + 200 U W \nu^2 - 32 W^2 \nu^2 - 64 U^3 - 20 U^2 W + 60 U^2 \nu + 72 U W \nu - 8 U \nu^2 - 32 W \nu^2 + 16 U^2 - 8 U \nu\big)\Big) \Big/ \Big(U (U\nu + U - 2) (W+1)^3 \big(8 U^3 \nu^2 + 16 U^3 \nu - 11 U^2 \nu^2 + 8 U^3 - 24 U^2 \nu + 4 U \nu^2 - 13 U^2 + 14 U \nu + 6 U - 4\nu\big)\Big) \tag{10.1.3}$$

> simplify(subs(Zt = ZtUW, x = XUW, eqZtUx));

$$0 \tag{10.1.4}$$

Asymptotic expansion in x of Z(nu ,tnu , tnu x) (Lemma 3.16)

nu < nuc

> $XUWsub := factor(subs(\mathrm{nu} = nuUsub, XUW));$

$$XUWsub := -\frac{1}{(-2+3U)\left(6U^2-10U+3\right)(W+1)^2}\left(24(U-1)W\left(6U^2W^2 - 12U^2W - 6UW^2 - 6U^2 + 12UW + W^2 + 10U - 2W - 3\right)\right) \quad \textbf{(10.2.1.1)}$$

> $ZtUWsub := factor(subs(\mathrm{nu} = nuUsub, ZtUW));$

$$ZtUWsub := \frac{1}{(-2+3U)\left(6U^2-10U+3\right)(W+1)^3}\left(\left(18U^3W^4 - 162U^3W^3 - 30U^2W^4 + 198U^3W^2 + 294U^2W^3 + 15UW^4 + 90U^3W - 330U^2W^2 - 159UW^3 - 2W^4 - 270U^2W + 161UW^2 + 22W^3 + 219UW - 18W^2 + 12U - 46W - 4\right)W\right) \quad \textbf{(10.2.1.2)}$$

Critical points in W of XUWsub:

> $factor(diff(XUWsub, W));$

$$-\frac{1}{(-2+3U)\left(6U^2-10U+3\right)(W+1)^3}\left(24(U-1)(W-1)\left(6U^2W^2 + 24U^2W - 6UW^2 + 6U^2 - 24UW + W^2 - 10U + 4W + 3\right)\right) \quad \textbf{(10.2.1.3)}$$

The critical point corresponding to the radius of convergence will be W=1. We want to rule out the roots of the polynomial of degree 2:

> $BadPol := collect\left(6U^2W^2 + 24U^2W - 6UW^2 + 6U^2 - 24UW + W^2 - 10U + 4W + 3, W, factor\right); Wsub1, Wsub2 := solve(BadPol, W);$

$$BadPol := \left(6U^2-6U+1\right)W^2 + \left(24U^2-24U+4\right)W + 6U^2 - 10U + 3$$

$$Wsub1, Wsub2 := \frac{-12U^2 + \sqrt{108U^4 - 192U^3 + 108U^2 - 20U + 1} + 12U - 2}{6U^2 - 6U + 1}, -\frac{12U^2 + \sqrt{108U^4 - 192U^3 + 108U^2 - 20U + 1} - 12U + 2}{6U^2 - 6U + 1} \quad \textbf{(10.2.1.4)}$$

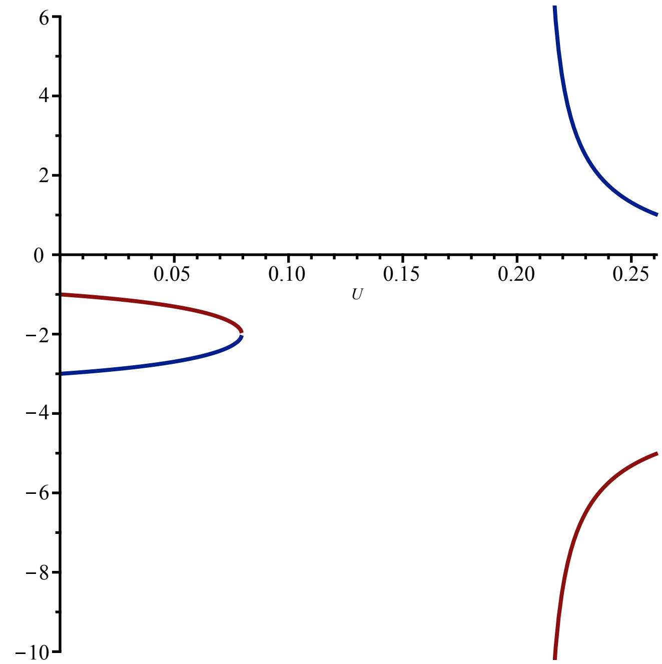
We first have to check that these two roots are never in [0,1]. (see Chen-Turunen prop 21)

> $factor(discrim(BadPol, W)); fsolve(\%);$

$$4\left(18U^2 - 14U + 1\right)\left(6U^2 - 6U + 1\right)$$

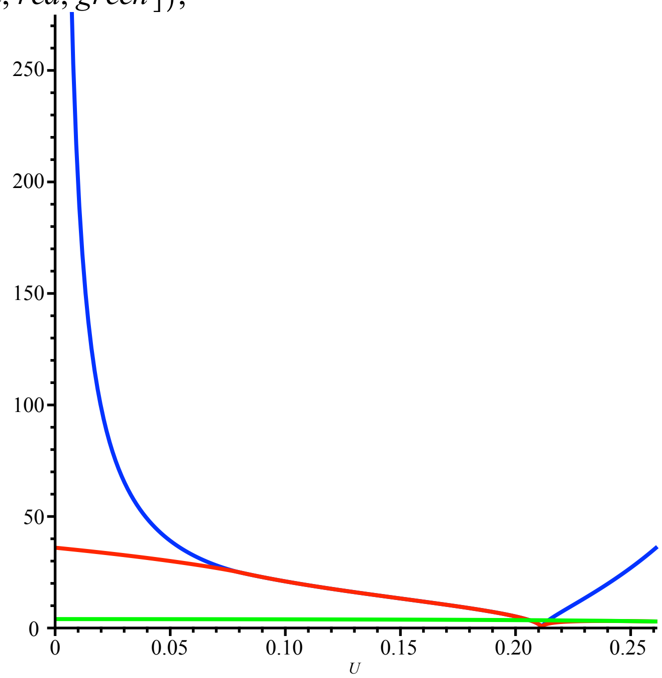
$$0.07956864651, 0.2113248654, 0.6982091313, 0.7886751346 \quad \textbf{(10.2.1.5)}$$

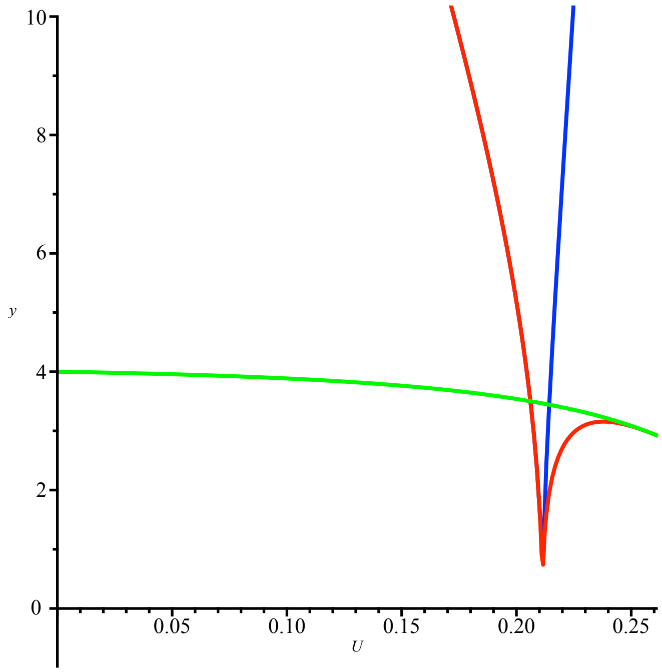
> $plot(\{Wsub1, Wsub2\}, U = 0..Uc);$



The radius of convergence is indeed given by W = 1. We then have to check that the roots of the polynomial of degree 2 don't give other dominant singularities. We can directly excluse them when they are real since none of them are in (-1,1) in this case ans \hat X is bijective on this interval. When the roots are imaginary, only one value of U gives |\hat X(Wi)| = \hat X(1):

```
> plot([abs(subs(W = Wsub1, XUWsub)), abs(subs(W = Wsub2, XUWsub)), subs(W = 1 ,
    XUWsub)], U = 0 ..Uc, color = [blue, red, green]); plot({abs(subs(W = Wsub1,
    XUWsub)), abs(subs(W = Wsub2, XUWsub)), subs(W = 1 , XUWsub)}, U = 0 ..Uc, y = -1
    ..10, color = [blue, red, green]);
```





We cannot compute explicitely the corresponding value of U

> factor(rationalize(factor(subs(W = Wsub1, XUWsub)·subs(W = Wsub2, XUWsub) − subs(W = 1, XUWsub)^2))); fsolve(243 U^6 − 1026 U^5 + 1686 U^4 − 1364 U^3 + 569 U^2 − 116 U + 9);

$$-\frac{576\left(243 U^6-1026 U^5+1686 U^4-1364 U^3+569 U^2-116 U+9\right)(U-1)^2}{U(-2+3 U)^3\left(6 U^2-10 U+3\right)^2}$$

$$0.2060759672, 0.7835199713 \quad \textbf{(10.2.1.6)}$$

Fortunately, we can still prove that Q is non singular at the corresponding values of \hat X when U is close to 0.2. We start by computing the values of \hat X:

> Xsubbad1 := factor(expand(rationalize(factor(subs(W = Wsub1, XUWsub)))));
Xsubbad2 := factor(expand(rationalize(factor(subs(W = Wsub2, XUWsub)))));

$$Xsubbad1 := -\frac{1}{U(-2+3 U)^2\left(6 U^2-10 U+3\right)}\Big(12(U-1)\Big(-180 U^4 + 18\sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}\, U^2+288 U^3 - 14\sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}\, U-132 U^2 + \sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}+12 U+1\Big)\Big)$$

$$Xsubbad2 := \frac{1}{U(-2+3 U)^2\left(6 U^2-10 U+3\right)}\Big(12(U-1)\Big(180 U^4 + 18\sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}\, U^2-288 U^3 - 14\sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}\, U+132 U^2 + \sqrt{\left(18 U^2-14 U+1\right)\left(6 U^2-6 U+1\right)}-12 U-1\Big)\Big) \quad \textbf{(10.2.1.7)}$$

And we directly calculate the development of Z at these values (we only do one, the other is the complex conjugate).

> $eqZtUxsub := op(5, factor(subs(\mathrm{nu} = nuUsub, eqZtUx))) : indets(\%);$

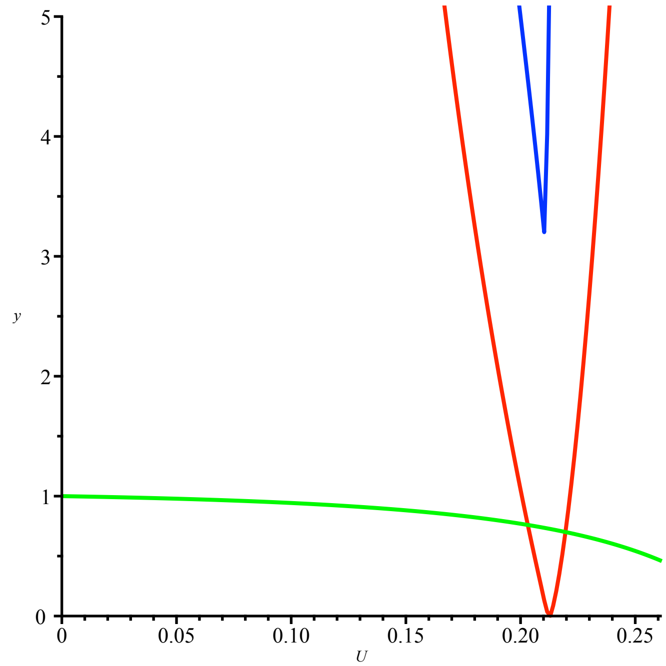
$$\{U, Zt, x\} \tag{10.2.1.8}$$

> $algeqtoseries(subs(x = Xsubbad1 \cdot (1 - XX), eqZtUxsub), XX, Zt, 1);$

$$\left[\left(-12528 U^6 + 1188 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^4 + 31320 U^5 - 1896 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^3 - 29268 U^4 + 924 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^2 + 12456 U^3 - 128 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U - 2328 U^2 + \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1} + 150 U + 1\right) \Big/ \left(2 \left(54 U^4 - 162 U^3 + 171 U^2 - 76 U + 12\right) U^2\right) + \mathrm{O}(XX), -\left(-3942 U^6 + 378 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^4 + 9450 U^5 - 564 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^3 - 8091 U^4 + 216 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U^2 + 2868 U^3 - 2 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1}\, U - 360 U^2 - 2 \sqrt{108 U^4 - 192 U^3 + 108 U^2 - 20 U + 1} + 18 U - 2\right) \Big/ \left(\left(54 U^4 - 162 U^3 + 171 U^2 - 76 U + 12\right) U^2\right) + \mathrm{O}\left(\sqrt{XX}\right)\right] \tag{10.2.1.9}$$

To decide which branch is the right one: if |\hat X| = \hat X(1), the the modulus of Zt at this value of x has to be smaller than the value at the radius of convergence (the series Zt has positive coefficients). We see that the right branch is the first one, which is non singular.

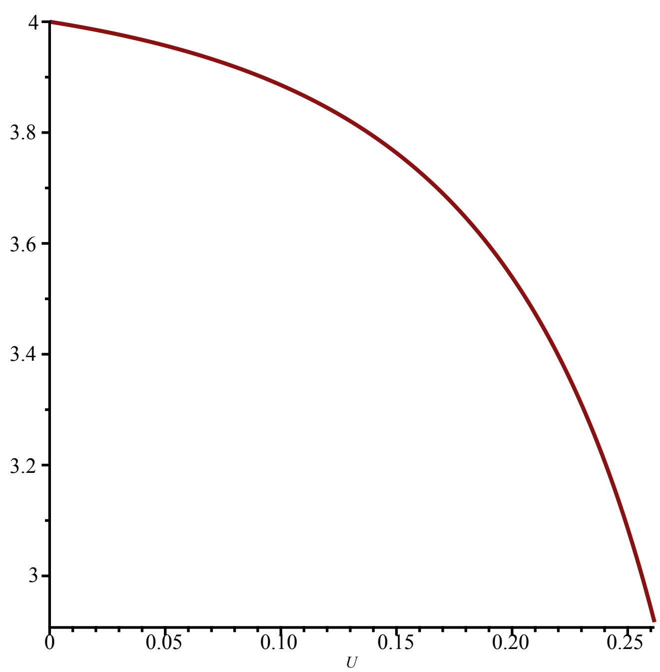
> $plot([\mathrm{abs}(convert(op(1, \mathbf{(10.2.1.9)}), polynom)), \mathrm{abs}(convert(op(2, \mathbf{(10.2.1.9)}), polynom)), subs(W = 1, ZtUWsub)], U = 0..Uc, y = 0..5, color = [red, blue, green]);$



Now we can produce the expansion of Zt at its unique dominant singularity:

> $xcritsub := factor(subs(W = 1, XUWsub)); plot(\%, U = 0 .. Uc);$

$$xcritsub := \frac{24\,(3\,U-1)\,(U-1)^2}{(-2+3\,U)\,(6\,U^2-10\,U+3)}$$



> $algeqtoseries(numer(xcritsub \cdot (1 - XX) - XUWsub), XX, W, 2)$

$$\left[-\frac{3\,U^2-4\,U+1}{6\,U^2-6\,U+1} + \frac{(9\,U^2-12\,U+4)\,U^2\,(3\,U^2-4\,U+1)}{(81\,U^4-180\,U^3+136\,U^2-40\,U+4)\,(6\,U^2-6\,U+1)}\,XX + \mathrm{O}(XX^2), 1 \right. \tag{10.2.1.10}$$

$$+ RootOf\left(\left(9 U^2 - 10 U + 2\right) \_Z^2 - 12 U^2 + 16 U - 4\right) \sqrt{XX} + O(XX)\Big]$$

> *Wsersubc* := *convert*(*op*(2, *algeqtoseries*(*numer*(*xcritsub*· (1 − *XX*) - *XUWsub*), *XX*, *W*, 4)), *polynom*);

$$Wsersubc := 1 + RootOf\left(\left(9 U^2 - 10 U + 2\right) \_Z^2 - 12 U^2 + 16 U - 4\right) \sqrt{XX} + \frac{2 (3 U - 1) \left(3 U^3 - 7 U^2 + 5 U - 1\right) XX}{\left(9 U^2 - 10 U + 2\right)^2} + \frac{1}{2 \left(9 U^2 - 10 U + 2\right)^3} \left(RootOf\left(\left(9 U^2 - 10 U + 2\right) \_Z^2 - 12 U^2 + 16 U - 4\right) \left(54 U^6 - 180 U^5 + 240 U^4 - 164 U^3 + 61 U^2 - 12 U + 1\right) XX^{3/2}\right) \quad \textbf{(10.2.1.11)}$$

> *Zplusubcser* := *collect*(*expand*(*rationalize*(*simplify*(*convert*(*series*(*subs*(*W* = *Wsersubc*, *ZtUWsub*), *XX*, 2), *polynom*)))), *XX*, *factor*);

$$Zplusubcser := -\frac{1}{\left(9 U^2 - 10 U + 2\right) \left(6 U^2 - 10 U + 3\right)} \left(2 (3 U - 1)^2 (U - 1)^2 RootOf\left(\left(9 U^2 - 10 U + 2\right) \_Z^2 - 12 U^2 + 16 U - 4\right) XX^{3/2}\right) - \frac{12 (3 U - 1) (U - 1)^2 XX}{(-2 + 3 U) \left(6 U^2 - 10 U + 3\right)} + \frac{18 U^3 - 42 U^2 + 31 U - 6}{(-2 + 3 U) \left(6 U^2 - 10 U + 3\right)} \quad \textbf{(10.2.1.12)}$$

The corresponding mean for mu:

$$> \quad factor\left(\frac{12 (U - 1)^2 (3 U - 1)}{(3 U - 2) \left(6 U^2 - 10 U + 3\right)} \cdot \frac{1}{1 + \frac{18 U^3 - 42 U^2 + 31 U - 6}{(3 U - 2) \left(6 U^2 - 10 U + 3\right)}}\right);$$

$$1 \quad \textbf{(10.2.1.13)}$$

Expansion of the singular term with NN = (nuc -nu):

$$> \quad algeqtoseries\left(subs\left(U = Uc - UU, \mathrm{nu} = 1 + \frac{\mathrm{sqrt}(7)}{7} - NN, numer(nuUsub - \mathrm{nu})\right), NN, UU, 1\right);$$

$$\left[\frac{1}{9} - \frac{2\sqrt{7}}{9} + O(NN), \left(-\frac{7}{243} + \frac{14\sqrt{7}}{243}\right) NN + O\left(NN^2\right)\right] \quad \textbf{(10.2.1.14)}$$

$$> \quad factor\left(expand\left(rationalize\left(simplify\left(convert\left(series\left(subs\left(U = Uc - \left(-\frac{7}{243} + \frac{14\sqrt{7}}{243}\right) NN, -\frac{2 (3 U - 1)^2 (U - 1)^2 RootOf\left(\left(9 U^2 - 10 U + 2\right) \_Z^2 - 12 U^2 + 16 U - 4\right)}{\left(9 U^2 - 10 U + 2\right) \left(6 U^2 - 10 U + 3\right)}\right),\right.\right.\right.\right.\right.\right.$$

$NN, 2\Big), polynom\Big)\Big)\Big)\Big)\Big);$

$$\frac{\sqrt{-392+406\sqrt{7}}\,\left(2\sqrt{7}+1\right)}{1715\,NN^{3/2}} \qquad \mathbf{(10.2.1.15)}$$

nu = nuc

> xcritcrit := expand(rationalize(subs(U = Uc, xcritsub)));

$$xcritcrit := \frac{4}{5}+\frac{4\sqrt{7}}{5} \qquad \mathbf{(10.2.2.1)}$$

with XX = 1-x/xcrit

> algeqtoseries(numer(xcritcrit · (1 − XX) - subs(U = Uc, XUWsub)), XX, W, 5);

$$\left[1+RootOf\left(\_Z^3+4\right)XX^{1/3}+\frac{RootOf\left(\_Z^3+4\right)^2 XX^{2/3}}{3}-\frac{XX}{3}-\frac{5\,RootOf\left(\_Z^3+4\right)XX^{4/3}}{81}+O\left(XX^{5/3}\right)\right] \qquad \mathbf{(10.2.2.2)}$$

> Wsercrit := convert(op(1, algeqtoseries(numer(xcritcrit · (1 − XX) - subs(U = Uc, XUWsub)), XX, W, 5)), polynom);

$$Wsercrit := 1+RootOf\left(\_Z^3+4\right)XX^{1/3}+\frac{RootOf\left(\_Z^3+4\right)^2 XX^{2/3}}{3}-\frac{XX}{3}-\frac{5\,RootOf\left(\_Z^3+4\right)XX^{4/3}}{81} \qquad \mathbf{(10.2.2.3)}$$

> expand(rationalize(simplify(convert(series(subs(W = Wsercrit, U = Uc, ZtUWsub), XX, 2), polynom))));

$$\frac{2\,XX^{5/3}\,RootOf\left(\_Z^3+4\right)^2}{15}-\frac{RootOf\left(\_Z^3+4\right)XX^{4/3}}{5}-\frac{2\sqrt{7}\,XX}{5}+\frac{2\sqrt{7}}{5}-\frac{3}{5} \qquad \mathbf{(10.2.2.4)}$$

The corresponding mean for the offspring distribution mu:

> $simplify\left(\frac{2\sqrt{7}}{5}\cdot\frac{1}{1+\frac{2\sqrt{7}}{5}-\frac{3}{5}}\right)$; rationalize(expand(%));

$$\frac{\sqrt{7}}{1+\sqrt{7}}$$

$$\frac{\sqrt{7}\left(-1+\sqrt{7}\right)}{6} \qquad \mathbf{(10.2.2.5)}$$

nu > nuc

> $XUWsupc := factor(subs(\mathrm{nu} = nusupK, U = UsupK, XUW));$

$$XUWsupc := -\frac{1}{\left(K\sim^2 + 8\,K\sim + 13\right)\,(W+1)^2\,\left(K\sim^2 - 3\right)^3}\left(8\,\left(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11\right)\,(K\sim + 1)\,W\,\left(K\sim^4\,W^2 - 2\,K\sim^4\,W - K\sim^4 - 24\,K\sim^3\,W - 6\,K\sim^2\,W^2 - 8\,K\sim^3 - 68\,K\sim^2\,W - 10\,K\sim^2 - 56\,K\sim\,W + 9\,W^2 + 24\,K\sim - 2\,W + 39\right)\right) \quad \mathbf{(10.2.3.1)}$$

We start by locating the critical points of \hat X:

> $factor(diff(XUWsupc, W));$

$$-\frac{1}{\left(K\sim^2 + 8\,K\sim + 13\right)\,(W+1)^3\,\left(K\sim^2 - 3\right)^3}\left(8\,(K\sim + 1)\,\left(K\sim^3 + 3\,K\sim^2 + 9\,K\sim + 11\right)\,\left(K\sim^2\,W - K\sim^2 - 8\,K\sim - 3\,W - 13\right)\,\left(K\sim^2\,W^2 + 4\,K\sim^2\,W + K\sim^2 + 8\,K\sim\,W - 3\,W^2 + 4\,W - 3\right)\right) \quad \mathbf{(10.2.3.2)}$$

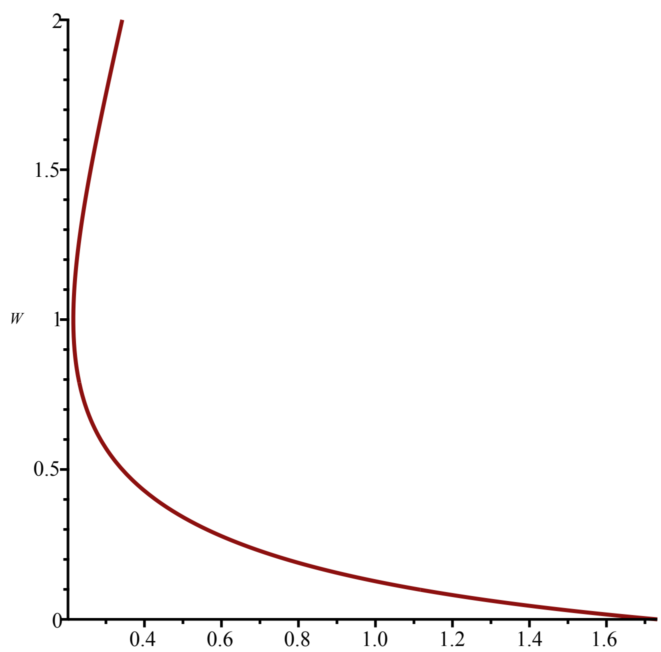
The root of the polynomial of degree 1 is < -1 (which we recall is the pole of \hat X):

> $factor\left(solve\left(K^2\,W - K^2 - 8\,K - 3\,W - 13, W\right) + 1\right);$

$$\frac{2\,\left(K\sim^2 + 4\,K\sim + 5\right)}{K\sim^2 - 3} \quad \mathbf{(10.2.3.3)}$$

The roots of the polynomial of degree 2 are positive and the smallest gives the radius of convergence.
There is no other non real singularity.

> $implicitplot\left(\left(\left(K^2\,W^2 + 4\,K^2\,W + K^2 + 8\,K\,W - 3\,W^2 + 4\,W - 3\right)\right), K = Kc\,..Kinfini, W = -2\,..2\right);$



> $WKsupccrit := -\frac{2\,K^2 + 4\,K - \sqrt{\left(K^2 + 4\,K + 5\right)\,\left(3\,K^2 + 4\,K - 1\right)} + 2}{K^2 - 3}:$

> $simplify(subs(W = WKsupccrit, ((K^2 W^2 + 4 K^2 W + K^2 + 8 K W - 3 W^2 + 4 W - 3))));$

$$0 \tag{10.2.3.4}$$

The corresponding radius of convergence:

> $XWsupccrit := simplify(subs(W = WKsupccrit, XUWsupc), symbolic);$

$$XWsupccrit := \Big(16 \left(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\right) \left(\sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 2 (K\sim + 1)^2\right) \left(K\sim^2 + 4 K\sim + 5\right) (K\sim + 1) \left(\left(3 K\sim^2 + 4 K\sim - 1\right)^{3/2} \sqrt{K\sim^2 + 4 K\sim + 5} - 5 K\sim^4 - 20 K\sim^3 - 26 K\sim^2 - 4 K\sim + 11\right)\Big) \Big/ \Big(\left(-K\sim^2 - 4 K\sim + \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} - 5\right)^2 \left(K\sim^2 + 8 K\sim + 13\right) \left(K\sim^2 - 3\right)^3\Big) \tag{10.2.3.5}$$

We compute the development of W at the radius of convergence. (XX = (1-x/xc)^1/2

> $simplify(algeqtoseries(numer(XWsupccrit \cdot (1 - XX^2) - XUWsupc), XX, W, 4));$

$$\left[\frac{\left(K\sim^2 + 4 K\sim + 5\right)^{3/2} \sqrt{3 K\sim^2 + 4 K\sim - 1} + K\sim^4 + 12 K\sim^3 + 34 K\sim^2 + 28 K\sim + 1}{\left(K\sim^2 - 3\right)^2} \right. \tag{10.2.3.6}$$

$$+ 44 \left(\left(\sqrt{K\sim^2 + 4 K\sim + 5}\, (K\sim + 1)^2 \sqrt{3 K\sim^2 + 4 K\sim - 1} + \frac{7 K\sim^4}{4} + 8 K\sim^3 + \frac{27 K\sim^2}{2} + 8 K\sim - \frac{1}{4}\right) \left(K\sim^2 + 4 K\sim + 5\right)\right.$$

$$\left(\frac{1}{11}\left(\sqrt{3 K\sim^2 + 4 K\sim - 1}\, \left(11 K\sim^4 + 40 K\sim^3 + 46 K\sim^2 + 8 K\sim - 13\right) \sqrt{K\sim^2 + 4 K\sim + 5}\right) - \frac{19 K\sim^6}{11} - \frac{120 K\sim^5}{11} - \frac{293 K\sim^4}{11} - \frac{304 K\sim^3}{11}\right.$$

$$- \frac{65 K\sim^2}{11} + \frac{72 K\sim}{11} + \frac{17}{11} \Big) \Big) \Big/ \Big(\big(K\sim^2 - 3 \big)^4 \big(3 K\sim^2 + 4 K\sim$$

$$- 1 \big) \big(3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 6 K\sim^2 - 20 K\sim - 22 \big) \Big) XX^2 +$$

$$\mathrm{O}\big(XX^4\big), \Big(27 \Big(K\sim^2 + \frac{16}{3} K\sim + \frac{23}{3} \Big)^2 \big(K\sim^2 - 3 \big)^2 \Big(K\sim^2 + \frac{4}{3} K\sim$$

$$- \frac{1}{3} \Big) XX\, RootOf\Big(\_Z^2 \big(9 K\sim^{10} + 60 K\sim^9 + 49 K\sim^8 - 464 K\sim^7 - 950 K\sim^6 + 936 K\sim^5$$

$$+ 3474 K\sim^4 + 432 K\sim^3 - 4131 K\sim^2 - 2052 K\sim + 621 \big) + 114 K\sim^{10}$$

$$- 66 K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1592 K\sim^9$$

$$- 744 K\sim^7 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 9634 K\sim^8$$

$$- 3480 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 32608 K\sim^7$$

$$- 8536 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 66164 K\sim^6$$

$$- 11228 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 78672 K\sim^5$$

$$- 6520 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 45700 K\sim^4$$

$$+ 520 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 672 K\sim^3$$

$$+ 1784 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 13830 K\sim^2$$

$$+ 142 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 3272 K\sim + 1210 \Big)$$

$$- 39 \sqrt{K\sim^2 + 4 K\sim + 5} \Big(\Big(XX^2 - \frac{9}{13} \Big) K\sim^8 + \Big(\frac{524 XX^2}{39} - \frac{108}{13} \Big) K\sim^7$$

$$+ \Big(\frac{972 XX^2}{13} - \frac{492}{13} \Big) K\sim^6 + \Big(\frac{8468 XX^2}{39} - \frac{2716}{39} \Big) K\sim^5 + \Big(\frac{13090 XX^2}{39}$$

$$+ \frac{178}{13}\Big) K\sim^4 + \left(\frac{9028\,XX^2}{39} + \frac{3228}{13}\right) K\sim^3 + \left(-\frac{68\,XX^2}{13} + \frac{12940}{39}\right) K\sim^2 + \Big($$

$$-\frac{212\,XX^2}{3} + \frac{1380}{13}\Big) K\sim - \frac{433\,XX^2}{39} - \frac{529}{13}\Big) \sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$+ 27 \left(K\sim^2 + \frac{16}{3} K\sim + \frac{23}{3}\right)^2 \left(K\sim^2 - 3\right)^2 \left(K\sim^2 + \frac{4}{3} K\sim - \frac{1}{3}\right) \mathrm{O}\left(XX^{5/2}\right)$$

$$+ \left(69\,XX^2 - 54\right) K\sim^{10} + \left(1100\,XX^2 - 756\right) K\sim^9 + \left(7581\,XX^2 - 4302\right) K\sim^8$$

$$+ \left(29232\,XX^2 - 11984\right) K\sim^7 + \left(67794\,XX^2 - 12748\right) K\sim^6 + \Big(92872\,XX^2$$

$$+ 16072\Big) K\sim^5 + \left(63418\,XX^2 + 65684\right) K\sim^4 + \left(1072\,XX^2 + 79408\right) K\sim^3 + \Big($$

$$-23527\,XX^2 + 39266\Big) K\sim^2 + \left(-5300\,XX^2 + 1932\right) K\sim + 4057\,XX^2 - 3174\Big) \Big/$$

$$\left(27 \left(K\sim^2 + \frac{16}{3} K\sim + \frac{23}{3}\right)^2 \left(K\sim^2 - 3\right)^2 \left(K\sim^2 + \frac{4}{3} K\sim - \frac{1}{3}\right)\right)\Bigg)\Bigg]$$

The singular branch is the second one:

```
> devWsupc := collect(convert(op(2, simplify(algeqtoseries(numer(XWsupccrit·(1 − XX^2)
      − XUWsupc), XX, W, 6))), polynom), XX, factor); degree(%, XX);
```

$$devWsupc := -\Big(\big(-224605 + 57053\,K\sim^{12} + 374712\,K\sim^{11} + 1600211\,K\sim^{10}$$

$$- 1460700\,K\sim^6 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$+ 285028\,K\sim^3 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$+ 346046\,K\sim^2 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$+ 161028\,K\sim \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 207\,K\sim^{14} + 5124\,K\sim^{13}$$

$$+ 4606300\,K\sim^9 + 8898169\,K\sim^8 + 10895504\,K\sim^7$$

$$- 489861\,K\sim^8 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 1051512\,K\sim^7 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 150316\,K\sim^9 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 3372\,K\sim^{11} \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 29506\,K\sim^{10} \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 171\,K\sim^{12} \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1} + 568996\,K\sim$$

$$- 1194264\,K\sim^5 \sqrt{K\sim^2 + 4\,K\sim + 5}\,\sqrt{3\,K\sim^2 + 4\,K\sim - 1}$$

$$- 355845 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 37589 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 655928 K\sim^{3} + 2157313 K\sim^{2}$$

$$+ 6883741 K\sim^{6} - 168324 K\sim^{5} - 2447289 K\sim^{4} \Big) \mathit{RootOf}\Big(\_Z^{2} \big(9 K\sim^{10} + 60 K\sim^{9}$$

$$+ 49 K\sim^{8} - 464 K\sim^{7} - 950 K\sim^{6} + 936 K\sim^{5} + 3474 K\sim^{4} + 432 K\sim^{3} - 4131 K\sim^{2}$$

$$- 2052 K\sim + 621\big) + 114 K\sim^{10} - 66 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 1592 K\sim^{9} - 744 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 9634 K\sim^{8}$$

$$- 3480 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 32608 K\sim^{7}$$

$$- 8536 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 66164 K\sim^{6}$$

$$- 11228 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 78672 K\sim^{5}$$

$$- 6520 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 45700 K\sim^{4}$$

$$+ 520 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 672 K\sim^{3}$$

$$+ 1784 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 13830 K\sim^{2}$$

$$+ 142 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 3272 K\sim + 1210\Big) XX^{3}\Big) \Big/ \Big(4 \big(K\sim^{2}$$

$$- 3\big) \left(3 K\sim^{2} + 16 K\sim + 23\right)^{3} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{2} \left(K\sim^{2} + 4 K\sim + 5\right)\Big)$$

$$+ \Big(\Big(69 K\sim^{10} - 39 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 1100 K\sim^{9} - 524 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim +}$$

$$- 8468 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 67794 K\sim^{6}$$

$$- 13090 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 92872 K\sim^{5}$$

$$- 9028 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 63418 K\sim^{4}$$

$$+ 204 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 1072 K\sim^{3}$$

$$+ 2756 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 23527 K\sim^{2}$$

$$+ 433 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 5300 K\sim + 4057\Big) XX^{2}\Big) \Big/$$

$$\Big(\left(3 K\sim^{2} + 16 K\sim + 23\right)^{2} \left(3 K\sim^{2} + 4 K\sim - 1\right) \left(K\sim^{2} - 3\right)^{2}\Big) + \mathit{RootOf}\Big(\_Z^{2} \big(9 K\sim^{10}$$

$$+ 60 K\sim^{9} + 49 K\sim^{8} - 464 K\sim^{7} - 950 K\sim^{6} + 936 K\sim^{5} + 3474 K\sim^{4} + 432 K\sim^{3}$$

$$- 4131 K\sim^{2} - 2052 K\sim + 621\big) + 114 K\sim^{10}$$

$$- 66 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 1592 K\sim^{9}$$

$$- 744 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 9634 K\sim^{8}$$

$$- 3480 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 32608 K\sim^{7}$$

$$- 8536 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 66164 K\sim^6$$
$$- 11228 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 78672 K\sim^5$$
$$- 6520 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 45700 K\sim^4$$
$$+ 520 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 672 K\sim^3$$
$$+ 1784 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 13830 K\sim^2$$
$$+ 142 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 3272 K\sim + 1210\Big) XX$$
$$- \frac{2 K\sim^2 + 4 K\sim - \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 2}{K\sim^2 - 3}$$

3 **(10.2.3.7)**

> *Zplussupcser* := *collect*(*expand*(*rationalize*(*convert*(*simplify*(*series*(*subs*(*W* = *devWsupc*, *subs*(nu = *nusupK*, *U* = *UsupK*, *ZtUW*)), *XX*, 4)), *polynom*))), *XX*, *factor*); *degree*(%, *XX*);

$$Zplussupcser := \Big(8 \left(K\sim^2 + 4 K\sim + 5\right) \Big(3 K\sim^6 + 40 K\sim^5$$
$$+ 4 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 213 K\sim^4$$
$$+ 32 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 560 K\sim^3$$
$$+ 84 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 721 K\sim^2$$
$$+ 72 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 360 K\sim - 1\Big) RootOf\Big(\_Z^2 \big(9 K\sim^{10}$$
$$+ 60 K\sim^9 + 49 K\sim^8 - 464 K\sim^7 - 950 K\sim^6 + 936 K\sim^5 + 3474 K\sim^4 + 432 K\sim^3$$
$$- 4131 K\sim^2 - 2052 K\sim + 621\big) + 114 K\sim^{10}$$
$$- 66 K\sim^8 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 1592 K\sim^9$$
$$- 744 K\sim^7 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 9634 K\sim^8$$
$$- 3480 K\sim^6 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 32608 K\sim^7$$
$$- 8536 K\sim^5 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 66164 K\sim^6$$
$$- 11228 K\sim^4 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 78672 K\sim^5$$
$$- 6520 K\sim^3 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 45700 K\sim^4$$
$$+ 520 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 672 K\sim^3$$
$$+ 1784 K\sim \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 13830 K\sim^2$$
$$+ 142 \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} - 3272 K\sim + 1210\Big) XX^3\Big) \Big/ \Big(\big(K\sim^2$$
$$- 3\big) \left(K\sim^2 + 8 K\sim + 13\right) \left(3 K\sim^2 + 16 K\sim + 23\right)^2\Big) - \Big(2 \Big(183 K\sim^{10}$$

$$- 105 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 2440 K\sim^{9}$$

$$- 1120 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 14163 K\sim^{8}$$

$$- 5000 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 46480 K\sim^{7}$$

$$- 11984 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 93606 K\sim^{6}$$

$$- 16354 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 116608 K\sim^{5}$$

$$- 11776 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 84070 K\sim^{4}$$

$$- 2576 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 25456 K\sim^{3}$$

$$+ 1392 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 8045 K\sim^{2}$$

$$+ 435 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 10248 K\sim - 3273 \Big) XX^{2} \Big) \Big/ \Big(\big(3 K\sim^{2}$$

$$+ 16 K\sim + 23 \big) \left(K\sim^{2} + 8 K\sim + 13\right) \left(K\sim^{2} - 3\right)^{3} \Big) + \Big(-5212350$$

$$- 55118056 K\sim^{12} + 116275392 K\sim^{11} + 269553168 K\sim^{10}$$

$$- 5569280751 K\sim^{6} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 63446596 K\sim^{3} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 75376799 K\sim^{2} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 6251410 K\sim \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 405350 \big(K\sim^{2} + 4 K\sim$$

$$+ 5 \big)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2} - 1158360 K\sim^{18} + 1348704 K\sim^{17} + 9795262 K\sim^{16}$$

$$+ 12563776 K\sim^{15} - 18628784 K\sim^{14} - 74766528 K\sim^{13} + 146 K\sim^{24} + 2336 K\sim^{23}$$

$$+ 14568 K\sim^{22} + 34336 K\sim^{21} + 107911872 K\sim^{9} - 298256850 K\sim^{8} - 455494752 K\sim^{7}$$

$$+ 121675125666 K\sim^{14} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 75985384274 K\sim^{8} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 14360027568 K\sim^{7} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 176278975876 K\sim^{9} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 319697376664 K\sim^{11} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 274849334678 K\sim^{10} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 209374781412 K\sim^{13} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$

$$+ 289880536806 K\sim^{12} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 22044960 K\sim$$

$$- 63756 K\sim^{20} - 595104 K\sim^{19}$$

$$- 4753091686 K\sim^{5} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 1152681213 K\sim^{4} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$- 1155275 \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} + 133786080 K\sim^{3}$$
$$- 5406264 K\sim^{2} - 135842616 K\sim^{6} + 263816352 K\sim^{5} + 323169588 K\sim^{4}$$
$$+ 92541 K\sim^{22} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$+ 2488834 K\sim^{21} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$+ 31903335 K\sim^{20} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$+ 258673604 K\sim^{19} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$+ 1484891035 K\sim^{18} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1}$$
$$+ 6401006954 K\sim^{17} \sqrt{K\sim^{2} + 4 K\sim + 5} \sqrt{3 K\sim^{2} + 4 K\sim - 1} - 1495 K\sim^{14} \left(K\sim^{2}\right.$$
$$\left. + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2} - 23038 K\sim^{13} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/}$$
$$^{2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2} - 164537 K\sim^{12} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim\right.$$
$$\left. - 1\right)^{5/2} - 382507 K\sim^{4} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2}$$
$$+ 98655 K\sim^{2} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2}$$
$$- 26390 K\sim^{18} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 572284 K\sim^{17} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 5872022 K\sim^{16} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 37754304 K\sim^{15} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 169865240 K\sim^{14} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 565728496 K\sim^{13} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 6650365 K\sim^{8} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2}$$
$$- 5298341 K\sim^{6} \left(K\sim^{2} + 4 K\sim + 5\right)^{5/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{5/2}$$
$$- 1432734776 K\sim^{6} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$- 312185520 K\sim^{5} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$+ 61554600 K\sim^{4} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$+ 63731584 K\sim^{3} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$
$$+ 14858810 K\sim^{2} \left(K\sim^{2} + 4 K\sim + 5\right)^{3/2} \left(3 K\sim^{2} + 4 K\sim - 1\right)^{3/2}$$

$$- 186780\,K\sim\,(K\sim^2+4\,K\sim+5)^{3/2}(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$+ 21446541609\,K\sim^{16}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}$$
$$- 1438617976\,K\sim^{12}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$+ 57016140464\,K\sim^{15}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}$$
$$- 2838897152\,K\sim^{11}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$- 4370666708\,K\sim^{10}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$- 5224726856\,K\sim^{9}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$- 4765934388\,K\sim^{8}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$- 3191389632\,K\sim^{7}\,(K\sim^2+4\,K\sim+5)^{3/2}\,(3\,K\sim^2+4\,K\sim-1)^{3/2}$$
$$- 716996\,K\sim^{11}\,(K\sim^2+4\,K\sim+5)^{5/2}\,(3\,K\sim^2+4\,K\sim-1)^{5/2}$$
$$- 2113699\,K\sim^{10}\,(K\sim^2+4\,K\sim+5)^{5/2}\,(3\,K\sim^2+4\,K\sim-1)^{5/2}$$
$$- 4416402\,K\sim^{9}\,(K\sim^2+4\,K\sim+5)^{5/2}\,(3\,K\sim^2+4\,K\sim-1)^{5/2}$$
$$- 7162488\,K\sim^{7}\,(K\sim^2+4\,K\sim+5)^{5/2}(3\,K\sim^2+4\,K\sim-1)^{5/2}$$
$$- 2393538\,K\sim^{5}\,(K\sim^2+4\,K\sim+5)^{5/2}(3\,K\sim^2+4\,K\sim-1)^{5/2}$$
$$+ 191868\,K\sim^{3}\,(K\sim^2+4\,K\sim+5)^{5/2}(3\,K\sim^2+4\,K\sim-1)^{5/2} + 5522\,K\sim\,(K\sim^2$$
$$+ 4\,K\sim+5)^{5/2}\,(3\,K\sim^2+4\,K\sim-1)^{5/2} - 2783\,(K\sim^2+4\,K\sim+5)^{5/2}\,(3\,K\sim^2$$
$$+ 4\,K\sim-1)^{5/2}\Big)\Big/\Big(2\,(K\sim^2+8\,K\sim+13)\,(K\sim^2+4\,K\sim+5)^2\,(K\sim^2-3)^9\Big)$$

$$3 \qquad \textbf{(10.2.3.8)}$$

The singularity is in XX^3 = (1-x/xc)^3/2:

```
> coeff(Zplussupcser, XX, 1);
```

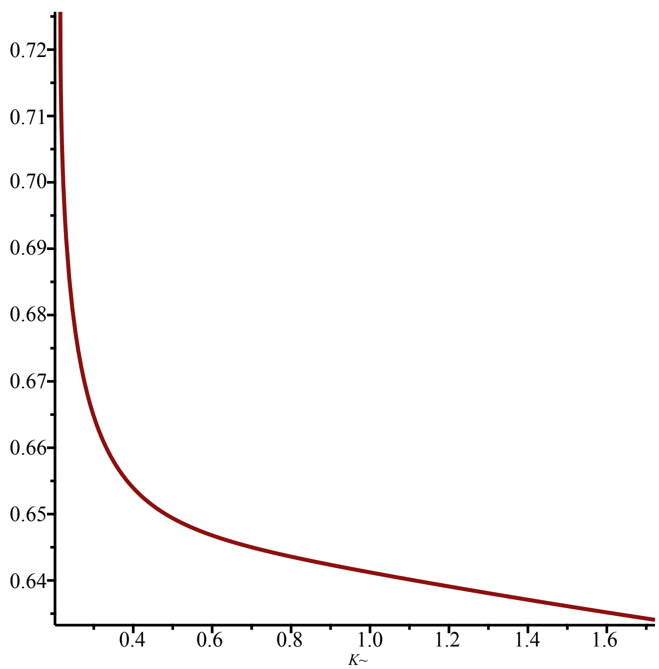
$$0 \qquad \textbf{(10.2.3.9)}$$

The corresponding mean for mu:

```
> dermuK := factor(simplify(rationalize(simplify(factor(-coeff(Zplussupcser, XX, 2)/(1 + coeff(Zplussupcser, XX, 0))),
      symbolic)), symbolic)); plot(%, K = Kc..Kinfini - 0.01);
```

$$dermuK := \Big(\Big(183\,K\sim^{10} - 105\,K\sim^{8}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 2440\,K\sim^{9}$$
$$- 1120\,K\sim^{7}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 14163\,K\sim^{8}$$
$$- 5000\,K\sim^{6}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 46480\,K\sim^{7}$$
$$- 11984\,K\sim^{5}\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1} + 93606\,K\sim^{6}$$

$$
\begin{aligned}
&- 16354\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 116608\, K\sim^{5} \\
&- 11776\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 84070\, K\sim^{4} \\
&- 2576\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 25456\, K\sim^{3} \\
&+ 1392\, K\sim \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 8045\, K\sim^{2} \\
&+ 435 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 10248\, K\sim - 3273\Big) \Big(37\, K\sim^{8} \\
&+ 21\, K\sim^{6} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 296\, K\sim^{7} \\
&+ 112\, K\sim^{5} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 1016\, K\sim^{6} \\
&+ 255\, K\sim^{4} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 1896\, K\sim^{5} \\
&+ 304\, K\sim^{3} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 2058\, K\sim^{4} \\
&+ 159\, K\sim^{2} \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} + 1304\, K\sim^{3} + 320\, K\sim^{2} \\
&- 11 \sqrt{K\sim^{2} + 4\, K\sim + 5}\, \sqrt{3\, K\sim^{2} + 4\, K\sim - 1} - 296\, K\sim - 247\Big)\Big) \Big/ \Big(2\, \big(K\sim^{2} + 6\, K\sim \\
&+ 7\big) \big(K\sim^{2} + 2\, K\sim - 1\big) \big(23\, K\sim^{6} + 184\, K\sim^{5} + 593\, K\sim^{4} + 1008\, K\sim^{3} + 989\, K\sim^{2} + 568\, K\sim \\
&+ 163\big) \big(K\sim^{2} - 3\big)^{3} \big(3\, K\sim^{2} + 16\, K\sim + 23\big)\Big)
\end{aligned}
$$



The limit at Kc is the right one:

> *expand*(*rationalize*(*limit*(*dermuK*, *K* = *Kc*, *right*))); *evalf*(%);

$$\frac{7}{6} - \frac{\sqrt{7}}{6}$$

$$0.7257081151 \qquad \textbf{(10.2.3.10)}$$

> *expand*(*rationalize*(*limit*(*dermuK*, *K* = *Kinfini*, *left*))); *evalf*(%);

$$-\frac{\sqrt{3}}{2}+\frac{3}{2}$$

$$0.6339745960 \quad \textbf{(10.2.3.11)}$$

When nu -> nuc, the derivative has a behavior in (nu-nuc)^1/2:

```
> algeqtoseries(subs(nu = nuc + NN, K = Kc + KK, numer(nusupK − nu)), NN, KK, 2);
```

$$\left[RootOf\left(9\_Z^2+\left(9\sqrt{7}+9\right)\_Z+2\sqrt{7}+26\right)+\left(-\frac{28\sqrt{7}}{81}+\frac{14}{81}+\frac{7\,RootOf\left(9\_Z^2+\left(9\sqrt{7}+9\right)\_Z+2\sqrt{7}+26\right)\sqrt{7}}{27}-\frac{7\,RootOf\left(9\_Z^2+\left(9\sqrt{7}+9\right)\_Z+2\sqrt{7}+26\right)}{9}\right)NN+O\left(NN^2\right),\left(\frac{56}{81}+\frac{14\sqrt{7}}{81}\right)NN-\frac{196}{729}\frac{\left(4+\sqrt{7}\right)\left(2\sqrt{7}+1\right)}{7+\sqrt{7}}NN^2+O\left(NN^3\right)\right] \quad \textbf{(10.2.3.12)}$$

```
> collect(subs(KK = (56/81 + 14*sqrt(7)/81) NN, expand(rationalize(convert(simplify(series(subs(K = Kc + KK, dermuK), KK, 1)), polynom)))), NN, factor);
```

$$\frac{7^{3/4}\sqrt{14}\sqrt{\left(4+\sqrt{7}\right)NN}\sqrt{2}}{81}-\frac{4\,7^{1/4}\sqrt{14}\sqrt{\left(4+\sqrt{7}\right)NN}\sqrt{2}}{81}-\frac{\sqrt{7}}{6}+\frac{7}{6} \quad \textbf{(10.2.3.13)}$$

Proof of proposition 4.14 : asymptotic behavior of the weights q_k

We start from the algebraic equation satisfied by Q(t,ty):

```
> eqQt := Qt^3 ν^2 w^2 y^5 − w ν y^2 (ν y^2 − 2 ν y − y^2 − ν + 3 y) Qt^2 − (2 ν^2 w y^3 tZ1 + ν^2 w y^3 − 2 ν w y^3 + ν^2 y − ν y^2 − 2 ν^2 + ν y + y^2 + 2 ν − 2 y) Qt − (2 ν^2 tZ1^2 + 2 ν^2 t2Z2 − ν^2 tZ1 + w ν^2 − 2 ν tZ1^2 − 2 ν t2Z2 − ν tZ1 + ν + 2 tZ1 − 1) y^2 − (2 ν tZ1 − ν − 2) (ν − 1) y − 2 (ν − 1) ν;
```

$$eqQt := Qt^3\nu^2w^2y^5-w\nu y^2\left(\nu y^2-2\nu y-y^2-\nu+3y\right)Qt^2-\left(2\nu^2wy^3tZ1+\nu^2wy^3-2\nu wy^3+\nu^2y-\nu y^2-2\nu^2+\nu y+y^2+2\nu-2y\right)Qt-\left(2\nu^2tZ1^2+2\nu^2t2Z2-\nu^2tZ1+\nu^2w-2\nu tZ1^2-2\nu t2Z2-\nu tZ1+\nu+2tZ1-1\right)y^2-(\nu-1)\left(2\nu tZ1-\nu-2\right)y-2\nu(\nu-1) \quad \textbf{(11.1)}$$

By a change of varibles, this readily gives a algebraic equation for \tilde F:

> $eqtildeF := collect\left(factor\left(numer\left(factor\left(subs\left(Qt = FF\cdot(1-z), y = \frac{1}{1-z}, eqQt\right)\right)\right)\right), z, factor\right);$

$$eqtildeF := -2\,FF\,\nu\,(\nu-1)\,z^3 + \left(FF^2\nu^2 w + 5\,FF\,\nu^2 - 7\,FF\,\nu - 2\,\nu^2 + 2\,FF + 2\,\nu\right)z^2 + \left(-4\,FF^2\nu^2 w + 3\,w\,\nu\,FF^2 - 4\,FF\,\nu^2 + 2\,\nu^2\,tZ1 + 7\,FF\,\nu + 3\,\nu^2 - 2\,\nu\,tZ1 - 3\,FF - 5\,\nu + 2\right)z + FF^3\nu^2 w^2 + 2\,FF^2\nu^2 w - 2\,FF\,\nu^2\,tZ1\,w - 2\,w\,\nu\,FF^2 - FF\,\nu^2 w - 2\,\nu^2\,tZ1^2 + FF\,\nu^2 + 2\,FF\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2 w + 2\,\nu\,tZ1^2 - 2\,FF\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + FF + 2\,\nu - 2\,tZ1 - 1 \tag{11.2}$$

From the definition of \tilde F, we know that its constant term (as a formal power series in z) is equal to Q (t,t). We would like to get from the previous equation and equation of the form (\tilde F-Q(t,t)) Pol_1 = z* Pol_2.

We start from the algebraic equation satisfied by Q(t,t) (=Qty1)

> $eqQty1 := collect(subs(y = 1, Qt = Qty1, eqQt), Qt, factor):$

we check whether there exists another solution of the previous equation which is also a formal power series in z but with a different constant term. Its constant term FFz0 should be solution of the following equation:

> $subs(FF = FFz0, coeff(eqtildeF, z, 0));$

$$FFz0^3\nu^2 w^2 + 2\,FFz0^2\nu^2 w - 2\,FFz0\,\nu^2\,tZ1\,w - 2\,FFz0^2\,\nu\,w - FFz0\,\nu^2 w - 2\,\nu^2\,tZ1^2 + FFz0\,\nu^2 + 2\,FFz0\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2 w + 2\,\nu\,tZ1^2 - 2\,FFz0\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + FFz0 + 2\,\nu - 2\,tZ1 - 1 \tag{11.3}$$

And Q(t,t) is solution to the following algebraic equation:

> $collect(subs(y = 1, Qt = Qty1, eqQt), Qt, factor);$

$$Qty1^3\nu^2 w^2 + 2\,w\,\nu^2\,Qty1^2 - 2\,Qty1\,\nu^2\,tZ1\,w - 2\,w\,\nu\,Qty1^2 - Qty1\,\nu^2 w - 2\,\nu^2\,tZ1^2 + Qty1\,\nu^2 + 2\,Qty1\,\nu\,w - 2\,\nu^2\,t2Z2 - \nu^2\,tZ1 - \nu^2 w + 2\,\nu\,tZ1^2 - 2\,Qty1\,\nu - \nu^2 + 2\,\nu\,t2Z2 + 3\,\nu\,tZ1 + Qty1 + 2\,\nu - 2\,tZ1 - 1 \tag{11.4}$$

Hence the constant term FFz0 of a solution of the algebraic equation satisfied by eqtilde F, must be solution of:

> $factor(simplify((11.3)-(11.4)));$

$$(FFz0 - Qty1)\left(FFz0^2\nu^2 w^2 + FFz0\,Qty1\,\nu^2 w^2 + Qty1^2\nu^2 w^2 + 2\,FFz0\,\nu^2 w + 2\,Qty1\,\nu^2 w - 2\,\nu^2\,tZ1\,w - 2\,FFz0\,\nu\,w - 2\,Qty1\,\nu\,w - \nu^2 w + \nu^2 + 2\,\nu\,w - 2\,\nu + 1\right) \tag{11.5}$$

The first factor when FFz0 = Qty1 is the derivative of the equation satistied by Qt:

> $simplify(subs(FFz0 = Qty1, op(1, (11.5)))) - factor(subs(y = 1, Qt = Qty1, diff(eqQt, Qt)));$

$$-1 + \left(-1 - 3\,w^2\,Qty1^2 + (2\,tZ1 - 4\,Qty1 + 1)\,w\right)\nu^2 + (2 + (4\,Qty1 - 2)\,w)\,\nu \tag{11.6}$$

The series Delta when nu< nu_c

The singular term in the asymptotic expansion of Q(t,ty)

> *subs*(*Usubc* = *U*, *coeff*(*Qtsubcsing3*, *XX*, 3));

$$-\left(4\,(Vsub+1)\,Vsub\,\sqrt{6}\,\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\,\left(\left(Vsub^3-7\,Vsub^2-Vsub-1\right)U-\frac{2\,Vsub^3}{3}+6\,Vsub^2+2\,Vsub+\frac{2}{3}\right)\right)\Big/\left(9\left(U-\frac{2}{3}\right)\left(Vsub^2+4\,Vsub+1\right)(Vsub-1)^4\right) \quad \textbf{(11.1.1)}$$

The singular term in the asymptotic expansion of the partition function \mathcal{Z}:

> *coeff*(*Zpsubcdevt*, *XX*, 3);

$$\frac{12\,\sqrt{\frac{6\,U^2-10\,U+3}{9\,U^2-10\,U+2}}\,\sqrt{6}\,\left(U^2-U+\frac{1}{3}\right)(U-2)}{54\,U^3-126\,U^2+87\,U-18} \quad \textbf{(11.1.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vsub:

> $AlephDeltaSubc := factor\left(simplify\left(\frac{\textbf{(11.1.1)}}{\textbf{(11.1.2)}}\right)\right);$

$$AlephDeltaSubc := -\left(\left(6\,U^2-10\,U+3\right)\left(3\,U\,Vsub^3-21\,U\,Vsub^2-2\,Vsub^3-3\,U\,Vsub+18\,Vsub^2-3\,U+6\,Vsub+2\right)Vsub\,(Vsub+1)\right)\Big/\left(3\,(U-2)\left(3\,U^2-3\,U+1\right)(Vsub-1)^4\left(Vsub^2+4\,Vsub+1\right)\right) \quad \textbf{(11.1.3)}$$

The series Delta and B when n = nu_c

The singular term in the asymptotic expansion of Q(t,ty)

> *coeff*(*Qtcsing4*, *XX*, 4);

$$\frac{1}{36\,(Vc-1)^4\left(Vc^2+4\,Vc+1\right)}\left(\left(1240\,\sqrt{7}-1700\right)^{1/3}\left(2\,\sqrt{7}+1\right)\left(2\,\sqrt{7}\,Vc^2-Vc^3+2\,\sqrt{7}\,Vc+5\,Vc^2-Vc+1\right)Vc\,(Vc+1)\right) \quad \textbf{(11.2.1)}$$

The singular term in the asymptotic expansion of the partition function \mathcal{Z}:

> *coeff*(*Zpscritdevt*, *XX*, 4);

$$\frac{3\left(1240\,\sqrt{7}-1700\right)^{1/3}\sqrt{7}}{20} \quad \textbf{(11.2.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vc:

> $AlephDeltaCrit := factor\left(simplify\left(\frac{\textbf{(11.2.1)}}{\textbf{(11.2.2)}}\right)\right);$

$AlephDeltaCrit :=$ **(11.2.3)**

$$\frac{5\left(14+\sqrt{7}\right)\left(2\sqrt{7}\,Vc^2-Vc^3+2\sqrt{7}\,Vc+5\,Vc^2-Vc+1\right)Vc\,(Vc+1)}{189\,(Vc-1)^4\left(Vc^2+4\,Vc+1\right)}$$

We can verify that it is the same expression as the subcritical one:

> $factor\left(\frac{subs(U=Uc,\,Vsub=Vc,\,AlephDeltaSubc)}{AlephDeltaCrit}\right);$

$$1 \qquad \textbf{(11.2.4)}$$

The series Delta when n > nu_c

> $coeff(Qtsupsing, XX, 3);$

$$\Big(32\,RootOf\big(\left(1296\,K\sim^4+6048\,K\sim^3+8928\,K\sim^2+3360\,K\sim-1200\right)\_Z^2-K\sim^8-10\,K\sim^7-24\,K\sim^6+26\,K\sim^5+158\,K\sim^4+114\,K\sim^3-192\,K\sim^2-306\,K\sim-117\big)\,Vsup\,\Big(\left(K\sim^2-3\right)^2 Vsup^3+\left(-7\,K\sim^4-40\,K\sim^3-110\,K\sim^2-136\,K\sim-55\right)Vsup^2-\left(K\sim^2-8\,K\sim-11\right)\left(K\sim^2-3\right)Vsup-\left(K\sim^2-3\right)^2\Big)\Big((K\sim+1)^2\,Vsup^2+\left(K\sim^2-3\right)Vsup+(K\sim+1)^2\Big)\left(K\sim+\frac{5}{3}\right)(Vsup+1)\Big)\Big/\Big((K\sim+1)\Big(\left(K\sim^2-3\right)Vsup^2+\left(-2\,K\sim^2-8\,K\sim-10\right)Vsup+K\sim^2-3\Big)\Big(\left(K\sim^2-3\right)Vsup^2+4\,(K\sim+1)^2\,Vsup+K\sim^2-3\Big)^3\Big) \qquad \textbf{(11.3.1)}$$

> $coeff(Zpsupcdevt, XX, 3);$

$$\frac{1}{3\left(K\sim^2+8\,K\sim+13\right)(K\sim+1)^4\left(K\sim^2-3\right)}\Big(8\left(21\,K\sim^6+242\,K\sim^5+1083\,K\sim^4+2388\,K\sim^3+2695\,K\sim^2+1410\,K\sim+225\right)RootOf\big(\left(1296\,K\sim^4+6048\,K\sim^3+8928\,K\sim^2+3360\,K\sim-1200\right)\_Z^2-K\sim^8-10\,K\sim^7-24\,K\sim^6+26\,K\sim^5+158\,K\sim^4+114\,K\sim^3-192\,K\sim^2-306\,K\sim-117\big)\Big) \qquad \textbf{(11.3.2)}$$

The series AlephQplus(nu,y)/AlephZps of the proposition parametrized by Vsup:

> $AlephDeltaSupc := factor\left(simplify\left(\frac{\textbf{(11.3.1)}}{\textbf{(11.3.2)}}\right)\right);$

$$AlephDeltaSupc := \Big(4\left(K\sim^2-3\right)\left(K\sim^2+8\,K\sim+13\right)(K\sim+1)^3\,(Vsup+1)\left(K\sim^2\,Vsup^2+K\sim^2\,Vsup+2\,K\sim\,Vsup^2+K\sim^2+Vsup^2+2\,K\sim-3\,Vsup+1\right)\left(K\sim^4\,Vsup^3-7\,K\sim^4\,Vsup^2-K\sim^4\,Vsup-40\,K\sim^3\,Vsup^2-6\,K\sim^2\,Vsup^3-K\sim^4+8\,K\sim^3\,Vsup-110\,K\sim^2\,Vsup^2+14\,K\sim^2\,Vsup-136\,K\sim\,Vsup^2+9\,Vsup^3+6\,K\sim^2-24\,K\sim\,Vsup-55\,Vsup^2-33\,Vsup-9\right)Vsup\Big)\Big/\Big((K\sim+3)\left(K\sim^2\,Vsup^2+4\,K\sim^2\,Vsup+K\sim^2+8\,K\sim\,Vsup-3\,Vsup^2+4\,Vsup-3\right)^3\left(K\sim^2+4\,K\sim+1\right)\left(7\,K\sim^2+20\,K\sim\right. \qquad \textbf{(11.3.3)}$$

$$+ 15)\left(K{\sim}^2\, Vsup^2 - 2\, K{\sim}^2\, Vsup + K{\sim}^2 - 8\, K{\sim}\, Vsup - 3\, Vsup^2 - 10\, Vsup - 3\right)\right)$$

The value for Kc is also the critical value:

> $factor\left(\dfrac{subs(K = Kc, Vsup = Vc, AlephDeltaSupc)}{AlephDeltaCrit}\right);$

$$1 \qquad \textbf{(11.3.4)}$$

Hypergeometric functions and their singular expansions in Lemma 4.17

> $simplify\left(\dfrac{1}{\mathrm{Pi}} \cdot int\left((x \cdot (1 - x))^{-\frac{1}{2}} \cdot (1 - z \cdot x)^{\frac{2}{3}}, x = 0..1\right)\right)$ assuming $z < 1$ **and** $z > 0$;
> $series(\%, z = 1, 2);$

$$\mathrm{hypergeom}\left(\left[-\frac{2}{3}, \frac{1}{2}\right], [1], z\right)$$

$$\frac{\sqrt{\pi}}{2\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{\sqrt{\pi}\,(-1+z)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} + \frac{12\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(-1+z)^{7/6}}{7\,\pi^{3/2}} + \mathrm{O}\left((-1+z)^2\right) \qquad \textbf{(12.1)}$$

> $simplify\left(\dfrac{1}{\mathrm{Pi}} \cdot int\left((x \cdot (1 - x))^{-\frac{1}{2}} \cdot (1 - z \cdot x)^{\frac{4}{3}}, x = 0..1\right)\right)$ assuming $z < 1$ **and** $z > 0$;
> $series(\%, z = 1, 2);$

$$\mathrm{hypergeom}\left(\left[-\frac{4}{3}, \frac{1}{2}\right], [1], z\right)$$

$$\frac{15\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{16\,\pi^{3/2}} - \frac{3\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1+z)}{4\,\pi^{3/2}} - \frac{32\sqrt{\pi}\,(-1)^{5/6}(-1+z)^{11/6}}{55\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} + \mathrm{O}\left((-1+z)^2\right) \qquad \textbf{(12.2)}$$

> $simplify\left(\dfrac{1}{\mathrm{Pi}} \cdot int\left((x \cdot (1 - x))^{-\frac{1}{2}} \cdot (1 - z \cdot x)^{\frac{5}{3}}, x = 0..1\right)\right)$ assuming $z < 1$ **and** $z > 0$;
> $series(\%, z = 1, 2);$

$$\mathrm{hypergeom}\left(\left[-\frac{5}{3}, \frac{1}{2}\right], [1], z\right)$$

(12.3)

$$\frac{7\sqrt{\pi}}{20\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{1}{4}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}\,(-1+z) + \mathrm{O}\left((-1+z)^2\right) \tag{12.3}$$

Cluster volume expectation (proof of Theorem 1.4)

The numerator of the expected volume as calculated in the paper:

> $EVol := \frac{1}{wU}\cdot\left(\frac{3}{8}\cdot\left(\frac{1}{yp^2}+\frac{1}{ym^2}\right)+\frac{1}{4}\cdot\frac{1}{yp\cdot ym}\right);$

$$EVol := \left(32\,(-1+2\,U)^2\,\nu^3\left(\frac{3}{8\,yp^2}+\frac{3}{8\,ym^2}+\frac{1}{4\,yp\,ym}\right)\right)\Big/\left((U(\nu+1)-2)\,U\left(8(\nu+1)^2U^3-(11\,\nu+13)(\nu+1)\,U^2+2(\nu+3)(2\,\nu+1)\,U-4\,\nu\right)\right) \tag{13.1}$$

nu < nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

> $ypsubsing;\ ymsubsing;$

$$2-\frac{1}{9\,Usubc-9}\left(3\left(-\frac{2}{3}+Usubc\right)\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,RootOf\left(-2\sqrt{6}\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}+3\,\_Z^2\right)XX^{3\,|\,2}\right)+\mathrm{O}(XX^2)$$

$$-\frac{4\,(Usubc-1)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)}{(21\,Usubc-16)\sqrt{3}-37\,Usubc+28} - 4\,\frac{\left(6\,Usubc^2-10\,Usubc+3\right)(-2+3\,Usubc)\left(780\sqrt{3}-1351\right)(Usubc-1)}{\left(21\sqrt{3}\,Usubc-16\sqrt{3}-37\,Usubc+28\right)^2(2\,Usubc-1)\left(2\sqrt{3}-3\right)^3}\,XX^2 - \frac{8}{9}\left(\left(6\,Usubc^2-10\,Usubc+3\right)\sqrt{2}\,(-2+3\,Usubc)\sqrt{\frac{6\,Usubc^2-10\,Usubc+3}{9\,Usubc^2-10\,Usubc+2}}\,(Usubc-1)\left(1380661\sqrt{3}\,Usubc-2391375\,Usubc-1048348\sqrt{3}+1815792\right)\right)\Big/\left((2\,Usubc-1)\left(21\sqrt{3}\,Usubc-16\sqrt{3}-37\,Usubc+28\right)^3\left(2\sqrt{3}-3\right)^5\right)XX^3+\mathrm{O}(XX^4) \tag{13.1.1}$$

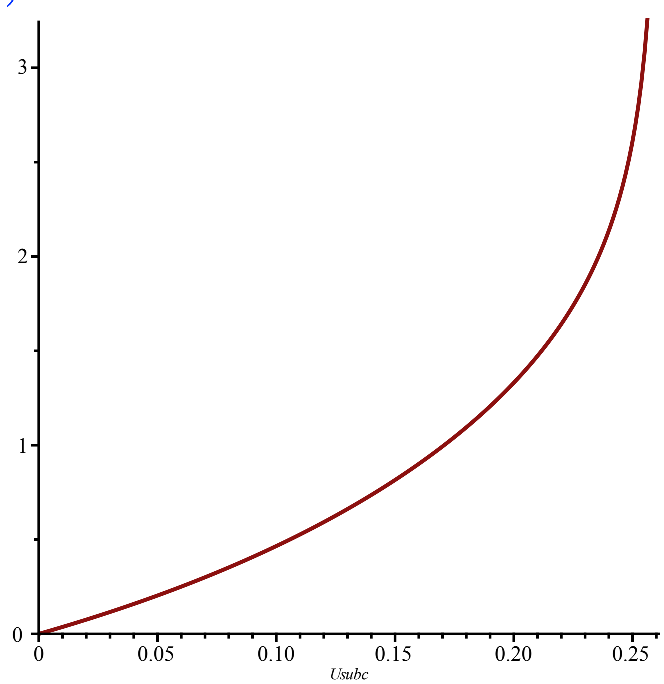
> $Evolsubcnumser := simplify\left(series\left(subs\left(subs(U=Usubc, subs(\mathrm{nu}=nuUsub)), U = Usubcsing3, yp = ypsubsing, ym = ymsubsing, XX = XX^2, EVol\right), XX, 4\right)\right)$ assuming $XX > 0;$

$$Evolsubcnumser := \Big(27 \left(2943 \sqrt{3}\, Usubc^2 - 4660 \sqrt{3}\, Usubc - 5098\, Usubc^2 + 1852 \sqrt{3} + 8072\, Usubc - 3208\right) Usubc\, (Usubc - 1)\Big) \Big/ \Big(2 \left(\sqrt{3} - 1\right)^2 (-2 + 3\, Usubc)^2 \left(-2 + \sqrt{3}\right)^2 \left(6\, Usubc^2 - 10\, Usubc + 3\right)\Big) + \frac{3}{2} \Bigg(Usubc\, (Usubc - 1) \sqrt{3} \sqrt{2} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}}\, RootOf\left(-2 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}} + 3\, \_Z^2\right) \left(39 \sqrt{3}\, Usubc - 34 \sqrt{3} - 67\, Usubc + 58\right)\Bigg) \Big/ \Big(\left(18\, Usubc^3 - 42\, Usubc^2 + 29\, Usubc - 6\right) \left(\sqrt{3} - 1\right) \left(-2 + \sqrt{3}\right)\Big) XX^3 + O\left(XX^4\right) \quad \textbf{(13.1.2)}$$

The constant in the asymptotics of the Expected volume:

> $simplify\left(\frac{coeff\,(Evolsubcnumser, XX, 3)}{subs\,(U = Usubc, coeff\,(Zpsubcdevt, XX, 3))}\right); plot(\%, Usubc = 0 ..Uc);$

$$\Bigg(3\, Usubc\, (Usubc - 1)\, RootOf\left(-2 \sqrt{6} \sqrt{\frac{6\, Usubc^2 - 10\, Usubc + 3}{9\, Usubc^2 - 10\, Usubc + 2}} + 3\, \_Z^2\right) \left(39 \sqrt{3}\, Usubc - 34 \sqrt{3} - 67\, Usubc + 58\right)\Bigg) \Big/ \Big(8 \left(\sqrt{3} - 1\right) \left(-2 + \sqrt{3}\right) \left(Usubc^2 - Usubc + \frac{1}{3}\right) (Usubc - 2)\Big)$$



nu =nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

> *ypcsing*; *ymcsing*;

$$2+\frac{1}{\left(-5+\sqrt{7}\right)^2\left(7+\sqrt{7}\right)^2\left(7+13\sqrt{7}\right)}\left(\left(-812\sqrt{7}+784\right)XX^{3/2}\left(1240\sqrt{7}-1700\right)^{1/3}RootOf\left(-2\left(1240\sqrt{7}-1700\right)^{1/3}\sqrt{7}+27\_Z^2-\left(1240\sqrt{7}-1700\right)^{1/3}\right)\right)+O\left(XX^2\right)$$

$$-\frac{4\left(-\frac{1}{2}+\sqrt{7}\right)\left(7+\sqrt{7}\right)\left(-2+\sqrt{3}\right)\left(\sqrt{3}-1\right)}{\left(24\sqrt{7}+231\right)\sqrt{3}-38\sqrt{7}-413}+\frac{27440}{3}\left(\left(362\sqrt{3}-627\right)\left(78806\sqrt{7}-181693\right)\right)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^2\left(7+\sqrt{7}\right)^2\left(-5+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^2\left(2\sqrt{3}-3\right)^2\right)XX^3+\frac{1}{8}\Bigg(\Big(\left(\left(330082753200510240\sqrt{3}-571720105508325550\right)\sqrt{7}-1024391999457256185\sqrt{3}+1774299006738717515\right)\left(1240\sqrt{7}-1700\right)^{1/3}+15704812680490206\left(\left(\sqrt{3}-\frac{33518496652}{19351912887}\right)\sqrt{7}-\frac{8999600785\sqrt{3}}{4300425086}+\frac{140289893863}{38703825774}\right)50^{1/3}\left(1240\sqrt{7}-1700\right)^{2/3}+\left(-586902892127647914\,50^{2/3}\sqrt{3}+1016545638114735092\,50^{2/3}\right)\sqrt{7}+1392843856906107333\,50^{2/3}\sqrt{3}-2412476352951155491\,50^{2/3}\Big)\Big/\left(\left(24\sqrt{7}\sqrt{3}-38\sqrt{7}+231\sqrt{3}-413\right)^3\left(7+\sqrt{7}\right)^3\left(-14+\sqrt{7}\right)^3\left(2\sqrt{3}-3\right)^3\left(-5+\sqrt{7}\right)^4\right)XX^4+O\left(XX^5\right) \qquad \textbf{(13.2.1)}$$

> *Evolcnumser* := *simplify*(*series*(*subs*(nu = *nuc*, *U* = *Ucsing4*, *yp* = *ypcsing*, *ym* = *ymcsing*, *XX* = *XX*^2, *EVol*), *XX*, 4))assuming *XX* > 0;

$$Evolcnumser := \frac{\left(13860\sqrt{7}+56979\right)\sqrt{3}-23984\sqrt{7}-98762}{10\left(-\frac{1}{2}+\sqrt{7}\right)\left(\sqrt{3}-1\right)^2\left(-5+\sqrt{7}\right)\left(-2+\sqrt{3}\right)^2} \qquad \textbf{(13.2.2)}$$

$$+\frac{1}{5}\left(\left(\left(192825780\sqrt{3}-332594948\right)\sqrt{7}+564274788\sqrt{3}\right.\right.$$

$$- 947353652) (1240\sqrt{7} - 1700)^{1/3} \mathit{RootOf}\Big(-2 (1240\sqrt{7} - 1700)^{1/3} \sqrt{7} + 27\,\_Z^2 - (1240\sqrt{7} - 1700)^{1/3}\Big)\Big) \Big/ \Big((7+\sqrt{7})^4 (-5+\sqrt{7})^3 (-1 + 2\sqrt{7})^2 (7 + 13\sqrt{7}) (-2+\sqrt{3}) (\sqrt{3} - 1)\Big) XX^3 + \mathrm{O}(XX^4)$$

The constant in the asymptotics of the Expected volume:

> *Zpscritdevt*

$$\frac{3\sqrt{7}\,(1240\sqrt{7} - 1700)^{1/3}\, XX^4}{20} + \left(-\frac{476}{25} + \frac{148\sqrt{7}}{25}\right) XX^3 + \frac{263\sqrt{7}}{50} - \frac{308}{25} \qquad \textbf{(13.2.3)}$$

> *simplify*(*expand*(*rationalize*(*simplify*(coeff(Evolcnumser, XX, 3) / coeff(Zpscritdevt, XX, 3))))); *evalf*(%);

$$\frac{1}{109872}\Big(\big((760\sqrt{7} + 2135)\sqrt{3} - 3860\sqrt{7} - 9940\big) (1240\sqrt{7} - 1700)^{1/3} \mathit{RootOf}\Big(-2 (1240\sqrt{7} - 1700)^{1/3} \sqrt{7} + 27\,\_Z^2 - (1240\sqrt{7} - 1700)^{1/3}\Big)\Big)$$

$$-2.265903514 \qquad \textbf{(13.2.4)}$$

nu>nuc

We have the developments of the singularities of y+(\nu,t) and y-(nu,t)

> *ypsupsing*; *ymsupsing*;

$$-\Big(16 (3K\sim + 5) (3K\sim^2 + 8K\sim + 7) (K\sim^3 + 3K\sim^2 + 9K\sim + 11) \Big(36K\sim^{10} + 31K\sim^8 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 384K\sim^9 + 248K\sim^7 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 1996K\sim^8 + 844K\sim^6 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 6624K\sim^7 + 1544K\sim^5 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 14952K\sim^6 + 1818K\sim^4 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 22176K\sim^5 + 2088K\sim^3 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 19160K\sim^4 + 2508K\sim^2 \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} + 6560K\sim^3 + 1816K\sim \sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} - 1804K\sim^2 + 479\sqrt{K\sim^2 + 4K\sim + 5}\sqrt{3K\sim^2 + 4K\sim - 1} - 1184K\sim + 220\Big) \mathit{RootOf}\Big((1296K\sim^4 + 6048K\sim^3 + 8928K\sim^2 + 3360K\sim - 1200)\_Z^2 - K\sim^8 - 10K\sim^7 - 24K\sim^6 + 26K\sim^5 + 158K\sim^4 + 114K\sim^3 - 192K\sim^2 - 306K\sim - 117\Big) XX\Big) \Big/ \Big((K\sim^2 + 4K\sim + 5) (23K\sim^6 + 184K\sim^5 + 593K\sim^4 + 1008K\sim^3 + 989K\sim^2 + 568K\sim + 163)^2 (K\sim^2 - 3)^2\Big) - \Big(4 (K\sim + 1) (K\sim^3 + 3K\sim^2 + 9K\sim + 11) \Big(2K\sim^4$$

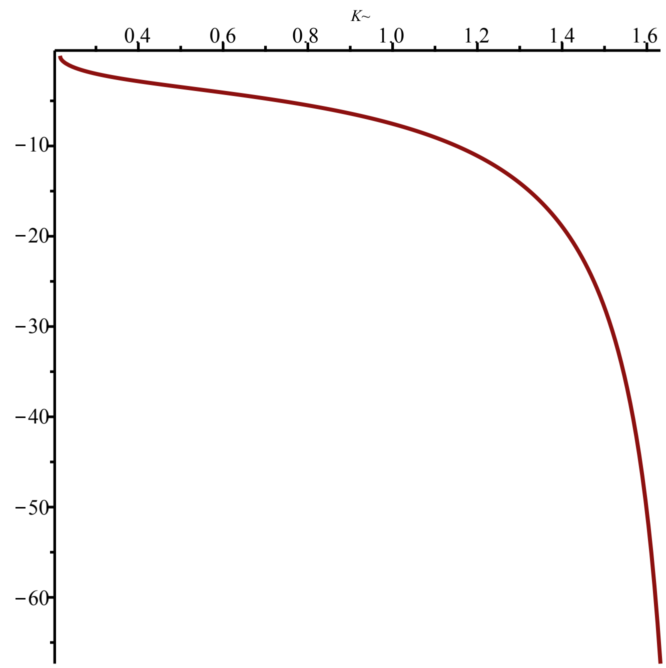
$$+ 3 K\sim^2 \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 8 K\sim^3 + 4 K\sim \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 24 K\sim^2 - \sqrt{K\sim^2 + 4 K\sim + 5}\, \sqrt{3 K\sim^2 + 4 K\sim - 1} + 40 K\sim + 22\Big)\Big) \Big/ \Big(\big(23 K\sim^6 + 184 K\sim^5 + 593 K\sim^4 + 1008 K\sim^3 + 989 K\sim^2 + 568 K\sim + 163\big) \big(K\sim^2 - 3\big)\Big)$$

$$\Big(4\, RootOf\big(\big(1296 K\sim^4 + 6048 K\sim^3 + 8928 K\sim^2 + 3360 K\sim - 1200\big)\, \_Z^2 - K\sim^8 - 10 K\sim^7 - 24 K\sim^6 + 26 K\sim^5 + 158 K\sim^4 + 114 K\sim^3 - 192 K\sim^2 - 306 K\sim - 117\big) (3 K\sim + 5)\, (K\sim + 1)\, \big(K\sim^2 + 8 K\sim + 13\big) \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big) \big(3 K\sim^4 + 12 K\sim^3 + 22 K\sim^2 + 28 K\sim + 19\big) \big(K\sim^6 \sqrt{2} + 6 K\sim^5 \sqrt{2} + 27 K\sim^4 \sqrt{2} + 16 K\sim^4 \sqrt{2 + K\sim} + 108 K\sim^3 \sqrt{2} + 80 K\sim^3 \sqrt{2 + K\sim} + 339 K\sim^2 \sqrt{2} + 240 K\sim^2 \sqrt{2 + K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2 + K\sim}\, K\sim + 377 \sqrt{2} + 352 \sqrt{2 + K\sim}\big) XX^3\Big) \Big/ \Big(3 \sqrt{2 + K\sim}\, \big(3 K\sim^2 + 8 K\sim + 7\big)^3 \big(K\sim^2 - 3\big)^2 \big(K\sim^4 + 6 K\sim^3 + 30 K\sim^2 + 62 K\sim + 45\big)^2\Big) + \Big(\big(K\sim^2 + 8 K\sim + 13\big) \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big) (K\sim + 1)^2 \big(K\sim^6 \sqrt{2} + 6 K\sim^5 \sqrt{2} + 27 K\sim^4 \sqrt{2} + 16 K\sim^4 \sqrt{2 + K\sim} + 108 K\sim^3 \sqrt{2} + 80 K\sim^3 \sqrt{2 + K\sim} + 339 K\sim^2 \sqrt{2} + 240 K\sim^2 \sqrt{2 + K\sim} + 582 K\sim \sqrt{2} + 464 \sqrt{2 + K\sim}\, K\sim + 377 \sqrt{2} + 352 \sqrt{2 + K\sim}\big) XX^2\Big) \Big/ \Big(2 \sqrt{2 + K\sim}\, \big(K\sim^2 - 3\big) \big(K\sim^4 + 6 K\sim^3 + 30 K\sim^2 + 62 K\sim + 45\big)^2 \big(3 K\sim^2 + 8 K\sim + 7\big)\Big) + \Big(2 \big(K\sim^3 + 3 K\sim^2 + 9 K\sim + 11\big) \big(K\sim^3 + 4 \sqrt{2}\, \sqrt{2 + K\sim}\, K\sim + 3 K\sim^2 + 8 \sqrt{2}\, \sqrt{2 + K\sim} + 9 K\sim + 11\big)\Big) \Big/ \Big(\big(K\sim^2 - 3\big) \big(K\sim^4 + 6 K\sim^3 + 30 K\sim^2 + 62 K\sim + 45\big)\Big) \qquad \textbf{(13.3.1)}$$

```
> Evolsupcnumser := simplify(series(subs(nu = nusupK, U = Usupcsing, yp = ypsupsing, ym
      = ymsupsing, EVol), XX, 4)) :
```

The constant in the asymptotics of the Expected volume:

$$> \quad simplify\left(\frac{coeff(Evolsupcnumser, XX, 3)}{coeff(Zpsupcdevt, XX, 3)}\right) : plot(\%, K = Kc..Kinfini - 0.1);$$



Percolation probability (proof of Theorem 1.1)

Finite clusters in the high temperature regime (nu \leq nu_c)

We start from the rational parametrization of y in terms of U_nu and V (in which nu has been replaced by its expression in terms of U_\nu).

> *yUVsubc*;

$$-\frac{24\,(U-1)\,V\,(V+1)}{3\,U\,V^3-21\,U\,V^2-2\,V^3-3\,V\,U+18\,V^2-3\,U+6\,V+2} \qquad \textbf{(14.1.1)}$$

The value of the negative singularity y- corresponds to V equal to Vsubl (computed in **(4.1.6)**):

> *ymsub* := *factor*(*subs*(*V*=-2 + sqrt(3), *yUVsubc*));

$$ymsub := -\frac{4\left(3\sqrt{3}+2\right)(U-1)}{23\,U-14+2\sqrt{3}} \qquad \textbf{(14.1.2)}$$

To perform the change of variables in the integral, we compute the new bounds by solving the following equations, (recall that y- = ymsub and y+ = 2)

> $factor\left(\frac{yUVsubc-1}{yUVsubc}-\frac{1}{ymsub}\right)$;

$$\frac{(-2+3\,U)\left(V-7+4\sqrt{3}\right)\left(V+2+\sqrt{3}\right)^2}{24\,(U-1)\,V\,(V+1)} \qquad \textbf{(14.1.3)}$$

> $factor\left(\frac{1}{2}-\frac{yUVsubc-1}{yUVsubc}\right)$;

(14.1.4)

$$-\frac{(V-1)^3\,(-2+3\,U)}{24\,(U-1)\,V\,(V+1)} \qquad \textbf{(14.1.4)}$$

In the integral, V varies between -2+sqrt(3) and 1 so the square root factor is given by:

> $rootfactorsubc := \frac{(2-3\,U)}{24\cdot(1-U)}\cdot\frac{(V+2+\mathrm{sqrt}(3))\cdot(1-V)}{V\cdot(V+1)}\cdot\mathrm{sqrt}\big((1-V)\cdot\big(V-7+4\sqrt{3}\big)\big);$

$$rootfactorsubc := \frac{(2-3\,U)\,\big(V+2+\sqrt{3}\big)\,(1-V)\,\sqrt{(1-V)\,\big(V-7+4\sqrt{3}\big)}}{(-24\,U+24)\,V\,(V+1)} \qquad \textbf{(14.1.5)}$$

Recall that our expression for AlephDeltaSubc is also valid at nuc :

> $factor\left(\frac{1}{yUVsubc}\cdot subs(Usubc=U, Vsub=V, AlephDeltaSubc)\cdot diff(yUVsubc, V)\right);$

$$\frac{\big(6\,U^2-10\,U+3\big)\,(-2+3\,U)}{3\,(V-1)^2\,\big(3\,U^2-3\,U+1\big)\,(U-2)} \qquad \textbf{(14.1.6)}$$

> $factor\left(\frac{yUVsubc-1}{yUVsubc}+\frac{1}{2}\cdot\left(\frac{1}{ymsub}+\frac{1}{2}\right)\right);$

$$-\frac{1}{24\,(U-1)\,V\,(V+1)}\big(9\sqrt{3}\,U\,V^2-3\,U\,V^3+9\sqrt{3}\,U\,V-6\sqrt{3}\,V^2-15\,U\,V^2+2\,V^3-6\sqrt{3}\,V-33\,V\,U+18\,V^2+3\,U+30\,V-2\big) \qquad \textbf{(14.1.7)}$$

The following is the probability that the cluster is finite:

> $simplify\left(\frac{1}{2\cdot\mathrm{Pi}\cdot nuUsub\cdot subs(\mathrm{nu}=nuUsub, wU)}\ int(\textbf{(14.1.6)}\cdot\textbf{(14.1.7)}\cdot\textbf{(14.1.5)}, V=7-4\,\mathrm{sqrt}(3)\,..1)\right);$

$$1 \qquad \textbf{(14.1.8)}$$

Percolation probability when nu>nuc and critical exponent beta:

The symmetry in 1/V of \hat y:

> $simplify\left(subs\left(V=\frac{1}{V}, yUV\right)-\frac{yUV}{yUV-1}\right);$

$$0 \qquad \textbf{(14.2.1)}$$

The values of V for the singularities of Q(t,ty) at t_\nu were computed in Section 4 (equations **(4.2.11)** and below).

> $nusupK;\ UsupK;\ factor(numer(diff(yUVsupc, V)));$

$$-\frac{K\sim^3+3\,K\sim^2+9\,K\sim+11}{(K\sim+3)\,\big(K\sim^2-3\big)}$$

$$-\frac{K\sim^2-3}{6\,K\sim+10}$$

$$8\,(K\sim+1)\,\big(K\sim^3+3\,K\sim^2+9\,K\sim+11\big)\,\big(V^2\,K\sim^2+4\,K\sim^2\,V+K\sim^2+8\,K\sim\,V-3\,V^2+4\,V \qquad \textbf{(14.2.2)}$$

$-3)\ (V^2 K\sim^2 - 2 K\sim^2 V + K\sim^2 - 8 K\sim V - 3 V^2 - 10 V - 3)$

> $collect\left(\left(K\sim^2 V^2 + 4 K\sim^2 V + K\sim^2 + 8 K\sim V - 3 V^2 + 4 V - 3\right), V, factor\right); collect\left(\left(K\sim^2 V^2 - 2 K\sim^2 V + K\sim^2 - 8 K\sim V - 3 V^2 - 10 V - 3\right), V, factor\right);$

$$(K\sim^2 - 3)\ V^2 + 4\ (K\sim + 1)^2\ V + K\sim^2 - 3$$

$$(K\sim^2 - 3)\ V^2 + (-2 K\sim^2 - 8 K\sim - 10)\ V + K\sim^2 - 3 \tag{14.2.3}$$

> $VK11; VK22;$

$$-\frac{2 K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} + 2}{K\sim^2 - 3}$$

$$\frac{K\sim^2 + 4 K\sim - 2\sqrt{2}\ (K\sim + 1)\sqrt{2 + K\sim} + 5}{K\sim^2 - 3} \tag{14.2.4}$$

We first compute the bounds for the integral. We have to factorize:

> $factor\left(numer\left(factor\left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V = \frac{1}{VK11}, yUVsupc\right)}\right)\right)\right);$

$$\begin{aligned}
&45 + 15 V - 1188 K\sim V + 3132 V^2 K\sim^2 + 5 K\sim^8 + 28 K\sim^7 \\
&- 3 K\sim^6 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} \\
&- 8 K\sim^5 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} \\
&+ 11 K\sim^4 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} \\
&+ 48 K\sim^3 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} \\
&+ 15 K\sim^2 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} \\
&- 72 K\sim \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} + 396 K\sim + 3\ ((K\sim^2 + 4 K\sim \\
&+ 5)\ (3 K\sim^2 + 4 K\sim - 1))^{3/2} V^2 + 3\ ((K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim \\
&- 1))^{3/2} V + 63 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)}\ V^3 \\
&- 52 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)}\ V^2 \\
&+ 102 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)}\ V - 12 K\sim^3 + 492 K\sim^2 + 28 K\sim^6 \\
&- 124 K\sim^5 - 298 K\sim^4 + 298 K\sim^4 V^3 + 10950 K\sim^4 V^2 - 1630 K\sim^4 V + 8620 K\sim^3 V^2 \\
&- 492 K\sim^2 V^3 - 45 V^3 - 63 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)} + 125 V^2 \\
&- 4188 K\sim^3 V + 12 K\sim^3 V^3 - 396 K\sim V^3 - 3692 K\sim^2 V + 340 K\sim V^2 - 5 K\sim^8 V^3 \\
&+ 69 K\sim^8 V^2 - 28 K\sim^7 V^3 + 39 K\sim^8 V + 708 K\sim^7 V^2 - 28 K\sim^6 V^3 + 300 K\sim^7 V \\
&+ 3148 K\sim^6 V^2 + 124 K\sim^5 V^3 + 836 K\sim^6 V + 7708 K\sim^5 V^2 + 628 K\sim^5 V \\
&+ 3 K\sim^6 \sqrt{(K\sim^2 + 4 K\sim + 5)\ (3 K\sim^2 + 4 K\sim - 1)}\ V^3
\end{aligned} \tag{14.2.5}$$

$- 36\, K\sim^6 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2$

$+ 8\, K\sim^5 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^3$

$- 18\, K\sim^6 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

$- 288\, K\sim^5 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2$

$- 11\, K\sim^4 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^3$

$- 96\, K\sim^5 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

$- 940\, K\sim^4 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2$

$- 48\, K\sim^3 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^3 - K\sim^2\, ((K\sim^2 + 4\, K\sim$

$+ 5)\, (3\, K\sim^2 + 4\, K\sim - 1))^{3/2}\, V^2$

$- 142\, K\sim^4 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

$- 1536\, K\sim^3 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2 - K\sim^2\, ((K\sim^2 + 4\, K\sim$

$+ 5)\, (3\, K\sim^2 + 4\, K\sim - 1))^{3/2}\, V$

$- 15\, K\sim^2 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^3$

$+ 128\, K\sim^3 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

$- 1276\, K\sim^2 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2$

$+ 72\, K\sim \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^3$

$+ 554\, K\sim^2 \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

$- 480\, K\sim \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V^2$

$+ 480\, K\sim \sqrt{(K\sim^2 + 4\, K\sim + 5)\, (3\, K\sim^2 + 4\, K\sim - 1)}\, V$

1/V+ is a double root:

> $simplify\left(rem\left(\textbf{(14.2.5)}, \left(V - \frac{1}{VK11}\right)^2, V\right)\right);$

$$0 \qquad \textbf{(14.2.6)}$$

VK11^2 is the third root:

> $simplify(subs(V = VK11^2, \textbf{(14.2.5)}));$

$$0 \qquad \textbf{(14.2.7)}$$

Same for the other bound, we want to factorize:

> $factor\left(numer\left(factor\left(\left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V = \frac{1}{VK22}, yUVsupc\right)}\right)\right)\right)\right);$

$117 + 753\, V + 8\, K\sim^4 \sqrt{2}\, (2 + K\sim)^{3/2}\, V^2 + 3\, K\sim^6 \sqrt{2} \sqrt{2 + K\sim}\, V^3 + 8\, K\sim^4 \sqrt{2}\, (2$ **(14.2.8)**

$$+ K\sim)^{3/2} V + 16 K\sim^3 \sqrt{2} (2+K\sim)^{3/2} V^2 - 9 K\sim^6 \sqrt{2} \sqrt{2+K\sim} V^2 + 8 K\sim^5 \sqrt{2} \sqrt{2+K\sim} V^3 + 16 K\sim^3 \sqrt{2} (2+K\sim)^{3/2} V - 16 K\sim^2 \sqrt{2} (2 + K\sim)^{3/2} V^2 + 9 K\sim^6 \sqrt{2} \sqrt{2+K\sim} V - 120 K\sim^5 \sqrt{2} \sqrt{2+K\sim} V^2 - 11 K\sim^4 \sqrt{2} \sqrt{2+K\sim} V^3 - 16 K\sim^2 \sqrt{2} (2+K\sim)^{3/2} V - 48 K\sim \sqrt{2} (2 + K\sim)^{3/2} V^2 + 72 K\sim^5 \sqrt{2} \sqrt{2+K\sim} V - 623 K\sim^4 \sqrt{2} \sqrt{2+K\sim} V^2 - 48 K\sim^3 \sqrt{2} \sqrt{2+K\sim} V^3 - 48 K\sim \sqrt{2} (2+K\sim)^{3/2} V + 175 K\sim^4 \sqrt{2} \sqrt{2+K\sim} V - 1648 K\sim^3 \sqrt{2} \sqrt{2+K\sim} V^2 - 15 K\sim^2 \sqrt{2} \sqrt{2+K\sim} V^3 + 16 K\sim^3 \sqrt{2} \sqrt{2+K\sim} V - 2339 K\sim^2 \sqrt{2} \sqrt{2+K\sim} V^2 + 72 K\sim \sqrt{2} \sqrt{2+K\sim} V^3 - 509 K\sim^2 \sqrt{2} \sqrt{2+K\sim} V - 1656 K\sim \sqrt{2} \sqrt{2+K\sim} V^2 - 696 K\sim \sqrt{2} \sqrt{2+K\sim} V + 1857 K\sim V + 5633 V^2 K\sim^2 + 11 K\sim^4 \sqrt{2} \sqrt{2+K\sim} + 48 K\sim^3 \sqrt{2} \sqrt{2+K\sim} + 15 K\sim^2 \sqrt{2} \sqrt{2+K\sim} + K\sim^7 - 8 K\sim^5 \sqrt{2} \sqrt{2+K\sim} + 189 K\sim - 117 K\sim^3 + 3 K\sim^2 - 24 \sqrt{2} (2+K\sim)^{3/2} V^2 - 24 \sqrt{2} (2+K\sim)^{3/2} V + 63 \sqrt{2} \sqrt{2+K\sim} V^3 - 445 \sqrt{2} \sqrt{2+K\sim} V^2 - 291 \sqrt{2} \sqrt{2+K\sim} V - 63 \sqrt{2} \sqrt{2+K\sim} + 9 K\sim^6 + 15 K\sim^5 - 41 K\sim^4 + 41 K\sim^4 V^3 + 1893 K\sim^4 V^2 - 421 K\sim^4 V + 4401 K\sim^3 V^2 - 3 K\sim^2 V^3 - 117 V^3 + 1039 V^2 - 72 \sqrt{2} \sqrt{2+K\sim} K\sim + 143 K\sim^3 V + 117 K\sim^3 V^3 - 189 K\sim V^3 + 1471 K\sim^2 V + 3775 K\sim V^2 - K\sim^7 V^3 + 3 K\sim^7 V^2 - 9 K\sim^6 V^3 - 3 K\sim^7 V + 51 K\sim^6 V^2 - 15 K\sim^5 V^3 - 51 K\sim^6 V + 437 K\sim^5 V^2 - 245 K\sim^5 V - 3 K\sim^6 \sqrt{2} \sqrt{2+K\sim}$$

1/V- is a double root:

> $simplify\left(rem\left(\mathbf{(14.2.8)}, \left(V - \frac{1}{VK22}\right)^2, V\right)\right);$

$$0 \qquad \mathbf{(14.2.9)}$$

and V-^2 is the third root:

> $simplify(subs(V = VK22^2, \mathbf{(14.2.8)}));$

$$0 \qquad \mathbf{(14.2.10)}$$

Since we identify all the roots, the numerator of the terms under the square root is fully factorized. We now look at its denominator:

> $factor\left(denom\left(factor\left(\left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V = \frac{1}{VK22}, yUVsupc\right)}\right) \cdot \left(\frac{1}{yUVsupc} - \frac{1}{subs\left(V = \frac{1}{VK11}, yUVsupc\right)}\right)\right)\right)\right);$

$$64 \left(2 K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} + 2\right) \left(K\sim^2 + 4 K\sim - \sqrt{(K\sim^2 + 4 K\sim + 5)(3 K\sim^2 + 4 K\sim - 1)} + 5\right) \left(-2 \sqrt{2} \sqrt{2+K\sim} K\sim + K\sim^2\right. \qquad \mathbf{(14.2.11)}$$

$$-2\sqrt{2}\sqrt{2+K\sim}+4K\sim+5\big)(K\sim+1)^2\left(-\sqrt{2}\sqrt{2+K\sim}+K\sim+1\right)(V+1)^2V^2\left(K\sim^3+3K\sim^2+9K\sim+11\right)^2$$

We can hence write that the root factor in the integral is equal to f1K * rootfactorsupc, where f1K depends only on K (and not on V) and:

$$> rootfactorsupc := \frac{\left(\frac{1}{Vp}-V\right)\cdot\left(V-\frac{1}{Vm}\right)\cdot\mathrm{sqrt}\left(\left(Vp^2-V\right)\cdot\left(V-Vm^2\right)\right)}{V\cdot(V+1)};$$

$$rootfactorsupc := \frac{\left(\frac{1}{Vp}-V\right)\left(V-\frac{1}{Vm}\right)\sqrt{\left(Vp^2-V\right)\left(-Vm^2+V\right)}}{V(V+1)} \qquad \textbf{(14.2.12)}$$

To fully factorize the square root in the integral, we want to compute f1K:

$$> factor\left(expand\left(rationalize\left(factor\left(\left(coeff\left(numer\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V}, yUVsupc\right)}-\frac{1}{subs(V=VK22, yUVsupc)}\right)\right), V, 3\right)\right)\Big/\left(coeff\left(denom\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V}, yUVsupc\right)}-\frac{1}{subs(V=VK22, yUVsupc)}\right)\right), V, 2\right)\right)\right)\right)\right)\right);$$

$$factor\left(expand\left(rationalize\left(factor\left(\left(coeff\left(numer\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V}, yUVsupc\right)}-\frac{1}{subs(V=VK11, yUVsupc)}\right)\right), V, 3\right)\right)\Big/\left(coeff\left(denom\left(factor\left(\frac{1}{subs\left(V=\frac{1}{V}, yUVsupc\right)}-\frac{1}{subs(V=VK11, yUVsupc)}\right)\right), V, 2\right)\right)\right)\right)\right)\right);$$

$$\frac{\left(K\sim^2-3\right)^2}{8(K\sim+1)\left(K\sim^3+3K\sim^2+9K\sim+11\right)}$$

$$\frac{\left(K\sim^2-3\right)^2}{8(K\sim+1)\left(K\sim^3+3K\sim^2+9K\sim+11\right)} \qquad \textbf{(14.2.13)}$$

> $f1K := \frac{(3-K^2)^2}{8\,(K+1)\,(K^3+3K^2+9K+11)};$

$$f1K := \frac{(-K\sim^2+3)^2}{8\,(K\sim+1)\,(K\sim^3+3K\sim^2+9K\sim+11)} \quad \textbf{(14.2.14)}$$

We turn our attention to the other factors of the integral.
Another factor of the integral will give poles at V+ and 1/V+:

> $factor\left(\frac{diff(yUVsupc, V)}{yUVsupc}\cdot subs(Vsup = V, AlephDeltaSupc)\right);$

$$-(4\,(K\sim+1)^3\,(K\sim^2-3)\,(K\sim^2+8K\sim+13)\,(V^2K\sim^2+K\sim^2V+2K\sim V^2+K\sim^2+V^2+2K\sim-3V+1))\Big/((7K\sim^2+20K\sim+15)\,(K\sim^2+4K\sim+1)\,(V^2K\sim^2+4K\sim^2V+K\sim^2+8K\sim V-3V^2+4V-3)^2\,(K\sim+3)) \quad \textbf{(14.2.15)}$$

The roots of the polynomial (in V) of the denominator are VK11 and 1/VK11. Indeed, we have:

> $simplify(subs(V = VK11, (K^2V^2+4K^2V+K^2+8KV-3V^2+4V-3)));$
$simplify\left(subs\left(V = \frac{1}{VK11}, (K^2V^2+4K^2V+K^2+8KV-3V^2+4V-3)\right)\right);$

$$0$$
$$0 \quad \textbf{(14.2.16)}$$

We can also rewrite the polynomial (in V) of the numerator as:

> $collect(V^2K^2+K^2V+2KV^2+K^2+V^2+2K-3V+1, V, factor);$
$factor(solve(\%, V)[1]);$

$$(K\sim+1)^2V^2+(K\sim^2-3)\,V+(K\sim+1)^2$$
$$-\frac{K\sim^2-I\sqrt{(K\sim^2+4K\sim+5)\,(3K\sim^2+4K\sim-1)}-3}{2\,(K\sim+1)^2} \quad \textbf{(14.2.17)}$$

So that the expression in **(14.2.15)** can be factorized as f2K * AlephFactor, where:

> $f2K := \frac{4\,(K+1)^5\,(K^2+8K+13)}{(3-K^2)\,(7K^2+20K+15)\,(K^2+4K+1)\,(K+3)};$

$$f2K := \frac{4\,(K\sim+1)^5\,(K\sim^2+8K\sim+13)}{(-K\sim^2+3)\,(7K\sim^2+20K\sim+15)\,(K\sim^2+4K\sim+1)\,(K\sim+3)} \quad \textbf{(14.2.18)}$$

> $AlephFactor := \frac{\left(V^2-\frac{3-K^2}{(K+1)^2}V+1\right)}{\left((V-Vp)\cdot\left(V-\frac{1}{Vp}\right)\right)^2};$

$$AlephFactor := \frac{V^2-\frac{(-K\sim^2+3)\,V}{(K\sim+1)^2}+1}{(V-Vp)^2\left(V-\frac{1}{Vp}\right)^2} \quad \textbf{(14.2.19)}$$

The last factor is equal to:

> $lastfactor := \left(factor\left(\frac{yUVsupc-1}{yUVsupc} + \frac{1}{2}\cdot\left(subs\left(V=Vp, \frac{1}{yUVsupc}\right) + subs\left(V=Vm, \frac{1}{yUVsupc}\right)\right)\right)\right):$
> $denom(lastfactor);$

$$16\,(Vm+1)\,Vm\,(Vp+1)\,Vp\,\left(K\sim^3+3\,K\sim^2+9\,K\sim+11\right)\,(K\sim+1)\,V\,(V+1) \qquad \textbf{(14.2.20)}$$

>

>

To compute the integral, we perform a partial fraction decomposition of the denominator:

> $convert\left(\frac{1}{V^2\cdot(V+1)^2\cdot(V-Vp)^2\cdot\left(\frac{1}{Vp}-V\right)}, fullparfrac, V, factor\right);$

$$\frac{-Vp^3+Vp}{\left(Vp^2-1\right)^2 Vp^2\,(Vp+1)^2\,(V-Vp)^2} + \frac{5\,Vp^2-2\,Vp-2}{\left(Vp^2-1\right)^2 Vp^2\,(Vp+1)^2\,(V-Vp)} + \frac{1}{Vp\,V^2} + \frac{Vp^2-2\,Vp+2}{Vp^2\,V} + \frac{Vp}{\left(Vp^3+3\,Vp^2+3\,Vp+1\right)(V+1)^2} + \frac{Vp\,(4+3\,Vp)}{\left(Vp^4+4\,Vp^3+6\,Vp^2+4\,Vp+1\right)(V+1)} - \frac{Vp^6}{\left(Vp^2-1\right)^2 (Vp+1)^2 \left(V-\frac{1}{Vp}\right)} \qquad \textbf{(14.2.21)}$$

> $coefVp2 := \frac{-Vp^3+Vp}{\left(Vp^2-1\right)^2 Vp^2\,(1+Vp)^2}:$
> $coefVp1 := \frac{5\,Vp^2-2\,Vp-2}{\left(Vp^2-1\right)^2 Vp^2\,(1+Vp)^2}:$
> $coefInvVp := -\frac{Vp^6}{(1+Vp)^2\left(Vp^2-1\right)^2}:$
> $coef02 := \frac{1}{Vp}:$
> $coef01 := \frac{Vp^2-2\,Vp+2}{Vp^2}:$
> $coefMinus12 := \frac{Vp}{\left(Vp^3+3\,Vp^2+3\,Vp+1\right)}:$
> $coefMinus11 := \frac{Vp\,(4+3\,Vp)}{\left(Vp^4+4\,Vp^3+6\,Vp^2+4\,Vp+1\right)}:$

The elementary integrals that appear in the calculation (For some reason Maple simplifies better if we tell it z>0, but the result is the same if z<0):

> psi := [seq([seq(simplify(1/Pi · int(sqrt(x·(1 − x))/(1 − z·x)^i ·x^j, x = 0..1)), j = 0..6)], i = 1..2)] assuming z < 1 **and** z > 0 :

> psineg := [seq([seq(simplify(1/Pi · int(sqrt(x·(1 − x))/(1 − z·x)^i ·x^j, x = 0..1)), j = 0..6)], i = 1..2)] assuming z < 1 **and** z < 0 :

> psi − psineg;

$$[[0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0]] \qquad \textbf{(14.2.22)}$$

For each pole with multiplicity, we will replace z by:

> zpole := (Vp^2 − Vm^2)/(pole − Vm^2);

$$zpole := \frac{-Vm^2 + Vp^2}{-Vm^2 + pole} \qquad \textbf{(14.2.23)}$$

After the change of variables, the numerator becomes:

> numerproba := collect(simplify(subs(V = Vm^2 + (Vp^2 − Vm^2)·x, (V^2 − (3 − K^2)/(K + 1)^2 V + 1)·(V- 1/Vm)·numer(lastfactor)·(Vp^2 − Vm^2)^2)), x, factor) :

The contributions of each monomial for the integral:

> psiint := [seq(simplify(coefVp2·subs(z = zpole, pole = Vp, psi[2][j]/(Vm^2 − pole)^2) + coefVp1 ·subs(z = zpole, pole = Vp, psi[1][j]/(Vm^2 − pole)) + coefInvVp · subs(z = zpole, pole = 1/Vp, psi[1][j]/(Vm^2 − pole)) + coefMinus12·subs(z = zpole, pole = -1, psi[2][j]/(Vm^2 − pole)^2) + coefMinus11·subs(z = zpole, pole = -1, psi[1][j]/(Vm^2 − pole)) + coef02·subs(z = zpole, pole = 0, psi[2][j]/(Vm^2 − pole)^2) + coef01·subs(z = zpole, pole = 0, psi[1][j]/(Vm^2 − pole))), j = 1..7)] :

We just have to compute the global prefactor of the integral (not forgetting the term nu t^3 in the denominator and the denomiator of 'lastfactor':

> factor(subs(nu = nusupK, U = UsupK, nu·wU));

$$\frac{(K\sim^2 - 3)^2 (K\sim + 1) (K\sim^2 + 8 K\sim + 13)}{16 (K\sim + 3) (K\sim^3 + 3 K\sim^2 + 9 K\sim + 11)^2} \qquad \textbf{(14.2.24)}$$

> $factor\left(\frac{denom(lastfactor)}{V\cdot(V+1)}\right);$

$$16\,(Vm+1)\,Vm\,(Vp+1)\,Vp\,\left(K\sim^3+3\,K\sim^2+9\,K\sim+11\right)\,(K\sim+1) \qquad \textbf{(14.2.25)}$$

> $prefactor := factor\left(\frac{f1K\cdot f2K}{\textbf{(14.2.25)}\cdot 2\cdot\textbf{(14.2.24)}}\right);$

$$prefactor := \qquad \textbf{(14.2.26)}$$

$$-\frac{(K\sim+1)^2}{4\,\left(K\sim^2-3\right)\,Vp\,(Vp+1)\,Vm\,(Vm+1)\,\left(K\sim^2+4\,K\sim+1\right)\,\left(7\,K\sim^2+20\,K\sim+15\right)}$$

We finally have the probability that the cluster is infinite:

> $Probaperco := 1-(prefactor\cdot add(coeff(numerproba, x, j-1)\cdot psiint[j], j=1..7)):$

> $Probaperco2 := simplify(Probaperco):$

> $Probaperco3 := simplify(expand(rationalize(simplify(subs(Vp = VK11, Vm = VK22, Probaperco2))))));$

$$Probaperco3 := \Bigg(\bigg(-24\,\bigg(-\frac{1}{8}\,\bigg(51\,\bigg(K\sim^8+\frac{704}{51}\,K\sim^7+\frac{4204}{51}\,K\sim^6+\frac{4800}{17}\,K\sim^5$$

$$+\,614\,K\sim^4+\frac{134080}{153}\,K\sim^3+\frac{124292}{153}\,K\sim^2+\frac{69440}{153}\,K\sim+\frac{17977}{153}\bigg)\,\sqrt{2}\,(K\sim$$

$$+\,1)\,\sqrt{2+K\sim}\bigg)+K\sim^{10}+\frac{475\,K\sim^9}{12}+\frac{2453\,K\sim^8}{6}+\frac{6467\,K\sim^7}{3}+7048\,K\sim^6$$

$$+\,\frac{93281\,K\sim^5}{6}+\frac{71735\,K\sim^4}{3}+25489\,K\sim^3+\frac{53953\,K\sim^2}{3}+\frac{90683\,K\sim}{12}+\frac{8629}{6}\bigg)$$

$$\sqrt{K\sim^2+4\,K\sim+5}\,\sqrt{3\,K\sim^2+4\,K\sim-1}+81\,\left(K\sim^2+4\,K\sim+5\right)\,(K\sim+1)\,\bigg($$

$$-\frac{1}{9}\,\bigg(35\,\bigg(K\sim^8+\frac{512}{45}\,K\sim^7+\frac{17452}{315}\,K\sim^6+\frac{16384}{105}\,K\sim^5+\frac{18106}{63}\,K\sim^4$$

$$+\frac{112384}{315}K\sim^3+\frac{29924}{105}K\sim^2+\frac{5888}{45}K\sim+\frac{8443}{315}\Big)\sqrt{2}\sqrt{2+K\sim}\Big)+K\sim^9$$

$$+\frac{67K\sim^8}{3}+\frac{4540K\sim^7}{27}+\frac{18332K\sim^6}{27}+\frac{15430K\sim^5}{9}+\frac{25894K\sim^4}{9}$$

$$+\frac{264596K\sim^3}{81}+\frac{66172K\sim^2}{27}+\frac{29611K\sim}{27}+\frac{17377}{81}\Big)\Big)$$

$$\Big(\Big(\big(-15K\sim^4-64K\sim^3-102K\sim^2-64K\sim-7\big)\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1}+27\Big(K\sim^2+$$

$$+K\sim^4+16K\sim^3+58K\sim^2+80K\sim+41\big)\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1}$$

$$+3\Big(-\frac{8\sqrt{2}(K\sim+1)^3\sqrt{2+K\sim}}{3}+\big(K\sim^2+4K\sim+1\big)\Big(K\sim^2+4K\sim$$

$$+\frac{11}{3}\Big)\Big)\big(K\sim^2+4K\sim+5\big)\Big)\Big)^{1/2}-96\big(K\sim^3+3K\sim^2+9K\sim+11\big)\Big(\frac{1}{3}\Big(4\Big($$

$$-\frac{\big(-2\sqrt{2+K\sim}+\sqrt{2}(K\sim+1)\big)\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1}}{4}+\big(\big($$

$$-K\sim-1)\sqrt{2+K\sim}+\sqrt{2}(2+K\sim)\big)(K\sim+1)\Big)$$

$$\sqrt{-\sqrt{K\sim^2+4K\sim+5}\sqrt{3K\sim^2+4K\sim-1}+2(K\sim+1)^2}$$

$$\sqrt{\big(K\sim^2+4K\sim-2\sqrt{2}(K\sim+1)\sqrt{2+K\sim}+5\big)\big(K\sim^2+4K\sim+5\big)}\Big)-\frac{1}{3}\big(\big($$

$$-4\sqrt{2}(K\sim+1)\big(K\sim^2+4K\sim+5\big)\sqrt{2+K\sim}+K\sim^4+16K\sim^3+58K\sim^2+80K\sim$$

$$+ 41\Big) \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1}\Big) + \Bigg(-\frac{8\sqrt{2}\ (K\sim + 1)^3 \sqrt{2 + K\sim}}{3}$$

$$+ \left(K\sim^2 + 4 K\sim + 1\right) \left(K\sim^2 + 4 K\sim + \frac{11}{3}\right)\Bigg) \left(K\sim^2 + 4 K\sim + 5\right)\Bigg) (K\sim + 1)^3\Bigg)$$

$$\Bigg/ \Bigg(84 \left(K\sim^2 + 4 K\sim + 1\right) \left(K\sim^2 - 3\right) \left(K\sim^2 + \frac{20}{7} K\sim + \frac{15}{7}\right) \Bigg(-\frac{1}{3}\Big(\Big($$

$$-4\sqrt{2}\ (K\sim + 1) \left(K\sim^2 + 4 K\sim + 5\right) \sqrt{2 + K\sim} + K\sim^4 + 16 K\sim^3 + 58 K\sim^2 + 80 K\sim$$

$$+ 41\Big) \sqrt{K\sim^2 + 4 K\sim + 5}\ \sqrt{3 K\sim^2 + 4 K\sim - 1}\Big) + \Bigg(-\frac{8\sqrt{2}\ (K\sim + 1)^3 \sqrt{2 + K\sim}}{3}$$

$$+ \left(K\sim^2 + 4 K\sim + 1\right) \left(K\sim^2 + 4 K\sim + \frac{11}{3}\right)\Bigg) \left(K\sim^2 + 4 K\sim + 5\right)\Bigg)\Bigg)$$

> *Probaperco4* := *simplify*(*expand*(*rationalize*(*Probaperco3*)));

$$Probaperco4 := \Bigg(57 \left(K\sim^2\right. \qquad \textbf{(14.2.28}$$

$$\left. - 3\right)$$

$$\Bigg(\frac{1}{19}\Bigg(\sqrt{K\sim^2 + 4 K\sim + 5}\ \Bigg(\frac{1}{3}\Big(16\sqrt{2}\ (K\sim + 1) \left(K\sim^4 + K\sim^3 + 2 K\sim^2 + 13 K\sim\right.$$

$$\left. + 13\right) \sqrt{2 + K\sim}\Big) + K\sim^6 + \frac{16 K\sim^5}{3} - \frac{107 K\sim^4}{3} - \frac{544 K\sim^3}{3} - 261 K\sim^2 - 144 K\sim$$

$$- \frac{97}{3}\Bigg) \sqrt{3 K\sim^2 + 4 K\sim - 1}\Bigg) - \frac{1}{19}\Bigg(18\sqrt{2}\ \Bigg(K\sim^6 + \frac{136}{27} K\sim^5 + 7 K\sim^4$$

$$- \frac{368}{27} K\sim^3 - \frac{1639}{27} K\sim^2 - \frac{680}{9} K\sim - \frac{841}{27} \Big) (K\sim + 1) \sqrt{2 + K\sim} \Big) - \frac{4516 K\sim^2}{19}$$

$$+ \frac{104 K\sim^5}{19} - \frac{8920 K\sim}{57} + 8 K\sim^7 - \frac{2711}{57} + K\sim^8 - \frac{11624 K\sim^3}{57} - \frac{254 K\sim^4}{3}$$

$$+ \frac{1204 K\sim^6}{57} \Big) \big(K\sim^2 + 4 K\sim$$

$$+ 5\big)$$

$$\Bigg(\Bigg(\big(-15 K\sim^4 - 64 K\sim^3 - 102 K\sim^2 - 64 K\sim - 7 \big) \sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 27 \Big(K\sim^2$$

$$+ \frac{4}{3} K\sim - \frac{1}{3} \Big) \Big(K\sim^2 + \frac{8}{3} K\sim + \frac{7}{3} \Big)^2 \Bigg) \Bigg/ \Big(- \big(-4 \sqrt{2} (K\sim + 1) \big(K\sim^2 + 4 K\sim$$

$$+ 5 \big) \sqrt{2 + K\sim} + K\sim^4 + 16 K\sim^3 + 58 K\sim^2 + 80 K\sim + 41 \big)$$

$$\sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 3 \Big(- \frac{8 \sqrt{2} (K\sim + 1)^3 \sqrt{2 + K\sim}}{3} + \big(K\sim^2$$

$$+ 4 K\sim + 1 \big) \Big(K\sim^2 + 4 K\sim + \frac{11}{3} \Big) \Big) \big(K\sim^2 + 4 K\sim + 5 \big) \Big) \Big)^{1/2} - 64 \big(K\sim^3 + 3 K\sim^2$$

$$+ 9 K\sim + 11 \big) (K\sim + 1)^3 \Big(- \frac{1}{2} \Big(\big(2 \sqrt{2 + K\sim} + \sqrt{2} (K\sim$$

$$+ 1) \big) \sqrt{ -\sqrt{K\sim^2 + 4 K\sim + 5} \sqrt{3 K\sim^2 + 4 K\sim - 1} + 2 (K\sim + 1)^2 } \big(K\sim^2$$

$$+ \sqrt{K\sim^2 + 4K\sim + 5}\,\sqrt{3K\sim^2 + 4K\sim - 1} + 4K\sim + 5\Big)$$

$$\sqrt{\left(K\sim^2 + 4K\sim - 2\sqrt{2}\,(K\sim + 1)\sqrt{2 + K\sim} + 5\right)\left(K\sim^2 + 4K\sim + 5\right)}\,\Big) + \Big(K\sim^2$$

$$+ 4K\sim + 5\Big)\left(K\sim^2 - 3\right)^2\Big)\Big) \Bigg/ \Bigg(56\left(K\sim^2 - 3\right)^3\left(K\sim^2 + 4K\sim + 1\right)\Big(K\sim^2$$

$$+ \frac{20}{7}K\sim + \frac{15}{7}\Big)\left(K\sim^2 + 4K\sim + 5\right)\Bigg)$$

> *den4* := *denom*(*Probaperco4*);

$$den4 := 8\left(K\sim^2 - 3\right)^3\left(K\sim^2 + 4K\sim + 1\right)\left(7K\sim^2 + 20K\sim + 15\right)\left(K\sim^2 + 4K\sim + 5\right) \qquad \textbf{(14.2.29)}$$

> *num4* := *simplify*(*numer*(*Probaperco4*)); *nops*(%);

$$num4 := 57\Bigg(\Big(K\sim^2$$

$$- 3\Big)$$

$$\Bigg(\frac{1}{19}\Bigg(\sqrt{K\sim^2 + 4K\sim + 5}\,\Bigg(\frac{1}{3}\Big(16\sqrt{2}\,(K\sim + 1)\left(K\sim^4 + K\sim^3 + 2K\sim^2 + 13K\sim\right.$$

$$\left. + 13\right)\sqrt{2 + K\sim}\Big) + K\sim^6 + \frac{16K\sim^5}{3} - \frac{107K\sim^4}{3} - \frac{544K\sim^3}{3} - 261K\sim^2 - 144K\sim$$

$$- \frac{97}{3}\Bigg)\sqrt{3K\sim^2 + 4K\sim - 1}\Bigg) - \frac{1}{19}\Bigg(18\sqrt{2}\,\Bigg(K\sim^6 + \frac{136}{27}K\sim^5 + 7K\sim^4$$

$$- \frac{368}{27} K\sim^3 - \frac{1639}{27} K\sim^2 - \frac{680}{9} K\sim - \frac{841}{27} \Big) (K\sim + 1) \sqrt{2 + K\sim} \Big) - \frac{4516\, K\sim^2}{19}$$

$$+ \frac{104\, K\sim^5}{19} - \frac{8920\, K\sim}{57} + 8\, K\sim^7 - \frac{2711}{57} + K\sim^8 - \frac{11624\, K\sim^3}{57} - \frac{254\, K\sim^4}{3}$$

$$+ \frac{1204\, K\sim^6}{57} \Big)$$

$$\Big(\Big(\big(-15\, K\sim^4 - 64\, K\sim^3 - 102\, K\sim^2 - 64\, K\sim - 7 \big) \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 27 \Big(K\sim^2 +$$

$$+ 1) \big(K\sim^2 + 4\, K\sim + 5 \big) \sqrt{2 + K\sim} + K\sim^4 + 16\, K\sim^3 + 58\, K\sim^2 + 80\, K\sim + 41 \big)$$

$$\sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 3 \Big(- \frac{8 \sqrt{2}\, (K\sim + 1)^3 \sqrt{2 + K\sim}}{3} + \big(K\sim^2$$

$$+ 4\, K\sim + 1 \big) \Big(K\sim^2 + 4\, K\sim + \frac{11}{3} \Big) \Big) \big(K\sim^2 + 4\, K\sim + 5 \big) \Big) \Big)^{1/2} - \frac{1}{57} \Big(64 \big(K\sim^3$$

$$+ 3\, K\sim^2 + 9\, K\sim + 11 \big) \Big(- \frac{1}{2} \Big(\big(2 \sqrt{2 + K\sim} + \sqrt{2}\, (K\sim$$

$$+ 1) \big) \sqrt{K\sim^2 + 4\, K\sim - 2 \sqrt{2}\, (K\sim + 1) \sqrt{2 + K\sim} + 5}\, \Big(\sqrt{K\sim^2 + 4\, K\sim + 5}$$

$$+ \sqrt{3\, K\sim^2 + 4\, K\sim - 1} \Big)$$

$$\sqrt{ - \sqrt{K\sim^2 + 4\, K\sim + 5}\, \sqrt{3\, K\sim^2 + 4\, K\sim - 1} + 2\, (K\sim + 1)^2} \Big) + \big(K\sim^2 - 3 \big)^2 \Big)$$

$$(K\sim + 1)^3 \Big) \Big) \big(K\sim^2 + 4\, K\sim + 5 \big)$$

$$3 \tag{14.2.30}$$

The secondterm in the big factor is actually 0:

> $num42 := simplify(op(2, op(2, num4)))$ assuming $K > Kc$ **and** $K < Kinfini$;

$$num42 := \frac{1}{57}\left(32\left(\left(2\sqrt{2+K\sim}+\sqrt{2}\,(K\sim+1)\right)\sqrt{K\sim^2+4K\sim-2\sqrt{2}\,(K\sim+1)\sqrt{2+K\sim}+5}\left(\sqrt{K\sim^2+4K\sim+5}+\sqrt{3K\sim^2+4K\sim-1}\right)\sqrt{-\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+2(K\sim+1)^2}-2\left(K\sim^2-3\right)^2\right)\left(K\sim^3+3K\sim^2+9K\sim+11\right)(K\sim+1)^3\right) \quad \textbf{(14.2.31)}$$

> $factor\Big(expand\Big(simplify\Big(\Big(\big(2\sqrt{2+K}+\sqrt{2}\,(K+1)\big)\big(\sqrt{K^2+4K+5}+\sqrt{3K^2+4K-1}\big)\sqrt{K^2+4K-2\sqrt{2}\,(K+1)\sqrt{2+K}+5}\sqrt{-\sqrt{K^2+4K+5}\,\sqrt{3K^2+4K-1}+2(K+1)^2}\Big)^2\Big)\Big)\Big)$;

$$4\left(K\sim^2-3\right)^4 \quad \textbf{(14.2.32)}$$

Let's look at the first factor:

> $num41 := simplify(op(1, num4)\cdot op(3, num4)\cdot op(1, op(2, num4)))$ assuming $K > Kc$ **and** $K < Kinfini$;

$$num41 := \left(K\sim^2+4K\sim+5\right)\left(K\sim^2-3\right)\left(3\sqrt{K\sim^2+4K\sim+5}\left(\left(\frac{1}{3}\left(16\sqrt{2}\,(K\sim+1)\left(K\sim^4+K\sim^3+2K\sim^2+13K\sim+13\right)\sqrt{2+K\sim}\right)+K\sim^6+\frac{16K\sim^5}{3}-\frac{107K\sim^4}{3}-\frac{544K\sim^3}{3}-261K\sim^2-144K\sim-\frac{97}{3}\right)\sqrt{3K\sim^2+4K\sim-1}-54\sqrt{2}\left(K\sim^6+\frac{136}{27}K\sim^5+7K\sim^4-\frac{368}{27}K\sim^3-\frac{1639}{27}K\sim^2-\frac{680}{9}K\sim-\frac{841}{27}\right)(K\sim+1)\sqrt{2+K\sim}+57K\sim^8+456K\sim^7+1204K\sim^6+312K\sim^5-4826K\sim^4-11624K\sim^3-13548K\sim^2-8920K\sim-2711\right)\left(\left(\left(-15K\sim^4-64K\sim^3-102K\sim^2-64K\sim-7\right)\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+27\left(K\sim^2+41\right)\sqrt{K\sim^2+4K\sim+5}\,\sqrt{3K\sim^2+4K\sim-1}+3\left(-\frac{8\sqrt{2}\,(K\sim+1)^3\sqrt{2+K\sim}}{3}+\left(K\sim^2+4K\sim+1\right)\left(K\sim^2+4K\sim+\frac{11}{3}\right)\right)\left(K\sim^2+4K\sim+5\right)\right)^{1/2}\right)\right)$$

> rootProba := simplify(expand(rationalize(num41^2)));

$$rootProba := -6960\,(K\sim^2-3)^3\left(\left(\frac{1}{145}\left((2+K\sim)^{3/2}\,(145\,K\sim^8+1160\,K\sim^7 + 4612\,K\sim^6+11832\,K\sim^5+23430\,K\sim^4+39000\,K\sim^3+46916\,K\sim^2+31912\,K\sim + 8753)\sqrt{2}\right)+\frac{1}{145}\left(162\,(K\sim^3+3\,K\sim^2+9\,K\sim+11)\left(K\sim^8+8\,K\sim^7 + \frac{76}{3}\,K\sim^6+\frac{2872}{81}\,K\sim^5+\frac{518}{81}\,K\sim^4-\frac{3752}{81}\,K\sim^3-\frac{5308}{81}\,K\sim^2-\frac{3416}{81}\,K\sim - \frac{1039}{81}\right)\right)\right)\sqrt{3\,K\sim^2+4\,K\sim-1}\,(K\sim+1)\sqrt{K\sim^2+4\,K\sim+5} - \frac{1}{145}\left(288\,(K\sim^2+4\,K\sim+5)\left(K\sim^2+\frac{4}{3}\,K\sim-\frac{1}{3}\right)^2\left(\sqrt{2}\,(2+K\sim)^{3/2}\,(K\sim^3 + 3\,K\sim^2+9\,K\sim+11)\,(K\sim+1)^2+\frac{307\,K\sim^8}{256}+\frac{307\,K\sim^7}{32}+\frac{2179\,K\sim^6}{64} + \frac{2133\,K\sim^5}{32}+\frac{10185\,K\sim^4}{128}+\frac{2337\,K\sim^3}{32}+\frac{4211\,K\sim^2}{64}+\frac{1343\,K\sim}{32}+\frac{2579}{256}\right)\right)\right) \left(K\sim^2+\frac{8}{3}\,K\sim+\frac{7}{3}\right)(K\sim^2+4\,K\sim+5) \quad \textbf{(14.2.34)}$$

> den4;

$$8\,(K\sim^2-3)^3\,(K\sim^2+4\,K\sim+1)\,(7\,K\sim^2+20\,K\sim+15)\,(K\sim^2+4\,K\sim+5) \quad \textbf{(14.2.35)}$$

$$> Probapercosimple := \Bigg(\mathrm{sqrt}\Bigg(-6960\Bigg(\sqrt{3\,K\sim^2+4\,K\sim-1}\Bigg(\frac{1}{145}\left((2+K\sim)^{3/2}\,(145\,K\sim^8 + 1160\,K\sim^7+4612\,K\sim^6+11832\,K\sim^5+23430\,K\sim^4+39000\,K\sim^3+46916\,K\sim^2 + 31912\,K\sim+8753)\sqrt{2}\right)+\frac{1}{145}\left(162\,(K\sim^3+3\,K\sim^2+9\,K\sim+11)\left(K\sim^8 + 8\,K\sim^7+\frac{76}{3}\,K\sim^6+\frac{2872}{81}\,K\sim^5+\frac{518}{81}\,K\sim^4-\frac{3752}{81}\,K\sim^3-\frac{5308}{81}\,K\sim^2-\frac{3416}{81}\,K\sim - \frac{1039}{81}\right)\right)\Bigg)(K\sim+1)\sqrt{K\sim^2+4\,K\sim+5}-\frac{1}{145}\Big(288\,(K\sim^2+4\,K\sim$$

$$+ 5)\left(\sqrt{2}\,(2+K\sim)^{3/2}\,(K\sim^3+3K\sim^2+9K\sim+11)\,(K\sim+1)^2+\frac{307K\sim^8}{256}+\frac{307K\sim^7}{32}+\frac{2179K\sim^6}{64}+\frac{2133K\sim^5}{32}+\frac{10185K\sim^4}{128}+\frac{2337K\sim^3}{32}+\frac{4211K\sim^2}{64}+\frac{1343K\sim}{32}+\frac{2579}{256}\right)\left(K\sim^2+\frac{4}{3}K\sim-\frac{1}{3}\right)^2\Big)\Big)(K\sim^2+4K\sim+5)(K\sim^2-3)^3\left(K\sim^2+\frac{8}{3}K\sim+\frac{7}{3}\right)\Big)\Big)\Big/\Big(8(K\sim^2+4K\sim+5)(K\sim^2+4K\sim+1)(7K\sim^2+20K\sim+15)(3-K\sim^2)^3\Big):$$

We do an expansion at Kc:

```
> Vpser := collect(map(expand, map(rationalize, convert(series(subs(K = Kc + KK^4, VK11), KK, 9), polynom))), KK, factor) assuming KK > 0;
```

$$Vpser := 1+\left(\frac{103}{4}-\frac{37\sqrt{7}}{4}\right)KK^8+\left(\frac{65\sqrt{7+4\sqrt{7}}}{12}-\frac{367\sqrt{7+4\sqrt{7}}\sqrt{7}}{168}\right)KK^6+\left(2\sqrt{7}-\frac{7}{2}\right)KK^4+\left(-\frac{4\sqrt{7+4\sqrt{7}}}{3}+\frac{\sqrt{7+4\sqrt{7}}\sqrt{7}}{3}\right)KK^2 \quad \textbf{(14.2.36)}$$

```
> Vmser := collect(map(expand, map(rationalize, convert(series(subs(K = Kc + KK^4, VK22), KK, 9), polynom))), KK, factor) assuming KK > 0;
```

$$Vmser := \left(-\frac{97\sqrt{7}\sqrt{3}}{18}+\frac{1075\sqrt{3}}{72}-\frac{103}{4}+\frac{37\sqrt{7}}{4}\right)KK^8+\left(-\frac{7\sqrt{3}}{3}+\frac{4\sqrt{7}\sqrt{3}}{3}-2\sqrt{7}+\frac{7}{2}\right)KK^4-2+\sqrt{3} \quad \textbf{(14.2.37)}$$

```
> map(simplify, series(subs(Vp = Vpser, Vm = Vmser, K = Kc + KK^4, Probapercosimple), KK, 7)) assuming KK > 0;
```

$$\frac{1}{32\,(8\sqrt{7}+23)\,(-1+2\sqrt{7})\,(4+\sqrt{7})^2}\Big(3\,7^{1/8}\,2^{3/4}\sqrt{(2140679\sqrt{7}+5663177)\sqrt{3}-2344320\sqrt{7}-6203376}\,(1+\sqrt{7})^2\sqrt{3}\Big)KK-\frac{9}{1792}\Big(\sqrt{3}\,7^{1/8}\,2^{3/4}\,(86730850565683\sqrt{7}\sqrt{3}-104983600061232\sqrt{7}+229468259516419\sqrt{3}-277760499103584)\Big)\Big/\Big(\sqrt{(2140679\sqrt{7}+5663177)\sqrt{3}-2344320\sqrt{7}-6203376}\,(4+\sqrt{7})^6\,(-1+2\sqrt{7})^2\,(8\sqrt{7}+23)^2\Big)KK^5+O(KK^7) \quad \textbf{(14.2.38)}$$

> collect(simplify(expand(rationalize(convert((14.2.38), polynom)))), KK, factor);

$$\left(\frac{1}{35659927296}\left(1706786219\sqrt{3}\,7^{5/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right) - \frac{1}{5896152}\left(454837\,7^{5/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right) - \frac{1}{636784416}\left(80677003\sqrt{3}\,7^{1/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right) + \frac{1}{1474038}\left(300865\,7^{1/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right)\right) KK^5 + \left(\frac{1}{104976}\left(241\sqrt{3}\,7^{5/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right) - \frac{1}{52488}\left(311\sqrt{3}\,7^{1/8}\,2^{3/4}\sqrt{2140679\sqrt{7}\sqrt{3}-2344320\sqrt{7}+5663177\sqrt{3}-6203376}\right)\right) KK \quad \textbf{(14.2.39)}$$

We have the expansion of nu in terms of KK:

> expand(map(rationalize, series(subs(K = Kc + KK^4, nusupK), KK, 5)))

$$1+\frac{\sqrt{7}}{7}+\left(-\frac{9\sqrt{7}}{14}+\frac{18}{7}\right)KK^4+O\left(KK^8\right) \quad \textbf{(14.2.40)}$$

The coefficient in front of (nu - nu_c)^{1/4} in the expansion:

> simplify $\left(\text{expand}\left(\text{rationalize}\left(\frac{\text{coeff}((\mathbf{14.2.39}), KK, 1)}{\left(\frac{18}{7}-\frac{9\sqrt{7}}{14}\right)^{\frac{1}{4}}}\right)\right)\right)$;

$$\frac{1}{4408992}\left(\sqrt{3}\,7^{1/8}\,2^{3/4}\sqrt{\left(2140679\sqrt{7}+5663177\right)\sqrt{3}-2344320\sqrt{7}-6203376}\right. \quad \textbf{(14.2.41)}$$

$\sqrt{294-42\sqrt{7}}\,\left(355\sqrt{7}-889\right)\Big)$

A simpler expression for this coefficient:

> $simplify\left(\textbf{(14.2.41)} - \frac{\sqrt{3+2\sqrt{3}}\;2^{3/4}\,7^{3/8}\left(-10\sqrt{3}\sqrt{7}+18\sqrt{7}-23\sqrt{3}+63\right)}{144}\right);$

$$0 \qquad \textbf{(14.2.42)}$$

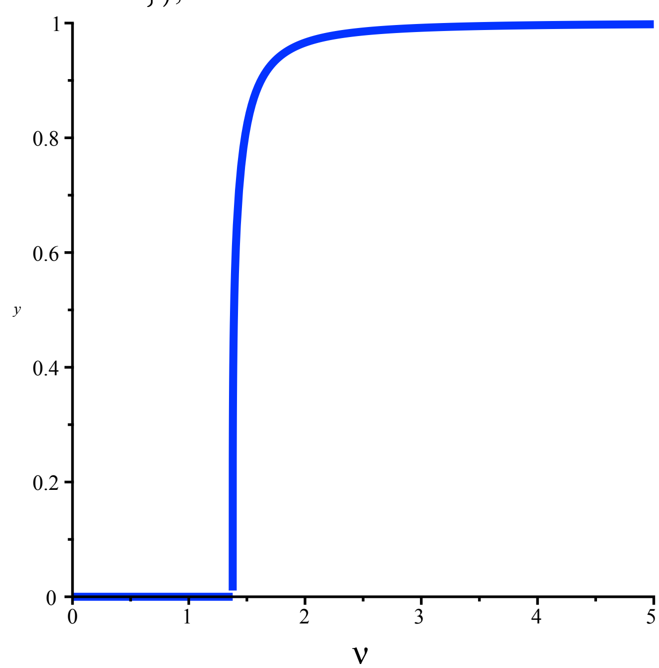
We want a plot of the probability in terms of nu, so we need K in terms of nu:

> $Knu := RootOf(numer(nusupK - \mathrm{nu}), K);$

$$Knu := RootOf\left((\nu+1)\,\_Z^3 + (3\,\nu+3)\,\_Z^2 + (-3\,\nu+9)\,\_Z - 9\,\nu+11\right) \qquad \textbf{(14.2.43)}$$

> $Plotnusupc := plot(subs(Vp = VK11, Vm = VK22, K = Knu, Probapercosimple), \mathrm{nu} = nuc\,..5, y = 0\,..1, color = \text{"Blue"}, thickness = 3) : Plotnusubc := plot(0, \mathrm{nu} = 0\,..\,nuc, color = \text{"Blue"}, thickness = 3) :$

> $display(\{Plotnusupc, Plotnusubc\});$



Hypergeometric functions and their singular expansion in Theorem 1.2

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{\frac{1}{2}}\cdot(1-z\cdot x)^{-\frac{4}{3}}, x = 0\,..1\right)\right)$ assuming $z < 1$ **and** $z > 0$;
$series(\%, z = 1, 2);$

$$\frac{\mathrm{hypergeom}\left(\left[\frac{4}{3}, \frac{3}{2}\right], [3], z\right)}{8}$$

(15.1)

$$\frac{3\sqrt{\pi}}{2\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{9\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(z-1)^{1/6}}{2\,\pi^{3/2}} - \frac{18\sqrt{\pi}\,(z-1)}{5\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} + \frac{135\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(z-1)^{7/6}}{14\,\pi^{3/2}} + \mathrm{O}\left((z-1)^2\right) \tag{15.1}$$

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{\frac{1}{2}}\cdot(1-z\cdot x)^{-\frac{2}{3}}, x=0..1\right)\right)$ assuming $z<1$ **and** $z>0$;
> $series(\%, z=1, 2)$;

$$\frac{\mathrm{hypergeom}\left(\left[\frac{2}{3}, \frac{3}{2}\right], [3], z\right)}{8}$$

$$\frac{9\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{16\,\pi^{3/2}} - \frac{6\sqrt{\pi}\,(-1)^{5/6}(z-1)^{5/6}}{5\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{27\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(z-1)}{8\,\pi^{3/2}} + \frac{126\sqrt{\pi}\,(-1)^{5/6}(z-1)^{11/6}}{55\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} + \mathrm{O}\left((z-1)^2\right) \tag{15.2}$$

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left((x\cdot(1-x))^{\frac{1}{2}}\cdot(1-z\cdot x)^{-\frac{1}{3}}, x=0..1\right)\right)$ assuming $z<1$ **and** $z>0$;
> $series(\%, z=1, 2)$;

$$\frac{\mathrm{hypergeom}\left(\left[\frac{1}{3}, \frac{3}{2}\right], [3], z\right)}{8}$$

$$\frac{3\sqrt{\pi}}{20\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} + \frac{9\sqrt{\pi}\,(z-1)}{20\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)} - \frac{9\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(z-1)^{7/6}}{7\,\pi^{3/2}} + \mathrm{O}\left((z-1)^2\right) \tag{15.3}$$