# THÈSE

*présentée pour obtenir*

## LE GRADE DE DOCTEUR EN SCIENCES DE L'UNIVERSITÉ PARIS XI

Spécialité : Mathématiques

*par*

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*Sujet :*

## Conditionnement de grands arbres aléatoires et configurations planes non-croisées

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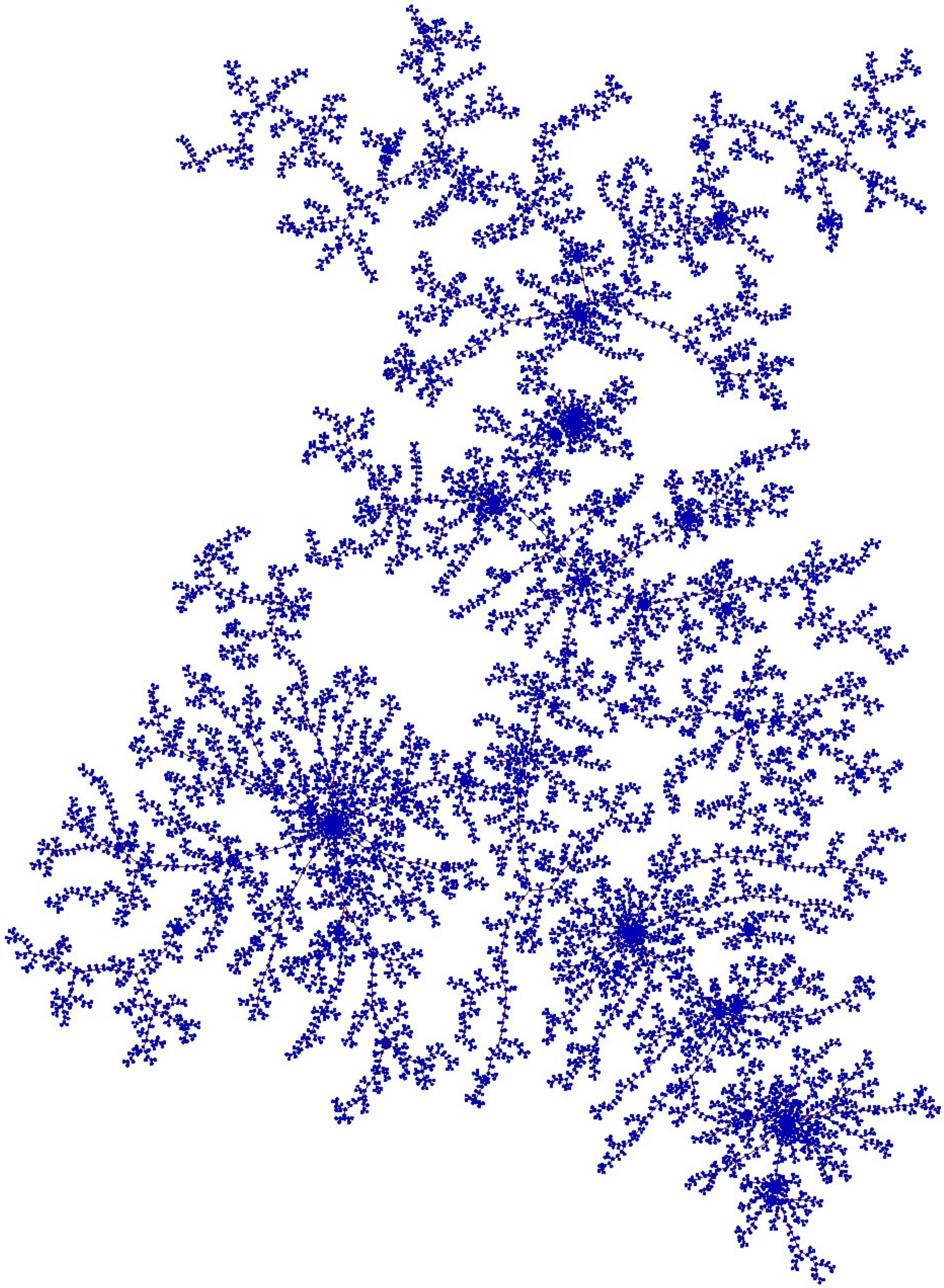

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*Я помню чудное мгновенье:*
*Передо мной явилась ты,*
*Как мимолетное виденье,*
*Как гений чистой красоты.*

# *Résumé*

Les limites d'échelle de grands arbres aléatoires jouent un rôle central dans cette thèse. Nous nous intéressons plus spécifiquement au comportement asymptotique de plusieurs fonctions codant des arbres de Galton-Watson conditionnés. Nous envisageons plusieurs types de conditionnements faisant intervenir différentes quantités telles que le nombre total de sommets ou le nombre total de feuilles, avec des lois de reproductions différentes. Lorsque la loi de reproduction est critique et appartient au domaine d'attraction d'une loi stable, un phénomène d'universalité se produit : ces arbres ressemblent à un même arbre aléatoire continu, l'arbre de Lévy stable. En revanche, lorsque la criticalité est brisée, la communauté de physique théorique a remarqué que des phénomènes de condensation peuvent survenir, ce qui signifie qu'avec grande probabilité, un sommet de l'arbre a un degré macroscopique comparable à la taille totale de l'arbre. Une partie de cette thèse consiste à mieux comprendre ce phénomène de condensation. Finalement, nous étudions des configurations non croisées aléatoires, obtenues à partir d'un polygône régulier en traçant des diagonales qui ne s'intersectent pas intérieurement, et remarquons qu'elles sont étroitement reliées à des arbres de Galton-Watson conditionnés à avoir un nombre de feuilles fixé. En particulier, ce lien jette un nouveau pont entre les dissections uniformes et les arbres de Galton-Watson, ce qui permet d'obtenir d'intéressantes conséquences de nature combinatoire.

# *Abstract*

Scaling limits of large random trees play an important role in this thesis. We are more precisely interested in the asymptotic behavior of several functions coding conditioned Galton-Watson trees. We consider several types of conditioning, involving different quantities such as the total number of vertices or leaves, as well as several types of offspring distributions. When the offspring distribution is critical and belongs to the domain of attraction of a stable law, a universality phenomenon occurs: these trees look like the same continuous random tree, the so-called stable Lévy tree. However, when the offspring distribution is not critical, the theoretical physics community has noticed that condensation phenomena may occur, meaning that with high probability there exists a unique vertex with macroscopic degree comparable to the total size of the tree. The goal of one of our contributions is to grasp a better understanding of this phenomenon. Last but not least, we study random non-crossing configurations consisting of diagonals of regular polygons, and notice that they are intimately related to Galton-Watson trees conditioned on having a fixed number of leaves. In particular, this link sheds new light on uniform dissections and allows us to obtain some interesting results of a combinatorial flavor.

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# Première Partie

# *Introduction*

*Cette introduction est articulée autour de deux parties : nous présentons d'abord nos contributions [68, 70, 71] centrées sur les arbres aléatoires, puis celles [69, 28] portant sur des applications à l'étude des configurations planes non croisées aléatoires. Nous insistons principalement sur les techniques et idées utilisées et renvoyons le lecteur intéressé aux parties ultérieures pour les détails. Dans l'introduction, les résultats originaux contenus dans cette thèse sont encadrés.*

Chapitre 1

# Arbres de Galton-Watson conditionnés

*Arbres de la forêt, vous connaissez mon âme !*
*Au gré des envieux, la foule loue et blâme ;*
*Vous me connaissez, vous ! - vous m'avez vu souvent,*
*Seul dans vos profondeurs, regardant et rêvant.*

— Victor Hugo, *Aux arbres*

## 1.1 Arbres de Galton-Watson

Nous commençons par introduire les définitions des objets qui nous intéresseront, puis nous expliquons les techniques usuelles et les résultats connus apparaissant dans l'étude des limites locales et limites d'échelle d'arbres aléatoires, et concluons par un bref historique de cette théorie. Notre contribution à l'étude des grands arbres de Galton-Watson conditionnés est présentée en Section 1.2.

### 1.1.1 Arbres et forêts

Nous suivons le formalisme introduit par Neveu [88] pour définir les arbres plans enracinés. Soit $\mathbb{N} = \{0, 1, \ldots\}$ l'ensemble des entiers naturels, et $\mathbb{N}^* = \{1, \ldots\}$. On désigne par $\mathbb{U}$ l'ensemble des **étiquettes** :

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

où, par convention, $(\mathbb{N}^*)^0 = \{\emptyset\}$. Nous noterons $<$ l'ordre lexicographique sur $\mathbb{U}$ (par exemple $\emptyset < 1 < 21 < 22$). Ainsi, un élément de $\mathbb{U}$ est une suite $u = u_1 \cdots u_j$ d'entiers strictement positifs. Par définition, la **génération** de $u = u_1 \cdots u_j$, notée $|u|$, est l'entier $j$. Lorsque $u = u_1 \cdots u_j$ et $v = v_1 \cdots v_k$ sont des éléments de $\mathbb{U}$, on note $uv = u_1 \cdots u_j v_1 \cdots v_k$ la concaténation de $u$ et $v$. En particulier, on a $u\emptyset = \emptyset u = u$. Finalement, un **arbre plan enraciné** $\tau$ est un sous-ensemble (fini ou infini) de $\mathbb{U}$ vérifiant les trois conditions suivantes :

1. $\emptyset \in \tau$,
2. si $v \in \tau$ et $v = uj$ pour un certain $j \in \mathbb{N}^*$, alors $u \in \tau$,
3. pout tout $u \in \tau$, il existe $k_u(\tau) \in \mathbb{N} \cup \{\infty\}$ tel que, pour chaque $j \in \mathbb{N}^*$, $uj \in \tau$ si et seulement si $1 \leqslant j \leqslant k_u(\tau)$.

Dans toute la suite, par **arbre** nous entendons toujours arbre plan enraciné. L'ensemble des arbres est noté $\mathbb{T}$. Nous visualiserons souvent les sommets d'un arbre $\tau$ comme les individus d'une population dont $\tau$ est l'arbre généalogique. À ce titre, pour $u \in \tau$, nous appellerons $k_u(\tau)$ le **nombre d'enfants** de $u$. La taille totale de $\tau$, ou nombre total de sommets, sera noté $\zeta(\tau) = \mathrm{Card}(\tau)$. Pour un arbre $\tau$ et $u \in \tau$, on note $T_u\tau = \{v \in \mathcal{U} ; uv \in \tau\}$ qui est lui même un arbre. La **hauteur** de $\tau$, notée $\mathcal{H}(\tau)$, est la plus grande génération d'un sommet de $\tau$. Une **feuille** de $\tau$ est un sommet $u \in \tau$ tel que $k_u(\tau) = 0$. Finalement, pour $j \geqslant 1$, une **forêt** de $j$ arbres est un élément de $\mathbb{T}^j$.

Insistons sur le fait que, contrairement au cadre habituel, nous autorisons ici $\tau$ et $k_u(\tau)$ à être infinis (cela sera en effet utile lorsque nous présenterons l'arbre de Galton-Watson critique conditionné à survivre, et serons amenés à considérer des arbres ayant un sommet de degré infini, voir la Section 1.3.1).

Nous rappelons maintenant la définition d'un arbre de Galton-Watson. Soit $\rho$ une mesure de probabilité sur $\mathbb{N}$ telle que $\rho(1) < 1$. La loi d'un **arbre de Galton-Watson de loi de reproduction** $\rho$ est l'unique mesure de probabilité $\mathbb{P}_\rho$ sur $\mathbb{T}$ vérifiant les deux conditions suivantes :

1. $\mathbb{P}_\rho[k_\emptyset = j] = \rho(j)$ pour $j \geqslant 0$,
2. pour tout $j \geqslant 1$ tel que $\rho(j) > 0$, conditionnellement à $\{k_\emptyset = j\}$, les arbres $T_1\tau, \dots, T_j\tau$ sont i.i.d. de loi $\mathbb{P}_\rho$.

Il s'ensuit que

$$\mathbb{P}_\rho[\tau] = \prod_{u \in \tau} \rho(k_u(\tau)). \tag{1.1}$$

Un arbre aléatoire de loi $\mathbb{P}_\rho$ sera appelé arbre de Galton-Watson de loi de reproduction $\rho$, ou plus simplement un $GW_\rho$ arbre. Il est bien connu qu'un $GW_\rho$ arbre est presque sûrement fini si et seulement si la moyenne de $\rho$ est au plus 1. Lorsque la moyenne de $\rho$ vaut exactement un, on dira que $\rho$ (et par extension un $GW_\rho$ arbre) est **critique**.

Pour un entier $j \geqslant 1$, $\mathbb{P}_{\rho,j}$ est la mesure de probabilité sur $\mathbb{T}^j$ qui est la loi de $j$ $GW_\rho$ arbres indépendants.

Pour un arbre $\tau$ et $k \geqslant 0$, on note $Z_k(\tau) = \mathrm{Card}\{u \in \tau ; |u| = k\}$ le nombre de sommets à hauteur $k$. Le processus $(Z_k(\tau), k \geqslant 0)$ sous $\mathbb{P}_\rho$ est appelé processus de Bienaymé-Galton-Watson.

### 1.1.2 Arbres de Galton-Watson conditionnés par le nombre de sommets

Fixons une loi de reproduction $\mu$. Dans toute la suite de cette Section 1.1, pour chaque $n \geqslant 1$ tel que $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$, $t_n$ désigne un arbre de Galton-Watson conditionné à avoir $n$ sommets, c'est-à-dire un arbre aléatoire de loi $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) = n]$.

Une partie de cette thèse consiste à comprendre la structure de $t_n$ lorsque $n$ est grand, en fonction des caractéristiques de $\mu$.

#### Arbres aléatoires vus comme arbres de Galton-Watson conditionnés

Plusieurs classes d'arbres aléatoires à $n$ sommets (souvent appelées « classes combinatoires ») peuvent être réalisées comme arbres de Galton-Watson conditionnés à avoir $n$ sommets pour des lois de reproduction particulières. Par exemple :

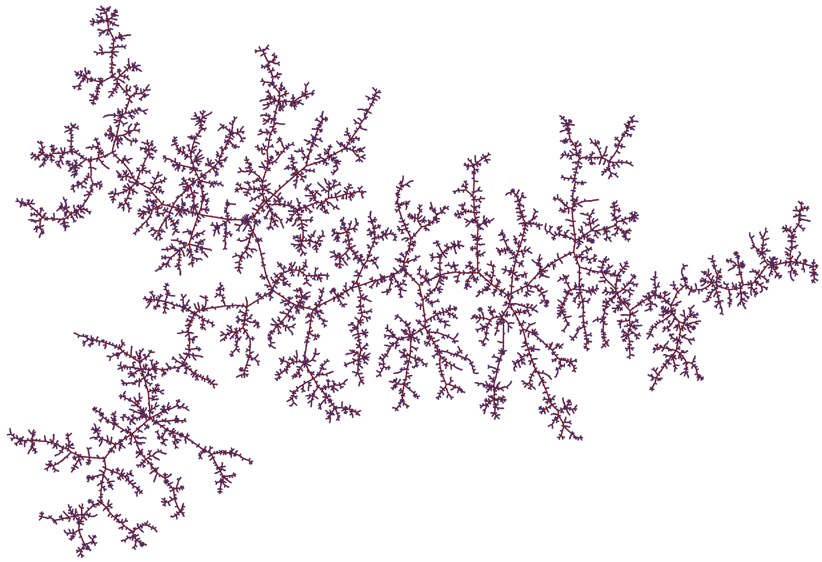

**FIGURE 1.1 –** Une réalisation de $t_{13068}$ pour une loi de reproduction critique et de variance finie.

- lorsque $\mu$ est la loi géométrique de paramètre $1/2$, $t_n$ est uniformément distribué sur l'ensemble des arbres à $n$ sommets,
- lorsque $\mu(0) = 1/2$ et $\mu(2) = 1/2$, $t_n$ est uniformément distribué sur l'ensemble des arbres binaires à $n$ sommets,
- lorsque $\mu$ est la loi de Poisson de paramètre $1$, $t_n$ est uniformément distribué sur l'ensemble des arbres étiquetés non ordonnés à $n$ sommets.

Nous renvoyons à [59, Section 10] pour des précisions et d'autres exemples. Mentionnons finalement que la Proposition 13 établit qu'une classe naturelle d'arbres aléatoires se réalise comme un arbre de Galton-Watson conditionné à avoir un nombre de feuilles fixé (et non pas un nombre de sommets fixé, comme nous en avons l'habitude).

#### Arbres simplement générés

On considère une suite $\mathbf{w} = (w_k)_{k \geqslant 0}$ de réels positifs telle que $w_0 > 0$ et telle qu'il existe $k > 1$ avec $w_k > 0$ (on dira que $\mathbf{w}$ est une suite de poids). Notons $\mathbb{T}_f$ l'ensemble des arbres finis ainsi que, pour tout $n \geqslant 1$, $\mathbb{T}_n$ l'ensemble des arbres à $n$ sommets. Pour tout $\tau \in \mathbb{T}_f$, on définit le poids $w(\tau)$ de $\tau$ par :

$$w(\tau) = \prod_{u \in \tau} w_{k_u(\tau)}.$$

On pose alors pour $n \geqslant 1$ :

$$Z_n = \sum_{\tau \in \mathbb{T}_n} w(\tau).$$

Pour tout $n \geqslant 1$ tel que $Z_n \neq 0$, soit $\mathcal{T}_n$ un arbre aléatoire à valeurs dans $\mathbb{T}_n$ tel que pour tout $\tau \in \mathbb{T}_n$ :

$$\mathbb{P}\left[\mathcal{T}_n = \tau\right] = \frac{w(\tau)}{Z_n}.$$

L'arbre aléatoire $\mathcal{T}_n$ est dit **simplement généré** (cette définition remonte à Meir & Moon [83]). Compte tenu de (1.1), il apparaît que les arbres de Galton-Watson conditionnés par le nombre de sommets sont un cas particulier d'arbres simplement générés.

#### Familles exponentielles

Soit $\lambda > 0$ un paramètre fixé tel que $Z_\lambda = \sum_{i \geqslant 0} \mu_i \lambda^i < \infty$. On pose $\mu_i^{(\lambda)} = \mu_i \lambda^i / Z_\lambda$ pour $i \geqslant 0$. En utilisant (1.1), il est aisé de voir qu'alors un $GW_\mu$ arbre conditionné à avoir $n$ sommets a la même loi qu'un $GW_{\mu^{(\lambda)}}$ arbre conditionné à avoir $n$ sommets. On dit que $\mu$ et $\mu^{(\lambda)}$ appartiennent à la même **famille exponentielle**. Ainsi, s'il existe $\lambda > 0$ tel que $Z_\lambda < \infty$ et $\mu^{(\lambda)}$ soit critique, alors étudier un arbre de Galton-Watson non critique conditionné revient à étudier un arbre de Galton-Watson critique conditionné (cette technique a été introduite par Kennedy [63]). Il est clair qu'on ne peut pas toujours se ramener à l'étude d'un arbre de Galton-Watson critique : le cas échéant, on dit que $\mu$ est **non générique** (par exemple si $\mu$ est sous-critique et le rayon de convergence de $\sum \mu_i z^i$ vaut 1). Nous verrons que dans ce cas des phénomènes intéressants surviennent.

Finalement, en utilisant ce procédé, il est possible de montrer que si $\mathcal{T}_n$ est un arbre simplement généré associé à la suite de poids $\mathbf{w}$, alors il existe une loi de reproduction $\mu$ telle que $\mathcal{T}_n$ a la même loi qu'un $GW_\mu$ arbre conditionné à avoir $n$ sommets si et seulement si le rayon de convergence de $\sum w_i z^i$ est strictement positif (voir [59, Section 4] pour une preuve).

### 1.1.3 Codage des arbres de Galton-Watson

L'étude des limites d'échelle d'arbres de Galton-Watson conditionnés passe par l'étude des convergences de fonctions renormalisées qui les codent. L'idée d'utiliser un codage par une fonction pour étudier des arbres aléatoires remonte à Harris [54]. L'intérêt réside dans les faits combinés que ces fonctions ont d'intéressantes propriétés probabilistes et qu'il existe de nombreuses stratégies disponibles permettant de prouver une convergence fonctionnelle. Nous allons présenter trois différentes fonctions (la marche de Lukasiewicz, la fonction de hauteur et enfin la fonction de contour) codant un même arbre, tout en présentant leurs spécificités.

#### Codage par la marche de Lukasiewicz

Soit $\tau$ un arbre fini, et considérons $u(0) < u(1) < \cdots < u(\zeta(\tau) - 1)$ ses sommets rangés dans l'ordre lexicographique. La **marche de Lukasiewicz** $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ de $\tau$ est définie par $\mathcal{W}_0(\tau) = 0$ et, pour $0 \leqslant n \leqslant \zeta(\tau) - 1$ :

$$\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1.$$

Voir Fig. 1.2 pour un exemple.

Pour tout arbre $\tau$, il est facile de voir (par exemple par récurrence) que $\mathcal{W}_n(\tau) \geqslant 0$ pour $0 \leqslant n \leqslant \zeta(\tau) - 1$, et que $\mathcal{W}_{\zeta(\tau)}(\tau) = -1$.

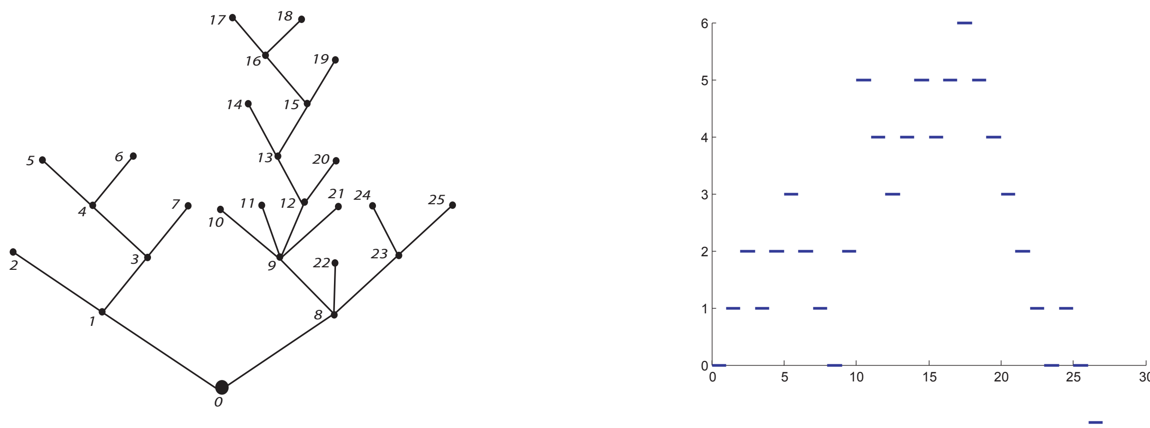


**FIGURE 1.2 –** Un arbre $\tau$ dont les sommets sont numérotés dans l'ordre lexicographique, et sa marche de Lukasiewicz associée. Ici $\zeta(\tau) = 26$.

La caractéristique cruciale de ce codage est le fait que la marche de Lukasiewicz d'un arbre de Galton-Watson se comporte comme une marche aléatoire conditionnée. Plus précisément, soit $(W_n\,; n \geqslant 0)$ une marche aléatoire sur $\mathbb{Z}$, issue de $0$ et dont la loi des sauts est donnée par $\nu(k) = \mu(k+1)$ pour $k \geqslant -1$. On note $\zeta_1 = \inf\{n \geqslant 0\,; W_n = -1\}$. La proposition suivante peut être trouvée dans Le Gall & Le Jan [76, Corollaire 2.2] ou Bennies & Kersting [13, Proposition 2].

**Proposition 1.1.1.** *Supposons que la moyenne de $\mu$ est au plus $1$. Alors la loi de $(W_0, W_1, \ldots, W_{\zeta_1})$ coincide avec celle de la marche de Lukasiewicz d'un $\mathrm{GW}_\mu$ arbre. En particulier, le nombre de sommets d'un $\mathrm{GW}_\mu$ arbre a la même loi que $\zeta_1$.*

Il est crucial de remarquer que la marche aléatoire $(W_n)_{n \geqslant 1}$ est centrée si et seulement si $\mu$ est critique.

Quand bien même le codage par la marche de Lukasiewicz d'un arbre $\tau$ est une bijection, certaines caractéristiques naturelles de $\tau$ ne peuvent pas lues « simplement » sur $\mathcal{W}(\tau)$ (par exemple le nombre d'individus présents à une génération donnée). Mentionnons tout de même que des exceptions existent : par exemple, pour trouver le nombre maximal d'enfants d'un sommet de $\tau$, il suffit d'ajouter un au saut maximal de $\mathcal{W}(\tau)$.

#### Marche de Lukasiewicz d'un arbre de Galton-Watson conditionné et transformée de Vervaat

Un corollaire extrêmement utile de la Proposition 1.1.1 permet d'identifier la loi de la marche de Lukasiewicz d'un arbre de Galton-Watson conditionné à avoir un nombre de sommets fixé. Plus précisément, soit $n \geqslant 1$ tel que $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ et notons $t_n$ un $\mathrm{GW}_\mu$ arbre conditionné à avoir $n$ sommets. Alors la loi de $(\mathcal{W}_0(t_n), \mathcal{W}_1(t_n), \ldots, \mathcal{W}_n(t_n))$ coïncide avec la loi de $(W_0, W_1, \ldots, W_n)$ sous la loi conditionnelle $\mathbb{P}[\,\cdot\,|\,\zeta_1 = n]$.

En utilisant un procédé appelé transformation de Vervaat, que nous allons maintenant décrire, il est possible de se ramener à étudier la loi de de $(W_0, W_1, \ldots, W_n)$ sous la loi conditionnelle plus simple $\mathbb{P}[\,\cdot\,|W_n = -1]$. Il n'y a plus de condition de positivité sous ce nouveau conditionnement, ce qui simplifie dans une grande mesure plusieurs points techniques. Pour définir la transformée de Vervaat, nous avons d'abord besoin d'introduire quelques notations.

Si $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ et $i \in \mathbb{Z}/n\mathbb{Z}$, on note $\mathbf{x}^{(i)}$ le $i$-ième translaté cyclique de $\mathbf{x}$ défini par $x_k^{(i)} = x_{i+k \bmod n}$ pour $1 \leqslant k \leqslant n$. Soient $n \geqslant 1$ un entier et $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. On pose $w_j = x_1 + \cdots + x_j$ pour $1 \leqslant j \leqslant n$ et on note $i_*(\mathbf{x})$ l'entier positif défini par $i_*(\mathbf{x}) = \inf\{j \geqslant 1; w_j = \min_{1 \leqslant i \leqslant n} w_i\}$. La **transformée de Vervaat** de $\mathbf{x}$, notée $\mathbf{V}(\mathbf{x})$, est par définition $\mathbf{x}^{(i_*(\mathbf{x}))}$. Il est également possible de définir la transformée de Vervaat d'une fonction de $\mathbb{D}([0,1], \mathbb{R})$ (voir Définition 4.6.6). La propriété évoquée au début de ce paragraphe est la suivante (il s'agit d'un fait bien connu, voir par exemple [90, Section 5]) :

**Proposition 1.1.2.** *Soit $(W_n, n \geqslant 0)$ la même marche aléatoire que dans la Proposition 1.1.1 et posons $R_k = W_k - W_{k-1}$ pour $k \geqslant 1$. Soit $n \geqslant 1$ un entier tel que $\mathbb{P}[W_n = -1] > 0$. Alors la loi de $\mathbf{V}(R_1, \ldots, R_n)$ sous $\mathbb{P}[\,\cdot\,| W_n = -1]$ coïncide avec la loi de $(R_1, \ldots, R_n)$ sous la loi conditionnée $\mathbb{P}[\,\cdot\,| \zeta_1 = n]$.*

En particulier, d'après la Proposition 1.1.1, la loi de $\mathbf{V}(R_1, \ldots, R_n)$ sous $\mathbb{P}[\,\cdot\,| W_n = -1]$ coïncide avec la loi de $(\mathcal{W}_1(\mathfrak{t}_n), \mathcal{W}_2(\mathfrak{t}_n) - \mathcal{W}_1(\mathfrak{t}_n), \ldots, \mathcal{W}_n(\mathfrak{t}_n) - \mathcal{W}_{n-1}(\mathfrak{t}_n))$.

### Codage par la fonction de contour

Pour définir la fonction de contour d'un arbre $\tau$, imaginons une particule qui explore l'arbre en partant de la racine à vitesse unité (on suppose que toutes les arêtes de l'arbre sont de longueur unité), de la façon suivante : lorsque la particule quitte un sommet $u$, celle-ci se dirige, si possible, vers le premier enfant de $u$ qu'elle n'a pas encore visité, sinon elle revient au parent de $u$. Pour $0 \leqslant t \leqslant 2(\zeta(\tau) - 1)$, $C_t(\tau)$ est alors défini comme étant égal à la distance entre la particule et la racine de $\tau$. Pour des raisons techniques, on pose $C_t(\tau) = 0$ lorsque $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. La fonction $C(\tau)$ est appelée **fonction de contour** de l'arbre $\tau$ (voir Fig. 1.3 pour un exemple). Si la particule se trouve à un sommet $u \in \tau$ à l'instant $t \in [0, 2(\zeta(\tau) - 1)]$, on dit que $u$ est visité par la fonction de contour à l'instant $u$.


**FIGURE 1.3 –** Respectivement les fonctions de contour et de hauteur de l'arbre $\tau$ de la Fig. 1.2.

Donnons une définition plus formelle de $C(\tau)$. À cet effet, pour $u, v \in \tau$, notons $u \wedge v$ l'ancêtre commun de plus grande génération parmi tous les ancêtres communs de $u$ et $v$. Si $l_1, \ldots, l_p$ désignent les feuilles de $\tau$, $C(\tau)$ est définie comme étant la fonction affine par morceaux sur $\mathbb{R}_+$, dont les pentes sont $-1$ ou $+1$, et dont les valeurs des extremas locaux sont successivement $0, |l_1|, |l_1 \wedge l_2|, |l_2|, \ldots, |l_{p-1} \wedge l_p|, |l_p|, 0$.

Utiliser le codage par la fonction de contour présente certains avantages : certaines propriétés se lisent plus aisément sur $C(\tau)$ que sur $\mathcal{W}(\tau)$ (par exemple la hauteur de $\tau$ est simplement le supremum de $C(\tau)$) et nous verrons dans la section 1.1.5 qu'une convergence des fonctions de contour renormalisées implique une convergence au sens de Gromov-Hausdorff des arbres, vus comme espaces métriques compacts.

Finalement, une propriété très importante dont jouit la fonction de contour d'un arbre de Galton-Watson est celle d'invariance par retournement du temps. Plus précisément, sous $\mathbb{P}_\mu$, les deux fonctions $(C_t(\tau), 0 \leqslant t \leqslant 2\zeta(\tau) - 2)$ et $(C_{2\zeta(\tau)-2-t}(\tau), 0 \leqslant t \leqslant 2\zeta(\tau) - 2)$ ont même loi. Expliquons comment cette propriété sera appliquée. Considérons $(B_n)_{n\geqslant 1}$ une suite de réels strictement positifs divergeant vers $+\infty$, $(t_n, n \geqslant 1)$ une suite d'arbres aléatoires et $C^{(n)}$ la fonction définie sur $[0, 1]$ par $C^{(n)}_t = \frac{1}{B_n} C_{(2\zeta(t_n)-2)t}$.

**Proposition.** *Supposons que, pour tout* $a \in (0, 1)$, $(C^{(n)}_t, 0 \leqslant t \leqslant a)$ *converge en loi lorsque* $n \to \infty$ *dans l'espace* $\mathcal{C}([0, a], \mathbb{R})$. *Alors* $(C^{(n)}_t, 0 \leqslant t \leqslant 1)$ *converge en loi lorsque* $n \to \infty$ *dans l'espace* $\mathcal{C}([0, 1], \mathbb{R})$.

Pour démontrer ce résultat, il suffit de remarquer que la convergence des marginales fini-dimensionnelles est une conséquence immédiate de l'hypothèse et que la tension au point $a = 1$ provient de l'invariance par retournement du temps évoquée précédemment.

Cependant, la fonction de contour d'un arbre de Galton-Watson ne satisfait pas en général à la propriété de Markov (sauf dans le cas particulier d'une loi de reproduction géométrique), ce qui rend son étude complexe.

#### Codage par la fonction de hauteur

La **fonction de hauteur** $H(\tau) = (H_n(\tau), 0 \leqslant n < \zeta(\tau))$ est définie par $H_n(\tau) = |u(n)|$ pour $0 \leqslant n < \zeta(\tau)$. Pour des raisons techniques, on pose $H_k(\tau) = 0$ pour $k \geqslant \zeta(\tau)$. On prolonge alors $H(\tau)$ à $\mathbb{R}_+$ par interpolation linéaire en posant $H_t(\tau) = (1-\{t\})H_{\lfloor t \rfloor}(\tau) + \{t\}H_{\lfloor t \rfloor+1}(\tau)$ pour $t \geqslant 0$, où $\{t\} = t - \lfloor t \rfloor$ (voir Fig. 1.3 pour un exemple).

L'étude de la fonction de contour se fera par le truchement de la fonction de hauteur. En effet, d'une part il est possible d'exprimer de manière exploitable $H(\tau)$ en fonction de $\mathcal{W}(\tau)$, et d'autre part $C(\tau)$ en fonction de $H(\tau)$. Précisons cela. La proposition suivante, obtenue indépendamment par Le Gall & Le Jan [76, Corollary 2.2] et Bennies & Kersting [13, Section 5], exprime le fait que $(H_0(\tau), \ldots, H_n(\tau))$ est une fonction mesurable de $(\mathcal{W}_0(\tau), \ldots, \mathcal{W}_n(\tau))$ pour tout entier $0 \leqslant n \leqslant \zeta(\tau) - 1$.

**Proposition.** *Pour tout entier* $0 \leqslant n \leqslant \zeta(\tau) - 1$, *on a :*

$$H_n(\tau) = \mathrm{Card}\left\{ j \,;\, 0 \leqslant j < n,\ \mathcal{W}_j(\tau) = \inf_{j \leqslant l \leqslant n} \mathcal{W}_l(\tau) \right\}. \tag{1.2}$$

Ainsi, bien que $H(\tau)$ ne satisfasse pas en général à la propriété de Markov sous $\mathbb{P}_\mu$, la fonction de hauteur est intimement liée à la marche de Lukasiewicz, qui elle est markovienne.

Afin de préciser le lien entre $H(\tau)$ et $C(\tau)$, il est nécessaire d'introduire quelques notations. Pour $0 \leqslant p < \zeta(\tau)$, notons $b_p = 2p - H_p(\tau)$, de sorte que $b_p$ représente le premier temps d'atteinte par la fonction de contour du $(p+1)$-ième sommet de $\tau$ dans l'ordre lexicographique (en effet, à cet instant-là la fonction de contour aura visité deux fois toutes les arêtes constituant le sous-arbre formé par les $(p+1)$ premier sommets de $\tau$, sauf les arêtes de l'unique chemin reliant la racine au $(p+1)$-ième sommet, qui sont au nombre de $H_p(\tau)$). Posons $b_{\zeta(\tau)} = 2(\zeta(\tau)-1)$. Alors pour $0 \leqslant n < \zeta(\tau)-1$ et $t \in (b_p, b_{p+1})$ :

$$C_t(\tau) = \begin{cases} H_p(\tau) - (t - b_p) & \text{si } t \in [b_p, b_{p+1}-1) \\ t - b_{p+1} + H_{p+1}(\tau) & \text{si } t \in [b_{p+1}-1, b_{p+1}), \end{cases}$$

et $C_t(\tau) = H_{\zeta(\tau)-1}(\tau) - (t - b_{\zeta(\tau)-1})$ si $t \in [b_{\zeta(\tau)-1}, b_{\zeta(\tau)})$.

### 1.1.4 Limites d'échelle d'arbres de Galton-Watson critiques

Soit $\mu$ une loi de reproduction critique, et comme précédemment, $t_n$ désigne un $GW_\mu$ arbre conditionné à avoir $n$ sommets pour chaque $n \geqslant 1$ tel que $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$. Dans cette partie, nous présentons les résultats connus concernant les limites d'échelle des différentes fonctions introduites codant $t_n$. Avant cela, nous introduisons les différents espaces fonctionnels mis en jeu.

**Espaces fonctionnels**

Soit I un intervalle. On note $\mathcal{C}(I, R)$ l'espace des fonctions continues de I dans $\mathbb{R}$ muni de la topologie de la convergence uniforme sur tous les compacts de I, qui en fait un espace polonais, c'est-à-dire un espace métrique complet et séparable. On note aussi $\mathbb{D}(I, \mathbb{R})$ l'espace des fonctions continues à droites et ayant des limites à gauche en tout point (càdlàg) de I dans $\mathbb{R}$, muni de la topologie $J_1$ de Skorokhod, qui en fait un espace polonais (voir [20, chap. 3], [57, chap. VI] pour les définitions et les propriétés usuelles de la topologie de Skorokhod).

**Variance finie : le théorème d'Aldous**

Commençons par rappeler la définition de l'excursion brownienne normalisée $(\mathbb{e}_t)_{0 \leqslant t \leqslant 1}$. À cet effet, considérons un mouvement brownien réel $(B_t)_{t \geqslant 0}$. Notons $g_1 = \sup\{t < 1 \,;\, B_t = 0\}$ et $d_1 = \inf\{t > 1 \,;\, B_t = 0\}$. Il est clair $d_1 - g_1 > 0$ presque sûrement. Pour $0 \leqslant t \leqslant 1$ on pose alors $\mathbb{e}_t = B_{g_1 + t(d_1 - g_1)}/\sqrt{d_1 - g_1}$.

Rappelons que $\sigma^2$ désigne la variance de $\mu$.

**Théorème 1.1.3** (Aldous). *On suppose que $\mu$ est critique et que $0 < \sigma^2 < \infty$. Alors la convergence*

$$\left( \frac{\sigma}{2\sqrt{n}} C_{2nt}(t_n) \,;\, 0 \leqslant t \leqslant 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbb{e}.$$

*a lieu en loi dans l'espace $\mathcal{C}([0,1], \mathbb{R})$.*

Un exemple d'application intéressante de ce théorème concerne la hauteur $\mathcal{H}(t_n)$ de $t_n$ : en remarquant que $H^{\text{exc}}(t_n)$ est le supremum de $C(t_n)$, il découle immédiatement du Théorème 1.1.3, que $\mathcal{H}(t_n)/\sqrt{n}$ converge en loi, lorsque $n \to \infty$, vers $\frac{2}{\sigma} \sup \mathbb{e}$. La fonction de répartition de $\sup \mathbb{e}$ est une fonction de Jacobi (voir [18]). Avec une hypothèse supplémentaire, Marckert & Mokkadem prouvent la convergence jointe suivante.

**Théorème** (Marckert & Mokkadem [79]). *Soit $\mu$ une loi de reproduction critique vérifiant $0 < \sigma^2 < \infty$. On suppose de plus qu'il existe $\alpha > 0$ tel que $\sum_{i \geqslant 0} e^{\alpha i} \mu_i < \infty$. Alors :*

$$\left( \frac{1}{\sigma\sqrt{n}} \mathcal{W}_{\lfloor nt \rfloor}(t_n), \frac{\sigma}{2\sqrt{n}} H_{nt}(t_n), \frac{\sigma}{2\sqrt{n}} C_{2nt}(t_n) \right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbb{e}_t, \mathbb{e}_t, \mathbb{e}_t)_{0 \leqslant t \leqslant 1}.$$

**Domaine d'attraction d'une loi stable : le théorème de Duquesne**

Une généralisation substantielle du théorème d'Aldous au cas où $\mu$ est dans le domaine d'attraction d'une loi stable a été obtenue par Duquesne. Rappelons que si $\theta \in (1, 2]$, on dit que $\mu$ est dans le domaine d'attraction d'une loi stable d'indice $\theta$ si $\theta = 2$ et la variance de $\mu$ est strictement positive et finie, ou bien si $\mu([j, \infty)) = L(j)/j^\theta$ pour $j \geqslant 1$, où $L : \mathbb{R}_+ \to \mathbb{R}_+$ est une fonction telle que $L(x) > 0$ pour $x$ suffisamment grand et $\lim_{x\to\infty} L(tx)/L(x) = 1$ pour tout $t > 0$ (une telle fonction est dite **à variation lente**). Dans la suite de cette partie, $\theta \in (1, 2]$ est fixé.

Dans le cas stable, la version continue de la marche de Lukasiewicz est l'excursion normalisée d'un processus de Lévy particulier. Plus précisément, soit $(X_t)_{t \geqslant 0}$ un processus de Lévy strictement stable spectralement positif d'indice $\theta$ (voir [15] pour des détails concernant les processus de Lévy). En d'autres termes, $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta)$ pour $t \geqslant 0$ et $\lambda > 0$ (lorsque $\theta = 2$, $X$ est simplement $\sqrt{2}$ fois le mouvement brownien réel). En revanche, lorsque $\theta \in (1, 2)$, la version continue des fonctions de hauteur et contour est un autre processus, appelé le **processus de hauteur** d'indice $\theta$, que nous allons décrire (nous renvoyons à [37, Chapitre 1]) pour les détails). Pour $0 \leqslant s \leqslant t$, on pose $I_s^t = \inf_{s \leqslant r \leqslant t} X_r$. Soit $t \geqslant 0$. Si $\theta = 2$, on pose $H_t = X_t - I_0^t$. Sinon, on pose

$$H_t := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{X_s < I_t^s + \epsilon\}} ds, \tag{1.3}$$

la limite existant en probabilité. Le processus $(H_t)_{t \geqslant 0}$ admet une modification continue que nous considérerons dorénavant. Donnons brièvement une motivation à cette définition. Pour $t > 0$ notons $\widehat{X}^{(t)}$ le processus défini par $\widehat{X}_s^{(t)} = X_t - X_{(t-s)-}$ pour $0 \leqslant s \leqslant t$ et posons $\widehat{S}_s^{(t)} = \sup_{0 \leqslant r \leqslant s} \widehat{X}_s^{(t)}$. Alors il est possible de prouver que $H_t$ a la même loi que le temps local au temps $t$ (convenablement normalisé) du processus $\widehat{S}^{(t)} - \widehat{X}^{(t)}$. Ainsi, $H_t$ correspond intuitivement à la « mesure » de l'ensemble $\{0 \leqslant s \leqslant t\,;\, X_s = \inf_{s \leqslant r \leqslant t} X_r\}$, par analogie avec (1.2). Par ailleurs, les excursions de $H$ au-dessus de $0$ coïncident avec les excursions de $X - I$ au dessus de $0$.

Les excursions normalisées $X^{exc}$ et $H^{exc}$ sont ensuite définies à partir de $X$ et $H$ comme suit. Pour $t \geqslant 0$, on pose $I_t = \inf_{[0,t]} X$. Soient $\underline{g}_1 = \sup\{s \leqslant 1\,;\, X_s = I_s\}$ et $\zeta_1 = \inf\{s > 1\,;\, X_s = I_s\} - \underline{g}_1$. Pour $0 \leqslant t \leqslant 1$ on pose alors :

$$(X_t^{exc}, H_t^{exc}) = \left( \zeta_1^{-\frac{1}{\theta}} (X_{\underline{g}_1 + \zeta_1 t} - X_{\underline{g}_1}), \zeta_1^{\frac{1}{\theta} - 1} H_{\underline{g}_1 + \zeta_1 t} \right).$$

Le processus $H^{exc}$ est appelé **excursion normalisée du processus de hauteur** d'indice $\theta$ et vérifie p.s. $H_0^{exc} = H_1^{exc} = 0$ et $H_t^{exc} > 0$ pour $t \in (0, 1)$ (voir Fig. 1.4 et 6.6 pour des simulations).

**Théorème 1.1.4** (Duquesne [36]). *Soit $\mu$ une loi de reproduction critique dans le domaine d'attraction d'une loi stable d'indice $\theta \in (1, 2]$. Il existe une suite de nombres réels strictement positifs $B_n \to \infty$ telle*

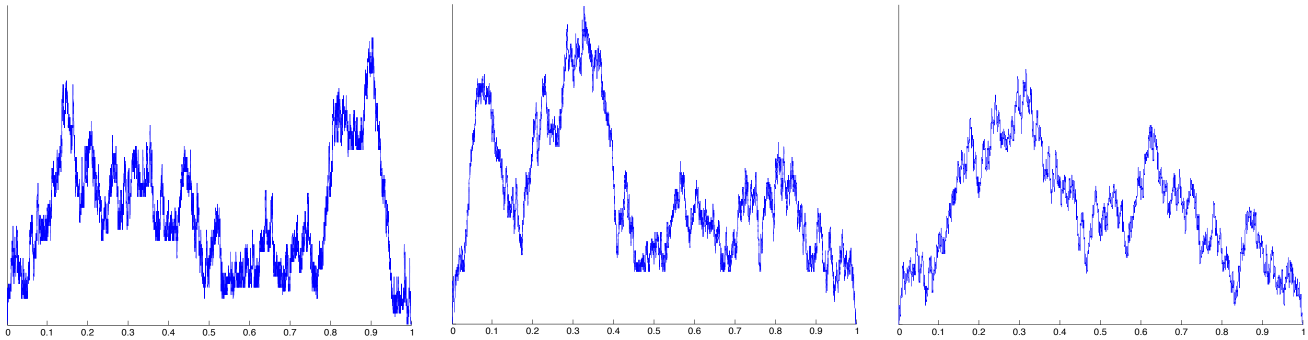

**FIGURE 1.4 –** Simulations de $H^{exc}$ pour respectivement $\theta = 1.1, 1.6$ et $2$.

*que*

$$\left(\frac{1}{B_n}\mathcal{W}_{\lfloor nt \rfloor}(t_n), \frac{B_n}{n}H_{nt}(t_n), \frac{B_n}{n}C_{2nt}(t_n)\right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t^{exc}, H_t^{exc}, H_t^{exc})_{0\leqslant t\leqslant 1}. \tag{1.4}$$

Cet énoncé mérite quelques remarques :

Il est possible d'expliciter $B_n$ en fonction de la loi de reproduction $\mu$. Si $\sigma^2$ est la variance de $\mu$ :

$$\begin{cases} B_n = \sigma\sqrt{n/2} & \text{si } \sigma^2 < \infty, \\ B_n = |\Gamma(1-\theta)|^{1/\theta} \inf\left\{x \geqslant 0\,;\ \mu([x,\infty)) \leqslant \dfrac{1}{n}\right\} & \text{si } \sigma^2 = \infty \text{ et } \theta < 2. \end{cases}$$

Lorsque $\sigma^2 = \infty$ et $\theta = 2$, la formule est légèrement plus compliquée et nous renvoyons au Théorème 4.1.10 pour un énoncé précis. Dans tous les cas, il est possible de montrer que $(B_n/n^{1/\theta})_{n\geqslant 1}$ est à variation lente, de sorte que $B_n$ est de l'ordre de $n^{1/\theta}$.

Lorsque $\theta = 2$, les deux processus $X^{exc}$ et $H^{exc}$ sont égaux et ont la loi de $\sqrt{2}\cdot\mathbb{e}$. Ainsi, dans ce cas, les trois processus limites sont identiques. En revanche, pour $\theta \neq 2$, $H^{exc}$ n'est pas un processus markovien. Nous avons déjà signalé que Marckert & Mokkadem [79] ont démontré le résultat dans le cas $\theta = 2$ dans le cas particulier où $\mu$ a un moment exponentiel.

### 1.1.5 $\mathbb{R}$-arbres compacts

Nous expliquons maintenant la construction d'arbres continus, appelés $\mathbb{R}$-arbres compacts, à partir d'une certaine classe de fonctions de type excursions. Par définition, un espace métrique $(\mathcal{T}, d)$ est un $\mathbb{R}$-arbre si pour tous $u, v \in \mathcal{T}$ les deux conditions suivantes sont vérifiées :

(i) Il existe une unique isométrie $f_{u,v} : [0, d(u,v)] \to \mathcal{T}$ telle qu'on ait $f_{u,v}(d(u,v)) = v$ et $f_{u,v}(0) = u$,

(ii) si $q : [0,1] \to \mathcal{T}$ est une application injective continue telle que $q(0) = u$ et $q(1) = v$, alors $q([0,1]) = f_{u,v}([0, d(u,v)])$.

Un $\mathbb{R}$-arbre enraciné est un triplet $(\mathcal{T}, \rho, d)$ où $(\mathcal{T}, d)$ est un $\mathbb{R}$-arbre $\rho = \rho(\mathcal{T})$ est un sommet distingué appelé racine, et sera souvent simplement noté $\mathcal{T}$ s'il n'y a pas d'ambiguité possible sur $\rho$ et $d$. Dans tout ce qui suit, tous les $\mathbb{R}$-arbres seront compacts.

On dit que deux $\mathbb{R}$-arbres compacts enracinés $\mathcal{T}$ et $\mathcal{T}$ sont équivalents s'il existe une bijection isométrique entre $\mathcal{T}$ et $\mathcal{T}'$ préservant la racine, et on note $\mathcal{T}_{\mathbb{R}}$ l'ensemble des classes d'équivalence de $\mathbb{R}$-arbres compacts enracinés.

Soit $g : [0,1] \to \mathbb{R}_+$ une fonction continue telle que $g(0) = g(1) = 1$. Pour $s, t \in [0,1]$, on pose

$$d_g(s,t) = g(s) + g(t) - 2 \min_{r \in [s \wedge t, s \vee t]} g(r).$$

Il est facile de voir que $d_g$ est une pseudo-distance sur $[0,1]$. On introduit alors la classe d'équivalence associée à $d_g$ : on pose $s \overset{g}{\sim} t$ si et seulement si $d_g(s,t) = 0$ (intuitivement, on identifie les points du graphe de $g$ qui sont en vis-à-vis), et on note $p_g : [0,1] \to \mathcal{T}$ la projection canonique. Si on note $\mathcal{T}_g := [0,1]/\overset{g}{\sim}$ l'espace métrique quotient, alors $(\mathcal{T}, p_g(0), d_g)$ est un $\mathbb{R}$-arbre compact enraciné (voir [38, Théorème 2.1]).

Dans le cas particulier où $g = \mathbb{e}$, où $\mathbb{e}$ est l'excursion brownienne normalisée, le $\mathbb{R}$-arbre $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ est appelé arbre brownien continu (abrégé en CRT pour « Continuum Random Tree ») et a été introduit par Aldous au début des années 1990. Plus généralement, pour un paramètre $\theta \in (1,2]$, l'arbre de Lévy stable de paramètre $\theta$ est par définition le $\mathbb{R}$-arbre $(\mathcal{T}_{H^{\mathrm{exc}}}, d_{H^{\mathrm{exc}}})$, qui a été introduit par Duquesne & Le Gall [37].

Finalement, il est possible de munir $\mathcal{T}_{\mathbb{R}}$ d'une topologie, appelée topologie de Gromov-Hausdorff, qui en fait un espace polonais, ce qui fournit un cadre privilégié pour étudier des convergences au sein de cet espace. Si $(E, \delta)$ est un espace métrique, on notera $\delta_H$ la distance de Hausdorff usuelle entre les compacts de $E$. Si $(\mathcal{T}, \rho, d)$ et $(\mathcal{T}', \rho', d')$ sont deux $\mathbb{R}$-arbres compacts enracinés, on note :

$$d_{GH}((\mathcal{T}, \rho, d), (\mathcal{T}', \rho', d')) = \inf \left( \delta_H(\phi(\mathcal{T}), \phi(\mathcal{T}')) \vee \delta(\phi(\rho), \phi'(\rho')) \right),$$

où l'infimum est pris sur toutes les injections isométriques $\phi : \mathcal{T} \to E$ et $\phi' : \mathcal{T}' \to E$ de $\mathcal{T}$ et $\mathcal{T}'$ dans un même espace métrique $(E, \delta)$ (nous renvoyons à [24] pour des précisions concernant la topologie de Gromov-Hausorff). Il est clair que $d_{GH}((\mathcal{T}, \rho, d), (\mathcal{T}', \rho', d')))$ ne dépend que des classes d'équivalence de $(\mathcal{T}, \rho, d)$ et $(\mathcal{T}', \rho', d')$ au sein de $\mathcal{T}_{\mathbb{R}}$. L'espace métrique $(\mathcal{T}_{\mathbb{R}}, d_{GH})$ est alors complet et séparable [42, Théorème 1].

D'après [38, Lemme 2.3], si $g, g' \in \mathcal{C}([0,1], \mathbb{R}_+)$ sont deux fonctions nulles en $0$ et en $1$, alors $d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leqslant 2 \|g - g'\|_\infty$. En utilisant cette propriété, il est possible de reformuler les théorèmes d'Aldous et Duquesne en termes de convergence de Gromov-Hausdorff. Plus précisément, pour un arbre $\tau$ et $\alpha > 0$, notons $(\tau, \alpha \cdot d_{gr})$ l'espace métrique obtenu en multipliant par $\alpha$ les distances de graphes au sein de $\tau$, la racine de $\tau$ étant un point distingué. Le Théorème 1.1.4, dont nous gardons les notations, fournit alors le corollaire suivant :

$$\left( t_n, \frac{B_n}{n} \cdot d_{gr} \right) \quad \xrightarrow[n \to \infty]{(d)} \quad (\mathcal{T}_{H^{\mathrm{exc}}}, d_{H^{\mathrm{exc}}}),$$

la convergence ayant lieu en loi dans l'espace $(\mathcal{T}_{\mathbb{R}}, d_{GH})$.

Pour conclure, signalons toutefois que la convergence au sens de Gromov-Hausdorff d'arbres n'implique pas nécessairement celle des fonctions de contour associées.

### 1.1.6 Historique et motivations

Nous concluons la partie consacrée à la présentation des objets qui nous intéressent par un bref historique présentant des travaux consacrés aux arbres de Galton-Watson conditionnés. Nous ne rentrons volontairement pas dans une description des résultats évoqués.

#### Au début furent les processus de Bienaymé-Galton-Watson

Les prémices des processus de Bienaymé-Galton-Watson remontent au milieu du XIX$^{\text{ème}}$ siècle, où ils sont introduits afin d'estimer la probabilité d'extinction de noms de familles nobles. En 1845, Bienaymé [19] affirme, sans justification mathématique, que la probabilité d'extinction vaut un si la moyenne de la loi de reproduction est inférieure à un. En 1875, Galton & Watson [98] proposent une approche fondée sur des méthodes de fonctions génératrices. Si la méthode est judicieuse, une erreur s'est glissée dans leur travail (ils concluent que la probabilité d'extinction vaut toujours $1$, voir [12, Chapitre 9]), et il faut attendre 1930 pour que Steffensen [97] donne une solution complète. Nous renvoyons à [64, 12] pour une étude historique détaillée.

Dès lors, le comportement asymptotique des processus de branchement en temps grand a suscité de nombreux travaux ; nous renvoyons aux ouvrages [78, Section 12] et [11] pour un descriptif des résultats en ce sens.

#### La naissance du CRT : l'arbre brownien continu

À partir de la deuxième moitié du XX$^{\text{ème}}$ siècle, différentes communautés se sont intéressées au comportement asymptotique d'arbres aléatoires tirés uniformément au sein d'une certaine classe ou bien conditionnés « à être grands », en étudiant certaines de leurs caractéristiques. Au frontières de la combinatoire et de l'informatique, en utilisant des méthodes de fonctions génératrices et de combinatoire analytique, diverses statistiques de ces arbres aléatoires ont été considérées, comme le degré maximal, le nombre de sommets de degré fixé ou encore le profil de l'arbre. Nous renvoyons à l'ouvrage [35] pour un traitement détaillé.

Dans le début des années 1990, au lieu de n'analyser que des propriétés spécifiques, Aldous a eu l'idée d'étudier la convergence de grands arbres aléatoires dans leur globalité. Plus précisément, Aldous [3] a expliqué comment voir des arbres aléatoires comme des sous-ensembles compacts aléatoires de l'espace $l_1$, et a prouvé dans ce cadre qu'un arbre de Galton-Watson dont la loi de reproduction est une loi de Poisson de paramètre $1$, conditionné à avoir $n$ sommets, converge, lorsque $n \to \infty$, vers un sous-ensemble compact aléatoire appelé *Continuum Random Tree*, abrégé en CRT. Un peu plus tard, Aldous [4, 5], a proposé une construction simple du CRT à partir de l'excursion brownienne normalisée $\mathbb{e}$, et a démontré que la fonction de contour renormalisée d'un arbre de Galton-Watson de loi de reproduction critique et de variance finie, conditionné à avoir $n$ sommets, converge, lorsque $n \to \infty$, vers $\mathbb{e}$. Le CRT apparaît ainsi comme un objet limite *universel*, en ce sens que des GW arbres de lois de reproduction différentes convergent vers le même objet continu.

Dans le cas particulier où la loi de reproduction admet un moment exponentiel, Marckert & Mokkadem [79] ont étendu le résultat d'Aldous en prouvant que la marche de Lukasiewicz, les fonctions de contour et de hauteur, convenablement renormalisées, convergent conjointement vers la même excursion brownienne normalisée.

En 2003, Evans, Pitman & Winter [43] suggèrent d'utiliser le formalisme des $\mathbb{R}$-arbres, introduits auparavant à de fins géométriques et algébriques (voir par exemple [89]), et de la topologie de Gromov-Hausdorff, introduite par Gromov [51] pour démontrer ce qui est connu sous le nom de Théorème de Gromov sur les groupes à croissance polynomiale. Ce point de vue, consistant à voir des arbres comme espaces métriques abstraits, est maintenant largement utilisé et a permis de donner un cadre naturel et efficace pour étudier des convergences de graphes aléatoires (qui ne sont pas nécessairement codés par une fonction de type excursion). Par exemple, Haas & Miermont [53] ont démontré que plusieurs types d'arbres aléatoires non planaires, convenablement renormalisés, convergeaient lorsque leur nombre de sommets tend vers l'infini vers le CRT.

#### Arbres de Lévy

Un pas important a été franchi dans la généralisation des résultats d'Aldous par Le Jan & Le Gall [76] qui ont entre autres étudié le cas où la loi de reproduction $\mu$ est dans le domaine d'attraction d'une loi stable d'indice $\theta \in (1, 2]$. Le Gall & Le Jan ont prouvé que la fonction de hauteur, convenablement renormalisée, d'une forêt de $GW_\mu$ arbres convergeait en loi vers le processus de hauteur d'indice $\theta$. Ce processus a ensuite été étudié en détail par Duquesne & Le Gall dans la monographie [37] et leur a permis d'introduire les arbres de Lévy. Duquesne [36] a démontré que la fonction de contour, convenablement renormalisée, d'un $GW_\mu$ arbre conditionné à avoir $n$ sommets, convergeait en loi, lorsque $n \to \infty$, vers le processus de hauteur normalisé $H^{exc}$ d'indice $\theta$ codant l'arbre de Lévy stable d'indice $\theta$. Ces travaux ont initié l'étude de nombreuses propriétés fines des arbres de Lévy, voir par exemple [38, 39, 1]

#### Autres types de conditionnements

Des conditionnements faisant intervenir d'autres quantités que la taille totale ont été considérés en vue de différentes applications. Sous l'hypothèse de variance finie, l'étude asymptotique d'arbres de Galton-Watson conditionnés à avoir une hauteur au moins $n$ a été initiée par Kesten [65] et a intéressé d'autres auteurs (voir Aldous & Pitman [8], Geiger [49] et [59, Section 22.2], ce dernier ne se limitant pas à la variance finie). Le conditionnement à avoir une hauteur $n$, plus délicat, a été étudié dans [50, 72]. D'autres types de conditionnements faisant intervenir des degrés particuliers ont récemment fait l'objet de plusieurs travaux. Ainsi, toujours dans le cas de la variance finie, Rizzolo [94] a introduit le conditionnement à avoir un nombre fixé de sommets de degrés appartenant à un sous-ensemble fixé de $\mathbb{N}^*$, tandis que Broutin & Marckert [23] et Addario-Berry [2] considèrent des arbres aléatoires ayant une suite fixée de degrés.

#### Arbres de Galton-Watson non génériques

L'étude des arbres de Galton-Watson conditionnés non critiques se ramenant souvent au cas critique (voir paragraphe 1.1.2), les lois de reproduction non critiques ont été longtemps délaissées, et ce n'est que très récemment que Jonsson & Steffánsson [62] ont approfondi le cas où on ne peut pas se ramener à une loi de reproduction critique, cas appelé non-générique. Jonsson & Steffánsson s'intéressent aux limites locales d'arbres de Galton-Watson non-génériques conditionnés, et d'autres travaux [60, 59] ont poursuivi cette étude.

## 1.2 Grands arbres de Galton-Watson critiques conditionnés

Une grande partie de cette thèse consiste à étudier les limites d'échelle d'arbres de Galton-Watson critiques. Notre première contribution donne une nouvelle preuve plus simple du théorème de Duquesne, et la seconde s'intéresse aux arbres de Galton-Watson conditionnés à avoir un (grand) nombre de feuilles fixé.

### 1.2.1 Une nouvelle preuve du théorème de Duquesne

Le Gall [72] a récemment donné une preuve alternative du théorème d'Aldous en utilisant une propriété d'absolue continuité entre deux mesures d'Itô conditionnées. Dans cette partie, nous expliquons comment ces idées peuvent être généralisées au cas où $\mu$ est dans le domaine d'attraction d'une loi stable, ce qui permet de donner une preuve plus simple du théorème de Duquesne. Nous présenterons les idées mises en jeu en détail, car elles seront cruciales dans l'étude du conditionnement par un nombre de feuilles fixé. Nous renvoyons au chapitre 3 pour les détails.

Un rôle clé est joué par la mesure d'Itô, que nous commençons par introduire rapidement.

#### Mesure d'Itô

Notons $I_t = \inf_{0 \leqslant s \leqslant t} X_s$ pour $t \geqslant 0$. Le processus $X - I$ vérifie la propriété de Markov forte, et $0$ est régulier vis-à-vis de lui-même pour ce processus. On peut donc choisir $-I$ comme temps local pour $X - I$ au niveau $0$ (comme $X$ est spectralement positif, le processus $-I$ est continu). Notons $(g_i, d_i), i \in \mathcal{I}$ les excursions de $X - I$ au-dessus de $0$. Pour tous $i \in \mathcal{I}$ et $s \geqslant 0$, on pose $\omega^i_s = X_{(g_i+s)\wedge d_i} - X_{g_i}$, qui est un élément de l'espace des excursions $\mathcal{E}$ défini par

$$\mathcal{E} = \{\omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)\,;\ \omega(0) = 0 \text{ et } \zeta(\omega) := \sup\{s > 0\,;\ \omega(s) > 0\} \in (0, \infty)\}.$$

Pour $\omega \in \mathcal{E}$, on appelle $\zeta(\omega)$ la durée de vie de l'excursion $\omega$. D'après la théorie des excursions, la mesure ponctuelle

$$\mathbf{N}(dt d\omega) = \sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, \omega^i)}$$

est une mesure de Poisson d'intensité $dt\mathbf{N}(d\omega)$ sur $\mathbb{R}_+ \times \mathcal{E}$, où $\mathbf{N}(d\omega)$ est une mesure $\sigma$-finie sur $\mathcal{E}$ appelée **mesure d'Itô**.

L'expression de la transformée de Laplace de $X$ indique une invariance par changement d'échelle : pour tout $c > 0$, le processus $(c^{-1/\theta} X_{ct}, t \geqslant 0)$ a la même loi que $X$. Ceci va permettre de donner un sens à la mesure conditionnée $\mathbf{N}\left(\,\cdot\,|\,\zeta = 1\right)$ (l'événement $\{\zeta = 1\}$ étant de mesure d'Itô nulle, le conditionnement considéré n'a pas de sens *a priori*) . En effet, pour $\lambda > 0$, soit $S^{(\lambda)}$ l'opérateur de changement d'échelle défini sur $\mathcal{E}$ par $S^{(\lambda)}(\omega) = \left(\lambda^{1/\theta}\omega(s/\lambda),\, s \geqslant 0\right)$. La propriété d'invariance par changement d'échelle de $X$ garantit que l'image de la mesure $\mathbf{N}(\,\cdot\,|\,\zeta > t)$ par $S^{(1/\zeta)}$ ne dépend pas de $t > 0$. Cette loi, à support sur l'ensemble des excursions de durée de vie unité, est notée $\mathbf{N}\left(\,\cdot\,|\,\zeta = 1\right)$. Intuitivement, il s'agit de la loi d'une excursion sous la mesure d'Itô conditionnée à avoir une durée de vie unité.

En utilisant la formule d'approximation (1.3), il est également possible de définir le processus de hauteur $H$ sous $\mathbf{N}$ ainsi que sous $\mathbf{N}\left(\,\cdot\,|\,\zeta = 1\right)$ (voir [36]). La loi de $(X, H)$ sous $\mathbf{N}\left(\,\cdot\,|\,\zeta = 1\right)$ coïncide avec la loi de $(X^{\text{exc}}, H^{\text{exc}})$ définie dans la Section 1.1.4 (voir [25, 36]).

On rappelle que $(W_n; n \geqslant 0)$ est une marche aléatoire sur $\mathbb{Z}$, issue de $0$ et dont la loi des sauts est donnée par $\nu(k) = \mu(k+1)$ pour $k \geqslant -1$. On rappelle également la notation $\zeta_1 = \inf\{n \geqslant 0\,;\, W_n = -1\}$. Suivant (1.2), on pose pour $n \geqslant 0$ :

$$H_n^W = \mathrm{Card}\left\{j\,;\, 0 \leqslant j < n,\, W_j = \inf_{j \leqslant l \leqslant n} W_l\right\}.$$

Pour simplifier l'exposition, nous allons d'abord expliquer comment prouver la convergence de la première composante dans (1.4), puis celle de la seconde.

**Convergence de la marche de Lukasiewicz renormalisée : transformée de Vervaat**

Compte tenu de la Proposition 1.1.1, la convergence de la première composante dans (1.4) est équivalente à

$$\left(\left(\frac{1}{B_n} W_{\lfloor nt \rfloor}\right)_{0 \leqslant t \leqslant 1} \middle|\, \zeta_1 = n\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t)_{0 \leqslant t \leqslant 1} \quad \text{sous } \mathbf{N}(\,\cdot \mid \zeta = 1).$$

Nous suivons l'approche de Duquesne [36] pour prouver cette convergence qui est maintenant standard, et renvoyons à [36] pour les détails. En utilisant un argument d'absolue continuité entre la loi de $(W_0, \ldots, W_{\lfloor na \rfloor})$ sous $\mathbb{P}[\,\cdot \mid W_n = -1]$ et $(W_0, \ldots, W_{\lfloor na \rfloor})$ valable pour tout $a \in (0,1)$, puis un argument de retournement du temps, on prouve la convergence

$$\left(\left(\frac{1}{B_n} W_{\lfloor nt \rfloor}\right)_{0 \leqslant t \leqslant 1} \middle|\, W_n = -1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t^{\mathrm{br}})_{0 \leqslant t \leqslant 1},$$

où $X^{\mathrm{br}}$ désigne le pont de Lévy stable d'indice $\theta$, qui peut être informellement vu comme le processus $X$ conditionné par $X_1 = 0$. On conclut en prenant la transformée de Vervaat, combinant la Proposition 1.1.2 avec le fait que la transformée de Vervaat de $X^{\mathrm{br}}$ est $X^{\mathrm{exc}}$.

**Convergence de la fonction de hauteur renormalisée : d'un théorème limite inconditionnel vers un théorème limite conditionnel**

Expliquons maintenant comment prouver la convergence de la seconde composante dans (1.4), chose bien plus délicate. D'après la Proposition 1.1.1, celle-ci est équivalente à

$$\left(\left(\frac{B_n}{n} H_{nt}^W\right)_{0 \leqslant t \leqslant 1} \middle|\, \zeta_1 = n\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t)_{0 \leqslant t \leqslant 1} \quad \text{sous } \mathbf{N}(\,\cdot \mid \zeta = 1). \tag{1.5}$$

**Étape préliminaire : théorème limite inconditionnel.** On part d'un théorème limite sans conditionnement dû à Duquesne & Le Gall [37] :

$$\left(\frac{1}{B_n} W_{\lfloor nt \rfloor}, \frac{B_n}{n} H_{nt}^W\right)_{t \geqslant 0} \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t, H_t)_{t \geqslant 0}. \tag{1.6}$$

La convergence de la première composante est une simple généralisation au cas stable du théorème d'invariance de Donsker, mais obtenir la convergence du couple est plus délicat.

**Étape 1 : théorème limite conditionné par un événement de mesure d'Itô non nulle.** Soit $A \subset \mathcal{E}$ un sous-ensemble mesurable d'excursions tel que $\mathbf{N}(A) > 0$. D'après le théorème de représentation de Skorokhod, nous pouvons supposer que la convergence (1.6) a lieu presque sûrement (la marche aléatoire $W$ apparaissant dans (1.6) dépendrait alors de $n$). Pour simplifier les notations, soit $H^{(n)}$ le processus défini par $H^{(n)}_t = \frac{B_n}{n} H^W_{nt}$ pour $t \geqslant 0$. Notons $H^{(n,A)}$ la première excursion de $H^{(n)}$ au-dessus de $0$ appartenant à $A$, ainsi que $H^{(A)}$ la première excursion de $H$ au-dessus de $0$ appartenant à $A$. Il est naturel de s'attendre à ce que $H^{(n,A)} \to H^{(A)}$ lorsque $n \to \infty$. Supposons un instant que cette dernière convergence ait lieu. Alors $H^{(n,A)}$ a la loi de la fonction de hauteur normalisée d'un $GW_\mu$ arbre conditionnée à être dans $A$, et, d'après les propriétés des mesures ponctuelles de Poisson, $H^{(A)}$ suit la loi conditionnelle $\mathbf{N}(\,\cdot\,|\,A)$. Ainsi, sous l'hypothèse $\mathbf{Hyp}_A$ : $H^{(n,A)} \to H^{(A)}$ lorsque $n \to \infty$, nous obtenons :

$$\left(H^{(n)} \middle| \, H^{(n)} \in A\right) \quad \xrightarrow[n\to\infty]{(d)} \quad H \quad \text{sous } \mathbf{N}(\,\cdot\,|\,A). \tag{1.7}$$

Duquesne & Le Gall [37, Section 2.5] ont prouvé que $\mathbf{Hyp}_A$ est vérifiée lorsqu'on prend $A = \{\omega\,;\,\zeta(\omega) > 1\}$, et Le Gall [72, Théorème 5.1] a démontré que, lorsque la variance de $\mu$ est finie, $\mathbf{Hyp}_A$ est vérifiée dès que $A$ est ouvert et que $\mathbf{N}(\{\omega \in \mathcal{E}\,;\, d(\omega, A) < \epsilon\}) \to \mathbf{N}(A)$ quand $\epsilon \to 0$. En particulier, en prenant $A = \{\omega\,;\,\zeta(\omega) > 1\}$, nous obtenons la convergence :

$$\left(\left(\frac{B_n}{n} H^W_{nt}\right)_{t\geqslant 0} \middle| \, \zeta_1 \geqslant n\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t)_{t\geqslant 0} \quad \text{sous } \mathbf{N}(\,\cdot\,|\,\zeta > 1).$$

Par ailleurs, en utilisant le fait les excursions de $W$ au-dessus de son minimum courant coïncident avec les excursions de $H$ au-dessus de $0$, il n'est pas difficile d'obtenir la convergence jointe suivante (voir [37, Section 2.5] et [72] pour les détails) :

$$\left(\left(\frac{1}{B_n} W_{\lfloor nt \rfloor}, \frac{B_n}{n} H^W_{nt}\right)_{t\geqslant 0} \middle| \, \zeta_1 \geqslant n\right) \quad \xrightarrow[n\to\infty]{(d)} \quad ((X_t, H_t)_{t\geqslant 0} | \zeta > 1). \tag{1.8}$$

**Étape 2 : théorème limite conditionné par un événement de mesure d'Itô nulle.** Comme l'ensemble d'excursions $A = \{\omega\,;\,\zeta(\omega) = 1\}$ est de $\mathbf{N}$ mesure nulle, il n'est pas possible d'utiliser (1.7) afin d'obtenir (1.5). Un autre argument est ainsi nécessaire. L'idée originelle de Duquesne consistait à utiliser une transformée de Vervaat pour les fonctions de hauteur et contour. Nous présentons une autre approche, qui, dans le cas de la variance finie, est due à Le Gall [72], et repose sur une relation d'absolue continuité entre $\mathbf{N}(\,\cdot\,|\,\zeta = 1)$ et $\mathbf{N}(\,\cdot\,|\,\zeta > 1)$.

On commence par établir que pour tout $a \in (0, 1)$,

$$\left(\left(\frac{B_n}{n} H^W_{\lfloor nt \rfloor}\right)_{0\leqslant t\leqslant a} \middle| \, \zeta_1 = n\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t)_{0\leqslant t\leqslant a} \quad \text{sous } \mathbf{N}(\,\cdot\,|\,\zeta = 1). \tag{1.9}$$

On fixe $F : \mathbb{D}([0, a], \mathbb{R}) \to \mathbb{R}_+$ une fonction continue bornée. En utilisant la propriété de Markov de $W$ au temps $\lfloor na \rfloor$, on montre qu'il existe une fonction $D^{(a)}_n : \mathbb{N} \to \mathbb{R}_+$ telle que

$$\mathbb{E}\left[F\left(\frac{B_n}{n} H^W_{\lfloor nt \rfloor}\,;\, 0 \leqslant t \leqslant a\right) \middle| \, \zeta_1 = n\right] = \mathbb{E}\left[F\left(\frac{B_n}{n} H^W_{\lfloor nt \rfloor}\,;\, 0 \leqslant t \leqslant a\right) D^{(a)}_n(W_{\lfloor nt \rfloor}) \middle| \, \zeta_1 \geqslant n\right].$$

L'utilisation d'une relation d'absolue continuité entre les lois conditionnelles $\mathbb{P}[\cdot\,|\,\zeta_1 = n]$ et $\mathbb{P}[\cdot\,|\,\zeta_1 \geqslant n]$ remonte à Le Gall & Miermont [75]. Des estimées techniques établissent ensuite l'existence d'une application $\Gamma_a : \mathbb{R}_+ \to \mathbb{R}_+$ telle que $D_n^{(a)}(j)$ est de l'ordre de $\Gamma_a(j/B_n)$ pour une certaine plage de valeurs de j. Par ailleurs, des propriétés de la mesure d'Itô donnent la relation d'absolue continuité suivante :

**Proposition 1.**

Pour $s > 0$, notons $p_s$ la densité de $X_s$. Pour $s, x > 0$ posons $q_s(x) = \frac{x}{s} p_s(-x)$ et $\Gamma_a(x) = \theta q_{1-a}(x) / \int_{1-a}^{\infty} ds\, q_s(x)$. Alors :

$$\mathbf{N}\left(F((H_t)_{0 \leqslant t \leqslant a}) \Gamma_a(X_a) \,|\, \zeta > 1\right) = \mathbf{N}\left(F((H_t)_{0 \leqslant t \leqslant a}) \,|\, \zeta = 1\right).$$

Ainsi, en utilisant (1.8), il vient :

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}_\mu &\left[ F\left( \frac{B_n}{n} H^W_{\lfloor nt \rfloor} \,;\, 0 \leqslant t \leqslant a \right) \middle| \, \zeta_1 = n \right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[ F\left( \frac{B_n}{n} H^W_{\lfloor nt \rfloor} \,;\, 0 \leqslant t \leqslant a \right) D_n^{(a)}\left(W_{\lfloor an \rfloor}\right) \middle| \, \zeta_1 \geqslant n \right] \\
&= \mathbf{N}\left(F(H_t \,;\, 0 \leqslant t \leqslant a) \Gamma_a(X_a) \mid \zeta > 1\right) \\
&= \mathbf{N}\left(F(H_t \,;\, 0 \leqslant t \leqslant a) \mid \zeta = 1\right).
\end{aligned}$$

Pour conclure, on montre que (1.9) a également lieu pour $a = 1$ en utilisant un argument fondé sur l'invariance par retournement du temps de la fonction de contour.

Finalement, signalons que dans le cas de la variance finie, Le Gall [72] utilise un théorème local limite fort pour estimer la densité $D_n^{(a)}$. Cependant, dans le cas stable, il n'y a pas de théorème connu de ce type, ce qui nous oblige à utiliser un autre argument.

### 1.2.2 Conditionnement par le nombre de feuilles

Nous nous intéressons également aux propriétés asymptotiques d'arbres de Galton-Watson conditionnés à avoir un (grand) nombre de feuilles fixé. On rappelle que $\lambda(\tau)$ désigne le nombre de feuilles de $\tau$. Considérons une loi de reproduction critique $\mu$ dans le domaine d'attraction d'une loi stable d'indice $\theta \in (1, 2]$, et pour chaque $n \geqslant 1$ tel que $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$, notons maintenant $t_n$ un arbre aléatoire de loi conditionnée $\mathbb{P}_\mu[\cdot\,|\,\lambda(\tau) = n]$. Afin de simplifier l'exposition, supposons que $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$ pour n suffisamment grand. On rappelle que $(B_n)$ est la suite de nombres réels tendant vers l'infini apparaissant dans le Théorème 1.1.4.

Notre contribution principale consiste à obtenir un équivalent de la probabilité $\mathbb{P}_\mu[\lambda(\tau) = n]$ lorsque $n \to \infty$, et à prouver la convergence de la marche de Lukasiewicz et des fonctions de hauteur et de contour de $t_n$ convenablement renormalisées et ainsi d'étendre le théorème de Duquesne au cas où on conditionne par le nombre de feuilles.

Nous présentons les résultats et idées principales, et renvoyons au chapitre 4 pour les détails. Les notations restent les mêmes que dans la Section 1.1.

**Concentration du nombre de feuilles**

Lorsque le nombre de feuilles d'un arbre est fixé, le nombre de sommets ne l'est généralement pas. Une première étape clé consiste ainsi à comprendre le comportement du nombre de feuilles d'un $GW_\mu$ arbre avec $n$ sommets, et, réciproquement, le nombre de sommets d'un $GW_\mu$ arbres avec $n$ feuilles :

**Proposition 2.**

Il existe une constante $c > 0$ telle que pour tout $n$ assez grand :

(i) $\mathbb{P}_\mu\left[\left|\frac{\lambda(\tau)}{n} - \mu_0\right| > \frac{1}{n^{1/4}} \text{ et } \zeta(\tau) = n\right] \leqslant e^{-c\sqrt{n}}$,

(ii) $\mathbb{P}_\mu\left[\lambda(\tau) = n \text{ et } \left|\zeta(\tau) - \frac{n}{\mu_0}\right| > \zeta(\tau)^{3/4}\right] \leqslant e^{-c\sqrt{n}}$.

D'après la Proposition 1.1.1, le nombre de feuilles d'un $GW_\mu$ arbre a la même loi que la somme $\sum_{k=1}^{\zeta_1} 1_{\{W_k - W_{k-1} = -1\}}$. En effet, les feuilles de l'arbre correspondent au sauts d'amplitude $-1$ de la marche de Lukasiewicz. En remarquant que cette dernière somme est constituée de variables aléatoires de Bernoulli indépendantes de paramètre $\mu_0$, la Proposition 2 est obtenue par des techniques de grandes déviations. Signalons qu'une technique similaire a été utilisée par Marckert & Mokkadem [79].

**Loi jointe** $(\zeta(\tau), \lambda(\tau))$

Un autre ingrédient important est la détermination de la loi jointe de $(\zeta(\tau), \lambda(\tau))$ sous $\mathbb{P}_\mu$ :

**Proposition 3.**

On a, pour $1 \leqslant n \leqslant p$,

$$\mathbb{P}_\mu[\zeta(\tau) = p, \lambda(\tau) = n] = \frac{1}{p}\,\mathbb{P}[S_p = n]\,\mathbb{P}[W'_{p-n} = n-1], \tag{1.10}$$

où $S_p$ est la somme de $p$ variables aléatoires de Bernoulli indépendantes et $W'$ est la marche aléatoire $W$ conditionnée à ne faire que des sauts positifs.

Avant de donner l'idée de la preuve de cette formule, nous avons besoin dintroduisons la notation $\Lambda(p) = \mathrm{Card}\{0 \leqslant i \leqslant p-1\,;\, W_{i+1} - W_i = -1\}$. D'après la Proposition 1.1.1, nous avons $\mathbb{P}_\mu\left[\zeta(\tau) = p, \lambda(\tau) = n\right] = \mathbb{P}\left[\Lambda(p) = n, \zeta_1 = p\right]$. Un argument de type « lemme cyclique » permet d'écrire l'égalité

$$\mathbb{P}\left[\Lambda(p) = n, \zeta_1 = p\right] = \frac{1}{p}\mathbb{P}\left[\Lambda(p) = n, W_p = -1\right].$$

La Proposition 3 en découle aisément : en effet, il reste à choisir la place de $n$ sauts d'amplitude $-1$, ce qui va donner la contribution $\mathbb{P}[S_p = n]$, et une fois qu'on a enlevé ces sauts, il nous reste une marche aléatoire conditionnée à ne pas faire de sauts d'amplitude $-1$ et devant valoir $n-1$ à l'instant $p-n$, donnant la contribution $\mathbb{P}\left[W'_{p-n} = n-1\right]$.

En sommant (1.10) par rapport à la plage de valeurs de $p$ (fournie par la Proposition 2) apportant une contribution significative, une transformation série-intégrale et l'utilisation de théorèmes locaux limites estimant les probabilités apparaissant dans le terme de droite de la formule de la Proposition 3, permettent d'obtenir l'estimée suivante, où $p_1$ désigne la densité de $X_1$.

**Théorème 4.**

On a $\quad \mathbb{P}_\mu[\lambda(\tau) = n] \underset{n\to\infty}{\sim} \mu_0^{1/\theta} p_1(0) \dfrac{1}{nB_n}.$

Ce résultat est à comparer avec l'équivalent classique de $\mathbb{P}_\mu[\zeta(\tau) = n]$ : lorsque $n \to \infty$, on a $\mathbb{P}_\mu[\zeta(\tau) = n] \sim p_1(0)/(nB_n)$ (cela découle immédiatement du théorème local limite et de l'égalité $\mathbb{P}_\mu[\zeta(\tau) = n] = \mathbb{P}[W_n = -1]/n$).

### Limites d'échelle

Citons maintenant l'extension du théorème de Duquesne au conditionnement par le nombre de feuilles :

**Théorème 5.**

Le triplet

$$\left(\frac{1}{B_{\zeta(t_n)}} \mathcal{W}_{\lfloor \zeta(t_n) t \rfloor}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} H_{\zeta(t_n)t}(t_n)\right)_{0\leqslant t\leqslant 1}$$

converge en loi, lorsque $n \to \infty$ vers $(X_t, H_t, H_t)_{0\leqslant t\leqslant 1}$ sous $\mathbf{N}(\,\cdot\,|\,\zeta = 1)$.

Lorsque $n \to \infty$, $\zeta(t_n)/n$ converge en probabilité vers $1/\mu_0$, de sorte qu'il est possible de remplacer les facteur $1/B_{\zeta(t_n)}$ et $B_{\zeta(t_n)}/\zeta(t_n)$ par respectivement $\mu_0^{1/\theta}/B_n$ et $\mu_0^{1-1/\theta}B_n/n$ sans changer le résultat du théorème. Ceci amène à plusieurs applications intéressantes :

- la suite d'arbres aléatoires $\left(t_n, \mu_0^{1-1/\theta}\frac{B_n}{n} \cdot d_{gr}\right)$ converge en loi au sens de la métrique de Gromov-Hausdorff, lorsque $n \to \infty$, vers $(\mathcal{T}_{H^{exc}}, d_{H^{exc}})$,
- lorsque $n \to \infty$, $\mu_0^{1-1/\theta}\frac{B_n}{n} \cdot \mathcal{H}(t_n)$ converge en loi vers $\sup_{0\leqslant t\leqslant 1} H_t^{exc}$,
- si on note $\Delta(t_n)$ le degré maximal de $t_n$, alors $\mu_0^{1/\theta}\frac{1}{B_n}\Delta(t_n)$ converge en loi vers la quantité $\sup_{0<t\leqslant 1}(X_t^{exc} - X_{t-}^{exc})$.

Finalement, la convergence de la marche de Lukasiewicz renormalisée joue un rôle important dans l'étude des laminations stables (voir section 2.1.2 et chapitre 6).

La preuve du Théorème 5 suit la structure générale exposée dans la partie 1.2.1. La première étape est de prouver le Théorème 5 lorsque $t_n$ suit la loi $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$. En reprenant les notations de la partie 1.2.1, il n'est malheureusement pas possible de trouver un sous-ensemble $A \subset E$ d'excursions tel que $\left(H^{(n)} \middle| H^{(n)} \in A\right)$ soit distribué selon $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$. Un autre argument est donc nécessaire. L'idée est d'utiliser le résultat de concentration de la Proposition 2 afin d'en déduire que les deux mesures conditionnées $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$ et $\mathbb{P}_\mu\left[\,\cdot\,|\,\zeta(\tau) \geqslant \mu_0 n - n^{3/4}\right]$ sont proches pour $n$ grand. On en tire le théorème limite souhaité sous $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$ à partir de celui sous $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) \geqslant n]$. La deuxième étape consiste à utiliser une relation d'absolue continuité entre les deux mesures conditionnées $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$ et $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$, puis de contrôler la densité obtenue.

Si le cheminement est similaire à celui de la preuve alternative du théorème de Duquesne, les estimées techniques nécessaires sont délicates et de multiples difficultés surviennent du fait que le nombre total de sommets n'est pas fixé.

## 1.3 Arbres de Galton-Watson non génériques

Nous nous intéressons maintenant aux limites d'échelle d'arbres de Galton-Watson de loi de reproduction $\mu$ non générique, c'est-à-dire, en reprenant les notations de la Section 1.1.2, pour laquelle il n'existe pas $\lambda > 0$ tel que $Z_\lambda < \infty$ et $\mu^{(\lambda)}$ soit critique.

L'étude de grands arbres de Galton-Watson non génériques conditionnés n'a été initiée que récemment par Jonsson & Steffánsson [62], qui se sont intéressés à leurs limites locales.

### 1.3.1 Limites locales d'arbres de Galton-Watson conditionnés

**Convergence locale**

Si $(T_n)_{n\geqslant 1} \in \mathbb{T}^{\mathbb{N}^*}$ est une suite d'arbres et $T \in \mathbb{T}$, on dit que $T_n$ **converge localement** vers $T$ si pour tout $u \in U$, $k_u(T_n) \to k_u(T)$ lorsque $n \to \infty$. Il est possible de munir $\mathbb{T}$ d'une distance $d_{loc}$ qui métrise cette notion de convergence et qui fait de $(\mathbb{T}, d_{loc})$ un espace métrique complet et séparable (voir [59, Section 6]).

Une définition équivalente de la topologie locale, plus usuelle, passe par la convergence des « boules » centrées en la racine. Plus précisément, soit $U^{[m]} = \bigcup_{k=0}^{m} \{1, \ldots, m\}^k$, et pour $T \in \mathbb{T}$ notons $T^{[m]} = T \cap U^{[m]}$. Ainsi, $T^{[m]}$ est obtenu à partir de $T$ en tronquant $T$ à la génération $m$ et en ne gardant que les $m$ premiers enfants des sommets qui restent. Il est alors possible de montrer que, lorsque $n \to \infty$, $T_n \to T$ si et seulement si $T_n^{[m]} \to T^{[m]}$ pour tout $m \geqslant 1$ (voir [59, Section 6]). La définition de convergence locale sous cette dernière forme a été proposée par Jonsson & Steffánsson [62]. La raison de l'introduction de $U^{[m]}$ provient du fait que les arbres considérés ne sont pas nécessairement localement finis.

**L'arbre de Galton-Watson conditionné à survivre**

Nous allons maintenant décrire la construction d'un arbre aléatoire infini qui sera la limite locale de grands arbres aléatoires. Soit $\nu$ une loi de reproduction critique ou sous-critique. On note $m_\nu$ sa moyenne. Soit $\zeta^*_\nu$ la variable aléatoire définie par $\mathbb{P}[\zeta^*_\nu = k] := k\nu_k/m_\nu$ pour $k \geqslant 0$. On introduit également la variable aléatoire $S_\nu$ définie comme suit. Si $m_\nu < 1$, on pose $\mathbb{P}[S_\nu = i] = (1-m_\nu)m_\nu^{i-1}$ pour $i \geqslant 1$, et si $m_\nu = 1$ on pose $S_\nu = \infty$.

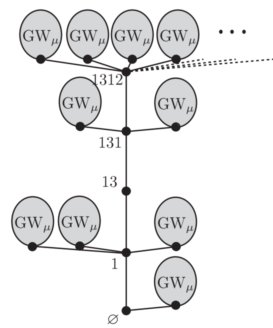


**FIGURE 1.5 –** Un exemple de réalisation de $\widehat{\mathcal{T}}_\nu$. Ici, l'épine dorsale est constituée des sommets $\emptyset, 1, 13, 131, 1312$.

Soit $\widehat{\mathcal{T}}_\nu$ l'arbre aléatoire infini construit informellement de la manière suivante (voir [59, Section 4] pour une définition plus précise). On commence avec une épine dorsale ayant un nombre aléatoire de sommets distribué selon $S_\nu$. On greffe ensuite des branches (voir figure 1.5) comme suit. D'une part, en haut de l'épine dorsale, on attache un nombre infini de branches, chaque branche étant un $\mathrm{GW}_\mu$ arbre (si $S_\nu = \infty$, l'épine dorsale est infinie et on n'effectue pas cette opération). D'autre part, à chaque autre sommet de l'épine dorsale, on attache un nombre aléatoire de branches distribué selon $\zeta^*_\nu - 1$ soit à gauche soit à droite de l'épine de dorsale, chaque branche étant un $\mathrm{GW}_\mu$ arbre. Si $k$ nouvelles branches ont été greffées sur un sommet, le nombre de branches greffées à gauche de ce sommet est uniformément distribué sur $\{0, \dots, k\}$. De plus, tous les choix aléatoires sont indépendants.

Janson [59, Théorème 7.1] a prouvé que pour n'importe quel arbre simplement généré $\mathcal{T}_n$ associé à une suite de poids, il existe une loi de reproduction critique ou sous-critique $\nu$ telle que $\mathcal{T}_n$ converge localement en loi vers $\widehat{\mathcal{T}}_\nu$. Nous renvoyons à [59, Théorème 7.1] pour la construction explicite de $\nu$ à partir de la suite de poids, mais précisons que si $\mathcal{T}_n$ est un $\mathrm{GW}_\mu$ arbre conditionné à avoir $n$ sommets, l'égalité $\nu = \mu$ n'est pas toujours vérifiée. Cependant, on a $\nu = \mu$ lorsque $\mu$ est critique ou bien $\mu$ est sous-critique et le rayon de convergence de $\sum \mu_i z^i$ vaut $1$.

Cette convergence très générale était connue dans plusieurs cas particuliers. En 1986, Kesten [65] a construit $\widehat{\mathcal{T}}_\nu$ lorsque $\nu$ est critique et a prouvé la convergence locale de $\mathcal{T}_n$ vers $\widehat{\mathcal{T}}_\mu$ lorsque $\mathcal{T}_n$ est un $\mathrm{GW}_\mu$ arbre conditionné à avoir hauteur au moins $n$, avec $\mu$ critique et de variance finie (mais sa méthode s'adapte aisément au cas où $\mathcal{T}_n$ est un $\mathrm{GW}_\mu$ arbre conditionné à avoir $n$ sommets avec $\mu$ critique et de variance finie). La construction de $\widehat{\mathcal{T}}_\nu$ lorsque $\nu$ est sous-critique est due à Jonsson & Steffánsson [62], qui ont démontré le résultat décrit précédemment lorsque $\mathcal{T}_n$ est un $GW_\mu$ arbre conditionné à avoir $n$ sommets, avec $\mu$ sous-critique et telle que $\mu_i \sim c/i^{1+\beta}$ lorsque $i \to \infty$, où $c > 0$, $\beta > 1$ sont des constantes.

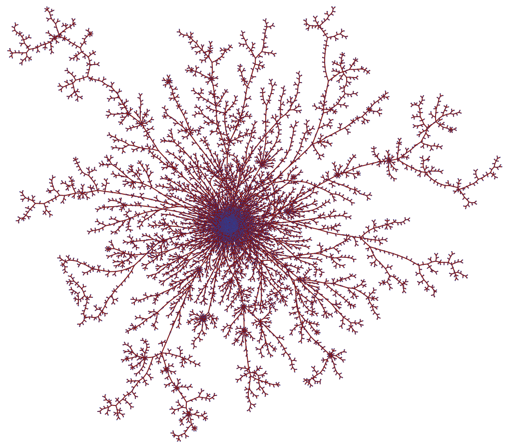

**FIGURE 1.6 –** Un exemple de réalisation de $t_{10004}$ lorsque $\mu$ est sous-critique et $\mu_i \sim c/i^{1+\beta}$ avec $\beta = 1.5$.

**Condensation**

Supposons que la loi de reproduction $\mu$ soit sous-critique et que $\mu_i \sim c/i^{1+\beta}$ lorsque $i \to \infty$, où $c > 0$, $\beta > 1$ sont des constantes. Notons $m$ la moyenne de $\mu$ et posons $\gamma = 1 - m$. Soit $t_n$ un arbre aléatoire distribué selon $\mathbb{P}_\mu [\,\cdot \mid \zeta(\tau) = n]$. Jonsson & Steffánsson [62] ont mis en évidence un phénomène de condensation en prouvant que si $\Delta(t_n)$ désigne le degré maximal de $t_n$, alors $\Delta(t_n)/n$ converge en probabilité vers $1 - m$, et qu'avec probabilité tendant vers 1 les autres degrés sont $o(n)$.

Cependant, ce phénomène de condensation n'est pas universel parmi toutes les lois de reproduction non-génériques. En effet, Janson [59, Example 19.37] a construit un exemple de loi de reproduction $\mu$ pour laquelle il existe deux sous-suites $(n_j)$ et $(n'_j)$ telles que d'une part $\Delta(t_{n_j}) = o(n_j)$, et d'autre part $t_{n'_j}$ a deux sommets de degré $n'_j/3$, avec probabilité tendant vers 1 lorsque $j \to \infty$.

### 1.3.2 Limites d'échelle

Soit $t_n$ un arbre aléatoire distribué selon $\mathbb{P}_\mu [\,\cdot \mid \zeta(\tau) = n]$, avec $\mu$ une loi de reproduction non-générique. Notre contribution à la théorie des arbres non génériques consiste à étudier, pour des lois de reproduction non génériques particulières, des propriétés globales de $t_n$ qui ne peuvent pas *a priori* être obtenues à partir de la seule convergence locale de $t_n$ lorsque $n \to \infty$.

Soit $\theta > 1$ fixé. Nous faisons les hypothèses suivantes sur $\mu$ :

(i) $\mu$ est sous-critique (c'est-à-dire $0 < \sum_{j=0}^{\infty} j\mu_j < 1$).

(ii) Il existe une fonction à variation lente $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}_+$ telle que $\mu_n = \mathcal{L}(n)/n^{1+\theta}$ pour $n \geqslant 1$.

L'hypothèse (ii) est légèrement plus générale que le cas $\mu_i \sim c/i^{1+\beta}$ étudié par Jonsson & Steffánsson [62] et Janson [59, Théorème 19.34]. Notre dessein est de répondre aux questions suivantes : les marches de Lukasiewicz, fonctions de contour et hauteur de $t_n$, convenablement renormalisées, convergent-elles ? Si on note $u_\star(t_n)$ le sommet ayant le plus d'enfants de $t_n$, que est le degré de $u_\star(t_n)$ (et quelles sont ses fluctuations) ? Quelle est la hauteur de $u_\star(t_n)$ ? Et enfin, quelle est la hauteur de $t_n$ ?

Les notations de la partie 1.1.3 sont gardées. En particulier, $(W_n; n \geqslant 0)$ désigne une marche aléatoire sur $\mathbb{Z}$, issue de $0$ et dont la loi des sauts est donnée par $\nu(k) = \mu(k+1)$ pour $k \geqslant -1$.

#### Convergence de la marche de Lukasiewicz : lois sous-exponentielles

Rappelons que l'on note $\Delta(\tau)$ le plus grand nombre d'enfants d'un sommet de $\tau$ (en particulier, le degré maximal de $\tau$ est soit $\Delta(\tau)$, soit $\Delta(\tau)+1$).

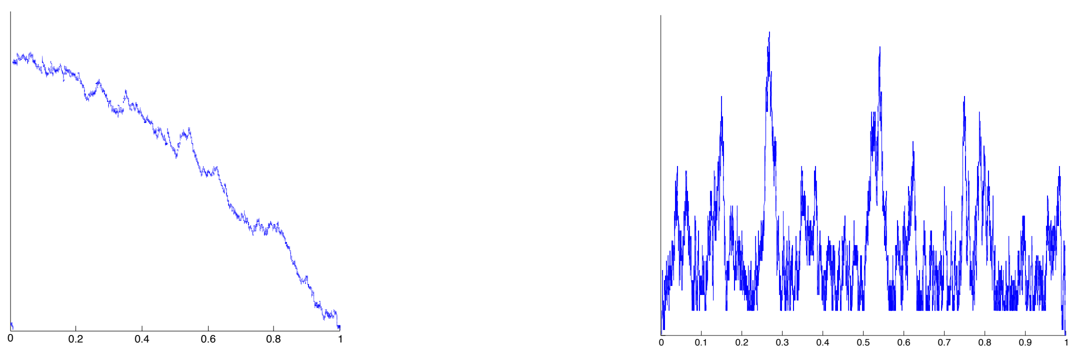


**FIGURE 1.7 –** Respectivement la marche de Lukasiewicz et la fonction de contour de l'arbre de la Fig. 1.6.

Le résultat suivant établit la convergence de la marche de Lukasiewicz normalisée de $t_n$.

**Théorème 6.**

Soit $U(t_n) = \min\{j \geqslant 0\,;\, \mathcal{W}_{j+1}(t_n) - \mathcal{W}_j(t_n) = \Delta(t_n) - 1\}$ l'indice du premier sommet de $t_n$ dans l'ordre lexicographique ayant le plus grand nombre d'enfants. Alors :

(i) $U(t_n)/n$ converge en probabilité vers $0$ lorsque $n \to \infty$.

(ii) On a $\displaystyle \sup_{0 \leqslant i \leqslant U(t_n)} \frac{\mathcal{W}_i(t_n)}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} 0.$

(iii) La convergence suivante a lieu en loi dans $\mathbb{D}([0,1],\mathbb{R})$ :

$$\left(\frac{\mathcal{W}_{\lfloor nt \rfloor \vee (U(t_n)+1)}(t_n)}{n}, 0 \leqslant t \leqslant 1\right) \xrightarrow[n\to\infty]{(d)} (\gamma(1-t), 0 \leqslant t \leqslant 1)\,.$$

(iv) On a $\displaystyle \frac{\Delta(t_n)}{\gamma n} \xrightarrow[n\to\infty]{(\mathbb{P})} 1.$

La propriété (ii) implique que $\left(\mathcal{W}_{\lfloor nt \rfloor}(t_n)/n, 0 \leqslant t \leqslant 1\right)$ ne converge pas en loi dans l'espace $\mathbb{D}([0,1],\mathbb{R})$ vers $(\gamma(1-t), 0 \leqslant t \leqslant 1)$, ce qui explique pourquoi on ne considère $\mathcal{W}(t_n)$ qu'à partir de l'instant $U(t_n)$ dans (iii). Ces résultats indiquent qu'avec probabilité tendant vers 1 lorsque $n \to \infty$, il existe un unique sommet de $t_n$ ayant approximativement $\gamma n$ enfants, dont l'indice lexicographique est $o(n)$, et tous les nombres d'enfants de tous les autres sommets de $t_n$ sont $o(n)$. Ceci est bien sûr cohérent avec les résultats précédemment mentionnés de Jonsson & Stefánsson et Janson.

Il est possible de comprendre intuitivement l'apparition de la quantité $\gamma$. En effet, soit $t'_n$ l'arbre aléatoire composé d'une racine sur laquelle on greffe $cn$ $\mathrm{GW}_\mu$ arbres indépendants, avec $c > 0$. Or, comme $m < 1$, le nombre moyen de sommets d'un $\mathrm{GW}_\mu$ arbre est $1/(1-m) = 1/\gamma$. Ainsi, $\mathbb{E}[\zeta(t'_n)] = cn/\gamma$. Si on admet un instant que $t_n$ et $t'_n$ sont proches, nous devons avoir $cn/\gamma = n$, ce qui donne $c = \gamma$.

La preuve du Théorème 6 passe par un lien avec des marches aléatoires conditionnées dont la loi des sauts est $(0,1]$– sous-exponentielle dans le sens suivant.

Soit $\Delta = (0,s]$ avec $s \in \mathbb{R}_+^* \cup \{\infty\}$. Pour $x > 0$, on pose $x + \Delta = (x, x+s)$. Soit $\rho$ une mesure de probabilité sur $\mathbb{R}$ et considérons une marche aléatoire $(Z_n)_{n \geqslant 0}$ issue de 0 dont la loi des sauts est $\rho$. On dit que $\rho$ est **$\Delta$ – sous-exponentielle** si $\rho(x+\Delta) > 0$ pour $x$ suffisamment grand, et, pour tous $y \in \mathbb{R}$ et $n \geqslant 1$ :

$$\lim_{x \to \infty} \frac{\rho(x+y+\Delta)}{\rho(x+\Delta)} = 1, \qquad \lim_{x \to \infty} \frac{\mathbb{P}[Z_n \in x+\Delta]}{n\rho(x+\Delta)} = 1.$$

Cette notion a été introduite par Asmussen, Foss & Korshunov [10]. Lorsque $\rho$ est $\Delta$–sous-exponentielle, il existe une suite $d_n \to \infty$ telle que

$$\lim_{n \to \infty} \sup_{x \geqslant d_n} \left| \frac{\mathbb{P}[Z_n \in x+\Delta]}{n\rho(x+\Delta)} - 1 \right| = 0,$$

et plusieurs travaux ont donné des conditions sur $(d_n)$ pour que cette formule soit vérifiée (voir [32]).

Armendáriz & Loulakis [9] ont étudié le comportement de marches aléatoires conditionnées dont la loi des sauts est $\Delta$ – sous-exponentielle. Avant de citer leur résultat, introduisons l'opérateur $T : \cup_{n \geqslant 1} \mathbb{R}^n \to \cup_{n \geqslant 1} \mathbb{R}^n$ qui échange la dernière composante et la (première) composante maximale d'une suite de nombre réels :

$$T(x_1, \ldots, x_n)_k = \begin{cases} \max\limits_{1 \leqslant i \leqslant n} x_i & \text{si } k = n \\ x_n & \text{si } x_k > \max\limits_{1 \leqslant i < k} x_i \text{ et } x_k = \max\limits_{k \leqslant i \leqslant n} x_i \\ x_k & \text{sinon.} \end{cases}.$$

**Théorème 1.3.1** (Armendáriz & Loulakis, Théorème 1 dans [9]). *Soit $\rho$ une mesure de probabilité $\Delta$ – sous-exponentielle et soit $(Z_n)_{n \geqslant 0}$ la marche aléatoire associée. Posons $R_n = Z_n - Z_{n-1}$ pour $n \geqslant 1$ de sorte que $Z_n = R_1 + \cdots + R_n$. Pour $n \geqslant 1$ et $x > 0$, soit $\mu_{n,x}$ la mesure de probabilité sur $\mathbb{R}^n$ qui est la loi de $(R_1, \ldots, R_n)$ sous la loi conditionnelle $\mathbb{P}[\,\cdot \mid Z_n \in x+\Delta]$. Alors il existe une suite $q_n \to \infty$ telle que :*

$$\lim_{n \to \infty} \sup_{x \geqslant q_n} \sup_{A \in \mathfrak{B}(\mathbb{R}^{n-1})} \left| \mu_{n,x} \circ T^{-1}[A \times \mathbb{R}] - \mu^{\otimes(n-1)}[A] \right| = 0.$$

Ce résultat signifie que si l'on considère une marche aléatoire sous $\mathbb{P}[\cdot \mid Z_n \in x + \Delta]$ et qu'on enlève le saut maximal, alors asymptotiquement il nous reste $n-1$ sauts qui se comportent comme des variables i.i.d de loi $\rho$. Nous renvoyons à [9] pour le lien entre la suite $(q_n)$ et la suite $(d_n)$ évoquée précédemment.

Expliquons maintenant comment utiliser le Théorème 1.3.1 pour étudier la marche de Lukasiewicz de $t_n$. D'après la Proposition 1.1.1, $(\mathcal{W}_1(t_n), \ldots, \mathcal{W}_n(t_n))$ a la même loi que $(W_1, \ldots, W_n)$ sous $\mathbb{P}[\cdot \mid \zeta_1 = -1]$. Étudions d'abord $(W_1, \ldots, W_n)$ sous le conditionnement $\mathbb{P}[\cdot \mid W_n = -1]$ qui est plus simple (il suffit ensuite d'appliquer une transformée de Vervaat). Un calcul simple donne $\mathbb{E}[W_1] = -\gamma = m - 1$. Comme il est plus commode de travailler avec variables aléatoires centrées, posons $\overline{W}_n = W_n + \gamma n$ et $\overline{R}_n = \overline{W}_n - \overline{W}_{n-1}$ pour $n \geqslant 1$, de sorte que $\overline{W}_n = \overline{R}_1 + \cdots + \overline{R}_n$.

Il s'agit donc d'étudier $(\overline{R}_1, \ldots, \overline{R}_n)$ sous $\mathbb{P}\left[\cdot \mid \overline{W}_n = \gamma n - 1\right]$. Or, d'après nos hypothèses sur $\mu$, on a $\mathbb{P}\left[\overline{W}_1 \in (x, x+1]\right] \sim \mathcal{L}(x)/x^{1+\theta}$ lorsque $x \to \infty$. Mais alors, d'après [32, Théorème 9.1], on a pour tout $\epsilon > 0$ fixé, uniformément en $x \geqslant \epsilon n$ :

$$\mathbb{P}\left[\overline{W}_n \in (x, x+1]\right] \underset{n \to \infty}{\sim} n\mathbb{P}\left[\overline{W}_1 \in (x, x+1]\right].$$

Ceci permet de dire que la loi de $\overline{R}_1$ est $(0,1]$ – sous-exponentielle et d'appliquer le résultat d'Armendáriz & Loulakis avec $q_n = \epsilon n$. Ceci est l'argument clé pour prouver le Théorème 6. Nous renvoyons au chapitre 5 pour les détails.

En particulier, le Théorème 6 reste vrai si on affaiblit la condition « $\mu_n = \mathcal{L}(n)/n^{1+\theta}$ » en la condition plus générale « $\mu$ est $(0,1]$ – sous-exponentielle et on peut appliquer le Théorème 1.3.1 avec $q_n = \epsilon n$ ». Cependant, tous les autres résultats qui vont suivre utilisent de manière cruciale le fait que $\mu$ est dans le domaine d'attraction d'une loi stable.

Le raisonnement précédent permet également de décrire les fluctuations autour de $\gamma n$ du nombre d'enfants du sommet ayant le plus grand nombre d'enfants. En effet, avec probabilité tendant vers 1 quand $n \to \infty$, $\gamma n - \Delta(t_n)$ se comporte comme la somme de $n-1$ variables aléatoires i.i.d. dans le domaine d'attraction d'une loi stable d'indice $2 \wedge \theta$. On rappelle que $(X_t)_{t \geqslant 0}$ est un processus de Lévy stable spectralement positif d'indice $2 \wedge \theta$.

**Théorème 7.**

Il existe une fonction à variation lente $L$ telle que

$$\frac{\Delta(t_n) - \gamma n}{L(n) n^{1/(2\wedge\theta)}} \xrightarrow[n \to \infty]{(d)} -X_1.$$

### Localisation du sommet de degré maximal

Rappelons la notation $u_\star(t_n)$ pour le (premier) sommet ayant le plus d'enfants de $t_n$ et $U(t_n)$ pour l'indice du premier sommet de $t_n$ dans l'ordre lexicographique ayant le plus grand nombre d'enfants. Il est possible de décrire la position de $u_\star(t_n)$ dans l'arbre $t_n$ :

**Théorème 8.**

Les deux convergences suivantes ont lieu :

(i) Pour $i \geqslant 0$, $\mathbb{P}[U(t_n) = i] \underset{n\to\infty}{\longrightarrow} \gamma \cdot \mathbb{P}_\mu[\zeta(\tau) \geqslant i+1]$.

(ii) Pour $i \geqslant 0$, $\mathbb{P}[|u_\star(t_n)| = i] \underset{n\to\infty}{\longrightarrow} (1-m)m^i$.

Il est crucial de remarquer que les lois limites obtenues pour $U(t_n)$ et $|u_\star(t_n)|$ sont des mesures de probabilité sur $\mathbb{N}$ (ne chargeant pas $+\infty$). Le point (i) se démontre à partir du Théorème 6 par un argument d'absolue continuité. Le point (ii) s'obtient en combinant (i) avec la convergence locale présentée en Section 1.3.1. Précisons qu'il ne suffit pas de dire que $u_\star(t_n)$ « converge » vers le haut de l'épine dorsale de la Section 1.3.1 : il faut encore pouvoir dire que $u_\star(t_n)$ ne s'échappe pas vers « l'infini », ce qui est exactement contenu dans le point (i).

**Hauteur de l'arbre**

La hauteur $\mathcal{H}(t_n)$ de $t_n$ ne croît pas comme une puissance de $n$, contrairement à ce qui se passe dans le cas critique :

**Théorème 9.**

Pour toute suite $(\lambda_n)_{n\geqslant 1}$ de nombres réels strictement positifs divergeant vers l'infini, on a

$$\mathbb{P}\left[\left|\mathcal{H}(t_n) - \frac{\ln(n)}{\ln(1/m)}\right| \leqslant \lambda_n\right] \underset{n\to\infty}{\longrightarrow} 1.$$

Cela vient du fait que $\mathcal{H}(t_n)$ est proche de la hauteur d'une forêt de $\gamma n$ $GW_\mu$ arbres indépendants et qu'il existe une constante $c > 0$ telle que $\mathbb{P}_\mu[\mathcal{H}(\tau) \geqslant k] \sim c \cdot m^k$ lorsque $k \to \infty$. Le Théorème 9 permet de régler la question de la convergence renormalisée des fonctions de hauteur et de contour de $t_n$ :

**Théorème 10.**

Soit $(r_n)_{n\geqslant 1}$ une suite de nombre réels strictement positifs. Pour tout $n \geqslant 1$, on pose soit $Y^{(n)} = (C_{2nt}(t_n)/r_n, 0 \leqslant t \leqslant 1)$, soit $Y^{(n)} = (H_{nt}(t_n)/r_n, 0 \leqslant t \leqslant 1)$.

(i) Si $r_n/\ln(n) \to \infty$, alors $Y^{(n)}$ converge en loi dans $\mathcal{C}([0,1],\mathbb{R})$ vers la fonction nulle sur $[0,1]$ lorsque $n \to \infty$.

(ii) Sinon, la suite $(Y^{(n)})_{n\geqslant 1}$ n'est pas tendue dans $\mathcal{C}([0,1],\mathbb{R})$.

Voir Fig. 1.7 pour une simulation de la fonction de contour de $t_n$.

Chapitre 2

# La triangulation brownienne : thème et variations

Au début des années 1990, Aldous s'est intéressé aux triangulations d'un polygône régulier inscrit dans le cercle unité, et a prouvé que lorsque son nombre de côtés tend vers l'infini, une triangulation choisie uniformément au hasard converge, en un certain sens, vers une triangulation « continue » aléatoire du disque unité, appelée triangulation brownienne. Dans ce chapitre, il s'agit de généraliser le résultat d'Aldous à diverses classes de configurations non croisées aléatoires construites à partir des sommets du polygône. Les arbres de Galton-Watson conditionnés à avoir un nombre de feuilles fixé seront la pierre angulaire de ces travaux.

## 2.1 Dissections et laminations aléatoires

### 2.1.1 Triangulations uniformes et triangulation brownienne

Soit, pour $n \geqslant 3$, $P_n$ le polygône régulier du plan dont les sommets sont les racines $n$-ièmes de l'unité. Par définition, une **dissection** de $P_n$ est l'union des côtés de $P_n$ et d'une collection de diagonales qui ne peuvent s'intersecter qu'en leurs extrémités. Les faces sont les composantes connexes du complémentaire de la dissection à l'intérieur de $P_n$. Une dissection est une **triangulation** lorsque toutes les faces sont des triangles.

Il est clair qu'une triangulation est un sous-ensemble compact du disque unité fermé $\overline{\mathbb{D}}$. Considérons une triangulation aléatoire $\mathfrak{T}_n$, choisie uniformément au hasard parmi toutes les triangulations de $P_n$. Aldous [6, 7] a eu l'idée d'étudier la convergence de $\mathfrak{T}_n$, lorsque $n \to \infty$, au sein de l'espace métrique des sous-ensembles compacts de $\overline{\mathbb{D}}$ muni de la distance de Hausdorff. Il a démontré que $\mathfrak{T}_n$ convergeait en loi, lorsque $n \to \infty$, vers un sous-ensemble compact aléatoire de $\overline{\mathbb{D}}$ appelé « triangulation brownienne », que nous définissons maintenant.

Soit $\mathbb{e}$ une excursion brownienne normalisée. Rappelons la relation d'équivalence $s \overset{\mathbb{e}}{\sim} t$ sur $[0,1]$ introduite dans la Section 1.1.5 : pour $s \leqslant t$, on pose $s \overset{\mathbb{e}}{\sim} t$ si et seulement si on a $\mathbb{e}_s = \mathbb{e}_t = \min_{r \in [s,t]} \mathbb{e}_r$. On définit alors (voir Fig. 2.1 pour un exemple) :

$$\mathbf{L}(\mathbb{e}) = \bigcup_{s \overset{\mathbb{e}}{\sim} t} [e^{-2i\pi s}, e^{-2i\pi t}]. \tag{2.1}$$

L'excursion brownienne étant invariante par retournement du temps, il est possible de remplacer $[e^{-2i\pi s}, e^{-2i\pi t}]$ par $[e^{2i\pi s}, e^{2i\pi t}]$ sans changer la loi de $\mathbf{L}(\mathbb{e})$, mais nous gardons les signes négatifs car nous préférons parcourir le cercle dans le sens horaire.

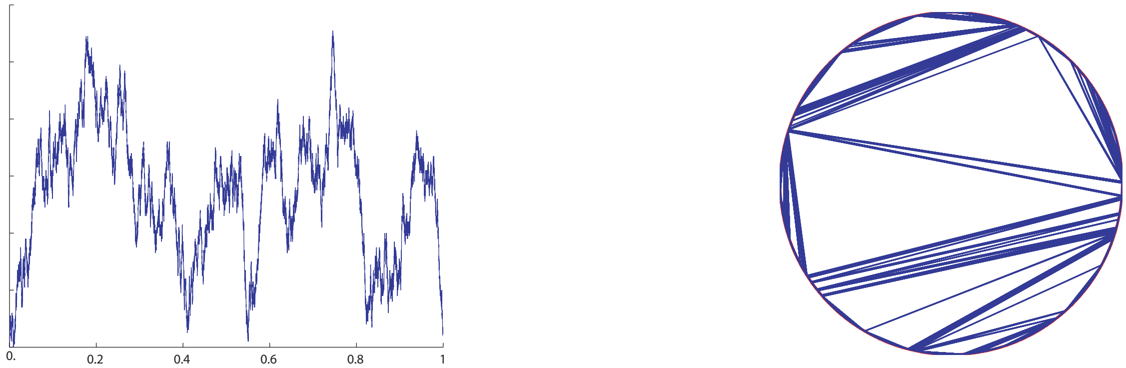


**FIGURE 2.1 –** Une excursion brownienne $\mathbb{e}$ et la triangulation brownienne associée $\mathbf{L}(\mathbb{e})$.

En utilisant le fait que lesminimas locaux de $\mathbb{e}$ sont presque sûrement différents, il est aisé de voir que $\mathbf{L}(\mathbb{e})$ est p.s. fermé et composé de cordes qui ne s'intersectent pas, et que $\mathbf{L}(\mathbb{e})$ est effectivement une triangulation, en ce sens que son complémentaire à l'intérieur de $\overline{\mathbb{D}}$ est p.s. constitué d'une union disjointe de triangles dont les sommets sont sur le cercle unité.

**Théorème 2.1.1** (Aldous [6])**.** *La convergence suivante a lieu :*

$$\mathfrak{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \boldsymbol{L}(\mathbb{e}).$$

En conséquence, la longueur de la plus longue diagonale de $\mathfrak{T}_n$ converge en loi vers la longueur de la plus longue corde de $\mathbf{L}(\mathbb{e})$, et l'aire du triangle de plus grande aire de $\mathfrak{T}_n$ converge vers l'aire du plus grand triangle de $\mathbf{L}(\mathbb{e})$. Les lois de ces deux quantités dépendant de $\mathbf{L}(\mathbb{e})$ sont explicites (voir [6]).

Par ailleurs, Aldous [6] et Le Gall & Paulin [77] ont prouvé que $\mathbf{L}(\mathbb{e})$ est p.s. de dimension de Hausdorff $3/2$.

Finalement, signalons que deux autres modèles de triangulations aléatoires continues ont récemment été étudiés : l'un construit récursivement, introduit par Curien & Le Gall [27] et l'autre invariant par transformations conformes, introduit par Curien & Werner [29].

Il s'agit maintenant de donner une idée des techniques utilisées pour démontrer le Théorème 2.1.1 et d'expliquer l'apparition de la définition (2.1). Le point clé est d'associer à une triangulation uniforme de $P_n$ un arbre binaire uniforme à $2n-3$ sommets par dualité, comme représenté sur la Figure 2.2.

Si $\mathfrak{T}$ est une triangulation, on note $\phi(\mathfrak{T})$ son arbre binaire associé. Considérons un triangle fixé de $\mathfrak{T}_n$, subdivisant les côtés de $P_n$ en trois ensembles. Intéressons-nous à la proportion de côtés dans chacun de ces ensembles. Sur la Figure 2.2, le triangle spécifié subdivise les côtés de $P_8$ en trois ensembles dont les proportions sont respectivement $4/8, 2/8, 2/8$.

Ces proportions peuvent également être lues sur la fonction de contour de l'arbre binaire : si on considère un sommet $u \in \mathfrak{T}_n$ qui n'est pas une feuille et qu'on note $s_1(u), s_2(u), s_3(u)$ les temps successifs de visites de $u$ par la fonction de contour (l'arbre étant binaire, $u$ est visité exactement trois fois), alors il est clair que le nombre total d'arêtes visitées par la fonction

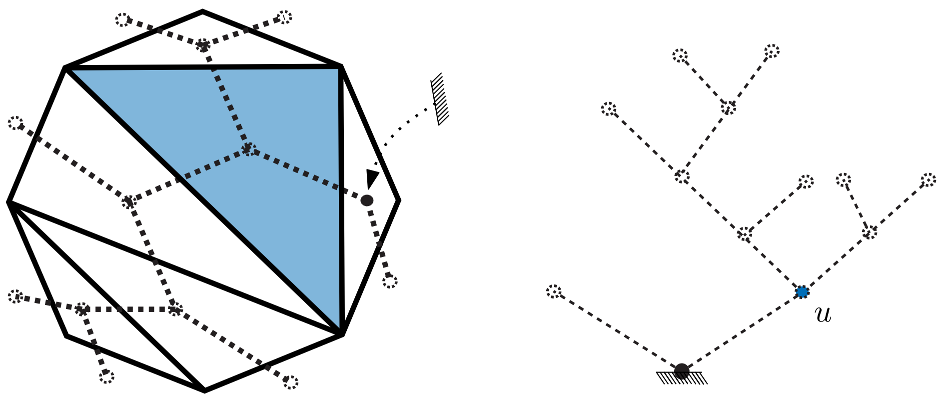


**FIGURE 2.2 –** Une triangulation de $P_8$ et son arbre binaire associé par dualité.

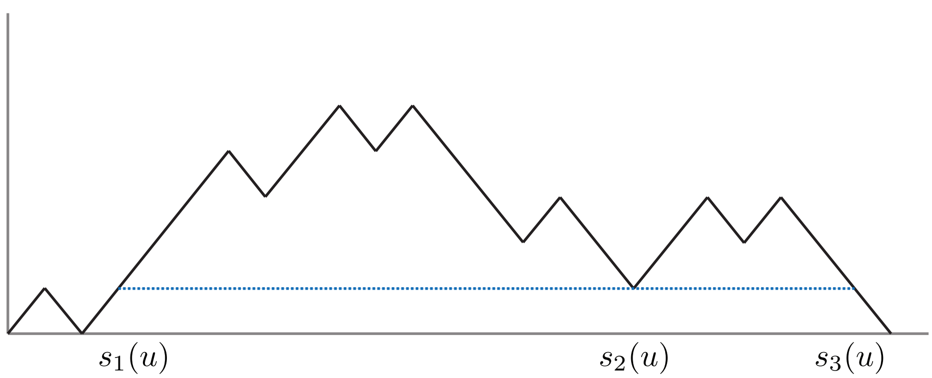


**FIGURE 2.3 –** La fonction de contour de l'arbre binaire de la Figure 2.2

de contour entre les instants $s_1(u)$ et $s_2(u)$ (resp. $s_2(u)$ et $s_3(u)$) est $(s_2(u) - s_1(u))/2$ (resp. $(s_3(u) - s_2(u))/2$), car chaque arête est visitée deux fois.

Or un arbre binaire a $k$ arêtes si et seulement si il a $k/2+1$ feuilles. Ainsi, le triangle associé à $u$ par dualité subdivise les côtés de $P_n$ en trois ensembles dont les proportions sont respectivement

$$\left( \frac{s_2(u) - s_1(u) + 2}{4n}, \frac{s_3(u) - s_2(u) + 2}{4n}, 1 - \frac{s_3(u) - s_1(u) + 4}{4n} \right).$$

Or $\zeta(\phi(\mathfrak{T}_n)) = 2n - 3$, de sorte que $\phi(\mathfrak{T}_n)$ est un arbre binaire uniforme à $2n-3$ sommets, autrement dit un arbre de Galton-Watson de reproduction $\mu$ donné par $\mu_0 = \mu_2 = 1/2$, conditionné à avoir $2n-3$ sommets. Le Théorème 1.1.3 nous donne alors :

$$\left(\frac{1}{2\sqrt{2n-3}} C_{2(2n-3)t}(\phi(\mathfrak{T}_n))\right)_{0\leqslant t\leqslant 1} \xrightarrow[n\to\infty]{(d)} \mathbb{e}. \tag{2.2}$$

D'après le théorème de représentation de Skorokhod, on peut supposer que cette convergence a lieu presque sûrement. Soit alors un instant $s_2$ de minimum local de $\mathbb{e}$. Il existe $s_1, s_3 \in [0,1]$ tels que $s_1 < s_2 < s_3$ et $\mathbb{e}_{s_1} = \mathbb{e}_{s_2} = \mathbb{e}_{s_3} = \min_{r\in[s_1,s_3]} \mathbb{e}_r$. Considérons le triangle

$$\bigcup_{1\leqslant i,j\leqslant 3} [e^{-2i\pi s_i}, e^{-2i\pi s_j}]. \tag{2.3}$$

Par définition, chaque triangle de $\mathbf{L}(\mathbb{e})$ peut être représenté sous cette forme. Nous allons maintenant voir que le triangle (2.3) peut être approché par des triangles de $\mathfrak{T}_n$ lorsque $n \to \infty$.

D'après (2.2), il existe des suites $s_1^{(n)}, s_2^{(n)}, s_3^{(n)}$ telles que $s_i^{(n)}/(4n-6) \to s_i$ pour $i = 1,2,3$ et $\mathbb{e}_{s_1^{(n)}} = \mathbb{e}_{s_2^{(n)}} = \mathbb{e}_{s_3^{(n)}} = \min_{r\in[s_1^{(n)},s_3^{(n)}]} \mathbb{e}_r$. D'après la discussion précédente, il existe un triangle de $\mathfrak{T}_n$ dont les proportions sont respectivement

$$\left(\frac{s_2^{(n)}(u) - s_1^{(n)}(u) + 2}{4n}, \frac{s_3^{(n)}(u) - s_2^{(n)}(u) + 2}{4n}, 1 - \frac{(s_3^{(n)}(u) - s_1^{(n)}(u)) + 4}{4n}\right).$$

Or, lorsque $n \to \infty$, ce triplet converge vers $(s_2 - s_1, s_3 - s_2, 1 - (s_3 - s_1))$, qui sont exactement les proportions du cercle délimitées par le triangle (2.3).

Cet argument permet de montrer que chaque triangle de $\mathbf{L}(\mathbb{e})$ est proche, au sens de la distance de Hausdorff, d'un triangle de $\mathfrak{T}_n$ lorsque $n \to \infty$, ce qui implique que $\mathbf{L}(\mathbb{e})$ est contenue dans n'importe quelle valeur d'adhérence de la suite $(\mathfrak{T}_n)$. Or $\mathbf{L}(\mathbb{e})$ est une triangulation maximale, en ce sens qu'il n'est pas possible de tracer une nouvelle corde sans en intersecter aucune autre. On en déduit alors la convergence $\mathfrak{T}_n \to \mathbf{L}(\mathbb{e})$.

Il est important de voir que dans cette esquisse de preuve le fait qu'on travaille avec des arbres binaires est crucial : d'une part, on utilise le fait que chaque sommet d'un arbre binaire qui n'est pas une feuille (sauf la racine) est visité exactement trois fois par la fonction de contour et, d'autre part, on utilise le fait qu'il existe une relation explicite entre le nombre de sommets et le nombre de feuilles d'un arbre binaire.

### 2.1.2 Laminations stables

Notre but est maintenant de généraliser le résultat d'Aldous à des dissections aléatoires qui ne sont pas nécessairement des triangulations.

L'équivalent continu d'une dissection est une lamination géodésique : par définition, une **lamination géodésique** (ou lamination) de $\overline{\mathbb{D}}$ est un sous-ensemble fermé L de $\overline{\mathbb{D}}$ qui est l'union d'une collection de cordes qui ne s'intersectent pas. La lamination L est dite **maximale** si elle est maximale au sens de l'inclusion parmi toutes les laminations géodésiques de $\overline{\mathbb{D}}$. Par exemple, la triangulation brownienne $\mathbf{L}(\mathbb{e})$ est une lamination maximale. Il est facile de voir que les laminations forment un sous-ensemble fermé vis-à-vis de la distance de Hausdorff.

En géométrie hyperbolique, les laminations géodésiques sont définies comme des sous-ensembles fermés du disque hyperbolique (voir [22]). Cependant, comme dans [27], nous voyons les laminations comme des sous-ensembles compacts de $\overline{\mathbb{D}}$ pour pouvoir étudier leur convergence au sens de la distance de Hausdorff.

Généralisons le modèle d'Aldous des triangulations uniformes en considérant une suite de nombres réels positifs $\boldsymbol{\mu} = (\mu_j)_{j \geqslant 0}$, appelée suite de poids. Pour $n \geqslant 2$, on note $\mathbb{L}_n$ l'ensemble des dissections de $P_{n+1}$, et on pose

$$Z_n = \sum_{\omega \in \mathbb{L}_n} \prod_{f \text{ face de } \omega} \mu_{\deg(f)-1},$$

où $\deg(f)$ est le degré de la face $f$, à savoir le nombre d'arêtes sur la frontière de $f$, et, pour chaque $n \geqslant 2$ tel que $Z_n \neq 0$, on introduit la mesure de probabilité de Boltzmann sur $\mathbb{L}_n$ associée à la suite de poids $\boldsymbol{\mu}$ :

$$\mathbb{P}_n^{\boldsymbol{\mu}}(\omega) = \frac{1}{Z_n} \prod_{f \text{ face de } \omega} \mu_{\deg(f)-1}, \qquad \omega \in \mathbb{L}_n.$$

Cette définition est similaire à celle des arbres aléatoires simplement générés introduits en Section 1.1.2. Deux cas sont particulièrement intéressants :

- pour un entier $p \geqslant 3$, si $\mu_0 = 1 - 1/(p-1)$, $\mu_{p-1} = 1/(p-1)$ et $\mu_i = 0$ si $i \notin \{0, p-1\}$, alors $\mathbb{P}_n^{\boldsymbol{\mu}}$ est la mesure uniforme sur l'ensemble des dissections de $\mathbb{L}_n$ dont toutes les faces ont degré $p$ (dans ce cas, il faut se restreindre aux valeurs de $n$ telles que $n-1$ est multiple de $p-2$). En particulier, pour $p = 3$ on retrouve le modèle des triangulations uniformes.
- Si $\mu_0 = 2 - \sqrt{2}$ et $\mu_i = ((2-\sqrt{2})/2)^{i-1}$ pour $i \geqslant 2$, nous verrons dans la section suivante que $\mathbb{P}_n^{\boldsymbol{\mu}}$ est la mesure uniforme sur $\mathbb{L}_n$.

Pour $n \geqslant 2$ tel que $Z_n \neq 0$, soit $\mathfrak{L}_n$ une dissection aléatoire de $\mathbb{L}_n$ de loi $\mathbb{P}_n^{\boldsymbol{\mu}}$. Est-ce que la suite $(\mathfrak{L}_n)$ converge en loi, lorsque $n \to \infty$, vers une lamination aléatoire du disque ?

Nous répondons par l'affirmative à cette question dans le cas où $\mu$ est critique et appartient au domaine d'attraction d'une loi stable d'indice $\theta \in (1, 2]$. Commençons par définir la lamination aléatoire limite. Rappelons la notation $H^{\text{exc}}$ pour le processus de hauteur normalisé d'indice $\theta$ (voir Section 1.1.4). Soit $\approx^{H^{\text{exc}}}$ la relation d'équivalence sur $[0, 1]$ définie par $s \approx^{H^{\text{exc}}} t$ si $H_s^{\text{exc}} = H_t^{\text{exc}}$ et $H_r^{\text{exc}} > H_s^{\text{exc}}$ pour tout $r \in (s \wedge t, s \vee t)$, ou bien si $(s, t)$ est la limite d'un couple satisfaisant à ces propriétés. Posons alors :

$$\mathbf{L}(H^{\text{exc}}) = \bigcup_{s \approx^{H^{\text{exc}}} t} [e^{-2i\pi s}, e^{-2i\pi t}].$$

Voir Fig. 2.4 pour des exemples.

Il est facile de vérifier que $\mathbf{L}(H^{\text{exc}})$ est une lamination aléatoire de $\overline{\mathbb{D}}$, appelée lamination stable d'indice $\theta$ du disque. Pour $\theta = 2$, $\mathbf{L}(H^{\text{exc}})$ a la même loi que $\mathbf{L}(\mathbb{e})$ (car $H^{\text{exc}}$ a la même loi que $\sqrt{2} \cdot \mathbb{e}$).

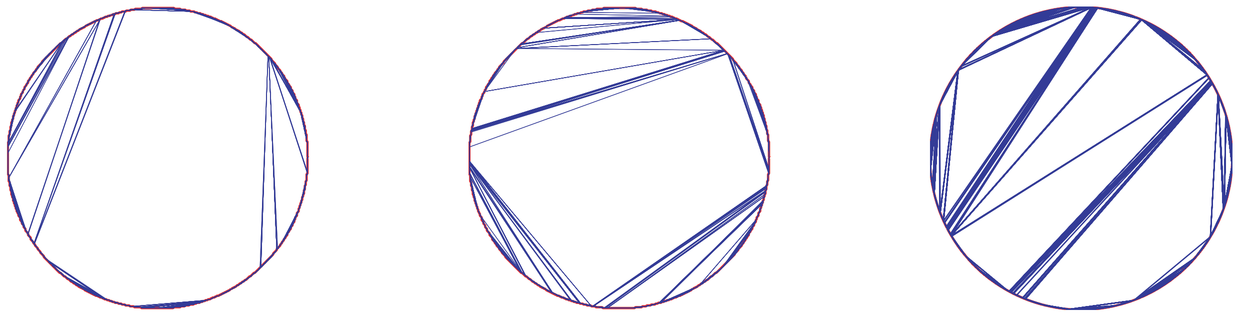

**FIGURE 2.4 –** Dissections aléatoires de $P_{27183}$ pour $\theta = 1.1$, $P_{11655}$ pour $\theta = 1.5$ et de $P_{20999}$ pour $\theta = 1.9$.

**Théorème 11.**

On suppose que $\mu$ est une mesure de probabilité critique, dans le domaine d'attraction d'une loi stable d'indice $\theta \in (1, 2]$ avec $\mu_1 = 0$. Alors :

$$\mathfrak{L}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}(H^{\mathrm{exc}}).$$

De plus, $\mathbf{L}(H^{\mathrm{exc}})$ est presque sûrement de dimension de Hausdorff $2 - 1/\theta$.

Ainsi, dès que $\mu$ est de variance finie et $\mu_1 = 0$, il y a convergence de $\mathfrak{L}_n$ vers la triangulation brownienne, qui vérifie donc une propriété d'universalité. En particulier, les dissections uniformes dont toutes les faces ont un degré $p \geqslant 3$ fixé convergent en loi vers la triangulation brownienne (ceci provient du fait que certaines arêtes dégénèrent à la limite pour ne donner que des triangles, et rappelle le fait que des arbres aléatoires conditionnés non nécessairement binaires convergent vers l'arbre brownien, dont tous les points de branchement sont binaires). En revanche, si $\theta \in (1, 2)$, des faces de degré infini apparaissent. Ceci est réminiscent du travail de Le Gall & Miermont [75] sur les cartes aléatoires tirées suivant des poids de Boltzmann, où des grandes faces demeurent à la limite lorsque la suite de poids a une queue lourde.

Le point clé pour établir le Théorème 11 est de remarquer que le dual $\phi(\mathfrak{L}_n)$ de $\mathfrak{L}_n$, construit comme dans la Figure 2.2, a la même loi qu'un arbre de Galton-Watson de loi de reproduction $\mu$, conditionné à avoir $n$ feuilles. L'idée directrice de la preuve suit alors celle du cas des triangulations uniformes en utilisant nos résultats concernant les arbres de Galton-Watson conditionnés à avoir un nombre de feuilles fixé.

Cependant, quelques différences majeures par rapport au cas des triangulations sont à noter, entraînant une complexification des arguments :

- l'arbre $\phi(\mathfrak{L}_n)$ n'étant plus forcément binaire, chaque sommet n'est pas nécessairement visité exactement trois fois par la fonction de contour. Au lieu d'utiliser la convergence de la fonction de contour renormalisée de $\phi(\mathfrak{L}_n)$, il faut alors plutôt utiliser la convergence de la marche de Lukasiewicz renormalisée de $\phi(\mathfrak{L}_n)$, garantie par le Théorème 5.

- lorsqu'un arbre n'est pas binaire, il n'y a pas de relation simple entre le nombre de sommets et le nombre de feuilles. Il s'agit alors d'utiliser les résultats de concentration établis par la Proposition 2 pour contrôler le nombre de feuilles d'un arbre de Galton-Watson conditionné.
- la lamination $\mathbf{L}(H^{exc})$ n'est pas maximale pour $\theta \neq 2$.

## 2.2 La triangulation brownienne : une limite universelle de configurations non croisées aléatoires

Après nous être intéressés à des dissections aléatoires tirées suivant des poids de Boltzmann, nous nous penchons sur d'autres modèles de configurations non-croisées tirées aléatoirement selon une loi uniforme. Un modèle que nous étudions en détail est celui des dissections choisies uniformément parmi toutes les dissections d'un polygône au nombre de côtés fixé. Nous verrons qu'il existe une suite de poids $\mu$ telle que ces dissections uniformes puissent être réalisées comme des dissections tirées suivant $\mu$, ce qui permet une approche nouvelle des dissections uniformes.

### 2.2.1 Convergence vers la triangulation brownienne

Par définition, un **graphe non croisé** de $P_n$ est un graphe tracé dans le plan dont les sommets sont les sommets de $P_n$ et les arêtes sont des segments qui ne se coupent pas intérieurement. Un **arbre non croisé** de $P_n$ est un graphe non croisé de $P_n$ qui est un arbre. Une **partition non croisée** de $P_n$ est un graphe non croisé de $P_n$ qui est une réunion disjointe de cycles ou de segments. Une **partition en paires non croisées** de $P_{2n}$ est un graphe non croisé constitué d'une union disjointe de segments tel qu'aucune composante connexe ne soit un singleton (voir Fig. 2.5 pour un exemple).

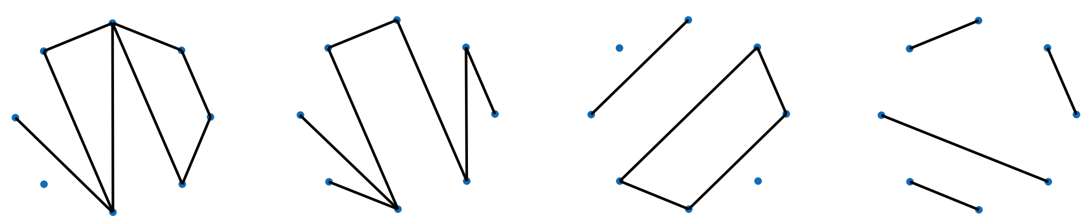

**FIGURE 2.5 –** Exemple de, successivement, un graphe non croisé, un arbre non croisé, une partition non croisée et une partition en paires non croisées de $P_8$.

Pour $n \geqslant 3$, soit $\mathfrak{N}_n$ une dissection tirée uniformément parmi toutes les dissections de $P_n$, ou bien un graphe non croisé de $P_n$ tiré uniformément parmi tous les graphes non croisés de $P_n$, ou bien un arbre non croisé de $P_n$ tirée uniformément parmi tous les arbres non croisés de $P_n$, ou bien une partition non croisée de $P_n$ tiré uniformément parmi toutes les partitions non croisées de $P_n$, ou bien une partition en paires non croisées de $P_{2n}$ tirée uniformément parmi toutes les partitions en paires non croisées de $P_{2n}$.

**Théorème 12** (avec N. Curien).

La convergence

$$\mathfrak{N}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}(\mathbb{e})$$

a lieu en loi dans l'espace métrique des sous-ensembles compacts de $\overline{\mathbb{D}}$ muni de la distance de Hausdorff.

Ainsi, lorsque le nombre de côtés du polygône est très grand, toutes ces configurations non croisées uniformes ressemblent à un même objet aléatoire continu : la triangulation brownienne (voir Fig. 2.6).

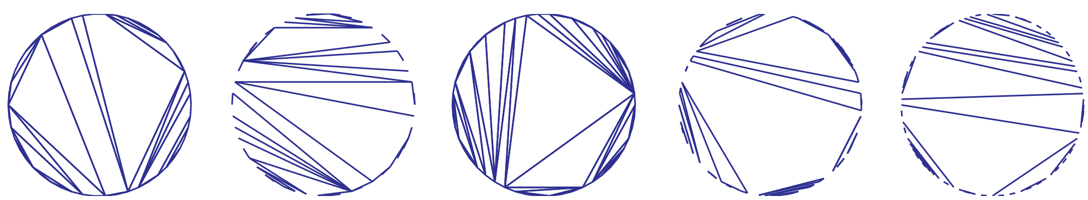

**FIGURE 2.6 –** Exemples de configurations non croisées uniformes : une dissection de $P_{50}$, un arbre non croisé de $P_{50}$, une triangulation de $P_{50}$, une partition non croisée de $P_{100}$ et une partition en paires non croisées de $P_{100}$.

Un conséquence particulièrement intéressante de ce phénomène d'universalité est que la longueur de la plus longue diagonale (resp. l'aire de la plus grande face à l'intérieur de $\overline{\mathbb{D}}$) de toutes ces configurations non croisées aléatoires va converger en loi vers la même loi de probabilité, qui est la longueur de la plus longue corde de la triangulation brownienne (resp. l'aire du triangle de plus grande aire de la triangulation brownienne).

L'idée clé pour démontrer le Théorème 12 est de remarquer que chacune de ces configurations non croisées uniformes peut être codée, en un certain sens, par un arbre de Galton-Watson conditionné, dont la loi de reproduction est de variance finie. Les fonctions de contour convenablement renormalisées de ces arbres conditionnés convergeant vers l'excursion brownienne, cela explique le Théorème 12 au niveau intuitif.

Présentons seulement ce codage pour les dissections uniformes (voir le chapitre 7 pour les autres modèles). À une dissection $\mathfrak{D}$ de $P_{n+1}$, on associe un arbre $\phi(\mathfrak{D})$ par dualité, comme sur la Figure 2.7.

Il est aisé de constater que $\phi$ réalise une bijection entre l'ensemble des dissections de $P_{n+1}$ et l'ensemble $\mathbb{T}_n^{(\ell)}$ des arbres avec $n$ feuilles tels qu'aucun sommet n'ait qu'un seul enfant. Ainsi, si $\mathfrak{D}_n$ est une dissection uniforme de $P_{n+1}$, alors $\phi(\mathfrak{D}_n)$ est uniformément distribué sur $\mathbb{T}_n^{(\ell)}$. Le résultat suivant, qui a également été prouvé indépendamment par Pitman & Rizzolo [91], établit que cette loi est en fait celle d'un arbre de Galton-Watson conditionné par le nombre de feuilles.

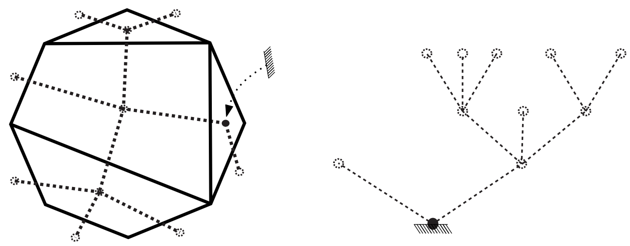

**FIGURE 2.7 –** Une dissection de $P_8$ et son arbre associé par dualité.

**Proposition 13** (Avec N. Curien).

Un arbre aléatoire de loi uniforme sur l'ensemble $\mathbb{T}_n^{(\ell)}$ a la même loi qu'un arbre de Galton-Watson de loi de reproduction $\mu^{(\ell)}$ donnée par

$$\mu_0^{(\ell)} = 2 - \sqrt{2}, \qquad \mu_1^{(\ell)} = 0, \qquad \mu_i^{(\ell)} = \left(\frac{2-\sqrt{2}}{2}\right)^{i-1} \qquad i \geqslant 2,$$

conditionné à avoir $n$ feuilles.

Ce résultat est une conséquence simple du fait qu'on a $\sum_{u\in\tau, k_u(\tau)>0}(k_u(\tau)-1) = \lambda(\tau)-1$ pour tout arbre $\tau$.

La convergence de $\mathfrak{D}_n$ vers la triangulation brownienne s'obtient alors aisément en utilisant notre théorème limite (Théorème 5 dans cette Introduction) concernant la convergence renormalisées des fonctions codant les arbres de Galton-Watson conditionnés à avoir un nombre de feuilles fixé.

### 2.2.2 Propriétés combinatoires de grandes dissections uniformes

Divers travaux ont concentré leur attention sur des propriétés combinatoires de configurations aléatoires non croisées de $P_n$ : Devroye, Flajolet, Hurtado, Noy & Steiger [34] se sont intéressés au degré maximal ainsi qu'à la longueur de la plus longue diagonale de triangulations uniformes, et Gao & Wormald [48] ont prouvé des résultats de concentration sur leur degré maximal. Deutsch & Noy [33] ont étudié de nombreuses statistiques d'arbres non croisés uniformes, et Marckert & Panholzer [81] ont établi qu'un arbre non croisé uniforme est « presque » un arbre de Galton-Watson. Finalement, Gao & Wormald [47] et Bernasconi, Panagiotou & Steger [14] obtiennent des résultats intéressants concernant la répartition des degrés de dissections uniformes.

Les arbres de Galton-Watson conditionnés à avoir un nombre de feuilles fixé apparaissent ainsi comme un nouvel outil pour étudier des propriétés combinatoires des dissections uniformes. Nous en déduisons des preuves probabilistes simples de certains résultats ayant été obtenus par des méthodes de combinatoire analytiques, et nous établissons également certains résultats nouveaux.

La première application consiste à obtenir un équivalent du nombre de dissections de $P_n$. Flajolet et Noy [45] établissent par des méthodes de combinatoire analytique l'équivalent

$$\#\mathbb{L}_{n-1} \underset{n\to\infty}{\sim} \frac{1}{4}\sqrt{\frac{99\sqrt{2}-140}{\pi}}n^{-3/2}(3+2\sqrt{2})^n. \tag{2.4}$$

Par des méthodes probabilistes utilisant les arbres de Galton-Watson conditionnés, nous prouvons le résultat suivant.

**Théorème 14** (Avec N. Curien)**.**

Soit $\mathcal{A}$ un sous-ensemble non vide de $\{3,4,5,\dots\}$. Soit $\mathbf{D}_n^{(\mathcal{A})}$ l'ensemble des dissections de $P_{n+1}$ dont le degré de toutes les faces appartient à $\mathcal{A}$. On se restreint aux valeurs de $n$ pour lesquelles $\mathbf{D}_n^{(\mathcal{A})} \neq \varnothing$. Il existe une loi de probabilité $\nu_{\mathcal{A}}$ sur $\mathbb{N}$ telle que si $\sigma_{\mathcal{A}}^2$ est la variance de $\nu_{\mathcal{A}}$, alors

$$\#\mathbf{D}_{n-1}^{(\mathcal{A})} \underset{n\to\infty}{\sim} \sqrt{\frac{\nu_{\mathcal{A}}(2)^4\nu_{\mathcal{A}}(0)^3}{2\pi\sigma_{\mathcal{A}}^2}} \cdot \frac{n^{-3/2}}{(\nu_{\mathcal{A}}(2)\nu_{\mathcal{A}}(0))^n}.$$

Lorsque $\mathcal{A}=\{3,4,5,\dots\}$, on a $\#\mathbf{D}_{n-1}^{(\mathcal{A})}=\#\mathbb{L}_{n-1}$, $\nu_{\mathcal{A}}=\mu^{(\ell)}$ et nous retrouvons (2.4).

Expliquons rapidement à quel moment interviennent les arbres de Galton-Watson dans la preuve du Théorème 14. Soit $\mathfrak{D}_n^{(\mathcal{A})}$ une dissection de $\mathbf{D}_n^{(\mathcal{A})}$ choisie uniformément au hasard. Comme dans la partie précédente, on voit que l'arbre dual $\phi(\mathfrak{D}_n^{(\mathcal{A})})$ est un arbre de Galton-Watson conditionné à avoir $n$ feuilles, pour une certaine loi de reproduction $\nu_{\mathcal{A}}$ critique et de variance finie. En particulier, pour n'importe quel arbre $\tau_0$ tel que $\mathbb{P}\left[\phi(\mathfrak{D}_n^{(\mathcal{A})})=\tau_0\right]>0$, on a :

$$\frac{1}{\#\mathbf{D}_{n-1}^{(\mathcal{A})}} = \frac{\mathbb{P}_{\nu_{\mathcal{A}}}[\tau=\tau_0]}{\mathbb{P}_{\nu_{\mathcal{A}}}[\lambda(\tau)=n]}.$$

Pour établir le Théorème 14, on choisit un arbre $\tau_0$ particulier, on évalue les probabilités mises en jeu (en utilisant le Théorème 4 pour le dénominateur), et on fait tendre $n\to\infty$.

Une deuxième application de cet outil consiste à étudier des propriétés de graphe d'une dissection uniforme $\mathfrak{D}_n$ de $P_{n+1}$. Introduisons quelques notations. Soient $\delta^{(n)}$ le degré de la face adjacente au côté $[1,e^{2i\pi/(n+1)}]$ dans la dissection $\mathcal{D}_n$ et $D^{(n)}$ le degré maximal d'une face de $\mathfrak{D}_n$. De même, on note $\partial^{(n)}$ le nombre de diagonales adjacentes au sommet d'affixe 1 dans $\mathfrak{D}_n$, et $\Delta^{(n)}$ le nombre maximal de diagonales adjacentes à un sommet de $\mathfrak{D}_n$. Finalement, pour $x,b>0$, on écrit $\log_b(x)$ pour $\ln(x)/\ln(b)$.

**Théorème 15** (Avec N. Curien)**.**

(i) On a, pour tout $k \geqslant 3$ :

$$\mathbb{P}\left[\delta^{(n)} = k\right] \underset{n\to\infty}{\longrightarrow} (k-1)\left(\frac{2-\sqrt{2}}{2}\right)^{k-2}.$$

(ii) On pose $\beta = 2+\sqrt{2}$. Alors pour tout $c > 0$, on a

$$\mathbb{P}\left(\log_\beta(n) - c\log_\beta\log_\beta(n) \leqslant D^{(n)} \leqslant \log_\beta(n) + c\log_\beta\log_\beta(n)\right) \underset{n\to\infty}{\longrightarrow} 1.$$

(iii) On a pour tout $k \geqslant 0$ :

$$\mathbb{P}(\partial^{(n)} = k) \underset{n\to\infty}{\longrightarrow} (k+1)(2-\sqrt{2})^2(\sqrt{2}-1)^k.$$

(iv) On pose $b = \sqrt{2}+1$. Alors, pour tout $c > 0$ :

$$\mathbb{P}\left(\Delta^{(n)} \geqslant \log_b(n) + (1+c)\log_b\log_b(n)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Le point (iv) résout une conjecture de Bernasconi, Panagiotou & Steger [14]. Tous ces résultats demeurent valides (avec des constantes différentes) lorsque $\mathfrak{D}_n$ est remplacé par $\mathfrak{D}_n^{(\mathcal{A})}$.

Commentons brièvement les techniques utilisées pour démontrer le Théorème 15. Pour les points (i) et (iii), on prouve que l'arbre dual $\phi(\mathfrak{D}_n)$, qui est un $\mathrm{GW}_{\mu^{(\ell)}}$ arbre conditionné à avoir $n$ feuilles, converge localement en loi, lorsque $n \to \infty$ vers l'arbre de Galton-Watson conditionné à survivre $\widehat{\mathcal{T}}_{\mu^{(\ell)}}$. Or, par dualité, $\delta^{(n)} - 1$ est le nombre d'enfants de la racine de $\phi(\mathfrak{D}_n)$, qui converge donc en loi vers le nombre d'enfants de la racine de $\widehat{\mathcal{T}}_{\mu^{(\ell)}}$, qui suit exactement la loi apparaissant dans (i). La preuve de (iii) est similaire, en remarquant que $\partial^{(n)}$ se lit aisément de manière « locale » sur $\phi(\mathfrak{D}_n)$. En revanche, les points (ii) et (iv), faisant intervenir des quantités qui ne sont pas locales, sont plus délicats à prouver.

*Les chapitres suivants présentent nos travaux [68, 70, 71, 69, 28], qui peuvent différer très légèrement des versions publiées ou soumises pour publication. Certaines figures ont été déplacées en Introduction, mais nous avons préféré ne pas supprimer les très brefs rappels sur les arbres afin que chaque chapitre puisse être lu de manière indépendante. Les notations restent globalement identiques, mais nous caressons l'espoir que le lecteur nous pardonnera certaines libertés (un processus de Lévy est tantôt noté* $X$, *tantôt* $Y$, *etc.).*

# Deuxième Partie

# *Théorèmes limites pour de grands arbres de Galton-Watson conditionnés*

Chapitre 3

# A simple proof of Duquesne's theorem on contour processes of conditioned Galton-Watson trees

*Les résultats de ce chapitre sont issus de l'article [68], soumis pour publication.*

## Contenu de ce chapitre



We give a simple new proof of a theorem of Duquesne, stating that the properly rescaled contour function of a critical aperiodic Galton-Watson tree, whose offspring distribution is in the domain of attraction of a stable law of index $\theta \in (1, 2]$, conditioned on having total progeny $n$, converges in the functional sense to the normalized excursion of the continuous-time height function of a strictly stable spectrally positive Lévy process of index $\theta$. To this end, we generalize an idea of Le Gall which consists in using an absolute continuity relation between the conditional probability of having total progeny exactly $n$ and the conditional probability of having total progeny at least $n$. This new method is robust and can be adapted to establish invariance theorems for Galton-Watson trees having $n$ vertices whose degrees are prescribed to belong to a fixed subset of the positive integers.

# Introduction

In this article, we are interested in the asymptotic behavior of critical Galton-Watson trees whose offspring distribution may have infinite variance. Aldous [5] studied the shape of large critical Galton-Watson trees whose offspring distribution has finite variance and proved that their properly rescaled contour functions converge in distribution in the functional sense to the Brownian excursion. This seminal result has motivated the study of the convergence of other rescaled paths obtained from Galton-Watson trees, such as the Lukasiewicz path (also known as the Harris walk) and the height function. In [79], under an additional exponential moment condition, Marckert & Mokkadem showed that the rescaled Lukasiewicz path, height function and contour function all converge in distribution to the same Brownian excursion. In parallel, unconditional versions of Aldous' result have been obtained in full generality. More precisely, when the offspring distribution is in the domain of attraction of a stable law of index $\theta \in (1, 2]$, Duquesne & Le Gall [37] showed that the concatenation of rescaled Lukasiewicz paths of a sequence of independent Galton-Watson trees converges in distribution to a strictly stable spectrally positive Lévy process $X$ of index $\theta$, and the concatenation of the associated rescaled height functions (or of the rescaled contour functions) converges in distribution to the so-called continuous-time height function associated to $X$. In the same monograph, Duquesne & Le Gall explained how to deduce a limit theorem for Galton-Watson trees conditioned on having at least $n$ vertices from the unconditional limit theorem. Finally, still in the stable case, Duquesne [36] showed that that the rescaled Lukasiewicz path of a Galton-Watson tree conditioned on having $n$ vertices converges in distribution to the normalized excursion of the Lévy process $X$ (thus extending Marckert & Mokkadem's result) and that the rescaled height and contour functions of a Galton-Watson tree conditioned on having $n$ vertices converge in distribution to the normalized excursion of the continuous-time height function $H^{exc}$ associated to $X$ (thus extending Aldous' result).

In this work, we give an alternative proof of Duquesne's result, which is based on an idea that appeared in the recent papers [72, 75]. Let us explain our approach after introducing some notation. For every $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to $x$. If $I$ is an interval, let $\mathcal{C}(I, \mathbb{R})$ be the space of all continuous functions $I \to \mathbb{R}$ equipped with the topology of uniform convergence on compact subsets of $I$. We also let $\mathbb{D}(I, \mathbb{R})$ be the space of all right-continuous with left limits (càdlàg) functions $I \to \mathbb{R}$, endowed with the Skorokhod $J_1$-topology (see [20, chap. 3], [57, chap. VI] for background concerning the Skorokhod topology). Denote by $\mathbb{P}_\mu$ the law of the Galton-Watson tree with offspring distribution $\mu$. The total progeny of a tree $\tau$ will be denoted by $\zeta(\tau)$. Fix $\theta \in (1, 2]$ and let $(X_t)_{t \geqslant 0}$ be the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta)$.

(0) We fix a critical offspring distribution $\mu$ in the domain of attraction of a stable law of index $\theta \in (1, 2]$. If $U_1, U_2, \ldots$ are i.i.d. random variables with distribution $\mu$, and $W_n = U_1 + \cdots + U_n - n$, there exist positive constants $(B_n)_{n \geqslant 0}$ such that $W_n / B_n$ converges in distribution to $X_1$.

(i) Fix $a \in (0, 1)$. To simplify notation, set $\mathcal{W}^{a,(n)} = (\mathcal{W}^{a,(n)}_j, 0 \leqslant j \leqslant \lfloor na \rfloor)$ where $\mathcal{W}^{a,(n)}_j = \mathcal{W}_j(\tau)/B_n$ and $\mathcal{W}(\tau)$ is the Lukasiewicz path of $\tau$. Then for every function $f_n : \mathbb{Z}^{\lfloor an \rfloor + 1} \to \mathbb{R}_+$, the following absolute continuity relation holds:

$$\mathbb{E}_\mu\left[f_n(\mathcal{W}^{a,(n)}) \,|\, \zeta(\tau) = n\right] = \mathbb{E}_\mu\left[f_n(\mathcal{W}^{a,(n)}) D^{(a)}_n\left(\mathcal{W}_{\lfloor an \rfloor}(\tau)\right) \,|\, \zeta(\tau) \geqslant n\right] \tag{3.1}$$

with a certain function $D_n^{(a)} : \mathbb{N} \to \mathbb{R}_+$.

(ii) We establish the existence of a measurable function $\Gamma_a : \mathbb{R}_+ \to \mathbb{R}_+$ such that the quantity $\left| D_n^{(a)}(j) - \Gamma_a(j/B_n) \right|$ goes to 0 as $n \to \infty$, uniformly in values of j such that $j/B_n$ stays in a compact subset of $\mathbb{R}_+^*$. Furthermore, if H denotes the continuous-time height process associated with X and **N** stands for the Itô excursion measure of X above its infimum, we have for every bounded measurable function $F : \mathbb{D}([0,a], \mathbb{R}) \to \mathbb{R}_+$:

$$\mathbf{N}\left(F((H_t)_{0 \leqslant t \leqslant a})\Gamma_a(X_a) \middle| \, \zeta > 1\right) = \mathbf{N}\left(F((H_t)_{0 \leqslant t \leqslant a}) \middle| \, \zeta = 1\right), \tag{3.2}$$

where $\zeta$ is the duration of the excursion under **N**.

(iii) We show that under $\mathbb{P}_\mu[\,\cdot\,|\zeta(\tau) = n]$, the rescaled height function converges in distribution on $[0,a]$ for every $a \in (0,1)$. To this end, we fix a bounded continuous function $F : \mathbb{D}([0,a], \mathbb{R}) \to \mathbb{R}_+$ and apply formula (3.1) with $f_n(\mathcal{W}^{a,(n)}) = F\left(\frac{B_n}{n} H_{\lfloor nt \rfloor}(\tau); 0 \leqslant t \leqslant a\right)$ where $H(\tau)$ is the height function of the tree $\tau$. Using the previously mentioned result of Duquesne & Le Gall concerning Galton-Watson trees conditioned on having at least n vertices, we show that we can restrict ourselves to the case where $\mathcal{W}_{\lfloor an \rfloor}(\tau)/B_n$ stays in a compact subset of $\mathbb{R}_+^*$, so that we can apply (ii) and obtain that:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\mu \left[ F\left(\frac{B_n}{n} H_{\lfloor nt \rfloor}(\tau);\, 0 \leqslant t \leqslant a\right) \middle| \, \zeta(\tau) = n \right] \\
&\qquad = \lim_{n\to\infty} \mathbb{E}_\mu \left[ F\left(\frac{B_n}{n} H_{\lfloor nt \rfloor}(\tau);\, 0 \leqslant t \leqslant a\right) D_n^{(a)}\left(\mathcal{W}_{\lfloor an \rfloor}(\tau)\right) \,|\, \zeta(\tau) \geqslant n \right] \\
&\qquad = \mathbf{N}\left(F(H_t; 0 \leqslant t \leqslant a)\Gamma_a(X_a) \mid \zeta > 1\right) \\
&\qquad = \mathbf{N}\left(F(H_t; 0 \leqslant t \leqslant a) \mid \zeta = 1\right).
\end{aligned}$$

(iv) By using a relationship between the contour function and the height function which was noticed by Duquesne & Le Gall in [37], we get that, under $\mathbb{P}_\mu[\,\cdot\,|\zeta(\tau) = n]$, the scaled contour function converges in distribution on $[0,a]$.

(v) By using the time reversal invariance property of the contour function, we deduce that under $\mathbb{P}_\mu[\,\cdot\,|\zeta(\tau) = n]$, the scaled contour function converges in distribution on the whole segment $[0,1]$.

(vi) Using once again the relationship between the contour function and the height function, we deduce that, under $\mathbb{P}_\mu[\,\cdot\,|\zeta(\tau) = n]$, the scaled height function converges in distribution on $[0,1]$.

In the case where the variance of $\mu$ is finite, Le Gall gave an alternative proof of Aldous' theorem in [72, Theorem 6.1] using a similar approach based on a strong local limit theorem. There are additional difficulties in the infinite variance case, since no such theorem is known in this case.

Let us finally discuss the advantage of this new method. Firstly, the proof is simpler and less technical. Secondly, we believe that this approach is robust and can be adapted to other situations. For instance, using the same ideas, we have established invariance theorems for Galton-Watson trees having n vertices whose degrees are prescribed to belong to a fixed subset of the nonnegative integers [70].

The rest of this text is organized as follows. In Section 1, we present the discrete framework by defining Galton-Watson trees and their codings. We explain how the local limit theorem gives information on the asymptotic behavior of large Galton-Watson trees and present the discrete absolute continuity relation appearing in (3.1). In Section 2, we discuss the continuous framework: we introduce the strictly stable spectrally positive Lévy process, its Itô excursion measure **N** and the associated continuous-time height process. We also prove the absolute continuity relation (3.3). Finally, in Section 3 we give the new proof of Duquesne's theorem by carrying out steps (i-vi).

**Acknowledgments.** I am deeply indebted to Jean-François Le Gall for insightful discussions and for making many useful suggestions on first versions of this manuscript.

**Notation and main assumptions.** Throughout this work $\theta \in (1, 2]$ is a fixed parameter. We consider a probability distribution $(\mu(j))_{j \geqslant 0}$ on the nonnegative integers satisfying the following three conditions:

(i) $\mu$ is critical, meaning that $\sum_{k=0}^{\infty} k\mu(k) = 1$.

(ii) $\mu$ is in the domain of attraction of a stable law of index $\theta \in (1, 2]$. This means that either the variance of $\mu$ is positive and finite, or $\mu([j, \infty)) = j^{-\theta} L(j)$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $L(x) > 0$ for $x$ large enough and $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$ (such a function is called slowly varying). We refer to [21] or [41, chapter 3.7] for details.

(iii) $\mu$ is aperiodic, which means that the additive subgroup of the integers $\mathbb{Z}$ spanned by $\{j;\ \mu(j) \neq 0\}$ is not a proper subgroup of $\mathbb{Z}$.

We introduce condition (iii) to avoid unnecessary complications, but our results can be extended to the periodic case.

In what follows, $(X_t)_{t\geqslant 0}$ will stand for the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^{\theta})$ where $t, \lambda \geqslant 0$ and $p_1$ will denote the density of $X_1$. Finally, $\nu$ will stand for the probability measure on $\mathbb{Z}$ defined by $\nu(k) = \mu(k+1)$ for $k \geqslant -1$. Note that $\nu$ has zero mean.

## 3.1 The discrete setting: Galton-Watson trees

### 3.1.1 Galton-Watson trees

**Definition 3.1.1.** Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of all nonnegative integers and $\mathbb{N}^* = \{1, \ldots\}$. Let also $U$ be the set of all labels:

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. An element of $U$ is a sequence $u = u_1 \cdots u_j$ of positive integers, and we set $|u| = j$, which represents the "generation" of $u$. If $u = u_1 \cdots u_j$ and $v = v_1 \cdots v_k$ belong to $U$, we write $uv = u_1 \cdots u_j v_1 \cdots v_k$ for the concatenation of $u$ and $v$. In particular, note that $u\emptyset = \emptyset u = u$. Finally, a *rooted ordered tree* $\tau$ is a finite subset of $U$ such that:

1. $\emptyset \in \tau$,
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,

3. for every $u \in \tau$, there exists an integer $k_u(\tau) \geqslant 0$ such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

In the following, by *tree* we will mean rooted ordered tree. The set of all trees is denoted by $\mathbb{T}$. We will often view each vertex of a tree $\tau$ as an individual of a population whose $\tau$ is the genealogical tree. The total progeny of $\tau$ will be denoted by $\zeta(\tau) = \mathrm{Card}(\tau)$. Finally, if $\tau$ is a tree and $u \in \tau$, we set $T_u\tau = \{v \in U;\ uv \in \tau\}$, which is itself a tree.

**Definition 3.1.2.** Let $\rho$ be a probability measure on $\mathbb{N}$ with mean less than or equal to 1 and such that $\rho(1) < 1$. The law of the Galton-Watson tree with offspring distribution $\rho$ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbb{T}$ such that:

1. $\mathbb{P}_\rho[k_\emptyset = j] = \rho(j)$ for $j \geqslant 0$,
2. for every $j \geqslant 1$ with $\rho(j) > 0$, conditionally on $\{k_\emptyset = j\}$, the shifted trees $T_1\tau, \dots, T_j\tau$ are i.i.d. with distribution $\mathbb{P}_\rho$.

A random tree whose distribution is $\mathbb{P}_\rho$ will be called a $\mathrm{GW}_\rho$ tree.

### 3.1.2 Coding Galton-Watson trees

We now explain how trees can be coded by three different functions. These codings are crucial in the understanding of large Galton-Watson trees.

**Definition 3.1.3.** We write $u < v$ for the lexicographical order on the labels $U$ (for example $\emptyset < 1 < 21 < 22$). Consider a tree $\tau$ and order the individuals of $\tau$ in lexicographical order: $\emptyset = u(0) < u(1) < \cdots < u(\zeta(\tau)-1)$. The height process $H(\tau) = (H_n(\tau), 0 \leqslant n < \zeta(\tau))$ is defined, for $0 \leqslant n < \zeta(\tau)$, by $H_n(\tau) = |u(n)|$. For technical reasons, we set $H_k(\tau) = 0$ for $k \geqslant \zeta(\tau)$. We extend $H(\tau)$ to $\mathbb{R}_+$ by linear interpolation by setting $H_t(\tau) = (1 - \{t\})H_{\lfloor t \rfloor}(\tau) + \{t\}H_{\lfloor t \rfloor + 1}(\tau)$ for $0 \leqslant t \leqslant \zeta(\tau)$, where $\{t\} = t - \lfloor t \rfloor$.

Consider a particle that starts from the root and visits continuously all edges at unit speed (assuming that every edge has unit length), going backwards as little as possible and respecting the lexicographical order of vertices. For $0 \leqslant t \leqslant 2(\zeta(\tau) - 1)$, $C_t(\tau)$ is defined as the distance to the root of the position of the particle at time $t$. For technical reasons, we set $C_t(\tau) = 0$ for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. The function $C(\tau)$ is called the contour function of the tree $\tau$. See Figure 1.3 for an example, and [36, Section 2] for a rigorous definition.

Finally, the Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), n \geqslant 0)$ of a tree $\tau$ is defined by $\mathcal{W}_0(\tau) = 0$, $\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1$ for $0 \leqslant n \leqslant \zeta(\tau) - 1$ and $\mathcal{W}_k(\tau) = 0$ for $k > \zeta(\tau)$ (see Figure 1.2) for an example.

Note that necessarily $\mathcal{W}_{\zeta(\tau)}(\tau) = -1$.

Let $(W_n; n \geqslant 0)$ be a random walk which starts at $0$ with jump distribution $\nu(k) = \mu(k+1)$ for $k \geqslant -1$. For $j \geqslant 1$, define $\zeta_j = \inf\{n \geqslant 0;\ W_n = -j\}$.

**Proposition 3.1.4.** $(W_0, W_1, \dots, W_{\zeta_1})$ *has the same distribution as the Lukasiewicz path of a* $\mathrm{GW}_\mu$ *tree. In particular, the total progeny of a* $\mathrm{GW}_\mu$ *tree has the same law as* $\zeta_1$.

*Proof.* See [73, Proposition 1.5]. □

We will also use the following well-known fact (see e.g. Lemma 6.1 in [90] and the discussion that follows).

**Proposition 3.1.5.** *For every integers* $1 \leqslant j \leqslant n$*, we have* $\mathbb{P}[\zeta_j = n] = \frac{j}{n}\mathbb{P}[W_n = -j]$.

### 3.1.3 Slowly varying functions

Slowly varying functions appear in the study of domains of attractions of stable laws. Here we recall some properties of these functions in view of future use.

Recall that a positive measurable function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be slowly varying if $L(x) > 0$ for $x$ large enough and, for all $t > 0$, $L(tx)/L(x) \to 1$ as $x \to \infty$. A useful result concerning these functions is the so-called Representation Theorem, which states that a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying if and only if it can be written in the form:

$$L(x) = c(x) \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), \qquad x \geqslant 0,$$

where $c$ is a nonnegative measurable function having a finite positive limit at infinity and $\epsilon$ is a measurable function tending to $0$ at infinity. See e.g. [21, Theorem 1.3.1] for a proof. The following result is then an easy consequence.

**Proposition 3.1.6.** *Fix* $\epsilon > 0$ *and let* $L : \mathbb{R}_+ \to \mathbb{R}$ *be a slowly varying function. There exist two constants* $C > 1$ *and* $N > 0$ *such that* $\frac{1}{C}x^{-\epsilon} \leqslant L(nx)/L(n) \leqslant Cx^{\epsilon}$ *for every integer* $n \geqslant N$ *and* $x \geqslant 1$.

### 3.1.4 The Local Limit Theorem

**Definition 3.1.7.** A subset $A \subset \mathbb{Z}$ is said to be lattice if there exist $b \in \mathbb{Z}$ and an integer $d \geqslant 2$ such that $A \subset b + d\mathbb{Z}$. The largest $d$ for which this statement holds is called the span of $A$. A measure on $\mathbb{Z}$ is said to be lattice if its support is lattice, and a random variable is said to be lattice if its law is lattice.

**Remark 3.1.8.** Since $\mu$ is supposed to be critical and aperiodic, using the fact that $\mu(0) > 0$, it is an exercise to check that the probability measure $\nu$ is non-lattice.

Recall that $p_1$ is the density of $X_1$. It is well known that $p_1(0) > 0$, that $p_1$ is positive, bounded and continuous, and that the absolute value of the derivative of $p_1$ is bounded over $\mathbb{R}$ (see e.g. [99, I. 4.]). The following theorem will allow us to find estimates for the probabilities appearing in Proposition 3.1.5.

**Theorem 3.1.9** (Local Limit Theorem)**.** *Let* $(Y_n)_{n\geqslant 0}$ *be a random walk on* $\mathbb{Z}$ *started from* $0$ *such that its jump distribution is in the domain of attraction of a stable law of index* $\theta \in (1, 2]$*. Assume that* $Y_1$ *is non-lattice, that* $\mathbb{E}[Y_1] = 0$ *and that* $Y_1$ *takes values in* $\mathbb{N} \cup \{-1\}$.

*(i) There exists an increasing sequence of positive real numbers* $(a_n)_{n\geqslant 1}$ *such that* $Y_n/a_n$ *converges in distribution to* $X_1$.

*(ii) We have* $\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| a_n \mathbb{P}[Y_n = k] - p_1\left(\frac{k}{a_n}\right)\right| = 0.$

*(iii) There exists a slowly varying function* $l : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $a_n = n^{1/\theta} l(n)$.

*Proof.* For (i), see [44, Section XVII.5, Theorem 3] and [21, Section 8.4]. The fact that $(a_n)$ may be chosen to be increasing follows from [41, Formula 3.7.2]. For (ii), see [56, Theorem 4.2.1]. For (iii), it is shown in [56, p. 46] that $a_{kn}/a_n$ converges to $k^{1/\theta}$ for every integer $k \geqslant 1$. Since $(a_n)$ is increasing, a theorem of de Haan (see [21, Theorem 1.10.7]) implies that there exists a slowly varying function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that $a_n = l(n)n^{1/\theta}$ for every positive integer $n$. □

Let $(W_n)_{n\geqslant 0}$ be as in Proposition 3.1.4 a random walk started from $0$ with jump distribution $\nu$. Since $\mu$ is in the domain of attraction of a stable law of index $\theta$, it follows that $\nu$ is in the same domain of attraction, and $W_1$ is not lattice by Remark 3.1.8. Since $\nu$ has zero mean, by the preceding theorem there exists an increasing sequence of positive integers $(B_n)_{n\geqslant 1}$ such that $B_n \to \infty$ and $W_n/B_n$ converges in distribution towards $X_1$ as $n \to \infty$. In what follows, the sequence $(B_n)_{n\geqslant 1}$ will be fixed, and $h : \mathbb{R}_+ \to \mathbb{R}_+$ will stand for a slowly varying function such that $B_n = h(n)n^{1/\theta}$.

**Lemma 3.1.10.** *We have:*

$$\text{(i)} \quad \mathbb{P}_\mu\left[\zeta(\tau) = n\right] \underset{n\to\infty}{\sim} \frac{p_1(0)}{n^{1/\theta+1}h(n)}, \qquad \text{(ii)} \quad \mathbb{P}_\mu\left[\zeta(\tau) \geqslant n\right] \underset{n\to\infty}{\sim} \frac{\theta p_1(0)}{n^{1/\theta}h(n)},$$

*where we write $a_n \sim b_n$ if $a_n/b_n \to 1$.*

*Proof.* We keep the notation of Proposition 3.1.4. Proposition 3.1.5 gives that:

$$\mathbb{P}_\mu[\zeta(\tau) = n] = \frac{1}{n}\mathbb{P}[W_n = -1]. \tag{3.3}$$

For (i), it suffices to notice that the local limit theorem (Theorem 3.1.9) and the continuity of $p_1$ entail $\mathbb{P}[W_n = -1] \sim p_1(0)/(h(n)n^{1/\theta})$. For (ii), we use (i) to write:

$$\mathbb{P}_\mu\left[\zeta(\tau) \geqslant n\right] = \sum_{k=n}^{\infty}\left(\frac{1}{h(k)k^{1+1/\theta}}p_1\left(0\right) + \frac{1}{h(k)k^{1+1/\theta}}\delta(k)\right),$$

where $\delta(k)$ tends to $0$ as $k \to \infty$. We can rewrite this in the form:

$$h(n)n^{1/\theta}\mathbb{P}_\mu\left[\zeta(\tau) \geqslant n\right] = \int_1^\infty du f_n(u), \tag{3.4}$$

where:

$$f_n(u) = \frac{h(n)n^{1/\theta+1}}{h(\lfloor nu\rfloor)\lfloor nu\rfloor^{1+1/\theta}}\left(p_1\left(0\right) + \delta(\lfloor nu\rfloor)\right).$$

For fixed $u \geqslant 1$, $f_n(u)$ tends to $\frac{p_1(0)}{u^{1/\theta+1}}$ as $n \to \infty$. Choose $\epsilon \in (0, 1/\theta)$. By Proposition 3.1.6, for every sufficiently large positive integer $n$ we have $f_n(u) \leqslant C/u^{1+1/\theta-\epsilon}$ for every $u \geqslant 1$, where $C$ is a positive constant. The dominated convergence theorem allows us to infer that:

$$\lim_{n\to\infty}\int_1^\infty du f_n(u) = \int_1^\infty du \frac{p_1(0)}{u^{1/\theta+1}} = \theta p_1(0),$$

and the desired result follows from (3.4). □

### 3.1.5 Discrete absolute continuity

The next lemma in another important ingredient of our approach.

**Lemma 3.1.11** (Le Gall & Miermont). *Fix $a \in (0,1)$. Then, with the notation of Proposition 3.1.5, for every $n \geqslant 1$ and for every bounded nonnegative function $f_n$ on $\mathbb{Z}^{\lfloor an \rfloor+1}$:*

$$\mathbb{E}\left[f_n(W_0,\dots,W_{\lfloor na \rfloor}) \,\middle|\, \zeta_1 = n\right] = \mathbb{E}\left[f_n(W_0,\dots,W_{\lfloor na \rfloor}) \frac{\phi_{n-\lfloor an \rfloor}(W_{\lfloor an \rfloor}+1)/\phi_n(1)}{\phi^*_{n-\lfloor an \rfloor}(W_{\lfloor an \rfloor}+1)/\phi^*_n(1)} \,\middle|\, \zeta_1 \geqslant n\right], \tag{3.5}$$

*where $\phi_p(j) = \mathbb{P}[\zeta_j = p]$ and $\phi^*_p(j) = \mathbb{P}[\zeta_j \geqslant p]$ for every integers $j \geqslant 1$ and $p \geqslant 1$.*

*Proof.* This result follows from an application of the Markov property to the random walk $W$ at time $\lfloor an \rfloor$. See [75, Lemma 10] for details in a slightly different setting. □

## 3.2 The continuous setting: stable Lévy processes

### 3.2.1 The normalized excursion of the Lévy process and the continuous-time height process

We follow the presentation of [36]. The underlying probability space will be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that X is a strictly stable spectrally positive Lévy process with index $\theta \in (1,2]$ such that for $\lambda > 0$:

$$\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta). \tag{3.6}$$

We denote the canonical filtration generated by X and augmented with the $\mathbb{P}$-negligible sets by $(\mathcal{F}_t)_{t\geqslant 0}$. See [15] for the proofs of the general assertions of this subsection concerning Lévy processes. In particular, for $\theta = 2$ the process X is $\sqrt{2}$ times the standard Brownian motion on the line. Recall that X has the following scaling property: for $c > 0$, the process $(c^{-1/\theta} X_{ct}, t \geqslant 0)$ has the same law as X. In particular, the density $p_t$ of the law of $X_t$ enjoys the following scaling property:

$$p_t(x) = t^{-1/\theta} p_1(x t^{-1/\theta}) \tag{3.7}$$

for $x \in \mathbb{R}$, $t > 0$. The following notation will be useful: for $s < t$, we set $I^s_t = \inf_{[s,t]} X$ and $I_t = \inf_{[0,t]} X$. Notice that the process I is continuous since X has no negative jumps.

The process $X - I$ is a strong Markov process and $0$ is regular for itself with respect to $X - I$. We may and will choose $-I$ as the local time of $X - I$ at level $0$. Let $(g_i, d_i), i \in \mathcal{I}$ be the excursion intervals of $X - I$ above $0$. For every $i \in \mathcal{I}$ and $s \geqslant 0$, set $\omega^i_s = X_{(g_i+s)\wedge d_i} - X_{g_i}$. We view $\omega^i$ as an element of the excursion space $\mathcal{E}$, which is defined by:

$$\mathcal{E} = \{\omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{R});\ \omega(0) = 0 \text{ and } \zeta(\omega) := \sup\{s > 0; \omega(s) > 0\} \in (0,\infty)\}.$$

From Itô's excursion theory, the point measure

$$\mathcal{N}(dt d\omega) = \sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, \omega^i)}$$

is a Poisson measure on $\mathbb{R}_+ \times \mathcal{E}$ with intensity $dt\mathbf{N}(d\omega)$, where $\mathbf{N}(d\omega)$ is a $\sigma$-finite measure on $\mathcal{E}$. By classical results, $\mathbf{N}(\zeta > t) = \Gamma(1-1/\theta)^{-1}t^{-1/\theta}$. Without risk of confusion, we will also use the notation X for the canonical process on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

We now define the normalized excursion of X. Let us first recall the Itô description of the excursion measure (see [25] or [15, Chapter VIII.4] for details). Define for $\lambda > 0$ the re-scaling operator $S^{(\lambda)}$ on $\mathcal{E}$ by $S^{(\lambda)}(\omega) = \left(\lambda^{1/\theta}\omega(s/\lambda), s \geqslant 0\right)$. Then there exists a unique collection of probability measures $(\mathbf{N}_{(a)}, a > 0)$ on $\mathcal{E}$ such that the following properties hold.

(i) For every $a > 0$, $\mathbf{N}_{(a)}(\zeta = a) = 1$.

(ii) For every $\lambda > 0$ and $a > 0$, we have $S^{(\lambda)}(\mathbf{N}_{(a)}) = \mathbf{N}_{(\lambda a)}$.

(iii) For every measurable subset $A$ of $\mathcal{E}$: $\mathbf{N}(A) = \displaystyle\int_0^\infty \mathbf{N}_{(a)}(A)\frac{da}{\theta\Gamma(1-1/\theta)a^{1/\theta+1}}$.

The probability distribution $\mathbf{N}_{(1)}$ on càdlàg paths with unit lifetime is called the law of the normalized excursion of X and will sometimes be denoted by $\mathbf{N}(\,\cdot\,|\zeta = 1)$. In particular, for $\theta = 2$ the process $X^{\text{exc}}$ is $\sqrt{2}$ times the normalized excursion of linear Brownian motion. Informally, $\mathbf{N}(\,\cdot\,|\zeta = 1)$ is the law of an excursion conditioned to have unit lifetime.

We will also use the so-called continuous-time height process H associated with X which was introduced in [76]. If $\theta = 2$, H is set to be equal to $X - I$. If $\theta \in (1,2)$, the process H is defined for every $t \geqslant 0$ by:

$$H_t := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{\{X_s < I_t^s + \epsilon\}} ds,$$

where the limit exists in $\mathbb{P}$-probability and in $\mathbf{N}$-measure on $\{t < \zeta\}$. The definition of H thus makes sense under $\mathbb{P}$ or under $\mathbf{N}$. The process H has a continuous modification both under $\mathbb{P}$ and under $\mathbf{N}$ (see [37, Chapter 1] for details), and from now on we consider only this modification. Using simple scale arguments one can also define H as a continuous random process under $\mathbf{N}(\,\cdot\,|\zeta = 1)$. For our purposes, we will need the fact that, for every $a \geqslant 0$, $(H_t)_{0 \leqslant t \leqslant a}$ is a measurable function of $(X_t)_{0 \leqslant t \leqslant a}$.

### 3.2.2 Absolute continuity property of the Itô measure

We now present the continuous counterpart of the discrete absolute continuity property appearing in Lemma 3.1.11. We follow the presentation of [72] but generalize it to the stable case. The following proposition is classical (see e.g. the proof of Theorem 4.1 in [93, Chapter XII], which establishes the result for Brownian motion).

**Proposition 3.2.1.** *Fix $t > 0$. Under the conditional probability measure $\mathbf{N}(\,\cdot\,|\zeta > t)$, the process $(X_{t+s})_{s \geqslant 0}$ is Markovian with the transition kernels of a strictly stable spectrally positive Lévy process of index $\theta$ stopped upon hitting 0.*

We will also use the following result (see [16, Corollary 2.3] for a proof).

**Proposition 3.2.2.** *Set $q_s(x) = \frac{x}{s} p_s(-x)$ for $x, s > 0$. For $x \geqslant 0$, let $T(x) = \inf\{t \geqslant 0;\, X_t < -x\}$ be the first passage time of $-X$ above $x$. Then $\mathbb{P}\left[T(x) \in dt\right] = q_t(x)dt$ for every $x > 0$.*

Note that $q_s$ is a positive continuous function on $(0, \infty)$, for every $s > 0$. It is also known that $q_s$ is bounded by a constant which is uniform when s varies over $[\epsilon, \infty)$, $\epsilon > 0$ (this follows from e.g. [99, I. 4.]).

**Proposition 3.2.3.** *For every* $a \in (0,1)$ *and* $x > 0$ *define:*

$$\Gamma_a(x) = \frac{\theta q_{1-a}(x)}{\int_{1-a}^{\infty} ds\, q_s(x)}.$$

*Then for every measurable bounded function* $G : \mathbb{D}([0,a],\mathbb{R}^2) \to \mathbb{R}_+$*:*

$$\mathbf{N}\left( G((X_t)_{0\leqslant t\leqslant a}, (H_t)_{0\leqslant t\leqslant a})\Gamma_a(X_a) \,|\, \zeta > 1 \right) = \mathbf{N}\left( G((X_t)_{0\leqslant t\leqslant a}, (H_t)_{0\leqslant t\leqslant a}) \,|\, \zeta = 1 \right).$$

*Proof.* Since $(H_t)_{0\leqslant t\leqslant a}$ is a measurable function of $(X_t)_{0\leqslant t\leqslant a}$, it is sufficient to prove that for every bounded measurable function $F : \mathbb{D}([0,a],\mathbb{R}) \to \mathbb{R}_+$:

$$\mathbf{N}\left( F((X_t)_{0\leqslant t\leqslant a})\Gamma_a(X_a) \,|\, \zeta > 1 \right) = \mathbf{N}\left( F((X_t)_{0\leqslant t\leqslant a}) \,|\, \zeta = 1 \right).$$

To this end, fix $r \in [0,a]$, let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ be two bounded continuous functions and let $h : \mathbb{R}_+^* \to \mathbb{R}_+$ be a continuous function. Using the notation of Proposition 3.2.2, we have:

$$\begin{aligned}
\mathbf{N}\left( f(X_r)h(X_a)g(\zeta)1_{\{\zeta>a\}} \right) &= \mathbf{N}\left( f(X_r)1_{\{\zeta>a\}}\mathbb{E}\left[h(x)g(a+T(x))\right]_{x=X_a} \right) \\
&= \int_0^{\infty} ds\, g(a+s)\mathbf{N}\left( f(X_r)h(X_a)q_s(X_a)1_{\{\zeta>a\}} \right) \\
&= \int_a^{\infty} du g(u)\mathbf{N}\left( f(X_r)h(X_a)q_{u-a}(X_a)1_{\{\zeta>a\}} \right), \qquad (3.8)
\end{aligned}$$

where we have used Proposition 3.2.1 in the first equality and Proposition 3.2.2 in the second equality. Moreover, by property (iii) in subsection 3.2.1:

$$\mathbf{N}\left( f(X_r)g(\zeta)1_{\{\zeta>a\}} \right) = \int_a^{\infty} du\, \frac{g(u)}{\theta\Gamma(1-1/\theta)u^{1/\theta+1}} \cdot \mathbf{N}_{(u)}(f(X_r)). \qquad (3.9)$$

Now observe that (3.8) (with $h = 1$) and (3.9) hold for any bounded continuous function $g$. Since both functions $u \mapsto \mathbf{N}\left( f(X_r)q_{u-a}(X_a)1_{\{\zeta>a\}} \right)$ and $u \mapsto \mathbf{N}_{(u)}\left( f(X_r) \right)$ are easily seen to be continuous over $(a,\infty)$, it follows that for every $u > a$:

$$\mathbf{N}\left( f(X_r)q_{u-a}(X_a)1_{\{\zeta>a\}} \right) = \frac{1}{\theta\Gamma(1-1/\theta)u^{1/\theta+1}}\mathbf{N}_{(u)}\left( f(X_r) \right).$$

In particular, for $u = 1$ we get:

$$\mathbf{N}\left( f(X_r)q_{1-a}(X_a)1_{\{\zeta>a\}} \right) = \frac{1}{\theta\Gamma(1-1/\theta)}\mathbf{N}_{(1)}\left( f(X_r) \right). \qquad (3.10)$$

On the other hand, applying (3.8) with $g \equiv 1$ and noting that $\mathbf{N}(\zeta > 1) = \frac{1}{\Gamma(1-1/\theta)}$, we get:

$$\mathbf{N}\left( f(X_r)h(X_a) \,|\zeta > 1 \right) = \Gamma(1-1/\theta)\mathbf{N}\left( f(X_r)h(X_a)1_{\{\zeta>a\}} \int_{1-a}^{\infty} ds\, q_s(X_a) \right). \qquad (3.11)$$

By combining (3.11) and (3.10) we conclude that:

$$\mathbf{N}\left( f(X_r)\frac{\theta q_{1-a}(X_a)}{\int_{1-a}^{\infty} ds\, q_s(X_a)} \,\middle|\, \zeta > 1 \right) = \mathbf{N}_{(1)}(f(X_r)).$$

One similarly shows that for $0 \leqslant r_1 < \cdots < r_n \leqslant a$ and $f_1, \ldots, f_n : \mathbb{R}_+ \to \mathbb{R}_+$ continuous bounded functions:

$$\mathbf{N}\left( f_1(X_{r_1}) \cdots f_n(X_{r_n}) \frac{\theta q_{1-a}(X_a)}{\int_{1-a}^{\infty} ds\, q_s(X_a)} \,\middle|\, \zeta > 1 \right) = \mathbf{N}_{(1)}(f_1(X_{r_1}) \cdots f_n(X_{r_n})).$$

The desired result follows since the Borel σ-field of $\mathbb{D}([0,a], \mathbb{R})$ is generated by the coordinate functions $X \mapsto X_r$ for $0 \leqslant r \leqslant a$ (see e.g. [20, Theorem 12.5 (iii)]). □

## 3.3 Convergence to the stable tree

### 3.3.1 An invariance theorem

We rely on the following theorem, which is similar in spirit to Donsker's invariance theorem (see the concluding remark of [37, Section 2.6] for a proof).

**Theorem 3.3.1** (Duquesne & Le Gall). *Let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot \mid \zeta(\tau) \geqslant n]$. We have:*

$$\left( \frac{1}{B_n} W_{\lfloor nt \rfloor}(t_n), \frac{B_n}{n} H_{nt}(t_n) \right)_{t \geqslant 0} \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t, H_t)_{0 \leqslant t \leqslant 1} \text{ under } \mathbf{N}(\,\cdot \mid \zeta > 1).$$

### 3.3.2 Convergence of the scaled contour and height functions

Recall the notation $\phi_n(j) = \mathbb{P}[\zeta_j = n]$ and $\phi_n^*(j) = \mathbb{P}[\zeta_j \geqslant n]$.

**Lemma 3.3.2.** *Fix $\alpha > 0$. We have:*

$$\text{(i)} \lim_{n\to\infty} \sup_{1 \leqslant k \leqslant \alpha B_n} \left| n\phi_n(k) - q_1\left(\frac{k}{B_n}\right) \right| = 0, \qquad \text{(ii)} \lim_{n\to\infty} \sup_{1 \leqslant k \leqslant \alpha B_n} \left| \phi_n^*(k) - \int_1^\infty ds\, q_s\left(\frac{k}{B_n}\right) \right| = 0.$$

This has been proved by Le Gall in [72] when μ has finite variance. In full generality, the proof is technical and is postponed to Section 3.3.

**Lemma 3.3.3.** *Fix $a \in (0,1)$. Let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot \mid \zeta(\tau) = n]$. Then the following convergence holds in distribution in the space $\mathcal{C}([0,a], \mathbb{R})$:*

$$\left( \frac{B_n}{n} H_{nt}(t_n);\, 0 \leqslant t \leqslant a \right) \qquad \xrightarrow[n\to\infty]{(d)} \qquad (H_t;\, 0 \leqslant t \leqslant a) \text{ under } \mathbf{N}(\,\cdot \mid \zeta = 1).$$

*Proof.* Recall the notation $\Gamma_a$ introduced in Proposition 3.2.3. We start by verifying that, for $\alpha > 1$, we have:

$$\lim_{n\to\infty} \left( \sup_{\frac{1}{\alpha} B_n \leqslant k \leqslant \alpha B_n} \left| \frac{\phi_{n-\lfloor an \rfloor}(k+1)/\phi_n(1)}{\phi^*_{n-\lfloor an \rfloor}(k+1)/\phi^*_n(1)} - \Gamma_a\left(\frac{k}{B_n}\right) \right| \right) = 0. \tag{3.12}$$

We first note that by Lemma 3.1.10, $\phi_n^*(1)/(n\phi_n(1)) \to \theta$ as $n \to \infty$. Then Lemma 3.3.2 implies that:

$$\lim_{n\to\infty}\left(\sup_{\frac{1}{\alpha}B_n\leqslant k\leqslant \alpha B_n}\left|\frac{\phi_{n-\lfloor an\rfloor}(k+1)/\phi_n(1)}{\phi^*_{n-\lfloor an\rfloor}(k+1)/\phi^*_n(1)} - \theta\frac{\frac{1}{1-a}q_1\left(\frac{k+1}{B_{n-\lfloor an\rfloor}}\right)}{\int_1^\infty ds q_s\left(\frac{k+1}{B_{n-\lfloor an\rfloor}}\right)}\right|\right) = 0 \qquad (3.13)$$

provided we can verify the existence of a constant $\delta > 0$ such that, for $n$ sufficiently large,

$$\inf_{\frac{1}{\alpha}B_n\leqslant k\leqslant \alpha B_n}\int_1^\infty ds q_s\left(\frac{k}{B_{n-\lfloor an\rfloor}}\right) > \delta.$$

This follows from the fact that, for every $\beta > 1$:

$$\inf_{\frac{1}{\beta}\leqslant x\leqslant \beta}\int_1^\infty ds q_s(x) > 0.$$

Details are left to the reader. Our claim (3.12) follows from (3.13) using the scaling property (3.7).

Now let $F : \mathbb{D}([0,a],\mathbb{R}) \to \mathbb{R}_+$ be a bounded continuous function. Fix $\alpha > 1$. To simplify notation, if $\tau$ is a tree, define the event $A_n^\alpha(\tau)$ by $A_n^\alpha(\tau) = \left\{\mathcal{W}_{\lfloor na\rfloor}(\tau) \in \left[\frac{1}{\alpha}B_n, \alpha B_n\right]\right\}$. We also set $F_n(\tau) = F\left(\frac{B_n}{n}H_{\lfloor nt\rfloor}(\tau);\, 0 \leqslant t \leqslant a\right)$. Since $(H_0(\tau), H_1(\tau), \dots, H_{\lfloor an\rfloor}(\tau))$ is a measurable function of $(\mathcal{W}_0(\tau), \mathcal{W}_1(\tau), \dots, \mathcal{W}_{\lfloor an\rfloor}(\tau))$ (see [73, Prop 1.2]), by Proposition 3.1.4, (3.5) and (3.12) we have:

$$\lim_{n\to\infty}\left|\mathbb{E}\left[F_n(t_n)1_{A_n^\alpha(t_n)}\right] - \mathbb{E}_\mu\left[F_n(\tau)1_{A_n^\alpha(\tau)}\Gamma_a\left(\frac{\mathcal{W}_{\lfloor an\rfloor}(\tau)}{B_n}\right)\middle|\, \zeta(\tau) \geqslant n\right]\right| = 0.$$

By Theorem 3.3.1, the law of

$$\left(\left(\frac{B_n}{n}H_{\lfloor nt\rfloor}(\tau);\, 0 \leqslant t \leqslant a\right), \frac{1}{B_n}\mathcal{W}_{\lfloor an\rfloor}(\tau)\right)$$

under $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) \geqslant n]$ converges towards the law of $((H_t; 0 \leqslant t \leqslant a), X_a)$ under $\mathbf{N}(\cdot\,|\,\zeta > 1)$ (for the convergence of the second component we have also used the fact that $X$ is almost surely continuous at $a$). Thus:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[F_n(t_n)1_{\{\mathcal{W}_{\lfloor na\rfloor}(t_n)\in[\frac{1}{\alpha}B_n,\alpha B_n]\}}\right] &= \mathbf{N}\left(F(H_t; 0 \leqslant t \leqslant a)\Gamma_a(X_a)1_{\{X_a\in[\frac{1}{\alpha},\alpha]\}}\,|\,\zeta > 1\right) \\ &= \mathbf{N}\left(F(H_t; 0 \leqslant t \leqslant a)1_{\{X_a\in[\frac{1}{\alpha},\alpha]\}}\,|\,\zeta = 1\right), \qquad (3.14)\end{aligned}$$

where we have used Proposition 3.2.3 in the second equality.

By taking $F \equiv 1$, we obtain:

$$\lim_{n\to\infty}\mathbb{P}\left[\mathcal{W}_{\lfloor na\rfloor}(t_n) \in \left[\frac{1}{\alpha}B_n, \alpha B_n\right]\right] = \mathbf{N}\left(X_a \in \left[\frac{1}{\alpha}, \alpha\right]\middle|\, \zeta = 1\right).$$

This last quantity tends to $1$ as $\alpha \to \infty$. By choosing $\alpha > 1$ sufficiently large, we easily deduce from the convergence (3.14) that:

$$\lim_{n\to\infty}\mathbb{E}\left[F\left(\frac{B_n}{n}H_{\lfloor nt\rfloor}(t_n);\, 0 \leqslant t \leqslant a\right)\right] = \mathbf{N}\left(F(H_t; 0 \leqslant t \leqslant a)\,|\,\zeta = 1\right).$$

The path continuity of $H$ under $\mathbf{N}(\,\cdot\,|\,\zeta = 1)$ then implies the claim of Lemma 3.3.3. □

**Theorem 3.3.4.** *Let* $t_n$ *be a random tree distributed according to* $\mathbb{P}_\mu[\,\cdot \mid \zeta(\tau) = n]$. *Then:*

$$\left(\frac{B_n}{n} H_{nt}(t_n), \frac{B_n}{n} C_{2nt}(t_n)\right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t, H_t)_{0 \leqslant t \leqslant 1} \text{ under } \mathbf{N}(\,\cdot \,|\zeta = 1).$$

*Proof.* The proof consists in showing that the scaled height process is close to the scaled contour process and then using a time-reversal argument in order to show that the convergence holds on the whole segment $[0, 1]$. To this end, we adapt [36, Remark 3.2] and [37, Section 2.4] to our context. For $0 \leqslant p < n$ set $b_p = 2p - H_p(t_n)$ so that $b_p$ represents the time needed by the contour process to reach the $(p + 1)$-st individual of $t_n$ (in the lexicographical order). Also set $b_n = 2(n - 1)$. Note that $C_{b_p} = H_p$. From this observation, we get:

$$\sup_{t \in [b_p, b_{p+1}]} |C_t(t_n) - H_p(t_n)| \leqslant |H_{p+1}(t_n) - H_p(t_n)| + 1. \tag{3.15}$$

for $0 \leqslant p < n$. Then define the random function $g_n : [0, 2n] \to \mathbb{N}$ by setting $g_n(t) = k$ if $t \in [b_k, b_{k+1})$ and $k < n$, and $g_n(t) = n$ if $t \in [2(n - 1), 2n]$ so that for $t < 2(n - 1)$, $g_n(t)$ is the index of the last individual which has been visited by the contour function up to time $t$ if the individuals are indexed $0, 1, \ldots, n - 1$ in lexicographical order. Finally, set $\widetilde{g}_n(t) = g_n(nt)/n$. Fix $a \in (0, 1)$. Then, by (3.15):

$$\sup_{t \leqslant \frac{b_{\lfloor an \rfloor}}{n}} \left| \frac{B_n}{n} C_{nt}(t_n) - \frac{B_n}{n} H_{n\widetilde{g}_n(t)}(t_n) \right| \leqslant \frac{B_n}{n} + \frac{B_n}{n} \sup_{k \leqslant \lfloor an \rfloor} |H_{k+1}(t_n) - H_k(t_n)|, \tag{3.16}$$

which converges in probability to $0$ by Lemma 3.3.3 and the path continuity of $(H_t)$. On the other hand it follows from the definition of $b_n$ that:

$$\sup_{t \leqslant \frac{b_{\lfloor an \rfloor}}{n}} \left| \widetilde{g}_n(t) - \frac{t}{2} \right| \leqslant \frac{1}{2B_n} \sup_{k \leqslant an} \frac{B_n}{n} H_k(t_n) + \frac{1}{n} \quad \xrightarrow{(\mathbb{P})} \quad 0$$

by Lemma 3.3.3. Finally, by the definition of $b_n$ and using Lemma 3.3.3 we see that $\frac{b_{\lfloor an \rfloor}}{n}$ converges in probability towards $2a$ and that $\frac{B_n}{n} \sup_{t \leqslant 2a} \left| H_{n\widetilde{g}_n(t)}(t_n) - H_{nt/2}(t_n) \right|$ converges in probability towards $0$ as $n \to \infty$. Using (3.16), we conclude that:

$$\frac{B_n}{n} \sup_{0 \leqslant t \leqslant a} |C_{2nt}(t_n) - H_{nt}(t_n)| \quad \xrightarrow{(\mathbb{P})} \quad 0. \tag{3.17}$$

Together with Lemma 3.3.3, this implies:

$$\left(\frac{B_n}{n} C_{2nt}(t_n);\, 0 \leqslant t \leqslant a\right) \quad \xrightarrow{(d)} \quad (H_t;\, 0 \leqslant t \leqslant a) \text{ under } \mathbf{N}(\,\cdot \mid \zeta = 1).$$

Since $(C_t(t_n);\, 0 \leqslant t \leqslant 2n - 2)$ and $(C_{2n-2-t}(t_n);\, 0 \leqslant t \leqslant 2n - 2)$ have the same distribution, it follows that:

$$\left(\frac{B_n}{n} C_{2nt}(t_n);\, 0 \leqslant t \leqslant 1\right) \quad \xrightarrow{(d)} \quad (H_t;\, 0 \leqslant t \leqslant 1) \text{ under } \mathbf{N}(\,\cdot \mid \zeta = 1). \tag{3.18}$$

See the last paragraph of the proof of Theorem 6.1 in [72] for details.

Finally, we show that this convergence in turn entails the convergence of the rescaled height function of $t_n$ on the whole segment $[0,1]$. To this end, we verify that convergence (3.17) remains valid for $a=1$. First note that:

$$\sup_{0\leqslant t\leqslant 2}\left|\widetilde{g}_n(t)-\frac{t}{2}\right| \leqslant \frac{1}{2n}\sup_{k\leqslant n} H_k(t_n)+\frac{1}{n}=\frac{1}{2B_n}\sup_{k\leqslant 2n}\frac{B_n}{n}C_k(t_n)+\frac{1}{n} \quad \xrightarrow{(\mathbb{P})} \quad 0 \tag{3.19}$$

by (3.18). Secondly, it follows from (3.15) that:

$$\begin{aligned}\sup_{0\leqslant t\leqslant 2}\left|\frac{B_n}{n}C_{nt}(t_n)-\frac{B_n}{n}H_{n\widetilde{g}_n(t)}\right| &\leqslant \frac{B_n}{n}+\frac{B_n}{n}\sup_{k\leqslant n-1}|H_{k+1}(t_n)-H_k(t_n)| \\ &= \frac{B_n}{n}+\frac{B_n}{n}\sup_{k\leqslant n-1}\left|C_{\frac{b_{k+1}}{n}n}(t_n)-C_{\frac{b_k}{n}n}(t_n)\right|.\end{aligned}$$

By (3.18), in order to prove that the latter quantity tends to $0$ in probability, it is sufficient to verify that $\sup_{k\leqslant n}\left|\frac{b_{k+1}}{n}-\frac{b_k}{n}\right|$ converges to $0$ in probability. But by the definition of $b_n$:

$$\sup_{k\leqslant n}\left|\frac{b_{k+1}}{n}-\frac{b_k}{n}\right|=\sup_{k\leqslant n}\left|\frac{2+H_k(t_n)-H_{k+1}(t_n)}{n}\right| \leqslant \frac{2}{n}+2\sup_{k\leqslant n}\frac{H_k(t_n)}{n}$$

which converges in probability to $0$ as in (3.19). As a consequence:

$$\frac{B_n}{n}\sup_{0\leqslant t\leqslant 1}|C_{2nt}(t_n)-H_{n\widetilde{g}_n(2t)}(t_n)| \quad \xrightarrow{(\mathbb{P})} \quad 0.$$

By (3.18), we get that:

$$\left(\frac{B_n}{n}H_{n\widetilde{g}_n(2t)}(t_n)\right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t)_{0\leqslant t\leqslant 1} \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta=1).$$

Combining this with (3.19), we conclude that:

$$\left(\frac{B_n}{n}C_{2nt}(t_n),\frac{B_n}{n}H_{nt}(t_n)\right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t,H_t)_{0\leqslant t\leqslant 1} \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta=1).$$

This completes the proof. □

**Remark 3.3.5.** If we see the tree $t_n$ as a finite metric space using its graph distance, this theorem implies that $t_n$, suitably rescaled, converges in distribution to the $\theta$-stable tree, in the sense of the Gromov-Hausdorff distance on isometry classes of compact metric spaces (see e.g. [73, Section 2] for details).

**Remark 3.3.6.** When the mean value of $\mu$ is greater than one, it is possible to replace $\mu$ with a critical probability distribution belonging to the same exponential family as $\mu$ without changing the distribution of $t_n$ (see [63]). Consequently, the theorem holds in the supercritical case as well. When $\mu$ is subcritical and the radius of convergence of $\sum_{i\geqslant 0}\mu(i)z^i$ is greater than $1$, this operation is still possible. The case where $\mu$ is subcritical and $\mu(i)\sim L(i)/i^{1+\theta}$ as $i\to\infty$ has been treated in [71]. However, in full generality, the non-critical subcritical case remains open.

### 3.3.3 Proof of the technical lemma

In this section, we prove Lemma 3.3.2.

*Proof of Lemma 3.3.2.* We first prove (i). By the local limit theorem (Theorem 3.1.9 (ii)), we have, for $k \geqslant 1$ and $j \in \mathbb{Z}$:

$$\left| B_n \mathbb{P}[W_n = j] - p_1\left(\frac{j}{B_n}\right) \right| \leqslant \epsilon(n),$$

where $\epsilon(n) \to 0$. By Proposition 3.1.5, we have $n\phi_n(j) = j\mathbb{P}[W_n = -j]$. Since $\frac{j}{B_n} p_1\left(-\frac{j}{B_n}\right) = q_1\left(\frac{j}{B_n}\right)$, we have for $1 \leqslant j \leqslant \alpha B_n$:

$$\left| n\phi_n(j) - q_1\left(\frac{j}{B_n}\right) \right| = \frac{j}{B_n} \left| B_n \mathbb{P}[W_n = -j] - p_1\left(\frac{j}{B_n}\right) \right| \leqslant \alpha\epsilon(n).$$

This completes the proof of (i).

For (ii), first note that by the definition of $q_s$ and the scaling property (3.7):

$$\int_1^\infty ds\, q_s\left(\frac{j}{B_n}\right) = \int_1^\infty \frac{j/B_n}{s^{1/\theta+1}} p_1\left(-\frac{j/B_n}{s^{1/\theta}}\right) ds.$$

By Proposition 3.1.5 and the local limit theorem:

$$\left| \phi_n^*(j) - \sum_{k=n}^\infty \frac{j}{kB_k} p_1\left(-\frac{j}{B_k}\right) \right| = \left| \sum_{k=n}^\infty \left( \frac{j}{k}\mathbb{P}[W_k = -j] - \frac{j}{kB_k} p_1\left(-\frac{j}{B_k}\right) \right) \right| \leqslant \sum_{k=n}^\infty \frac{j}{kB_k}\epsilon(k),$$

where $\epsilon(n) \to 0$. Then write:

$$\begin{aligned} &\left| \sum_{k=n}^\infty \frac{j}{kB_k} p_1\left(-\frac{j}{B_k}\right) - \int_1^\infty ds\, \frac{j/B_n}{s^{1/\theta+1}} p_1\left(-\frac{j/B_n}{s^{1/\theta}}\right) \right| \\ &\quad \leqslant \int_1^\infty ds \left| \frac{jn}{B_{\lfloor ns \rfloor}\lfloor ns \rfloor} - \frac{j/B_n}{s^{1/\theta+1}} \right| p_1\left(-\frac{j}{B_{\lfloor ns \rfloor}}\right) + \int_1^\infty ds\, \frac{j/B_n}{s^{1/\theta+1}} \left| p_1\left(-\frac{j}{B_{\lfloor ns \rfloor}}\right) - p_1\left(-\frac{j/B_n}{s^{1/\theta}}\right) \right|. \end{aligned}$$

Denote the first term of the right-hand side by $P(n,j)$ and the second term by $Q(n,j)$ . Since $p_1$ is bounded by a constant which we will denote by $M$, we have for $1 \leqslant j \leqslant \alpha B_n$:

$$P(n,j) \leqslant \alpha M \int_1^\infty ds\, \frac{1}{s^{1/\theta+1}} \left| \frac{nB_n s^{1/\theta+1}}{B_{\lfloor ns \rfloor}\lfloor ns \rfloor} - 1 \right|.$$

For fixed $s \geqslant 1$, $\frac{1}{s^{1/\theta+1}} \left| \frac{nB_n s^{1/\theta+1}}{B_{\lfloor ns \rfloor}\lfloor ns \rfloor} - 1 \right|$ tends to $0$ as $n \to \infty$, and using Proposition 3.1.6, the same quantity is bounded by an integrable function independent of $n$. The dominated convergence theorem thus shows that $P(n,j) \to 0$ uniformly in $1 \leqslant j \leqslant \alpha B_n$. Let us now bound $Q(n,j)$ for $1 \leqslant j \leqslant \alpha B_n$. Since the absolute value of the derivative of $p_1$ is bounded by a constant which we will denote by $M'$, we have:

$$Q(n,j) \leqslant M' \int_1^\infty ds\, \frac{j/B_n}{s^{1/\theta+1}} \left| \frac{j}{B_{\lfloor ns \rfloor}} - \frac{j/B_n}{s^{1/\theta}} \right| \leqslant \alpha^2 M' \int_1^\infty ds\, \frac{1}{s^{2/\theta+1}} \left| \frac{B_n s^{1/\theta}}{B_{\lfloor ns \rfloor}} - 1 \right|.$$

The right-hand side tends to $0$ by the same argument we used for $P(n,j)$. We have thus proved that:

$$\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left| \sum_{k=n}^{\infty} \frac{j}{kB_k} p_1\left(-\frac{k}{B_k}\right) - \int_1^{\infty} ds\, q_s\left(\frac{j}{B_n}\right) \right| = 0.$$

One finally shows that $\sum_{k=n}^{\infty} \frac{j}{kB_k} \epsilon(k)$ tends to $0$ as $n \to \infty$ uniformly in $1 \leqslant j \leqslant \alpha B_n$ by noticing that:

$$\sup_{n\geqslant 1} \sup_{1\leqslant j\leqslant \alpha B_n} \left( \sum_{k=n}^{\infty} \frac{j}{kB_k} \right) \leqslant \alpha \sup_{n\geqslant 1} \left( \sum_{k=n}^{\infty} \frac{B_n}{kB_k} \right) < \infty.$$

This completes the proof. □

Chapitre 4

# *Invariance principles for Galton-Watson trees conditioned on the number of leaves*

*Les résultats de ce chapitre sont issus de l'article publié [70].*

## Contenu de ce chapitre

We are interested in the asymptotic behavior of critical Galton-Watson trees whose offspring distribution may have infinite variance, which are conditioned on having a large fixed number of leaves. We first find an asymptotic estimate for the probability of a Galton-Watson tree having $n$ leaves. Second, we let $t_n$ be a critical Galton-Watson tree whose offspring distribution is in the domain of attraction of a stable law, and conditioned on having exactly $n$ leaves. We show that the rescaled Lukasiewicz path and contour function of $t_n$ converge respectively to $X^{exc}$ and $H^{exc}$, where $X^{exc}$ is the normalized excursion of a strictly stable spectrally positive Lévy process and $H^{exc}$ is its associated continuous-time height function. As an application, we investigate the distribution of the maximum degree in a critical Galton-Watson tree conditioned on having a large number of leaves. We also explain how these results can be generalized to the case of Galton-Watson trees which are conditioned on having a large fixed number of vertices with degree in a given set, thus extending results obtained by Aldous, Duquesne and Rizzolo.

# Introduction

In this article, we are interested in the asymptotic behavior of critical Galton-Watson trees whose offspring distribution may have infinite variance, and which are conditioned on having a large fixed number of vertices with degree in a given set. We focus in particular on Galton-Watson trees conditioned on having a large fixed number of leaves. Aldous [3, 5] studied the shape of large critical Galton-Watson trees whose offspring distribution has finite variance, under the condition that the total progeny is equal to $n$. Aldous' result has then been extended to the infinite variance case (see e.g. [36, 37]). In a different but related direction, the effect of conditioning a Galton-Watson tree on having height equal to $n$ has been studied [50, 66, 72], and Broutin & Marckert [23] have investigated the asymptotic behavior of uniformly distributed trees with prescribed degree sequence. In [69], we introduced a new type of conditioning involving the number of leaves of the tree in order to study a specific discrete probabilistic model, namely dissections of a regular polygon with Boltzmann weights. The results contained in the present article are important for understanding the asymptotic behavior of the latter model (see [28, 69]). The more general conditioning on having a fixed number of vertices with degree in a given set has been considered very recently by Rizzolo [94]. The results of the present work were obtained independently of [94] (see the end of this introduction for a discussion of the relation between the present work and [94]).

Before stating our main results, let us introduce some notation. If $\mu$ is a probability distribution on the nonnegative integers, $\mathbb{P}_\mu$ will be the law of the Galton-Watson tree with offspring distribution $\mu$ (in short the $GW_\mu$ tree). Let $\zeta(\tau)$ be the total number of vertices of a tree $\tau$ and let $\lambda(\tau)$ be its number of leaves, that is the number of individuals of $\tau$ without children. Let $\mathcal{A}$ be a non-empty subset of $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $\tau$ is a tree, denote the number of vertices $u \in \tau$ such that the number of children of $u$ is in $\mathcal{A}$ by $\zeta_{\mathcal{A}}(\tau)$. Note that $\zeta_{\mathbb{N}}(\tau) = \zeta(\tau)$ and $\zeta_{\{0\}}(\tau) = \lambda(\tau)$.

We now introduce three different coding functions which determine $\tau$ (see Definition 4.1.3 for details). Let $u(0), u(1), \ldots u(\zeta(\tau)-1)$ denote the vertices of $\tau$ in lexicographical order. The Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ is defined by $\mathcal{W}_0(\tau) = 0$ and for $0 \leqslant n \leqslant \zeta(\tau) - 1$, $\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_n - 1$, where $k_n$ is the number of children of $u(n)$. For $0 \leqslant$

$i \leqslant \zeta(\tau)-1$, define $H_i(\tau)$ as the generation of $u(i)$ and set $H_{\zeta(\tau)}(\tau)=0$. The height function $H(\tau)=(H_t(\tau); 0 \leqslant t \leqslant \zeta(\tau))$ is then defined by linear interpolation. To define the contour function $(C_t(\tau), 0 \leqslant t \leqslant 2\zeta(\tau))$, imagine a particle that explores the tree from the left to the right, moving at unit speed along the edges. Then, for $0 \leqslant t \leqslant 2(\zeta(\tau)-1)$, $C_t(\tau)$ is defined as the distance to the root of the position of the particle at time $t$ and we set $C_t(\tau)=0$ for $t \in [2(\zeta(\tau)-1), 2\zeta(\tau)]$. See Fig. 1.2 and 1.3 for an example.

Let $\theta \in (1,2]$ be a fixed parameter and let $(X_t)_{t\geqslant 0}$ be the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta)$. Let also $p_1$ be the density of $X_1$. For $\theta = 2$, note that $X$ is a constant times standard Brownian motion. Let $X^{exc} = (X^{exc}_t)_{0\leqslant t\leqslant 1}$ be the normalized excursion of $X$ and $H^{exc} = (H^{exc}_t)_{0\leqslant t\leqslant 1}$ its associated continuous-time height function (see Section 4.5.1 for precise definitions). Note that $H^{exc}$ is a random continuous function on $[0,1]$ that vanishes at $0$ and at $1$ and takes positive values on $(0,1)$, which codes the so-called $\theta$-stable random tree (see [36]).

We now state our main results. Fix $\theta \in (1,2]$. Let $\mu$ be an aperiodic probability distribution on the nonnegative integers. Assume that $\mu$ is critical (the mean of $\mu$ is $1$) and belongs to the domain of attraction of a stable law of index $\theta \in (1,2]$.

(I) Let $d \geqslant 1$ be the largest integer such that there exists $b \in \mathbb{N}$ such that $\mathrm{supp}(\mu)\backslash\{0\}$ is contained in $b + d\mathbb{Z}$, where $\mathrm{supp}(\mu)$ is the support of $\mu$. Then there exists a slowly varying function $h$ such that:

$$\mathbb{P}_\mu[\lambda(\tau)=n] \underset{n\to\infty}{\sim} \mu(0)^{1/\theta} p_1(0) \frac{\gcd(b-1,d)}{h(n)n^{1/\theta+1}}$$

for those values of $n$ such that $\mathbb{P}_\mu[\lambda(\tau)=n] > 0$. Here we write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.

(II) For every $n \geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau)=n] > 0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)=n]$. Then there exists a sequence of positive real numbers $(B_n)_{n\geqslant 1}$ converging to $\infty$ such that

$$\left(\frac{1}{B_{\zeta(t_n)}} \mathcal{W}_{\lfloor \zeta(t_n) t\rfloor}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} H_{\zeta(t_n)t}(t_n)\right)_{0\leqslant t\leqslant 1}$$

converges in distribution to $(X^{exc}, H^{exc}, H^{exc})$ as $n \to \infty$.

At the end of this work, we explain how to extend (I) and (II) when the condition "$\lambda(\tau)=n$" is replaced by the more general condition "$\zeta_{\mathcal{A}}(\tau)=n$" (see Theorem 4.8.1). However, we shall give detailed arguments only in the case of a fixed number of leaves. This particular case is less technical and suffices in view of applications to the study of random dissections.

We now explain the main steps and techniques used to establish (I) and (II) when $\mathcal{A}=\{0\}$. Let $\nu$ be the probability measure on $\mathbb{Z}$ defined by $\nu(k)=\mu(k+1)$ for $k \geqslant -1$. Our starting point is a well-known relation between the Lukasiewicz path of a $GW_\mu$ tree and an associated random walk. Let $(W_n; n \geqslant 0)$ be a random walk started at $0$ with jump distribution $\nu$ and set $\zeta = \inf\{n \geqslant 0; W_n = -1\}$. Then the Lukasiewicz path of a $GW_\mu$ tree has the same law as $(W_0, W_1, \ldots, W_\zeta)$. Consequently, the total number of leaves of a $GW_\mu$ tree has the same law as $\sum_{k=1}^{\zeta} 1_{\{W_k - W_{k-1} = -1\}}$. By noticing that this last sum involves independent identically distributed Bernoulli variables of parameter $\mu(0)$, large deviations techniques give:

$$\mathbb{P}_\mu\left[\lambda(\tau) = n \text{ and } \left|\zeta(\tau) - \frac{n}{\mu(0)}\right| > \zeta(\tau)^{3/4}\right] \leqslant e^{-c\sqrt{n}} \tag{4.1}$$

for some $c > 0$. This roughly says that a $GW_\mu$ tree with $n$ leaves has approximately $n/\mu(0)$ vertices with high probability. Since $GW_\mu$ trees conditioned on their total progeny are well known, this will allow us to study $GW_\mu$ trees conditioned on their number of leaves.

Let us now explain how an asymptotic estimate for $\mathbb{P}_\mu[\lambda(\tau) = n]$ can be derived. Define $\Lambda(n)$ by:

$$\Lambda(n) = \mathrm{Card}\{0 \leqslant i \leqslant n-1;\ W_{i+1} - W_i = -1\}.$$

The crucial step consists in noticing that for $n, p \geqslant 1$, the distribution of $(W_0, W_1, \dots, W_p)$ under the conditional probability measure $\mathbb{P}[\,\cdot\,|\,W_p = -1, \Lambda(p) = n]$ is cyclically exchangeable. The so-called Cyclic Lemma and the relation between the Lukasiewicz path of a $GW_\mu$ tree and the random walk $W$ easily lead to the following identity (Proposition 4.1.6):

$$\mathbb{P}_\mu[\zeta(\tau) = p, \lambda(\tau) = n] = \frac{1}{p}\,\mathbb{P}[\Lambda(p) = n, W_p = -1] = \frac{1}{p}\,\mathbb{P}[S_p = n]\,\mathbb{P}[W'_{p-n} = n-1], \tag{4.2}$$

where $S_p$ is the sum of $p$ independent Bernoulli random variables of parameter $\mu(0)$ and $W'$ is the random walk $W$ conditioned on having nonnegative jumps. From the concentration result (4.1) and using extensively a suitable local limit theorem, we deduce the asymptotic estimate (I).

The proof of (II) is more elaborate. The first step consists in proving the convergence on every interval $[0, a]$ with $a \in (0, 1)$. To this end, using the large deviation bound (4.1), we first prove an analog of (II) when $t_n$ is a tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\lambda(\tau) \geqslant n]$. We then use an absolute continuity relation between the conditional probability measure $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$ and the conditional probability measure $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$ to get the desired convergence on every interval $[0, a]$ with $a \in (0, 1)$. The second step is to extend this convergence to the whole interval $[0, 1]$ via a tightness argument based on a time-reversal property. In the case of the Lukasiewicz path, an additional argument using the Vervaat transformation is needed.

As an application of these techniques, we study the distribution of the maximum degree in a Galton-Watson tree conditioned on having many leaves. More precisely, if $\tau$ is a tree, let $\Delta(\tau)$ be the maximum number of children of a vertex of $\tau$. Let also $\overline{\Delta}(X^{\mathrm{exc}})$ be the largest jump of the càdlàg process $X^{\mathrm{exc}}$. Set $D(n) = \max\{k \geqslant 1;\ \mu([k, \infty)) \geqslant 1/n\}$. For every $n \geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. Then, under assumptions on the asymptotic behavior of the sequence $(\mu(n)^{1/n})_{n \geqslant 1}$ in the finite variance case (see Theorem 4.7.1):

(i) If the variance of $\mu$ is infinite, then $\mu(0)^{1/\theta}\Delta(t_n)/B_n$ converges in distribution towards $\overline{\Delta}(X^{\mathrm{exc}})$.

(ii) If the variance of $\mu$ is finite, then $\Delta(t_n)/D(n)$ converges in probability towards $1$.

The second case yields an interesting application to the maximum face degree in a large uniform dissection (see [28]). Let us mention that using generating functions and saddle-point techniques, similar results have been obtained by Meir and Moon [84] when $t_n$ is distributed according to $\mathbb{P}_\mu[\cdot \mid \zeta(\tau) = n]$. Our approach can be adapted to give a probabilistic proof of their result.

We now discuss the connections of the present article with earlier work. Using different arguments, formula (4.2) has been obtained in a different form by Kolchin [67]. The asymptotic behavior of $\mathbb{P}_\mu[\zeta_{\mathcal{A}}(\tau) = n]$ has been studied in [85, 86, 87] when $\mathrm{Card}(\mathcal{A}) = 1$ and the second moment of $\mu$ is finite. Absolute continuity arguments have often been used to derive invariance principles for random trees and forests, see e.g. [26, 36, 75, 72].

Let us now discuss the relationship between the present work and Rizzolo's recent article [94], which deals with similar conditionings of random trees. The main result of [94] considers a random tree distributed according to $\mathbb{P}_\mu[\cdot \mid \zeta_{\mathcal{A}}(\tau) = n]$, where it is assumed that $0 \in \mathcal{A}$. In the finite variance case, [94] gives the convergence in distribution in the rooted Gromov-Hausdorff-Prokhorov sense of the (suitably rescaled) tree $t_n$ viewed as a (rooted) metric space for the graph distance towards the Brownian CRT. Note that the convergence of the contour functions in (II), together with Corollary 4.3.3, does imply the Gromov-Hausdorff-Prokhorov convergence of trees viewed as metric spaces, but the converse is not true. Furthermore our results also apply to the infinite variance case and include the case where $0 \notin \mathcal{A}$.

The paper is organized as follows. In Section 1, we present the discrete framework and we define Galton-Watson trees and their codings. We prove (4.2) and explain how the local limit theorem gives information on the asymptotic behavior of large $GW_\mu$ trees. In Section 2, we present a law of large numbers for the number of leaves, which leads to the concentration formula (4.1). In Section 3, we prove (I). In Section 4, we establish an invariance principle under the conditional probability $\mathbb{P}_\mu[\cdot \mid \lambda(\tau) \geqslant n]$. In Sections 5 and 6, we refine this result by obtaining an invariance principle under the conditional probability $\mathbb{P}_\mu[\cdot \mid \lambda(\tau) = n]$, thus proving (II). As an application, we study in Section 7 the distribution of the maximum degree in a Galton-Watson tree conditioned on having many leaves. Finally, in Section 8, we explain how the techniques used to deal with the case $\mathcal{A} = \{0\}$ can be extended to general sets $\mathcal{A}$.

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**Notation and assumptions.** Throughout this work $\theta \in (1, 2]$ will be a fixed parameter. We say that a probability distribution $(\mu(j))_{j \geqslant 0}$ on the nonnegative integers satisfies hypothesis $(H_\theta)$ if the following three conditions hold:

(i) $\mu$ is critical, meaning that $\sum_{k=0}^{\infty} k\mu(k) = 1$, and $\mu(1) < 1$.

(ii) $\mu$ is in the domain of attraction of a stable law of index $\theta \in (1, 2]$. This means that either the variance of $\mu$ is finite, or $\mu([j, \infty)) = j^{-\theta} L(j)$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\lim_{x \to \infty} L(tx)/L(x) = 1$ for all $t > 0$ (such a function is called slowly varying). We refer to [21] or [41, chapter 3.7] for details.

(iii) $\mu$ is aperiodic, which means that the additive subgroup of the integers $\mathbb{Z}$ spanned by $\{j;\ \mu(j) \neq 0\}$ is not a proper subgroup of $\mathbb{Z}$.

We introduce condition (iii) to avoid unnecessary complications, but our results can be extended to the periodic case.

Throughout this text, $\nu$ will stand for the probability measure defined by $\nu(k) = \mu(k+1)$ for $k \geqslant -1$. Note that $\nu$ has zero mean. To simplify notation, we write $\mu_0$ instead of $\mu(0)$. Note that $\mu_0 > 0$ under $(H_\theta)$.

# 4.1 The discrete setting: Galton-Watson trees

## 4.1.1 Galton-Watson trees

**Definition 4.1.1.** Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of all nonnegative integers, $\mathbb{N}^* = \{1, 2, \ldots\}$ and $U$ the set of labels:

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. An element of $U$ is a sequence $u = u_1 \cdots u_m$ of positive integers, and we set $|u| = m$, which represents the "generation" of $u$. If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ belong to $U$, we write $uv = u_1 \cdots u_m v_1 \cdots v_n$ for the concatenation of $u$ and $v$. In particular, note that $u\emptyset = \emptyset u = u$. Finally, a *rooted ordered tree* $\tau$ is a finite subset of $U$ such that:

1. $\emptyset \in \tau$,
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,
3. for every $u \in \tau$, there exists an integer $k_u(\tau) \geqslant 0$ such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

In the following, by *tree* we will always mean rooted ordered tree. We denote by the set of all trees by $\mathbb{T}$. We will often view each vertex of a tree $\tau$ as an individual of a population whose $\tau$ is the genealogical tree. The total progeny of $\tau$ will be denoted by $\zeta(\tau) = \text{Card}(\tau)$. A leaf of a tree $\tau$ is a vertex $u \in \tau$ such that $k_u(\tau) = 0$. The total number of leaves of $\tau$ will be denoted by $\lambda(\tau)$. If $\tau$ is a tree and $u \in \tau$, we define the shift of $\tau$ at $u$ by $T_u\tau = \{v \in U;\ uv \in \tau\}$, which is itself a tree.

**Definition 4.1.2.** Let $\rho$ be a probability measure on $\mathbb{N}$ with mean less than or equal to 1 and, to avoid trivialities, such that $\rho(1) < 1$. The law of the Galton-Watson tree with offspring distribution $\rho$ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbb{T}$ such that:

1. $\mathbb{P}_\rho(k_\emptyset = j) = \rho(j)$ for $j \geqslant 0$,
2. for every $j \geqslant 1$ with $\rho(j) > 0$, the shifted trees $T_1\tau, \ldots, T_j\tau$ are independent under the conditional probability $\mathbb{P}_\rho(\,\cdot\,|\, k_\emptyset = j)$ and their conditional distribution is $\mathbb{P}_\rho$.

A random tree whose distribution is $\mathbb{P}_\rho$ will be called a Galton-Watson tree with offspring distribution $\rho$, or in short a $GW_\rho$ tree.

In the sequel, for an integer $j \geqslant 1$, $\mathbb{P}_{\mu,j}$ will stand for the probability measure on $\mathbb{T}^j$ which is the distribution of $j$ independent $GW_\mu$ trees. The canonical element of $\mathbb{T}^j$ will be denoted by $\mathbf{f}$. For $\mathbf{f} = (\tau_1, \ldots, \tau_j) \in \mathbb{T}^j$, set $\lambda(\mathbf{f}) = \lambda(\tau_1) + \cdots + \lambda(\tau_j)$ and $\zeta(\mathbf{f}) = \zeta(\tau_1) + \cdots + \zeta(\tau_j)$ for respectively the total number of leaves of $\mathbf{f}$ and the total progeny of $\mathbf{f}$.

### 4.1.2 Coding Galton-Watson trees

We now explain how trees can be coded by three different functions. These codings are crucial in the understanding of large Galton-Watson trees.

**Definition 4.1.3.** We write $u < v$ for the lexicographical order on the labels $\mathcal{U}$ (for example $\emptyset < 1 < 21 < 22$). Consider a tree $\tau$ and order the individuals of $\tau$ in lexicographical order: $\emptyset = u(0) < u(1) < \cdots < u(\zeta(\tau)-1)$. The height process $H(\tau) = (H_n(\tau), 0 \leqslant n < \zeta(\tau))$ is defined, for $0 \leqslant n < \zeta(\tau)$, by:

$$H_n(\tau) = |u(n)|.$$

For technical reasons, we set $H_{\zeta(\tau)}(\tau) = 0$.

Consider a particle that starts from the root and visits continuously all edges at unit speed (assuming that every edge has unit length), going backwards as little as possible and respecting the lexicographical order of vertices. For $0 \leqslant t \leqslant 2(\zeta(\tau)-1)$, $C_t(\tau)$ is defined as the distance to the root of the position of the particle at time $t$. For technical reasons, we set $C_t(\tau) = 0$ for $t \in [2(\zeta(\tau)-1), 2\zeta(\tau)]$. The function $C(\tau)$ is called the contour function of the tree $\tau$. See Figure 1.3 for an example, and [36, Section 2] for a rigorous definition.

Finally, the Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ of $\tau$ is defined by $\mathcal{W}_0(\tau) = 0$ and for $0 \leqslant n \leqslant \zeta(\tau)-1$:

$$\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1.$$

See Figure 1.2 for an example. Note that necessarily $\mathcal{W}_{\zeta(\tau)}(\tau) = -1$.

A forest is a finite or infinite ordered sequence of trees. The Lukasiewicz path of a forest is defined as the concatenation of the Lukasiewicz paths of the trees it contains (the word "concatenation" should be understood in the appropriate manner, see [36, Section 2] for a more precise definition). The following proposition explains the importance of the Lukasiewicz path.

**Proposition 4.1.4.** *Fix an integer $j \geqslant 1$. Let $(W_n; n \geqslant 0)$ be a random walk which starts at $0$ with jump distribution $\nu(k) = \mu(k+1)$ for $k \geqslant -1$. Define $\zeta_j = \inf\{n \geqslant 0;\ W_n = -j\}$. Then $(W_0, W_1, \ldots, W_{\zeta_j})$ is distributed as the Lukasiewicz path of a forest of $j$ independent $\mathrm{GW}_\mu$ trees. In particular, the total progeny of $j$ independent $\mathrm{GW}_\mu$ trees has the same law as $\zeta_j$.*

*Proof.* See [73, Proposition 1.5]. □

Note that the previous proposition applied with $j = 1$ entails that the Lukasiewicz path of a Galton-Watson tree is distributed as the random walk $W$ stopped when it hits $-1$ for the first time. We conclude this subsection by giving a link between the height function and the Lukasiewicz path (see [73, Prop. 1.2] for a proof).

**Proposition 4.1.5.** *Let $\tau$ be a tree. Then, for every $0 \leqslant n < \zeta(\tau)$:*

$$H_n(\tau) = \mathrm{Card}\left\{0 \leqslant j < n;\ \mathcal{W}_j(\tau) = \inf_{j \leqslant k \leqslant n} \mathcal{W}_k(\tau)\right\}. \tag{4.3}$$

### 4.1.3 The Cyclic Lemma

We now state the Cyclic Lemma which is crucial in the derivation of the joint law of $(\zeta(\tau), \lambda(\tau))$ under $\mathbb{P}_\mu$. For integers $1 \leqslant j \leqslant p$, define:

$$\mathcal{S}_p^{(j)} = \{(x_1, \dots, x_p) \in \{-1, 0, 1, 2, \dots\}^p; \sum_{i=1}^{p} x_i = -j\}$$

and

$$\overline{\mathcal{S}}_p^{(j)} = \{(x_1, \dots, x_p) \in \mathcal{S}_p^{(j)}; \sum_{i=1}^{m} x_i > -j \text{ for all } m \in \{0, 1, \dots, p-1\}\}.$$

For $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{S}_p^{(j)}$ and $i \in \mathbb{Z}/p\mathbb{Z}$, denote by $\mathbf{x}^{(i)}$ the $i$-th cyclic shift of $\mathbf{x}$ defined by $x_k^{(i)} = x_{i+k \mod p}$ for $1 \leqslant k \leqslant p$. For $\mathbf{x} \in \mathcal{S}_p^{(j)}$, finally set:

$$\mathcal{I}_{\mathbf{x}} = \left\{ i \in \mathbb{Z}/p\mathbb{Z};\ \mathbf{x}^{(i)} \in \overline{\mathcal{S}}_p^{(j)} \right\}.$$

The so-called Cyclic Lemma states that we have $\mathrm{Card}(\mathcal{I}_{\mathbf{x}}) = j$ for every $\mathbf{x} \in \mathcal{S}_p^{(j)}$ (see [90, Lemma 6.1] for a proof).

Let $(W_n; n \geqslant 0)$ and $\zeta_j$ be as in Proposition 4.1.4. Define $\Lambda(k)$ by $\Lambda(k) = \mathrm{Card}\{0 \leqslant i \leqslant k-1;\ W_{i+1} - W_i = -1\}$. Let finally $n, p \geqslant 1$ be positive integers. From the Cyclic Lemma and the fact that for all $k \in \mathbb{Z}/p\mathbb{Z}$ one has $\mathrm{Card}\{1 \leqslant i \leqslant p;\ x_i = -1\} = \mathrm{Card}\{1 \leqslant i \leqslant p;\ x_i^{(k)} = -1\}$, it is a simple matter to deduce that:

$$\mathbb{P}[\zeta_j = p, \Lambda(p) = n] = \frac{j}{p} \mathbb{P}[W_p = -j, \Lambda(p) = n]. \tag{4.4}$$

See e.g. [90, Section 6.1] for similar arguments. Note in particular that we have $\mathbb{P}[\zeta_j = p] = j\mathbb{P}[W_p = -j]/p$. This result allows us to derive the joint law of $(\zeta(\tau), \lambda(\tau))$ under $\mathbb{P}_\mu$:

**Proposition 4.1.6.** *Let $j$ and $n \leqslant p$ be positive integers. We have:*

$$\mathbb{P}_{\mu,j}[\zeta(f) = p, \lambda(f) = n] = \frac{j}{p}\, \mathbb{P}[S_p = n]\, \mathbb{P}[W'_{p-n} = n - j].$$

*where $S_p$ is the sum of $p$ independent Bernoulli random variables of parameter $\mu_0$ and $W'$ is the random walk started from $0$ with nonnegative jumps distributed according to $\eta(i) = \mu(i+1)/(1-\mu_0)$ for every $i \geqslant 0$.*

*Proof.* Using Proposition 4.1.4 and (4.4) , write $\mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = p, \lambda(\mathbf{f}) = n] = j\mathbb{P}[\Lambda(p) = n, W_p = -j]/p$. To simplify notation, set $X_i = W_i - W_{i-1}$ for $i \geqslant 1$ and note that:

$$\begin{aligned} \mathbb{P}[\Lambda(p) = n, W_p = -j] &= \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant p} \mathbb{P}\left[X_i = -1,\ \forall i \in \{i_1, \dots, i_n\}\right] \\ &\quad \cdot \mathbb{P}\left[ \sum_{i \notin \{i_1, \dots, i_n\}} X_i = n - j; \quad X_i > -1,\ \forall i \notin \{i_1, \dots, i_n\} \right]. \end{aligned}$$

The last probability is equal to $\mathbb{P}\left[W'_{p-n} = n - j\right] \mathbb{P}\left[X_i > -1,\ \forall i \notin \{i_1, \dots, i_n\}\right]$ and it follows that:

$$\mathbb{P}\left[\Lambda(p) = n, W_p = -j\right] = \mathbb{P}\left[W'_{p-n} = n - j\right] \mathbb{P}\left[S_p = n\right], \tag{4.5}$$

giving the desired result. □

### 4.1.4 Slowly varying functions

Slowly varying functions appear in the study of domains of attractions of stable laws. Here we recall some properties of these functions in view of future use.

Recall that a nonnegative measurable function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be slowly varying if, for every $t > 0$, $L(tx)/L(x) \to 1$ as $x \to \infty$. A useful result concerning these functions is the so-called Representation Theorem, which states that a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying if and only if it can be written in the form:

$$L(x) = c(x) \exp\left( \int_1^x \frac{\epsilon(u)}{u} du \right), \qquad x \geqslant 0,$$

where $c$ is a nonnegative measurable function having a finite positive limit at infinity and $\epsilon$ is a measurable function tending to $0$ at infinity. See e.g. [21, Theorem 1.3.1] for a proof. The following result is then an easy consequence.

**Proposition 4.1.7.** *Fix $\epsilon > 0$ and let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying function.*

*(i) We have $x^\epsilon L(x) \to \infty$ and $x^{-\epsilon} L(x) \to 0$ as $x \to \infty$.*

*(ii) There exists a constant $C > 1$ such that $\frac{1}{C} x^{-\epsilon} \leqslant L(nx)/L(n) \leqslant C x^\epsilon$ for every integer $n$ sufficiently large and $x \geqslant 1$.*

### 4.1.5 The Local Limit Theorem

**Definition 4.1.8.** A subset $A \subset \mathbb{Z}$ is said to be lattice if there exist $b \in \mathbb{Z}$ and $d \geqslant 2$ such that $A \subset b + d\mathbb{Z}$. The largest $d$ for which this statement holds is called the span of $A$. A measure on $\mathbb{Z}$ is said to be lattice if its support is lattice, and a random variable is said to be lattice if its law is lattice.

**Remark 4.1.9.** Since $\mu$ is supposed to be critical and aperiodic, using the fact that $\mu(0) > 0$, it is an exercise to check that the probability measure $\nu$ is non-lattice.

Recall that $(X_t)_{t \geqslant 0}$ is the spectrally positive Lévy process with Laplace exponent$\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta)$ and $p_1$ is the density of $X_1$. When $\theta = 2$, we have $p_1(x) = e^{-x^2/4}/\sqrt{4\pi}$. It is well known that $p_1$ is positive, continuous and bounded (see e.g. [99, I. 4]). The following theorem will allow us to find estimates for the probabilities appearing in Proposition 4.1.6.

**Theorem 4.1.10** (Local Limit Theorem). *Let $(W_n)_{n \geqslant 0}$ be a random walk on $\mathbb{Z}$ started from $0$ such that its jump distribution is in the domain of attraction of a stable law of index $\theta \in (1, 2]$. Assume that $W_1$ is non-lattice and that $\mathbb{P}[W_1 < -1] = 0$. Set $K(x) = \mathbb{E}[W_1^2 1_{|W_1| \leqslant x}]$ for $x \geqslant 0$. Let $\sigma^2$ be the variance of $W_1$ and set:*

$$\begin{cases} a_n = \sigma\sqrt{n/2} & \text{if } \sigma^2 < \infty, \\ a_n = |\Gamma(1-\theta)|^{1/\theta} \inf\left\{ x \geqslant 0;\, \mathbb{P}\left[W_1 > x\right] \leqslant \dfrac{1}{n} \right\} & \text{if } \sigma^2 = \infty \text{ and } \theta < 2, \\ a_n = \sqrt{n\, K\left( \sup\left\{ z \geqslant 0;\, \dfrac{K(z)}{z^2} \geqslant \dfrac{1}{n} \right\} \right)} & \text{if } \sigma^2 = \infty \text{ and } \theta = 2, \end{cases}$$

*with the convention $\sup \emptyset = 0$.*

*(i) The random variable* $(W_n - n\mathbb{E}[W_1])/a_n$ *converges in distribution towards* $X_1$.

*(ii) We have* $a_n = n^{1/\theta}L(n)$ *where* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *is slowly varying.*

*(iii) We have* $\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| a_n\mathbb{P}[W_n = k] - p_1\left(\frac{k - n\mathbb{E}[W_1]}{a_n}\right)\right| = 0.$

*Proof.* First note that $\mathbb{E}[|W_1|] < \infty$ since $\theta > 1$ (this is a consequence of [56, Theorem 2.6.1]).

We start with (i). The case $\sigma^2 < \infty$ is the classical central limit theorem. Now assume that $\sigma^2 = \infty$ and $\theta < 2$. Write $G(x) = \mathbb{P}[|W_1| > x]$ for $x \geqslant 0$ and introduce $a'_n = \inf\{x \geqslant 0;\ G(x) \leqslant 1/n\}$, so that $a_n = |\Gamma(1-\theta)|^{1/\theta}a'_n$ for $n$ sufficiently large. By [41, Formula 3.7.6], we have $nG(a'_n) \to 1$. By definition of the domain of attraction of a stable law, there exists a slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G(x) = L(x)/x^\theta$. Hence $G(a_n) \sim 1/(n|\Gamma(1-\theta)|)$. Next, by [44, Section XVII (5.21)] we have $K(x) \sim x^2G(x)\theta/(2-\theta)$ as $x \to \infty$. Hence:

$$\frac{nK(a_n)}{a_n^2} \quad \sim \quad \frac{n}{a_n^2}\frac{\theta}{2-\theta}a_n^2G(a_n) \quad \sim \quad \frac{\theta}{(2-\theta)|\Gamma(1-\theta)|}.$$

From [44, Section XVII.5, Theorem 3], we now get that $(W_n - n\mathbb{E}[W_1])/a_n$ converges in distribution to $X_1$. Finally, in the case $\sigma^2 = \infty$ and $\theta = 2$, assertion (i) is a straightforward consequence of the proof of Theorem 2.6.2 in [56].

We turn to the proof of (ii). By [56, p. 46], for every integer $k \geqslant 1$, $a_{kn}/a_n \to k^{1/\theta}$ as $n \to \infty$. Since $(a_n)$ is increasing, by a theorem of de Haan (see [21, Theorem 1.10.7]), this implies that there exists a slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that $a_n = L(n)n^{1/\theta}$ for every positive integer $n$.

Assertion (iii) is the classical local limit theorem (see [56, Theorem 4.2.1]). □

In the case $\sigma^2 = \infty$ and $\theta = 2$, note that $L(n) \to \infty$ as $n \to \infty$ and that $L$ can be chosen to be increasing.

Assume that $\mu$ satisfies $(H_\theta)$ for a certain $\theta \in (1, 2]$. Let $(W_n)_{n\geqslant 0}$ be a random walk started from $0$ with jump distribution $\nu$. Since $\mu$ is in the domain of attraction of a stable law of index $\theta$, it follows that $\nu$ is also in this domain of attraction. Moreover, $\mathbb{E}[W_1] = 0$ and $W_1$ is not lattice by Remark 4.1.9. Let $\sigma^2$ be the variance of $W_1$ and define $B_n$ to be equal to the quantity $a_n$ defined in Theorem 4.1.10. Then, as $n \to \infty$, $W_n/B_n$ converges in distribution towards $X_1$. In what follows, $h : \mathbb{R}_+ \to \mathbb{R}_+$ will stand for a slowly varying function such that $B_n = h(n)n^{1/\theta}$.

**Lemma 4.1.11.** *We have:*

$$\mathbb{P}_\mu[\zeta(\tau) = n] \underset{n\to\infty}{\sim} \frac{p_1(0)}{n^{1/\theta+1}h(n)}, \qquad \mathbb{P}_\mu[\zeta(\tau) \geqslant n] \underset{n\to\infty}{\sim} \frac{\theta p_1(0)}{n^{1/\theta}h(n)}.$$

*Proof.* This is an easy consequence of Theorem 4.1.10 (iii) together with the fact that we have $\mathbb{P}_\mu[\zeta(\tau) = n] = \mathbb{P}[W_n = -1]/n$, as noticed before Proposition 4.1.6. □

**Remark 4.1.12.** In particular, $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ for $n$ sufficiently large if $\mu$ is aperiodic. When $\mu$ is periodic, if $d$ is the span of the support of $\mu$, one can check that for $n$ sufficiently large, one has $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ if and only if $n = 1 \bmod d$.

## 4.2 A law of large numbers for the number of leaves

In the sequel, we fix $\theta \in (1, 2]$ and consider a probability distribution $\mu$ on $\mathbb{N}$ satisfying hypothesis $(H_\theta)$. In this section, we show that if a $GW_\mu$ tree has total progeny equal to $n$, then it has approximatively $\mu_0 n$ leaves with high probability. Intuitively, this comes from the fact that each individual of a $GW_\mu$ has a probability $\mu_0$ of being a leaf. Conversely, we also establish that if a $GW_\mu$ tree has $n$ leaves, than it has approximatively $n/\mu_0$ vertices with high probability.

**Definition 4.2.1.** Consider a tree $\tau \in \mathbb{T}$ and let $(u(i), 0 \leqslant i \leqslant \zeta(\tau) - 1)$ be the vertices of $\tau$ listed in lexicographical order and denote by $k_j$ the number of children of $u(j)$. For $0 \leqslant s < \zeta(\tau)$ define $\Lambda_\tau(s)$ by $\Lambda_\tau(s) = \sum_{j=0}^{\lfloor s \rfloor} 1_{\{k_j=0\}}$, where $\lfloor s \rfloor$ stands for the integer part of $s$. Set also $\Lambda_\tau(\zeta(\tau)) = \lambda(\tau)$.

**Lemma 4.2.2.** *Let $(X_i)_{i \geqslant 1}$ be a sequence of independent identically distributed Bernoulli random variables of parameter $\mu_0$. For $0 \leqslant x \leqslant 1$, define $\phi^*(x) = x \ln \frac{x}{\mu_0} + (1-x) \ln \frac{1-x}{1-\mu_0}$. The following two properties hold:*

*(i) For $a > 0$ and $n \geqslant 1$:*

$$\mathbb{P}\left[\frac{1}{n}\sum_{k=1}^{n} X_k > \mu_0 + a\right] \leqslant 2e^{-n\phi^*(\mu_0+a)}, \qquad \mathbb{P}\left[\frac{1}{n}\sum_{k=1}^{n} X_k < \mu_0 - a\right] \leqslant 2e^{-n\phi^*(\mu_0-a)}.$$

*(ii) We have $\phi^*(\mu_0 + x) = \dfrac{1}{2\mu_0(1-\mu_0)}x^2 + o(x^2)$ when $x \to 0$.*

*Proof.* For the first assertion, see [31, Remark (c) in Theorem 2.2.3]. The second one is a simple calculation left to the reader. □

**Definition 4.2.3.** Let $\epsilon > 0$. We say that a sequence of positive numbers $(x_n)$ is $oe_\epsilon(n)$ if there exist positive constants $c, C > 0$ such that $x_n \leqslant Ce^{-cn^\epsilon}$ for all $n$ and we write $x_n = oe_\epsilon(n)$.

**Remark 4.2.4.** It is easy to see that if $x_n = oe_\epsilon(n)$ for some $\epsilon > 0$ then the sequence $(y_n)_{n \geqslant 1}$ defined by $y_n = \sum_{k=n}^{\infty} x_k$ is also $oe_\epsilon(n)$.

**Lemma 4.2.5.** *Fix $0 < \eta < 1$ and $\delta > 0$.*

*(i) Let $(W_n = X_1 + \cdots + X_n; n \geqslant 0)$ be a random walk started at $0$ with jump distribution $\nu(k) = \mu(k+1)$, $k \geqslant -1$ under $\mathbb{P}$. Then:*

$$\mathbb{P}\left[\sup_{\eta \leqslant t \leqslant 1}\left|\frac{1}{nt}\sum_{j=0}^{\lfloor nt \rfloor} 1_{\{X_j=-1\}} - \mu_0\right| > \frac{\delta}{n^{1/4}}\right] = oe_{1/2}(n).$$

*(ii) For those values of $n$ such that $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ we have:*

$$\mathbb{P}_\mu\left[\sup_{\eta \leqslant t \leqslant 1}\left|\frac{\Lambda_\tau(nt)}{nt} - \mu_0\right| \geqslant \frac{\delta}{n^{1/4}} \;\middle|\; \zeta(\tau) = n\right] = oe_{1/2}(n).$$

*Proof.* For the first assertion, define $Z_k = \left| \frac{1}{k} \sum_{j=0}^{k} 1_{\{X_j=-1\}} - \mu_0 \right|$ for $k \geqslant 1$. By Lemma 4.2.2 (ii), for $n$ sufficiently large we have $\phi^* \left( \mu_0 \pm \delta n^{-1/4} \right) > c n^{-1/2}$, for some $c > 0$. Since the random variables $(1_{\{X_j=-1\}})_{j \geqslant 1}$ are independent Bernoulli random variables of parameter $\mu_0$, for large $n$ and $k \geqslant \lfloor \eta n \rfloor$ we have by Lemma 4.2.2 (i):

$$\mathbb{P}[Z_k > \delta n^{-1/4}] \leqslant 4 \exp\left( -c \frac{k}{n^{1/2}} \right) \leqslant 4 \exp\left( -c \frac{\eta n - 1}{n^{1/2}} \right) \leqslant 4 \exp\left( -\frac{c\eta}{2} n^{1/2} \right).$$

Therefore, for large enough $n$:

$$\begin{aligned} \mathbb{P}\left[ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{1}{nt} \sum_{j=0}^{\lfloor nt \rfloor} 1_{\{X_j=-1\}} - \mu_0 \right| > \frac{\delta}{n^{1/4}} \right] &\leqslant \mathbb{P}\left[ \exists k \in [\eta n - 1, n] \cap \mathbb{N} \text{ such that } Z_k > \frac{\delta}{n^{1/4}} \right] \\ &\leqslant \sum_{k=\lfloor \eta n \rfloor}^{n} \mathbb{P}\left[ Z_k > \frac{\delta}{n^{1/4}} \right] \\ &\leqslant 4(1-\eta) n \exp\left( -\frac{c\eta}{2} n^{1/2} \right), \end{aligned}$$

which is $oe_{1/2}(n)$.

For the second assertion, introduce $\zeta = \inf\{n \geqslant 0; W_n = -1\}$ and use Proposition 4.1.4 which tells us that:

$$\begin{aligned} &\mathbb{P}_\mu\left[ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{\Lambda_\tau(nt)}{nt} - \mu_0 \right| \geqslant \frac{\delta}{n^{1/4}} \,\middle|\, \zeta(\tau) = n \right] \\ &\qquad = \mathbb{P}\left[ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{1}{nt} \sum_{j=0}^{\lfloor nt \rfloor} 1_{\{X_j=-1\}} - \mu_0 \right| > \frac{\delta}{n^{1/4}} \,\middle|\, \zeta = n \right] \\ &\qquad \leqslant \frac{1}{\mathbb{P}[\zeta = n]} \mathbb{P}\left[ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{1}{nt} \sum_{j=0}^{\lfloor nt \rfloor} 1_{\{X_j=-1\}} - \mu_0 \right| > \frac{\delta}{n^{1/4}} \right]. \end{aligned}$$

By (i), the last probability in the right-hand side is $oe_{1/2}(n)$ and by Lemma 4.1.11 combined with Proposition 4.1.7 (ii), the quantity $\mathbb{P}[\zeta = n] = \mathbb{P}_\mu[\zeta(\tau) = n]$ is bounded below by $n^{-1/\theta - 2}$ for large $n$. The desired result follows. □

**Corollary 4.2.6.** *We have for every* $\eta \in (0,1)$ *and* $\delta > 0$*:*

$$\mathbb{P}_\mu\left[ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{\Lambda_\tau(\zeta(\tau)t)}{\zeta(\tau)t} - \mu_0 \right| \geqslant \frac{\delta}{n^{1/4}} \,\middle|\, \zeta(\tau) \geqslant n \right] = oe_{1/2}(n).$$

*Proof.* To simplify notation, set $A_n = \left\{ \sup_{\eta \leqslant t \leqslant 1} \left| \frac{\Lambda_\tau(\zeta(\tau)t)}{\zeta(\tau)t} - \mu_0 \right| \geqslant \frac{\delta}{n^{1/4}} \right\}$. It suffices to notice that:

$$\mathbb{P}_\mu[A_n \mid \zeta(\tau) \geqslant n] \leqslant \sum_{k=n}^{\infty} \frac{\mathbb{P}_\mu[\zeta(\tau) = k]}{\mathbb{P}_\mu[\zeta(\tau) \geqslant n]} \mathbb{P}_\mu[A_k \mid \zeta(\tau) = k],$$

observing that the quantities $\mathbb{P}_\mu[A_k | \zeta(\tau) = k]$ are bounded by Lemma 4.2.5 (ii). Details are left to the reader. □

We have just shown that if a $GW_\mu$ tree has total progeny $n$, then it has approximatively $\mu_0 n$ leaves and the deviations from this value have exponentially small probability. Part (ii) of the following crucial lemma provides a converse to this statement by proving that if a $GW_\mu$ tree has $n$ leaves, then the probability that its total progeny does not belong to $[n/\mu_0 - n^{3/4}, n/\mu_0 + n^{3/4}]$ decreases exponentially fast in $n$.

**Lemma 4.2.7.** *We have for $1 \leqslant j \leqslant n$ and $\delta > 0$:*

*(i)* $\mathbb{P}_{\mu,j}\left[\left|\frac{\lambda(f)}{n} - \mu_0\right| > \frac{\delta}{n^{1/4}} \text{ and } \zeta(f) = n\right] = oe_{1/2}(n)$, *uniformly in* $j$.

*(ii)* $\mathbb{P}_{\mu,j}\left[\lambda(f) = n \text{ and } \left|\zeta(f) - \frac{n}{\mu_0}\right| > \zeta(f)^{3/4}\right] = oe_{1/2}(n)$, *uniformly in* $j$.

*Proof.* The proof of assertion (i) is very similar to that of Lemma 4.2.5. The only difference is the fact that we are now considering a forest, but we can still use Proposition 4.1.4. We leave details to the reader.

Let us turn to the proof of the second assertion, which is a bit more technical. First write:

$$\mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) = n,\ \left|\zeta(\mathbf{f}) - \frac{n}{\mu_0}\right| > \zeta(\mathbf{f})^{3/4}\right]$$
$$= \mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) = n,\ \zeta(\mathbf{f}) > \frac{n}{\mu_0} + \zeta(\mathbf{f})^{3/4}\right] + \mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) = n,\ \zeta(\mathbf{f}) < \frac{n}{\mu_0} - \zeta(\mathbf{f})^{3/4}\right].$$

Denote the first term on the right-hand side by $I_n$ and the second term by $J_n$. We first deal with $I_n$ and show that $I_n = oe_{1/2}(n)$. We observe that:

$$I_n \leqslant \sum_{k=n}^{\infty} \mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) < \mu_0 k - \mu_0 k^{3/4},\ \zeta(\mathbf{f}) = k\right].$$

Assertion (i) implies that $\mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) < \mu_0 k - \mu_0 k^{3/4},\ \zeta(\mathbf{f}) = k\right] = oe_{1/2}(k)$, and this entails that $I_n = oe_{1/2}(n)$.

We complete the proof by showing that $J_n = oe_{1/2}(n)$. Write:

$$J_n \leqslant \sum_{k=n}^{\lfloor n/\mu_0 \rfloor} \mathbb{P}_{\mu,j}\left[\zeta(\mathbf{f}) = k,\ \frac{\lambda(\mathbf{f})}{k} - \mu_0 > \frac{\mu_0}{k^{1/4}}\right].$$

By Lemma 4.2.2 (ii), we have $\phi^*\left(\mu_0 + \mu_0 k^{-1/4}\right) > c_2 k^{-1/2}$ for some $c_2 > 0$ and for every $k \geqslant n$, provided that $n$ is sufficiently large. Then, using Proposition 4.1.6 and Lemma 4.2.2 (i):

$$J_n \leqslant \sum_{k=n}^{\lfloor n/\mu_0 \rfloor} \frac{j}{k}\mathbb{P}\left[\frac{1}{k}\sum_{p=1}^{k} 1_{\{X_p=-1\}} > \mu_0 + \frac{\mu_0}{k^{1/4}}\right] \leqslant \sum_{k=n}^{\lfloor n/\mu_0 \rfloor} 2\exp(-c_2 k^{1/2})$$

which is $oe_{1/2}(n)$. □

## 4.3 Estimate for the probability of having $n$ leaves

In this section, we give a precise asymptotic estimate for the probability that a $GW_\mu$ tree has $n$ leaves. This result is of independent interest, but will also be useful when proving an invariance principle for $GW_\mu$ trees conditioned on having $n$ leaves.

Recall that $\mu$ is a probability distribution on $\mathbb{N}$ satisfying hypothesis $(H_\theta)$ with $\theta \in (1, 2]$. Recall also that $h$ is the slowly varying function that was introduced just before Lemma 4.1.11.

**Theorem 4.3.1.** *Let supp$(\mu)$ be the support of $\mu$ and let $d \geqslant 1$ be the largest integer such that supp$(\mu)\backslash\{0\}$ is contained in $b + d\mathbb{Z}$ for some $b \in \mathbb{N}$. Then choose $b$ minimal such that the preceding property holds.*

*(i) There exists an integer $N > 0$ such that the following holds. For every $n \geqslant N$, $\mathbb{P}_\mu[\lambda(\tau) = n+1] > 0$ if, and only if, $n$ is a multiple of $\gcd(b-1, d)$.*

*(ii) We have:*

$$\mathbb{P}_\mu[\lambda(\tau) = n+1] \underset{n\to\infty}{\sim} \mu_0^{1/\theta} p_1(0) \frac{\gcd(b-1, d)}{h(n) n^{1/\theta+1}},$$

*when $n$ tends to infinity in the set of multiples of $\gcd(b-1, d)$. Here we recall that $p_1$ is the continuous density of the law of $X_1$, where $(X_t)_{t\geqslant 0}$ is the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta)$.*

*In particular, when the second moment of $\mu$ is finite :*

$$\mathbb{P}_\mu[\lambda(\tau) = n+1] \underset{n\to\infty}{\sim} \sqrt{\frac{\mu_0}{2\pi\sigma^2}} \frac{\gcd(b-1, d)}{n^{3/2}},$$

*when $n$ tends to infinity in the set of multiples of $\gcd(b-1, d)$.*

Note that supp$(\mu)\backslash\{0\}$ is non-lattice if and only if $d = 1$. It is crucial to keep in mind that even if $\mu$ is aperiodic, supp$(\mu)\backslash\{0\}$ can be lattice (for example if the support of $\mu$ is $\{0, 4, 7\}$).

**Remark 4.3.2.** In the case where $\mu$ has finite variance, Theorem 4.3.1 is a consequence of results contained in [85].

Before giving the proof of Theorem 4.3.1, let us mention a useful consequence.

**Corollary 4.3.3.** *Fix $\delta > 0$ and $\eta \in (0, 1)$. We have:*

$$\mathbb{P}_\mu\left[\sup_{\eta\leqslant t\leqslant 1}\left|\frac{\Lambda_\tau(\zeta(\tau)t)}{\zeta(\tau)t} - \mu_0\right| \geqslant \frac{\delta}{n^{1/4}} \,|\, \lambda(\tau) = n\right] = oe_{1/2}(n),$$

*when $n-1$ tends to infinity in the set of multiples of $\gcd(b-1, d)$.*

This bound is an immediate consequence of Corollary 4.2.6 once we know that $\mathbb{P}_\mu[\lambda(\tau) = n]$ decays like a power of $n$.

### 4.3.1 The Non-Lattice case

We consider a random variable Y on $\mathbb{N}$ with distribution:

$$\mathbb{P}[Y = i] = \frac{1}{1-\mu_0}\mu(i+1) = \frac{1}{1-\mu_0}\nu(i), \qquad i \geqslant 0. \tag{4.6}$$

We will first establish Theorem 4.3.1 when Y is non-lattice, that is $b = 0$ and $d = 1$ in the notation of Theorem 4.3.1.

In agreement with the notation of Proposition 4.1.6, we consider the random walk $W'$ defined as $W$ conditioned on having nonnegative jumps. In particular, $W'_n$ is the sum of $n$ independent copies of the random variable Y, which is in the domain of attraction of a stable law of index $\theta$. Indeed, when $\sigma^2 = \infty$, this follows from the characterization of the domain of attraction of stable laws (see [56, Theorem 2.6.1]). When $\sigma^2 < \infty$, formula (4.6) shows that Y has a finite second moment as well.

Consequently, if we write $B'_n$ for the quantity corresponding to $a_n$ in Theorem 4.1.10 when $W$ is replaced by $W'$, we have, by Theorem 4.1.10 (iii):

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| B'_n \mathbb{P}[W'_n = k] - p_1\left(\frac{k - n\mathbb{E}[Y]}{B'_n}\right)\right| = 0. \tag{4.7}$$

Moreover, there exists a slowly varying function $h' : \mathbb{R}_+ \to \mathbb{R}_+$ such that $B'_n = h'(n)n^{1/\theta}$, and $h'(n) \to \infty$ as $n \to \infty$ when both $\sigma^2 = \infty$ and $\theta = 2$. In the case where the second moment of $\mu$ is finite, we have $B'_n = \sigma'\sqrt{n/2}$ where $\sigma'^2$ is the variance of Y. Note also that $\mathbb{E}[Y] = \mu_0/(1-\mu_0)$.

The following lemma establishes an important link between $h$ and $h'$.

**Lemma 4.3.4.** *If $\sigma^2 = \infty$ we have $\lim_{n\to\infty} B'_n/B_n = \lim_{n\to\infty} h'(n)/h(n) = (1-\mu_0)^{-1/\theta}$.*

*Proof.* First assume that $\theta < 2$. Since $\mathbb{P}[Y \geqslant x] = \frac{1}{1-\mu_0}\mathbb{P}[W_1 \geqslant x]$ for $x \geqslant 0$, by Theorem 4.1.10 (i), we have for $n$ large enough:

$$B'_n = |\Gamma(1-\theta)^{1/\theta}| \inf\left\{x \geqslant 0;\ \mathbb{P}[Y \geqslant x] \leqslant \frac{1}{n}\right\} = |\Gamma(1-\theta)|^{1/\theta} \inf\left\{x \geqslant 0;\ \mathbb{P}[W_1 \geqslant x] \leqslant \frac{1-\mu_0}{n}\right\}.$$

Thus $B_{\lfloor n/(1-\mu_0)\rfloor} \leqslant B'_n \leqslant B_{\lceil n/(1-\mu_0)\rceil}$, and the conclusion easily follows. The proof in the case $\theta = 2$ is similar and is left to the reader. □

We will use the following refinement of the local limit theorem (see [96, Chapter 7, P10] for a proof).

**Theorem 4.3.5** (Strong Local Limit Theorem). *Let $Z = (Z_n)_{n\geqslant 0}$ be a random walk on $\mathbb{Z}$ with jump distribution $\rho$ started from 0, where $\rho$ is a non-lattice probability distribution on $\mathbb{Z}$. Assume that the second moment of $\rho$ is finite. Denote the mean of $\rho$ by $m$ and its variance by $\widetilde{\sigma}^2$. Set $\widetilde{a}_n = \widetilde{\sigma}\sqrt{n/2}$. Then:*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}} \left(1 \vee \frac{(x-mn)^2}{n}\right) \left| \widetilde{a}_n \mathbb{P}[Z_n = x] - p_1\left(\frac{x-mn}{\widetilde{a}_n}\right)\right| = 0.$$

*Proof of Theorem 4.3.1 when $Y$ is non-lattice.* We first show that $h'(n)n^{1/\theta+1}\mathbb{P}_\mu[\lambda(\tau)=n]$ converges to a positive real number. Fix $\epsilon \in (0,1/2)$ and write:

$$\mathbb{P}_\mu[\lambda(\tau)=n] = \mathbb{P}_\mu\left[\lambda(\tau)=n,\ (1-\epsilon)\frac{n}{\mu_0} \leqslant \zeta(\tau) \leqslant (1+\epsilon)\frac{n}{\mu_0}\right] + \mathbb{P}_\mu\left[\lambda(\tau)=n,\ \left|\frac{\mu_0\zeta(\tau)}{n}-1\right| > \epsilon\right].$$

By Proposition 4.1.7, there exists $C>0$ such that $h'(n) \leqslant Cn$ for every positive integer $n$. Moreover, for $n$ large enough, for every $x>0$, the property $|x\mu_0/n-1| \geqslant \epsilon$ implies $|x-n/\mu_0| \geqslant x^{3/4}$. Consequently:

$$h'(n)n^{1/\theta+1}\mathbb{P}_\mu\left[\lambda(\tau)=n,\ \left|\frac{\mu_0\zeta(\tau)}{n}-1\right|>\epsilon\right] \leqslant Cn^{1/\theta+2}\mathbb{P}_\mu\left[\lambda(\tau)=n,\ \left|\zeta(\tau)-\frac{n}{\mu_0}\right| \geqslant \zeta(\tau)^{3/4}\right],$$

which is $oe_{1/2}(n)$ by Lemma 4.2.7 (ii). It is thus sufficient to show that:

$$h'(n)n^{1/\theta+1}\mathbb{P}_\mu\left[\lambda(\tau)=n,\ (1-\epsilon)\frac{n}{\mu_0} \leqslant \zeta(\tau) \leqslant (1+\epsilon)\frac{n}{\mu_0}\right] \tag{4.8}$$

converges to a positive real number.

In the following, $S_p$ will denote the sum of $p$ independent Bernoulli variables of parameter $\mu_0$. Note that $S_1$ is non-lattice. The key idea is to write the quantity appearing in (4.8) as a sum, then rewrite it as an integral and finally use the dominated convergence theorem. For $x \in \mathbb{R}$, denote the smallest integer greater than or equal to $x$ by $\lceil x \rceil$. To simplify notation, we write $\mathcal{O}(1)$ for a bounded sequence indexed by $n$ and $o(1)$ for a sequence indexed by $n$ which tends to $0$. Using Proposition 4.1.6, we write:

$$\begin{aligned}
&h'(n)n^{1/\theta+1}\mathbb{P}_\mu\left[\lambda(\tau)=n,\ (1-\epsilon)\frac{n}{\mu_0} \leqslant \zeta(\tau) \leqslant (1+\epsilon)\frac{n}{\mu_0}\right] \\
&= h'(n)n^{1/\theta+1}\sum_{p=\lceil(1-\epsilon)n/\mu_0\rceil}^{\lfloor(1+\epsilon)n/\mu_0\rfloor}\frac{1}{p}\,\mathbb{P}[S_p=n]\,\mathbb{P}[W'_{p-n}=n-1] \\
&= \int_{-\frac{\epsilon}{\mu_0}n+\mathcal{O}(1)}^{\frac{\epsilon}{\mu_0}n+\mathcal{O}(1)} dx\,\frac{h'(n)n^{1/\theta+1}}{\lfloor n/\mu_0+x\rfloor}\,\mathbb{P}[S_{\lfloor n/\mu_0+x\rfloor}=n]\,\mathbb{P}[W'_{\lfloor n/\mu_0+x\rfloor-n}=n-1] \\
&= \int_{-\frac{\epsilon}{\mu_0}\sqrt{n}+o(1)}^{\frac{\epsilon}{\mu_0}\sqrt{n}+o(1)} du\,\frac{\sqrt{n}h'(n)n^{1/\theta+1}}{\lfloor n/\mu_0+u\sqrt{n}\rfloor}\mathbb{P}[S_{\lfloor n/\mu_0+u\sqrt{n}\rfloor}=n]\,\mathbb{P}[W'_{\lfloor n/\mu_0+u\sqrt{n}\rfloor-n}=n-1]
\end{aligned} \tag{4.9}$$

Using the case $\theta=2$ of Theorem 4.1.10 (iii), for fixed $u\in\mathbb{R}$, one sees that:

$$\lim_{n\to\infty}\sqrt{n}\mathbb{P}[S_{\lfloor n/\mu_0+u\sqrt{n}\rfloor}=n] = \frac{1}{\sqrt{2\pi(1-\mu_0)}}e^{-\frac{\mu_0^2}{2(1-\mu_0)}u^2}.$$

We now claim that there exists a bounded function $F:\mathbb{R}\to\mathbb{R}_+$ such that:

$$\lim_{n\to\infty} h'(n)n^{1/\theta}\mathbb{P}[W'_{\lceil n/\mu_0+u\sqrt{n}\rceil-n}=n-1] = F(u) \tag{4.10}$$

for every fixed $u \in \mathbb{R}$. We distinguish two cases. When $\sigma^2 = \infty$, we have by (4.7):

$$\begin{aligned}\lim_{n\to\infty} h'(n)n^{1/\theta}\mathbb{P}[W'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n} = n-1] \\ &= \left(\frac{\mu_0}{1-\mu_0}\right)^{\frac{1}{\theta}} \lim_{n\to\infty} p_1\left(\frac{n-1-\frac{\mu_0}{1-\mu_0}\left(\left\lfloor \frac{n}{\mu_0}+u\sqrt{n}\right\rfloor - n\right)}{B'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n}}\right) \\ &= \left(\frac{\mu_0}{1-\mu_0}\right)^{\frac{1}{\theta}} p_1(0).\end{aligned}$$

In the case $\theta = 2$, we use the property that $h'(n) \to \infty$ as $n \to \infty$. When $\sigma^2 < \infty$, (4.7) gives:

$$\begin{aligned}\lim_{n\to\infty} \sigma'\sqrt{n/2}\,\mathbb{P}[W'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n} = n-1] \\ &= \sqrt{\frac{\mu_0}{1-\mu_0}} \lim_{n\to\infty} p_1\left(\frac{n-1-\frac{\mu_0}{1-\mu_0}\left(\left\lfloor \frac{n}{\mu_0}+u\sqrt{n}\right\rfloor - n\right)}{\sigma'\sqrt{\left\lfloor \frac{n}{\mu_0}+u\sqrt{n}\right\rfloor - n}/\sqrt{2}}\right) \\ &= \sqrt{\frac{\mu_0}{1-\mu_0}}\, p_1\left(\frac{\sqrt{2}}{\sigma'}\cdot\left(\frac{\mu_0}{1-\mu_0}\right)^{3/2} u\right).\end{aligned}$$

In both cases, we have obtained our claim (4.10).

Next, for $n \geqslant 1$ and $u \in \mathbb{R}$, define :

$$f_n(u) = 1_{\{|u|<\frac{2\epsilon}{\mu_0}\sqrt{n}\}}\sqrt{n}\mathbb{P}[S_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor} = n] \quad , \quad g_n(u) = 1_{\{|u|<\frac{2\epsilon}{\mu_0}\sqrt{n}\}}B'_n\mathbb{P}[W'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n} = n-1].$$

The strong version of the Local Limit Theorem (Theorem 4.3.5) implies that there exists $C > 0$ such that $|f_n(u)| \leqslant C\min\left(1, \frac{1}{u^2}\right)$ for all $n > 1$ and $u \in \mathbb{R}$. To bound $g_n$, write:

$$g_n(u) = 1_{\{|u|<\frac{2\epsilon}{\mu_0}\sqrt{n}\}}\frac{B'_n}{B'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n}}B'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n}\mathbb{P}[W'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n} = n-1].$$

Proposition 4.1.7 (ii) implies that there exists $C' > 0$ such that $B'_n/B'_{\lfloor n/\mu_0+u\sqrt{n}\rfloor - n} \leqslant C'$ for every $n$ sufficiently large and $|u| < \frac{2\epsilon}{\mu_0}\sqrt{n}$, and then (4.7) entails that there exists $C > 0$ such that for all $n > 1$ and $u \in \mathbb{R}$ we have $|g_n(u)| \leqslant C$. By the preceding bounds on $f_n$ and $g_n$, we can apply the dominated convergence theorem to the right-hand side of (4.9) and we get:

$$\lim_{n\to\infty} h'(n)n^{1/\theta+1}\mathbb{P}_\mu[\lambda(\tau) = n] = \mu_0\int_{-\infty}^{+\infty} du\,F(u)\frac{1}{\sqrt{2\pi(1-\mu_0)}}e^{-\frac{\mu_0^2}{2(1-\mu_0)}u^2}. \tag{4.11}$$

Finally, we need to identify the value of the integral in (4.11) and to express $h'$ in terms of $h$. We again distinguish two cases. First suppose that $\sigma^2 < \infty$. An explicit computation of the right-hand side of (4.11) gives:

$$\frac{\sigma'}{\sqrt{2}}n^{3/2}\mathbb{P}_\mu[\lambda(\tau) = n] \quad \underset{n\to\infty}{\longrightarrow} \quad \sqrt{\frac{1}{4\pi}\cdot\frac{\mu_0\sigma'^2}{\mu_0/(1-\mu_0)+\sigma'^2(1-\mu_0)}}.$$

A simple calculation gives $\sigma'^2 = (\sigma^2 - \mu_0)/(1-\mu_0) - (\mu_0/(1-\mu_0))^2$, which entails:

$$\mathbb{P}_\mu[\lambda(\tau) = n] \underset{n\to\infty}{\sim} \sqrt{\frac{\mu_0}{2\pi\sigma^2}} n^{-3/2}.$$

When $\sigma^2 = \infty$, we have $F(u) = (\mu_0/(1-\mu_0))^{\frac{1}{\theta}} p_1(0)$ so that (4.11) immediately gives that $h'(n)n^{1/\theta+1}\mathbb{P}_\mu[\lambda(\tau) = n]$ converges towards $(\mu_0/(1-\mu_0))^{\frac{1}{\theta}} p_1(0)$ as $n \to \infty$. By Lemma 4.3.4, we conclude that:

$$\mathbb{P}_\mu[\lambda(\tau) = n] \underset{n\to\infty}{\sim} \mu_0^{\frac{1}{\theta}} p_1(0) \frac{1}{h(n)n^{1/\theta+1}}.$$

Note that this formula is still valid for $\theta = 2$. This concludes the proof in the non-lattice case. □

### 4.3.2 The Lattice case

We now sketch a proof of Theorem 4.3.1 when Y is lattice.

*Proof of Theorem 4.3.1 when* $Y$ *is lattice.* For (i), by Proposition 4.1.6, $\mathbb{P}_\mu[\lambda(\tau) = n+1] > 0$ if and only if there exists $k \geqslant 0$ such that $\mathbb{P}[W'_k = n] > 0$. As a consequence, $\mathbb{P}_\mu[\lambda(\tau) = n+1] > 0$ if and only if $n$ can be written as a sum of elements of $\mathrm{supp}(Y)$. Since $\mathrm{supp}(Y) \subset b - 1 + d\mathbb{Z}$, it follows that $\mathbb{P}_\mu[\lambda(\tau) = n+1] = 0$ if $n$ is not divisible by $\gcd(b-1, d)$, and it is an easy number theoretical exercise to show that there exists $N > 0$ such that for $n \geqslant N$, $\mathbb{P}_\mu[\lambda(\tau) = n+1] > 0$ if $n$ is a multiple of $\gcd(b-1, d)$.

The asymptotic estimate of (ii) is obtained exactly as in the non-lattice case by making use of the Local Limit Theorem for lattice random variables (see e.g. [56, Theorem 4.2.1]). We omit the argument to avoid technicalities. □

**Remark 4.3.6.** Let us briefly discuss the extension of the preceding results to the case where $\mu$ is periodic. In this case, $Y$ is necessarily lattice. Indeed, the property $\mathrm{supp}(\mu) \subset d\mathbb{Z}$ implies $\mathrm{supp}(Y) \subset d\mathbb{Z} - 1$. The same reasoning as above shows that Theorem 4.3.1 remains valid in this case. However, the span of $\mathrm{supp}(Y)$ is not necessarily equal to the span of $\mathrm{supp}(\mu)$. Consequently, $\mathbb{P}_\mu[\lambda(\tau) = n] = 0$ can hold for infinitely many $n$ (for example if the support of $\mu$ is $\{0, 28, 40, 52\}$) or for finitely many $n$ (for instance if the support of $\mu$ is $\{0, 3, 6\}$).

## 4.4 Conditioning on having at least $n$ leaves

In this section, we show that the scaling limit of a $GW_\mu$ tree conditioned on having at least $n$ leaves is the same (up to constants) as that of a $GW_\mu$ tree conditioned on having total progeny at least $n$. The argument goes as follows. By the large deviation result obtained in Section 2 (which states that if a $GW_\mu$ tree has $n$ leaves, then the probability that its total progeny does not belong to $[n/\mu_0 - n^{3/4}, n/\mu_0 + n^{3/4}]$ decreases exponentially fast in $n$), we establish that the probability measures $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) \geqslant n]$ and $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant \mu_0 n - n^{3/4}]$ are close to each other for large $n$. The fact that the rescaled contour function of a $GW_\mu$ tree under $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) \geqslant n]$ converges in distribution then allows us to conclude.

Henceforth, if $I$ is a closed subinterval of $\mathbb{R}_+$, $\mathcal{C}(I, \mathbb{R})$ stands for the space of all continuous functions from $I$ to $\mathbb{R}_+$, which is equipped with the topology of uniform convergence on every compact subset of $I$.

Recall that $\mu$ is a probability distribution on $\mathbb{N}$ satisfying the hypothesis $(H_\theta)$ for some $\theta \in (1, 2]$. Recall also the definition of the sequence $(B_n)$, introduced just before Lemma 4.1.11. Also recall the notation $C_t(\tau)$ for the contour function of a tree $\tau$ introduced in Definition 4.1.3.

**Theorem 4.4.1** (Duquesne). *There exists a random continuous function on $[0, 1]$ denoted by $H^{exc}$ such that if $t_n$ is a tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) = n]$:*

$$\left(\frac{B_n}{n} C_{2nt}(t_n);\ 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (H^{exc}_t;\ 0 \leqslant t \leqslant 1),$$

*where the convergence holds in the sense of weak convergence of the laws on $\mathcal{C}([0, 1], \mathbb{R})$.*

*Proof.* See [36, Theorem 3.1] or [68]. □

**Remark 4.4.2.** The random function $H^{exc}$, can be identified as the normalized excursion of the height process associated to the spectrally positive stable process $X$. The notion of the height process was introduced in [76] and studied in great detail in [37]; see Section 5.1 for a definition.

Using Theorem 4.4.1, we shall prove that for every bounded nonnegative continuous function $F$ on $\mathcal{C}([0, 1], \mathbb{R})$ the following convergence holds:

$$\mathbb{E}_\mu\left[F\left(\frac{B_{\zeta(\tau)}}{\zeta(\tau)} C_{2\zeta(\tau)t}(\tau);\ 0 \leqslant t \leqslant 1\right)\middle|\ \lambda(\tau) \geqslant n\right] \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[F(H^{exc})]. \tag{4.12}$$

Recall that $\mathbb{P}_{\mu,j}$ stands for the probability measure on $\mathbb{T}^j$ which is the distribution of $j$ independent $GW_\mu$ trees.

**Lemma 4.4.3.** *Fix $1 \leqslant j \leqslant n$. Let $U$ be a bounded nonnegative measurable function on $\mathbb{T}^j$. Then:*

$$\begin{aligned}&\left|\mathbb{E}_{\mu,j}\left[U(f)1_{\zeta(f)\geqslant n}\right] - \mathbb{E}_{\mu,j}\left[U(f)1_{\lambda(f)\geqslant \mu_0 n - n^{3/4}}\right]\right| \\ &\qquad\qquad \leqslant \quad \|U\|_\infty\, \mathbb{P}_{\mu,j}\left[n - (\mu_0^{-1}+1)n^{3/4} \leqslant \zeta(f) \leqslant n\right] + oe_{1/2}(n)\end{aligned}$$

*where the estimate $oe_{1/2}(n)$ is uniform in $j$.*

*Proof.* First note that:

$$\begin{aligned}\mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n}\right] &= \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n}1_{\lambda(\mathbf{f})\geqslant \mu_0\zeta(\mathbf{f})-\zeta(\mathbf{f})^{3/4}}\right] + \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n}1_{\lambda(\mathbf{f})< \mu_0\zeta(\mathbf{f})-\zeta(\mathbf{f})^{3/4}}\right] \\ &= \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n}1_{\lambda(\mathbf{f})\geqslant \mu_0\zeta(\mathbf{f})-\zeta(\mathbf{f})^{3/4}}\right] + oe_{1/2}(n),\end{aligned} \tag{4.13}$$

where we have used Lemma 4.2.7 (i) in the last equality.

Secondly, write:

$$\mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant \mu_0 n - n^{3/4}}\right] = \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant \mu_0 n - n^{3/4}}1_{\zeta(\mathbf{f})\geqslant n}\right] + \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant \mu_0 n - n^{3/4}}1_{\zeta(\mathbf{f})< n}\right]$$

Let $C_n$ and $D_n$ be respectively the first and the second term appearing in the right-hand side. To simplify notation, set $\alpha(n) = \mu_0 n - n^{3/4}$ for $n \geqslant 1$. Then, by Lemma 4.2.7 (ii), we have for $n$

large enough:

$$\begin{aligned} D_n &= \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant\alpha(n),\,\zeta(\mathbf{f})<n,\,\left|\zeta(\mathbf{f})-\frac{\lambda(\mathbf{f})}{\mu_0}\right|\leqslant\zeta(\mathbf{f})^{3/4}}\right]+oe_{1/2}(n)\\ &\leqslant \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant\alpha(n),\,\zeta(\mathbf{f})<n,\,\zeta(\mathbf{f})\geqslant\frac{\lambda(\mathbf{f})}{\mu_0}-\zeta(\mathbf{f})^{3/4}}\right]+oe_{1/2}(n)\\ &\leqslant \|U\|_\infty\,\mathbb{P}_{\mu,j}\left[n-(\mu_0^{-1}+1)n^{3/4}\leqslant\zeta(\mathbf{f})\leqslant n\right]+oe_{1/2}(n)\end{aligned}$$

We next consider $C_n$. Choose $n$ sufficiently large so that the function $x\mapsto\alpha(x)$ is increasing over $[\mu_0 n-n^{3/4},\infty)$ and write:

$$\begin{aligned} C_n &= \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n,\,\lambda(\mathbf{f})\geqslant\alpha(\zeta(\mathbf{f}))}\right]+\mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant\alpha(n),\,\zeta(\mathbf{f})\geqslant n,\,\lambda(\mathbf{f})<\alpha(\zeta(\mathbf{f}))}\right]\\ &\leqslant \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n,\,\lambda(\mathbf{f})\geqslant\alpha(\zeta(\mathbf{f}))}\right]+\mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n,\,\lambda(\mathbf{f})<\alpha(\zeta(\mathbf{f}))}\right]\\ &= \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n,\,\lambda(\mathbf{f})\geqslant\alpha(\zeta(\mathbf{f}))}\right]+oe_{1/2}(n)\end{aligned}$$

by Lemma 4.2.7 (i).

By the preceding estimates we have, for $n$ large:

$$\begin{aligned} 0 &\leqslant \mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\lambda(\mathbf{f})\geqslant\mu_0 n-n^{3/4}}\right]-\mathbb{E}_{\mu,j}\left[U(\mathbf{f})1_{\zeta(\mathbf{f})\geqslant n}1_{\lambda(\mathbf{f})\geqslant\mu_0\zeta(\mathbf{f})-\zeta(\mathbf{f})^{3/4}}\right]\\ &\leqslant \|U\|_\infty\,\mathbb{P}_{\mu,j}\left[n-(\mu_0^{-1}+1)n^{3/4}\leqslant\zeta(\mathbf{f})\leqslant n\right]+oe_{1/2}(n)\end{aligned}$$

and by combining this bound with (4.13) we get the desired estimate. □

**Proposition 4.4.4.** *Let $U,(U_n)_{n\geqslant1}:\mathbb{T}\to\mathbb{R}_+$ be uniformly bounded measurable functions, meaning that there exists $M>0$ such that for all $n\geqslant1$ and $\tau\in\mathbb{T}$, $U_n(\tau)\leqslant M$ and $U(\tau)\leqslant M$.*

*(i) If $\mathbb{E}_\mu\left[U(\tau)|\,\zeta(\tau)=n\right]$ converges as $n\to\infty$, then $\mathbb{E}_\mu\left[U(\tau)|\,\zeta(\tau)\geqslant n\right]$ converges to the same limit.*

*(ii) If $\mathbb{E}_\mu\left[U_n(\tau)|\,\zeta(\tau)\geqslant n\right]$ converges as $n\to\infty$, then $\mathbb{E}_\mu\left[U_n(\tau)|\,\lambda(\tau)\geqslant\lceil\mu_0 n-n^{3/4}\rceil\right]$ converges to the same limit.*

*Proof.* Using the formula:

$$\mathbb{E}_\mu\left[U(\tau)\,|\,\zeta(\tau)\geqslant n\right]=\frac{1}{\mathbb{P}_\mu[\zeta(\tau)\geqslant n]}\sum_{k=n}^{\infty}\mathbb{P}_\mu[\zeta(\tau)=k]\cdot\mathbb{E}_\mu\left[U(\tau)\,|\,\zeta(\tau)=k\right]$$

it is an easy exercise to verify that the first assertion is true.

We turn to the proof of (ii). Fix $0<\eta<1/4$. By Lemma 4.1.11, we may suppose that $n$ is sufficiently large so that $\mathbb{P}_\mu\left[\zeta(\tau)\geqslant n\right]\geqslant c_3 n^{-1/\theta-\eta}$ for a constant $c_3>0$. Next, setting again $\alpha(n)=\mu_0 n-n^{3/4}$, we have:

$$\begin{aligned} &\left|\mathbb{E}_\mu\left[U_n(\tau)|\zeta(\tau)\geqslant n\right]-\mathbb{E}_\mu\left[U_n(\tau)|\lambda(\tau)\geqslant\alpha(n)\right]\right|\\ &\leqslant \left|\frac{\mathbb{E}_\mu\left[U_n(\tau)1_{\zeta(\tau)\geqslant n}\right]}{\mathbb{P}_\mu\left[\zeta(\tau)\geqslant n\right]}-\frac{\mathbb{E}_\mu\left[U_n(\tau)1_{\lambda(\tau)\geqslant\alpha(n)}\right]}{\mathbb{P}_\mu\left[\zeta(\tau)\geqslant n\right]}\right|\\ &\qquad\qquad + \frac{\mathbb{E}_\mu\left[U_n(\tau)1_{\lambda(\tau)\geqslant\alpha(n)}\right]}{\mathbb{P}_\mu\left[\lambda(\tau)\geqslant\alpha(n)\right]}\left|\frac{\mathbb{P}_\mu[\lambda(\tau)\geqslant\alpha(n)]}{\mathbb{P}_\mu[\zeta(\tau)\geqslant n]}-1\right|\\ &\leqslant M\frac{n^{1/\theta+\eta}}{c_3}\mathbb{P}_\mu\left[n-(\mu_0^{-1}+1)n^{3/4}\leqslant\zeta(\tau)\leqslant n\right]+M\left|\frac{\mathbb{P}_\mu[\lambda(\tau)\geqslant\alpha(n)]}{\mathbb{P}_\mu[\zeta(\tau)\geqslant n]}-1\right|+oe_{1/2}(n),\end{aligned}$$

where we have used Lemma 4.4.3 in the last inequality. By Lemma 4.1.11, the first term of the right-hand side tends to $0$. From Theorem 4.3.1 (ii), it is easy to get that $\mathbb{P}_\mu[\lambda(\tau) \geqslant n] \sim \theta\mu_0^{1/\theta} p_1(0)/(n^{1/\theta}h(n))$ as $n \to \infty$. By combining this estimate with Lemma 4.1.11, we obtain that $\mathbb{P}_\mu[\lambda(\tau) \geqslant \alpha(n)]/\mathbb{P}_\mu[\zeta(\tau) \geqslant n]$ tends to $1$ as $n \to \infty$. This completes the proof. □

**Theorem 4.4.5.** *For $n \geqslant 1$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$. Then:*

$$\left(\frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n);\ 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (H^{exc}_t;\ 0 \leqslant t \leqslant 1), \tag{4.14}$$

*where the convergence holds in the sense of weak convergence of the laws on $\mathcal{C}([0,1],\mathbb{R})$.*

*Proof.* Let $F$ be a bounded nonnegative continuous function on $\mathcal{C}([0,1],\mathbb{R})$. By Theorem 4.4.1:

$$\mathbb{E}_\mu\left[F\left(\frac{B_{\zeta(\tau)}}{\zeta(\tau)} C_{2\zeta(\tau)t}(\tau);\ 0 \leqslant t \leqslant 1\right)\middle|\ \zeta(\tau) = n\right] \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[F(H^{exc})].$$

Proposition 4.4.4 (i) entails:

$$\mathbb{E}_\mu\left[F\left(\frac{B_{\zeta(\tau)}}{\zeta(\tau)} C_{2\zeta(\tau)t}(\tau);\ 0 \leqslant t \leqslant 1\right)\middle|\ \zeta(\tau) \geqslant n\right] \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[F(H^{exc})].$$

Proposition 4.4.4 (ii) then implies:

$$\mathbb{E}_\mu\left[F\left(\frac{B_{\zeta(\tau)}}{\zeta(\tau)} C_{2\zeta(\tau)t}(\tau);\ 0 \leqslant t \leqslant 1\right)\middle|\ \lambda(\tau) \geqslant \lceil \mu_0 n - n^{3/4}\rceil\right] \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[F(H^{exc})].$$

Since $\lceil \mu_0 n - n^{3/4}\rceil$ takes all positive integer values when $n$ varies, the proof is complete. □

**Remark 4.4.6.** When the second moment of $\mu$ is finite, $H^{exc} = \sqrt{2}\mathbb{e}$ where $\mathbb{e}$ denotes the normalized excursion of linear Brownian motion. Since the scaling constants $B_n = \sigma\sqrt{n/2}$ are known explicitly, in that case the theorem can be formulated as:

$$\left(\frac{\sigma}{2\sqrt{\zeta(t_n)}} C_{2\zeta(t_n)t}(t_n);\ 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbb{e}.$$

## 4.5 Conditioning on having exactly $n$ leaves

*To avoid technical issues, we suppose that* $\mathrm{supp}(\mu)\backslash\{0\}$ *is non-lattice, so that* $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$ *for* $n$ *large enough.*

Recall that we have obtained an invariance principle for $GW_\mu$ trees under the probability distribution $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) \geqslant n]$. Our goal is now to establish a similar result for trees under the probability distribution $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. The key idea is to use an "absolute continuity" property. Let us briefly sketch the main step of the argument.

Let $k \geqslant 1$. If $\tau$ is a tree and if $u(0), u(1), \ldots$ are the vertices of $\tau$ in lexicographical order, let $T_k(\tau)$ be the first index $j$ such that $\{u(0), u(1), \ldots, u(j)\}$ contains $k$ leaves and $T_k(\tau) = \infty$ if there is no such index. Fix $a \in (0,1)$ and recall the notation $\mathcal{W}(\tau)$ for the Lukasiewicz path of a tree $\tau$. Then there exists a positive function $D_a^n$ on $\mathbb{Z}_+$ such that, for every nonnegative function $f$ on the space of finite paths in $\mathbb{Z}$:

$$\mathbb{E}_\mu\left[f\left(\mathcal{W}_{\cdot\wedge T_{\lfloor an\rfloor}(\tau)}(\tau)\right) \mid \lambda(\tau)=n\right] = \mathbb{E}_\mu\left[f\left(\mathcal{W}_{\cdot\wedge T_{\lfloor an\rfloor}(\tau)}(\tau)\right) D_a^n(\mathcal{W}_{T_{\lfloor an\rfloor}(\tau)}(\tau)) \,|\, \lambda(\tau)\geqslant n\right].$$

By combining the invariance principle for trees under $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)\geqslant n]$ together with estimates for $D_a^n(\mathcal{W}_{T_{\lfloor an\rfloor}(\tau)}(\tau))$ as $n\to\infty$, we shall deduce an invariance principle for trees under the conditional probability measure $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)=n]$.

### 4.5.1 The normalized excursion of the Lévy process

We follow the presentation of [36]. The underlying probability space will be denoted by $(\Omega,\mathcal{F},\mathbb{P})$. Let $X$ be a process with paths in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$, the space of right-continuous with left limits (càdlàg) real-valued functions, endowed with the Skorokhod $J_1$-topology. We refer the reader to [20, chap. 3] and [57, chap. VI] for background concerning the Skorokhod topology. We denote the canonical filtration generated by $X$ and augmented with the $\mathbb{P}$-negligible sets by $(\mathcal{F}_t)_{t\geqslant 0}$. In agreement with the notation in the previous sections, we assume that $X$ is a strictly stable spectrally positive Lévy process with index $\theta\in(1,2]$ such that for $\lambda>0$:

$$\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta). \tag{4.15}$$

See [15] for the proofs of the general assertions of this subsection concerning Lévy processes. In particular, for $\theta=2$ the process $X$ is $\sqrt{2}$ times the standard Brownian motion on the line. Recall that $X$ has the following scaling property: for $c>0$, the process $(c^{-1/\theta}X_{ct}, t\geqslant 0)$ has the same law as $X$. In particular, if we denote by $p_t$ the density of $X_t$ with respect to the Lebesgue measure, $p_t$ enjoys the following scaling property:

$$p_{\lambda s}(x) = \lambda^{-1/\theta} p_s(x\lambda^{-1/\theta}) \tag{4.16}$$

for $x\in\mathbb{R}$ and $s,\lambda>0$. The following notation will be useful: for $s<t$ set

$$I_t^s = \inf_{[s,t]} X, \qquad I_t = \inf_{[0,t]} X.$$

Notice that the process $I$ is continuous since $X$ has no negative jumps.

The process $X-I$ is a strong Markov process and $0$ is regular for itself with respect to $X-I$. We may and will choose $-I$ as the local time of $X-I$ at level $0$. Let $(g_i,d_i), i\in\mathcal{I}$ be the excursion intervals of $X-I$ above $0$. For every $i\in\mathcal{I}$ and $s\geqslant 0$, set $\omega_s^i = X_{(g_i+s)\wedge d_i} - X_{g_i}$. We view $\omega^i$ as an element of the excursion space $\mathcal{E}$, which is defined by:

$$\mathcal{E} = \{\omega\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}_+);\ \omega(0)=0 \text{ and } \zeta(\omega):=\sup\{s>0;\omega(s)>0\}\in(0,\infty)\}.$$

From Itô's excursion theory, the point measure

$$\mathcal{N}(dtd\omega) = \sum_{i\in\mathcal{I}} \delta_{(-I_{g_i},\omega^i)}$$

is a Poisson measure with intensity $dt\mathbf{N}(d\omega)$, where $\mathbf{N}(d\omega)$ is a σ-finite measure on $\mathcal{E}$ which is called the Itô excursion measure. Without risk of confusion, we will also use the notation X for the canonical process on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

Let us define the normalized excursion of X. For every $\lambda > 0$, define the re-scaling operator $S^{(\lambda)}$ on the set of excursions by:

$$S^{(\lambda)}(\omega) = \left(\lambda^{1/\theta}\omega(s/\lambda),\ s \geqslant 0\right).$$

Note that $\mathbf{N}(\zeta > t) \in (0, \infty)$ for $t > 0$. The scaling property of X shows that the image of $\mathbf{N}(\cdot \mid \zeta > t)$ under $S^{(1/\zeta)}$ does not depend on $t > 0$. This common law, which is supported on the càdlàg paths with unit lifetime, is called the law of the normalized excursion of X and denoted by $\mathbf{N}(\cdot \mid \zeta = 1)$. We write $X^{\text{exc}} = (X^{\text{exc}}_s, 0 \leqslant s \leqslant 1)$ for a process distributed according to $\mathbf{N}(\cdot \mid \zeta = 1)$. In particular, for $\theta = 2$ the process $X^{\text{exc}}$ is $\sqrt{2}$ times the normalized excursion of linear Brownian motion. Informally, $\mathbf{N}(\cdot \mid \zeta = 1)$ is the law of an excursion under the Itô measure conditioned to have unit lifetime.

We will also use the so-called continuous-time height process H associated with X which was introduced in [76]. If $\theta = 2$, H is set to be equal to X. If $\theta \in (1, 2)$, the process H is defined for every $t \geqslant 0$ by:

$$H_t := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{\{X_s < I_t^s + \epsilon\}} ds,$$

where the limit exists in $\mathbb{P}$-probability and in $\mathbf{N}$-measure on $\{t < \zeta\}$. The definition of H thus makes sense under $\mathbb{P}$ or under $\mathbf{N}$. The process H has a continuous modification both under $\mathbb{P}$ and under $\mathbf{N}$ (see [37, Chapter 1] for details), and from now on we consider only this modification. Using simple scaling arguments one can also define H as a continuous random process under $\mathbf{N}(\cdot \mid \zeta = 1)$. Let us finally mention that the limiting process $H^{\text{exc}}$ in Theorem 4.4.1 has the distribution of H under $\mathbf{N}(\cdot \mid \zeta = 1)$.

### 4.5.2 An invariance principle

Recall that the Lukasiewicz path $\mathcal{W}(\tau)$ of a tree $\tau \in \mathbb{T}$ is defined up to time $\zeta(\tau)$. We extend it to $\mathbb{Z}_+$ by setting $\mathcal{W}_i(\tau) = 0$ for $i \geqslant \zeta(\tau)$. Similarly, we extend the height function $H(\tau)$ to $\mathbb{Z}_+$ by setting $H_i(\tau) = 0$ for $i \geqslant \zeta(\tau)$. We then extend $H(\tau)$ to $\mathbb{R}_+$ by linear interpolation,

$$H_t(\tau) = (1 - \{t\})H_{\lfloor t \rfloor}(\tau) + \{t\}H_{\lfloor t \rfloor + 1}(\tau), \qquad t \geqslant 0,$$

where $\{t\} = t - \lfloor t \rfloor$.

Recall that $\mu$ is a probability distribution on $\mathbb{N}$ satisfying the hypothesis $(H_\theta)$ for some $\theta \in (1, 2]$. Recall also the notation $h$, $B_n$ introduced just before Lemma 4.1.11. For technical reasons, we put $B_u = B_{\lfloor u \rfloor}$ for $u \geqslant 1$. It is useful to keep in mind that $B_n = \sigma\sqrt{n/2}$ when the variance $\sigma^2$ of $\mu$ is finite. We rely on the following theorem.

**Theorem 4.5.1** (Duquesne & Le Gall). *Let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\cdot \mid \zeta(\tau) \geqslant n]$. We have:*

$$\left(\frac{1}{B_n}\mathcal{W}_{\lfloor nt \rfloor}(t_n), \frac{B_n}{n}H_{nt}(t_n)\right)_{t \geqslant 0} \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t, H_t)_{0 \leqslant t \leqslant 1} \text{ under } \mathbf{N}(\cdot \mid \zeta > 1).$$

*Proof.* See the concluding remark of [37, Section 2.5]. □

### 4.5.3 Absolute continuity

Recall from the beginning of this section the definition of $T_k(\tau)$ for a tree $\tau$.

**Proposition 4.5.2.** *Let $n$ be a positive integer and let $k$ be an integer such that $1 \leqslant k \leqslant n-1$. To simplify notation, set $\mathcal{W}^{(k)}(\tau) = (\mathcal{W}_0(\tau), \ldots, \mathcal{W}_{T_k(\tau)}(\tau))$. For every bounded function $f : \cup_{i \geqslant 1} \mathbb{Z}^i \to \mathbb{R}_+$ we have:*

$$\mathbb{E}_\mu\left[f(W^{(k)}(\tau)) \middle| \lambda(\tau) = n\right] = \mathbb{E}_\mu\left[f(\mathcal{W}^{(k)}(\tau)) \frac{\psi_{n-k}(\mathcal{W}_{T_k(\tau)}(\tau))/\psi_n(1)}{\psi^*_{n-k}(\mathcal{W}_{T_k(\tau)}(\tau))/\psi^*_n(1)} \middle| \lambda(\tau) \geqslant n\right],$$

*where $\psi_p(j) = \mathbb{P}_{\mu,j}[\lambda(f) = p]$ and $\psi^*_p(j) = \mathbb{P}_{\mu,j}[\lambda(f) \geqslant p]$ for every integer $p \geqslant 1$.*

*Proof.* Let the random walk $W$ be as in Proposition 4.1.4. The result follows from the latter proposition and an application of the strong Markov property to the random walk $W$ at the first time it has made $k$ negative jumps. See [75, Lemma 10] for details of the argument in a slightly different context. □

We will also use the following continuous version of Proposition 4.5.2 (see [68, Proposition 2.3] for a proof).

**Proposition 4.5.3.** *For $s > 0$ and $x \geqslant 0$, set $q_s(x) = \frac{x}{s} p_s(-x)$. For every $a \in (0,1)$ and $x > 0$ define:*

$$\Gamma_a(x) = \frac{\theta q_{1-a}(x)}{\int_{1-a}^{\infty} ds\, q_s(x)}.$$

*Then for every measurable bounded function $F : \mathbb{D}([0,a], \mathbb{R}^2) \to \mathbb{R}_+$:*

$$\mathbf{N}\left(F((X_t)_{0\leqslant t\leqslant a}, (H_t)_{0\leqslant t\leqslant a}) \middle| \zeta = 1\right) = \mathbf{N}\left(F((X_t)_{0\leqslant t\leqslant a}, (H_t)_{0\leqslant t\leqslant a}) \Gamma_a(X_a) \middle| \zeta > 1\right).$$

We now control the Radon-Nikodym density appearing in Proposition 4.5.2. Recall that $p_s$ stands for the density of $X_s$. It is well known that $p_1$ is bounded over $\mathbb{R}$ and that the derivative of $q_u$ is bounded over $\mathbb{R}$ for every $u > 0$ (see e.g. [99, I. 4]).

**Lemma 4.5.4.** *Fix $\alpha > 0$. We have:*

$$(i) \lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left|\psi^*_n(j) - \int_1^\infty ds\, q_s\left(\frac{j}{B_{n/\mu_0}}\right)\right| = 0, \quad (ii) \lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left|n\psi_n(j) - q_1\left(\frac{j}{B_{n/\mu_0}}\right)\right| = 0.$$

The proof of Lemma 4.5.4 is technical and is postponed to Section 4.5.5.

**Corollary 4.5.5.** *Let $r_n$ be a sequence of positive integers such that $n/r_n \to \mu_0$ as $n \to \infty$.*

*(i) We have* $\displaystyle\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left|\psi^*_{n-\lfloor a\mu_0 r_n\rfloor}(j) - \int_{1-a}^\infty ds\, q_s\left(\frac{j}{B_{n/\mu_0}}\right)\right| = 0.$

*(ii) We have* $\displaystyle\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left|n\psi_{n-\lfloor a\mu_0 r_n\rfloor}(j) - q_{1-a}\left(\frac{j}{B_{n/\mu_0}}\right)\right| = 0.$

*Proof.* We shall only prove (i). The second assertion is easier and is left to the reader. By Lemma 4.5.4 (i):

$$\sup_{1\leqslant j\leqslant \alpha B_n} \left| \psi^*_{n-\lfloor a\mu_0 r_n \rfloor}(j) - \int_1^\infty ds\, q_s\left(\frac{j}{B_{(n-\lfloor a\mu_0 r_n\rfloor)/\mu_0}}\right)\right| = 0.$$

By (4.16) and the definition of $q_s(x)$:

$$\int_{1-a}^\infty ds\, q_s\left(\frac{j}{B_{n/\mu_0}}\right) = \int_1^\infty ds\, q_s\left(\frac{j}{(1-a)^{1/\theta}B_{n/\mu_0}}\right).$$

To simplify notation, set $a_1(n,j) = \frac{j}{(1-a)^{1/\theta}B_{n/\mu_0}}$ and $a_2(n,j) = \frac{j}{B_{(n-\lfloor a\mu_0 r_n\rfloor)/\mu_0}}$. It is thus sufficient to verify that for $n$ sufficiently large:

$$\sup_{1\leqslant j\leqslant \alpha B_n} \left| \int_1^\infty ds\, (q_s(a_1(n,j)) - q_s(a_2(n,j)))\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.17}$$

From (4.16), we have for $x \geqslant 0$:

$$\int_1^\infty ds\, q_s(x) = x\int_1^\infty \frac{ds}{s} p_s(-x) = x\int_1^\infty \frac{ds}{s^{1+1/\theta}} p_1(-xs^{-1/\theta}) = \theta\int_0^x p_1(-u)\, du,$$

so that

$$\left|\int_1^\infty ds\, (q_s(a_1(n,j)) - q_s(a_2(n,j)))\right| = \theta\left|\int_{a_1(n,j)}^{a_2(n,j)} p_1(-u)\, du\right| \leqslant \theta M'|a_2(n,j) - a_1(n,j)|,$$

where we have used the fact that $p_1$ is bounded by a positive real number $M'$. Thus we see that (4.17) will follow if we can verify that:

$$\sup_{1\leqslant j\leqslant \alpha B_n} |a_1(n,j) - a_2(n,j)| \underset{n\to\infty}{\longrightarrow} 0,$$

and to this end it is enough to establish that:

$$\left|\frac{B_n}{(1-a)^{1/\theta}B_{n/\mu_0}} - \frac{B_n}{B_{(n-\lfloor a\mu_0 r_n\rfloor)/\mu_0}}\right| \underset{n\to\infty}{\longrightarrow} 0.$$

The last convergence is however immediate from our assumption on the sequence $(r_n)$. □

### 4.5.4 Convergence of the scaled contour and height functions

We now aim at proving invariance theorems under the conditional probability measure $\mathbb{P}_\mu[\cdot \,|\, \lambda(\tau) = n]$.

Recall the notation $T_k(\tau)$ introduced in the beginning of this section. For $u \geqslant 0$, set $T_u(\tau) = T_{\lfloor u \rfloor}(\tau)$.

**Lemma 4.5.6.** *Fix $a \in (0,1)$ and $\alpha < \min(a/2, (1-a)/2)$.*

*(i) We have* $\lim_{n\to\infty} \mathbb{P}_\mu\left[\sup_{b\in(a-\alpha,a+\alpha)}\left|\frac{T_{\mu_0 bn}(\tau)}{n} - b\right| > \frac{1}{n^{1/4}} \,\middle|\, \zeta(\tau) \geqslant n\right] = 0.$

*(ii) We have* $\lim_{n\to\infty} \mathbb{P}_\mu \left[ \sup_{b\in(a-\alpha,a+\alpha)} \left| \frac{T_{bn}(\tau)}{n} - \frac{b}{\mu_0} \right| > \frac{1}{n^{1/4}} \,\middle|\, \lambda(\tau) = n \right] = 0.$

*Proof.* Both assertions are easy consequences of Corollaries 4.2.6 and 4.3.3. Details are left to the reader. □

**Lemma 4.5.7.** *Let* $d$ *be a positive integer. Fix* $a \in (0,1)$ *and consider a sequence* $(Z^n)_{n\geqslant 1}$ *of càdlàg processes with values in* $\mathbb{R}^d$*. Let also* $(K_n)_{n\geqslant 1}$ *and* $(S_n)_{n\geqslant 1}$ *be two sequences of positive random variables converging in probability towards* $1$*. Assume that* $(Z^n)_{n\geqslant 1}$ *converges in distribution in* $\mathbb{D}([0,\infty),\mathbb{R}^d)$ *towards a càdlàg process* $Z$ *such that a.s.* $Z$ *is continuous at* $a$*. Then* $(K_n Z^n_{S_n t};\, 0\leqslant t\leqslant a)$ *converges in distribution in* $\mathbb{D}([0,a],\mathbb{R})$ *towards* $(X_t;\, 0\leqslant t\leqslant a)$*.*

*Proof.* By the Skorokhod Representation Theorem (see e.g. [20, Theorem 6.7]), we can assume that $(X^n)_{n\geqslant 1}$ converges almost surely in $\mathbb{D}([0,\infty),\mathbb{R}^d)$ towards $(X_t; t\geqslant 0)$ and that both $(K_n)_{n\geqslant 1}$ and $(S_n)_{n\geqslant 1}$ converge almost surely towards $1$. The conclusion follows by standard properties of the Skorokhod topology (see e.g. [57, VI. Theorem 1.14]). □

**Lemma 4.5.8.** *For* $n\geqslant 1$*, let* $r_n$ *be the greatest positive integer such that* $\lceil \mu_0 r_n - r_n^{3/4}\rceil = n$*. Fix* $a\in(0,1)$*. Let* $t_n$ *be a random tree distributed according to* $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)=n]$*. Then the law of*

$$\left( \frac{1}{B_{r_n}} \mathcal{W}_{\lfloor T_{a\mu_0 r_n}(t_n)\frac{t}{a}\rfloor}(t_n), \frac{B_{r_n}}{r_n} H_{T_{a\mu_0 r_n}(t_n)\frac{t}{a}}(t_n) \right)_{0\leqslant t\leqslant a}$$

*converges to the law of* $(X_t, H_t)_{0\leqslant t\leqslant a}$ *under* $\mathbf{N}(\,\cdot\,|\,\zeta=1)$*.*

*Proof.* We start by proving that for every $\alpha>1$:

$$\lim_{n\to\infty} \left( \sup_{\frac{1}{\alpha}B_n\leqslant j\leqslant \alpha B_n} \left| \frac{\psi_{n-\lfloor a\mu_0 r_n\rfloor}(j)/\psi_n(1)}{\psi^*_{n-\lfloor a\mu_0 r_n\rfloor}(j)/\psi^*_n(1)} - \Gamma_a\left(\frac{j-1}{B_{n/\mu_0}}\right) \right| \right) = 0. \tag{4.18}$$

By Theorem 4.3.1, $\psi^*_n(1)/n\psi_n(1)\to\theta$ as $n\to\infty$. Using Corollary 4.5.5, it then suffices to verify that there exists $\delta>0$ such that for $n$ sufficiently large:

$$\inf_{\frac{1}{\alpha}B_n\leqslant j\leqslant\alpha B_n} \int_{1-a}^{\infty} ds\, q_s\left(\frac{j-1}{B_n}\right) > \delta.$$

This follows from the fact that there exists $\delta'>0$ such that $\int_{1-a}^{\infty} ds\, q_s(x) > \delta'$ for every $x\in[1/\alpha,\alpha]$ Details are left to the reader.

Fix a bounded continuous function $F:\mathbb{D}([0,a],\mathbb{R}^2)\to\mathbb{R}_+$. To simplify notation, for every tree $\tau$ with $\lambda(\tau)\geqslant n$, set $W^{(n)}(\tau) = (W^{(n)}_t(\tau))_{0\leqslant t\leqslant a}$ and $H^{(n)}(\tau) = (H^{(n)}_t(\tau))_{0\leqslant t\leqslant a}$, where for $0\leqslant t\leqslant a$:

$$W^{(n)}_t(\tau) = \frac{1}{B_{r_n}} \mathcal{W}_{\lfloor T_{a\mu_0 r_n}(\tau)\frac{t}{a}\rfloor}(\tau), \qquad H^{(n)}_t(\tau) = \frac{B_{r_n}}{r_n} H_{T_{a\mu_0 r_n}(\tau)\frac{t}{a}}(\tau).$$

Then set $G^{(n)}(\tau) = F\left(W^{(n)}(\tau), H^{(n)}(\tau)\right)$. Note that by (4.3), $H^{(n)}(\tau)$ is a measurable function of $W^{(n)}(\tau)$. Fix $\alpha>1$ and put:

$$A^\alpha_n(\tau) = \left\{ \frac{1}{\alpha} B_{n/\mu_0} < \mathcal{W}_{T_{a\mu_0 r_n}(\tau)}(\tau) < \alpha B_{n/\mu_0} \right\}.$$

By combining Proposition 4.5.2 and the estimate (4.18), we get:

$$\lim_{n\to\infty}\left|\mathbb{E}\left[G^{(n)}(t_n)\,1_{A_n^{\alpha}(t_n)}\right]-\mathbb{E}_{\mu}\left[G^{(n)}(\tau)\,1_{A_n^{\alpha}(\tau)}\Gamma_a\left(\frac{\mathcal{W}_{T_{a\mu_0 r_n}(\tau)}(\tau)}{B_{n/\mu_0}}\right)\middle|\;\lambda(\tau)\geqslant n\right]\right|=0. \quad (4.19)$$

We now claim that the law of $\left(W^{(n)}(\tau),H^{(n)}(\tau)\right)$ under $\mathbb{P}_{\mu}[\,\cdot\,|\,\lambda(\tau)\geqslant n]$ converges towards the law of $(X_t,H_t)_{0\leqslant t\leqslant a}$ under $\mathbf{N}(\cdot\,|\,\zeta>1)$. To establish this convergence, by Proposition 4.4.4 (ii), it is sufficient to show that the law of

$$\left(\frac{1}{B_n}\mathcal{W}_{\lfloor T_{a\mu_0 n}(\tau)\frac{t}{a}\rfloor}(\tau),\frac{B_n}{n}H_{T_{a\mu_0 n}(\tau)\frac{t}{a}}(\tau)\right)_{0\leqslant t\leqslant a}$$

under $\mathbb{P}_{\mu}[\,\cdot\,|\,\zeta(\tau)\geqslant n]$ converges towards the law of $(X_t,H_t)_{0\leqslant t\leqslant a}$ under $\mathbf{N}(\cdot\,|\,\zeta>1)$. Indeed, Proposition 4.4.4 (ii) will then imply that the same convergence holds if we replace $\mathbb{P}[\,\cdot\,|\zeta(\tau)\geqslant n]$ by $\mathbb{P}\left[\,\cdot\,|\lambda(\tau)\geqslant\lceil\mu_0 n-n^{3/4}\rceil\right]$ and we just have to replace $n$ by $r_n$. By Lemma 4.5.6, under $\mathbb{P}_{\mu}[\,\cdot\,|\,\zeta(\tau)\geqslant n]$, $T_{a\mu_0 n}(\tau)/(an)$ converges in probability towards $1$, and by Theorem 4.5.1, the law of $\left(\frac{1}{B_n}\mathcal{W}_{\lfloor nt\rfloor}(\tau),\frac{B_n}{n}H_{nt}(\tau)\right)_{t\geqslant 0}$ converges to the law of $(X_t,H_t)_{t\geqslant 0}$ under $\mathbf{N}(\,\cdot\,|\,\zeta>1)$. Our claim now follows from Lemma 4.5.7.

From the definition of $r_n$, we have $r_n/n\to 1/\mu_0$ as $n\to\infty$, which entails $B_{r_n}/B_{n/\mu_0}\to 1$. Thanks to (4.19) and the preceding claim, we get that:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[G^{(n)}(t_n)1_{A_n^{\alpha}(t_n)}\right] &= \mathbf{N}(F\left((X_t,H_t)_{0\leqslant t\leqslant a}\right)\Gamma_a(X_a)1_{\{\frac{1}{\alpha}<X_a<\alpha\}}\,|\,\zeta>1)\\ &= \mathbf{N}(F\left((X_t,H_t)_{0\leqslant t\leqslant a}\right)1_{\{\frac{1}{\alpha}<X_a<\alpha\}}\,|\,\zeta=1),\end{aligned} \quad (4.20)$$

where we have used Proposition 4.5.3 in the last equality. By taking $F\equiv 1$, we obtain:

$$\lim_{\alpha\to\infty}\lim_{n\to\infty}\mathbb{P}\left[A_n^{\alpha}(t_n)\right]=1.$$

By choosing $\alpha>0$ sufficiently large, we easily deduce from the convergence (4.20) that:

$$\lim_{n\to\infty}\mathbb{E}\left[G^{(n)}(t_n)\right]=\mathbf{N}(F\left((X_t,H_t)_{0\leqslant t\leqslant a}\right)\,|\,\zeta=1).$$

This completes the proof. □

Recall that $C(\tau)$ stands for the contour function of the tree $\tau$, introduced in Definition 4.1.3.

**Theorem 4.5.9.** *For every $n\geqslant 1$ such that $\mathbb{P}_{\mu}[\lambda(\tau)=n]>0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_{\mu}[\,\cdot\,|\,\lambda(\tau)=n]$. Then the following convergences hold.*

*(i) Fix $a\in(0,1)$. We have:*

$$\left(\frac{1}{B_{\zeta(t_n)}}\mathcal{W}_{\lfloor\zeta(t_n)t\rfloor}(t_n);\,0\leqslant t\leqslant a\right)\xrightarrow[n\to\infty]{(d)}(X_t;\,0\leqslant t\leqslant a)\text{ under }\mathbf{N}(\,\cdot\,|\,\zeta=1). \quad (4.21)$$

*(ii) We have:*

$$\left(\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_{2\zeta(t_n)t}(t_n),\frac{B_{\zeta(t_n)}}{\zeta(t_n)}H_{\zeta(t_n)t}(t_n)\right)_{0\leqslant t\leqslant 1}\xrightarrow[n\to\infty]{(d)}(H_t,H_t)_{0\leqslant t\leqslant 1}\text{ under }\mathbf{N}(\,\cdot\,|\,\zeta=1). \quad (4.22)$$

**Remark 4.5.10.** It is possible to replace the scaling factors $1/B_{\zeta(t_n)}$ and $B_{\zeta(t_n)}/\zeta(t_n)$ by respectively $\mu_0^{1/\theta}/B_n$ and $\mu_0^{1-1/\theta}B_n/n$ without changing the statement of the theorem. This follows indeed from the fact that $\zeta(t_n)/n$ converges in distribution towards $1/\mu_0$ under $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)=n]$.

The convergence of rescaled contour functions in (ii) implies that the tree $t_n$, viewed as a finite metric space for the graph distance and suitably rescaled, converges to the $\theta$-stable tree in distribution for the Gromov-Hausdorff distance on isometry classes of compact metric spaces (see e.g. [73, Section 2] for details).

The convergence (4.21) actually holds with $a=1$. This will be proved later in Section 6.

*Proof.* Recall that throughout this section we limit ourselves to the case where $\mathbb{P}_\mu[\lambda(\tau)=n]>0$ for all $n$ sufficiently large.

We start with (i). As in Lemma 4.5.8, let $r_n$ be the greatest positive integer such that $\lceil \mu_0 r_n - {r_n}^{3/4}\rceil = n$ and write:

$$\frac{1}{B_{\zeta(t_n)}}\mathcal{W}_{\lfloor \zeta(t_n)t\rfloor}(t_n) = K_n \cdot \frac{1}{B_{r_n}}\mathcal{W}_{\left\lfloor S_n\cdot T_{a\mu_0 r_n}(t_n)\frac{t}{a}\right\rfloor}(t_n),$$

where $K_n = B_{r_n}/B_{\zeta(t_n)}$ and $S_n = a\zeta(t_n)/T_{a\mu_0 r_n}(t_n)$. Recall that $r_n/n \to 1/\mu_0$. By Corollary 4.3.3, $\zeta(t_n)/n$ converges in probability to $1/\mu_0$. On the one hand, this entails that $K_n$ converges in probability towards $1$, and on the other hand, together with Lemma 4.5.6 (ii), this entails that $S_n$ converges in probability towards $1$. The convergence (4.21) then follows from Lemmas 4.5.8 and 4.5.7.

For the second assertion, we start by observing that:

$$\left(\frac{B_{\zeta(t_n)}}{\zeta(t_n)}H_{\zeta(t_n)t}(t_n);\ 0\leqslant t\leqslant a\right) \quad \xrightarrow{(d)} \quad (H_t;\ 0\leqslant t\leqslant a)\ \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta=1). \tag{4.23}$$

This convergence follows from Lemmas 4.5.8 and 4.5.7 by the same arguments we used to establish (i). To complete the proof we use known relations between the height process and the contour process (see e.g. [36, Remark 3.2]) to show that an analog of (4.23) also holds for the contour process. For $0\leqslant p<\zeta(t_n)$ set $b_p=2p-H_p(t_n)$ so that $b_p$ represents the time needed by the contour process to reach the $(p+1)$-th individual of $\zeta(t_n)$. Also set $b_{\zeta(t_n)}=2(\zeta(t_n)-1)$. Note that $C_{b_p}=H_p$ for every $p\in\{0,1,\ldots,\zeta(t_n)\}$. From this observation and the definitions of the contour function and the height function of a tree, we easily get:

$$\sup_{t\in[b_p,b_{p+1}]}|C_t(t_n)-H_p(t_n)| \leqslant |H_{p+1}(t_n)-H_p(t_n)|+1. \tag{4.24}$$

for $0\leqslant p<\zeta(t_n)$. Then define the random function $g_n:[0,2\zeta(t_n)]\to\mathbb{N}$ by setting $g_n(t)=k$ if $t\in[b_k,b_{k+1})$ and $k<\zeta(t_n)$, and $g_n(t)=\zeta(t_n)$ if $t\in[2(\zeta(t_n)-1),2\zeta(t_n)]$. If $t<2(\zeta(t_n)-1)$, $g_n(t)$ is the largest rank of an individual that has been visited before time $t$ by the contour function, if the individuals are listed $0,1,\ldots,\zeta(t_n)-1$ in lexicographical order. Finally, set $\widetilde{g}_n(t)=g_n(\zeta(t_n)t)/\zeta(t_n)$. Fix $\alpha\in(0,1)$. Then, by (4.24):

$$\sup_{t\leqslant\frac{b_{\lfloor\alpha\zeta(t_n)\rfloor}}{\zeta(t_n)}}\left|\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_{\zeta(t_n)t}(t_n)-\frac{B_{\zeta(t_n)}}{\zeta(t_n)}H_{\zeta(t_n)\widetilde{g}_n(t)}\right| \leqslant \frac{B_{\zeta(t_n)}}{\zeta(t_n)}+\frac{B_{\zeta(t_n)}}{\zeta(t_n)}\sup_{k\leqslant\lfloor\alpha\zeta(t_n)\rfloor}|H_{k+1}(t_n)-H_k(t_n)|,$$

which converges in probability to 0 by (4.23) and the path continuity of H. On the other hand, it follows from the definition of $g_n$ that

$$\begin{aligned}\sup_{t\leqslant \frac{b_{\lfloor \alpha\zeta(t_n)\rfloor}}{\zeta(t_n)}} \left|\widetilde{g}_n(t)-\frac{t}{2}\right| &\leqslant \frac{1}{\zeta(t_n)}\left(\sup_{k\leqslant \lfloor \alpha\zeta(t_n)\rfloor}\left|k-\frac{b_k}{2}\right|+1\right)\\ &\leqslant \frac{1}{2B_{\zeta(t_n)}}\sup_{k\leqslant \alpha\zeta(t_n)}\frac{B_{\zeta(t_n)}}{\zeta(t_n)}H_k(t_n)+\frac{1}{\zeta(t_n)} \quad \xrightarrow{(\mathbb{P})} \quad 0\end{aligned}$$

by (4.23). Finally, by the definition of $b_n$ and using (4.23) we see that $\frac{b_{\lfloor \alpha\zeta(t_n)\rfloor}}{\zeta(t_n)}$ converges in probability towards $2\alpha$. By applying the preceding observations with $\alpha$ replaced by $\alpha' \in (\alpha, 1)$, we conclude that:

$$\frac{B_{\zeta(t_n)}}{\zeta(t_n)}\sup_{0\leqslant t\leqslant \alpha}|C_{2\zeta(t_n)t}(t_n)-H_{\zeta(t_n)t}(t_n)| \quad \xrightarrow{(\mathbb{P})} \quad 0. \tag{4.25}$$

Together with (4.23), this implies:

$$\left(\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_{2\zeta(t_n)t}(t_n);\, 0\leqslant t\leqslant a\right) \quad \xrightarrow{(d)} \quad (H_t;\, 0\leqslant t\leqslant a) \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta=1). \tag{4.26}$$

We now use a time-reversal argument in order to show that the convergence holds on the whole segment $[0,1]$. To this end, we adapt [36, Remark 3.2] and [37, Section 2.4] to our context. See also [68], where we used the same argument to give another proof of Duquesne's Theorem 4.4.1. Observe that $(C_t(t_n);\, 0\leqslant t\leqslant 2(\zeta(t_n)-1))$ and $(C_{2(\zeta(t_n)-1)-t}(t_n);\, 0\leqslant t\leqslant 2(\zeta(t_n)-1))$ have the same distribution. From this convergence and the convergence (4.26), it is an easy exercise to obtain that:

$$\left(\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_{2\zeta(t_n)t}(t_n);\, 0\leqslant t\leqslant 1\right) \quad \xrightarrow{(d)} \quad (H_t;\, 0\leqslant t\leqslant 1) \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta=1). \tag{4.27}$$

See the last paragraph of the proof of Theorem 6.1 in [72] for additional details in a similar argument.

Finally, we verify that (4.22) can be derived from (4.27). To this end, we show that the convergence (4.25) also holds for $\alpha=1$. First note that:

$$\sup_{0\leqslant t\leqslant 2}\left|\widetilde{g}_n(t)-\frac{t}{2}\right| \leqslant \frac{1}{\zeta(t_n)}\left(\frac{1}{2}\sup_{k\leqslant \zeta(t_n)}H_k(t_n)+1\right) = \frac{1}{2B_{\zeta(t_n)}}\sup_{k\leqslant 2\zeta(t_n)}\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_k(t_n)+\frac{1}{\zeta(t_n)} \xrightarrow{(\mathbb{P})} 0 \tag{4.28}$$

by (4.27). Secondly, by (4.24):

$$\begin{aligned}\sup_{0\leqslant t\leqslant 2}\left|\frac{B_{\zeta(t_n)}}{\zeta(t_n)}C_{\zeta(t_n)t}(t_n)-\frac{B_{\zeta(t_n)}}{\zeta(t_n)}H_{\zeta(t_n)\widetilde{g}_n(t)}(t_n)\right| &\leqslant \frac{B_{\zeta(t_n)}}{\zeta(t_n)}+\frac{B_{\zeta(t_n)}}{\zeta(t_n)}\sup_{k<\zeta(t_n)}|H_{k+1}(t_n)-H_k(t_n)|\\ &= \frac{B_{\zeta(t_n)}}{\zeta(t_n)}+\frac{B_{\zeta(t_n)}}{\zeta(t_n)}\sup_{k<\zeta(t_n)}\left|C_{b_{k+1}}(t_n)-C_{b_k}(t_n)\right|.\end{aligned}$$

By (4.27), in order to show that the latter quantity tends to 0 in probability, it is sufficient to verify that $\sup_{k<\zeta(t_n)}\zeta(t_n)^{-1}|b_{k+1}-b_k|$ converges to 0 in probability. But by the definition of

$b_p$:

$$\sup_{k<\zeta(t_n)} \left| \frac{b_{k+1}}{\zeta(t_n)} - \frac{b_k}{\zeta(t_n)} \right| = \sup_{k<\zeta(t_n)} \left| \frac{2 + H_k(t_n) - H_{k+1}(t_n)}{\zeta(t_n)} \right| \leqslant \frac{2}{\zeta(t_n)} + 2 \sup_{k<\zeta(t_n)} \frac{H_k(t_n)}{\zeta(t_n)}$$

which converges in probability to 0 by the same argument as in (4.28). We have thus obtained that

$$\frac{B_{\zeta(t_n)}}{\zeta(t_n)} \sup_{0\leqslant t\leqslant 1} |C_{2\zeta(t_n)t}(t_n) - H_{\zeta(t_n)\tilde{g}_n(2t)}(t_n)| \quad \xrightarrow{(\mathbb{P})} \quad 0.$$

Combining this with (4.27), we conclude that:

$$\left( \frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} H_{\zeta(t_n)\tilde{g}_n(2t)}(t_n) \right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t, H_t)_{0\leqslant t\leqslant 1} \text{ under } \mathbf{N}(\,\cdot \mid \zeta = 1).$$

The convergence (4.28) then entails:

$$\left( \frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} H_{\zeta(t_n)t}(t_n) \right)_{0\leqslant t\leqslant 1} \quad \xrightarrow[n\to\infty]{(d)} \quad (H_t, H_t)_{0\leqslant t\leqslant 1} \text{ under } \mathbf{N}(\,\cdot \mid \zeta = 1).$$

This completes the proof. □

### 4.5.5 Proof of the technical lemma

In this section, we control the Radon-Nikodym densities appearing in Proposition 4.5.2. We heavily rely on the strong version of the Local Limit Theorem (Theorem 4.3.5).

Throughout this section, $(W_n)_{n\geqslant 0}$ will stand for the same random walk as in Proposition 4.1.4. Recall also the notation $q_s$ introduced in Proposition 4.5.3.

**Proof of Lemma 4.5.4 (i)**

We will use two lemmas to prove Lemma 4.5.4 (i): the first one gives an estimate for the quantity $\mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) \geqslant n]$ and the second one shows that the quantity $\mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) \geqslant n]$ is close to $\mathbb{P}_{\mu,j}\left[\lambda(\mathbf{f}) \geqslant \mu_0 n - n^{3/4}\right]$.

**Lemma 4.5.11.** *We have* $\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left| \mathbb{P}_{\mu,j}[\zeta(f) \geqslant n] - \int_1^\infty ds\, q_s\left(\frac{j}{B_n}\right) \right| = 0.$

*Proof.* Le Gall established this result in the case where the variance of $\mu$ is finite in [72]. See [68, Lemma 3.2 (ii)] for the proof in the general case, which is a generalization of Le Gall's proof. □

**Lemma 4.5.12.** *Fix* $\alpha > 0$. *We have* $\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} |\mathbb{P}_{\mu,j}[\zeta(f) \geqslant n] - \mathbb{P}_{\mu,j}[\lambda(f) \geqslant \mu_0 n - n^{3/4}]| = 0.$

*Proof.* To simplify notation, set $\gamma = \mu_0^{-1} + 1$. By Lemma 4.4.3, it is sufficient to show that:

$$\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \mathbb{P}_{\mu,j}[n - \gamma n^{3/4} \leqslant \zeta(\mathbf{f}) \leqslant n] = 0.$$

From the local limit theorem (Theorem 4.1.10), we have, for every $j \in \mathbb{Z}$:

$$\left| B_k \mathbb{P}[W_k = j] - p_1\left(\frac{j}{B_k}\right) \right| \leqslant \epsilon(k),$$

where $\epsilon(k) \to 0$. The function $x \mapsto |xp_1(-x)|$ is bounded over $\mathbb{R}$ by a real number which we will denote by $M$ (see e.g. [99, I. 4]). Set $M_n(j) = \mathbb{P}_{\mu,j}[n - \gamma n^{3/4} \leqslant \zeta(\mathbf{f}) \leqslant n]$ and $\delta(n) = \lfloor n - \gamma n^{3/4} \rfloor + 1$. Fix $\epsilon > 0$ and suppose that $n$ is sufficiently large so that $n - \gamma n^{3/4} \leqslant k \leqslant n$ implies $|\epsilon(k)| \leqslant \epsilon$ and $B_k \geqslant B_n/2$. Then, for $1 \leqslant j \leqslant \alpha B_n$, by (4.4):

$$\begin{aligned} M_n(j) &= \sum_{k=\delta(n)}^{n} \mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = k] = \sum_{k=\delta(n)}^{n} \frac{j}{k} \mathbb{P}_\mu[W_k = -j] \\ &\leqslant \sum_{k=\delta(n)}^{n} \frac{j}{kB_k}\left(p_1\left(-\frac{j}{B_k}\right) + \epsilon(k)\right) \\ &\leqslant \sum_{k=\delta(n)}^{n} \frac{M + 2\alpha\epsilon}{k}, \end{aligned}$$

which tends to $0$ as $n \to \infty$. □

*Proof of Lemma 4.5.4 (i).* By Lemmas 4.5.11 and 4.5.12:

$$\lim_{n\to\infty}\left(\sup_{1\leqslant j\leqslant \alpha B_n}\left|\mathbb{P}_{\mu,j}[\lambda(\mathbf{f}) \geqslant \mu_0 n - n^{3/4}] - \int_1^\infty ds\, q_s\left(\frac{j}{B_n}\right)\right|\right) = 0. \quad (4.29)$$

Let $r_n$ be the greatest positive integer such that $\lceil \mu_0 r_n - {r_n}^{3/4} \rceil = n$. We apply (4.29) with $n$ replaced by $r_n$, and we see that the desired result will follow if we can prove that

$$\lim_{n\to\infty}\left(\sup_{1\leqslant j\leqslant \alpha B_n}\left|\int_1^\infty ds\, q_s\left(\frac{j}{B_{r_n}}\right) - \int_1^\infty ds\, q_s\left(\frac{j}{B_{n/\mu_0}}\right)\right|\right) = 0.$$

The proof of the latter convergence is similar to that of (4.17) noting that:

$$\lim_{n\to\infty}\left|\frac{B_n}{B_{r_n}} - \frac{B_n}{B_{n/\mu_0}}\right| = 0.$$

This completes the proof of Lemma 4.5.4 (i). □

**Proof of Lemma 4.5.4 (ii)**

The proof of Lemma 4.5.4 (ii) is very technical, so we will sometimes only sketch arguments.

As previously, denote by $S_n$ the sum of $n$ independent Bernoulli random variables of parameter $\mu_0$, and by $W'$ the random walk $W$ conditioned on having nonnegative jumps. More precisely, $\mathbb{P}[W'_1 = i] = \mu(i+1)/(1-\mu_0)$ for $i \geqslant 0$. Recall that $\mathbb{E}[W'_1] = \mu_0/(1-\mu_0)$ and that $\sigma'^2$ is the variance of $W'_1$.

Fix $0 < \epsilon < 1$. By Lemma 4.2.7 (ii):

$$n\psi_n(j) = n\mathbb{P}_{\mu,j}\left[\lambda(\tau) = n,\ \frac{n}{\mu_0} - \epsilon n \leqslant \zeta(\mathbf{f}) \leqslant \frac{n}{\mu_0} + \epsilon n\right] + oe_{1/2}(n), \tag{4.30}$$

where the estimate $oe_{1/2}(n)$ is uniform in j. It is thus sufficient to control the first term in the last expression. For $|u| \leqslant \epsilon\sqrt{n}$ and $1 \leqslant j \leqslant n$ set:

$$r_n(u) = \left\lfloor n/\mu_0 + u\sqrt{n} \right\rfloor, \qquad a_n(u) = \sqrt{n}\mathbb{P}[S_{r_n(u)} = n], \qquad b_n(u,j) = B'_n\mathbb{P}[W'_{r_n(u)-n} = n-j],$$

and using Proposition 4.1.6 write:

$$\begin{aligned} n\mathbb{P}_{\mu,j}&\left[\lambda(\tau) = n,\ \frac{n}{\mu_0} - \epsilon n \leqslant \zeta(\mathbf{f}) \leqslant \frac{n}{\mu_0} + \epsilon n\right] \\ &= n \sum_{p=\lceil n/\mu_0 - \epsilon n\rceil}^{\lfloor n/\mu_0 + \epsilon n\rfloor} \frac{j}{p}\, \mathbb{P}[S_p = n]\, \mathbb{P}[W'_{p-n} = n-j] \\ &= n \int_{n/\mu_0 - \epsilon n + O(1)}^{n/\mu_0 + \epsilon n + O(1)} dx \frac{j}{\lfloor x\rfloor} \mathbb{P}\left[S_{\lfloor x\rfloor} = n\right] \mathbb{P}\left[W'_{\lfloor x\rfloor - n} = n-j\right] \\ &= \frac{j}{B'_n} \int_{-\epsilon\sqrt{n}+o(1)}^{\epsilon\sqrt{n}+o(1)} du\, \frac{n}{r_n(u)} a_n(u) b_n(u,j). \end{aligned} \tag{4.31}$$

Let us introduce the following notation. Set $c = \mu_0/(1-\mu_0)$ and for $u, x \in \mathbb{R}$:

$$F(u) = \frac{1}{\sqrt{2\pi\mu_0(1-\mu_0)}} e^{-\frac{1}{2\mu_0(1-\mu_0)}u^2}, \qquad G_0(u,x) = c^{1/\theta} p_1\left(-c^{1/\theta}x - \frac{\sqrt{2}c^{3/2}u}{\sigma'} 1_{\{\sigma^2<\infty\}}\right).$$

Put $F_0(u) = \sqrt{\mu_0}F(\mu_0^{3/2}u)$. Fix $\alpha > 0$. Set finally $\alpha' = \alpha(1 + (1-\mu_0)^{1/\theta})$. By Lemma 4.3.4, for $n$ sufficiently large, we have $\alpha B_n \leqslant \alpha' B'_n$. We suppose in the following that $n$ is sufficiently large so that the latter inequality holds.

**Lemma 4.5.13.** *For fixed* $u \in \mathbb{R}$*, we have:*

$$a_n(u) \underset{n\to\infty}{\longrightarrow} F_0(u), \qquad \sup_{1\leqslant j\leqslant \alpha B_n} |b_n(u,j) - G_0(u, j/B'_n)| \underset{n\to\infty}{\longrightarrow} 0.$$

*Proof.* The first convergence is an immediate consequence of Theorem 4.1.10 (iii) after noting that $\mathbb{E}[S_1] = \mu_0$ and that the variance of $S_1$ is $\mu_0(1-\mu_0)$. The second convergence is more technical. To simplify notation, set:

$$q_n(u) = r_n(u) - n, \qquad Q_n(u,j) = c^{1/\theta} p_1\left(\frac{n - j - cq_n(u)}{B'_{q_n(u)}}\right).$$

Note that $q_n(u) = n/c + u\sqrt{n} + \mathcal{O}(1)$. In particular, $B'_n \sim c^{1/\theta} B'_{q_n(u)}$ as $n \to \infty$. Consequently, by (4.7), $|b_n(u,j) - Q_n(u,j)| \to 0$ as $n \to \infty$, uniformly in $0 \leqslant j \leqslant \alpha B_n$. It thus remains to show that

$$\sup_{1\leqslant j\leqslant \alpha B_n} |Q_n(u,j) - G_0(u, j/B'_n)| \longrightarrow 0. \tag{4.32}$$

To this end, introduce:

$$K_n(u,j) = \left| \frac{n-j-cq_n(u)}{B'_{q_n(u)}} + c^{1/\theta}\frac{j}{B'_n} + \frac{\sqrt{2}c^{3/2}u}{\sigma'}1_{\{\sigma^2<\infty\}} \right|.$$

Recall that the absolute value of the derivative of $p_1$ is bounded by a constant which will be denoted by $M'$, giving $|Q_n(u,j) - G_0(u,j/B'_n)| \leqslant M'K_n(u,j)$. It is thus sufficient to show that $K_n(u,j) \to 0$ as $n \to \infty$, uniformly in $0 \leqslant j \leqslant \alpha B_n$.

We first treat the case where $\sigma^2 < \infty$, so that $\theta = 2$. In this case, $B'_n = \sigma'\sqrt{n/2}$, where $\sigma'^2$ is the variance of $W'_1$. Simple calculations show that $K_n(u,j) \leqslant A/\sqrt{n}$ for some $A \geqslant 1$ depending only on $u$, so that $K_n(u,j) \to 0$ as $n \to \infty$, uniformly in $0 \leqslant j \leqslant \alpha B_n$.

Let us now suppose that $\sigma^2 = \infty$. First assume that $\theta < 2$. Choose $\eta > 0$ such that $\epsilon' := 1/\theta - \eta - 1/2 > 0$. By Proposition 4.1.7 (i), for $n$ sufficiently large, $B'_{q_n(u)} \geqslant n^{1/\theta-\eta}$. Moreover, we can write $B'_n/(c^{1/\theta}B'_{q_n(u)}) = 1 + \epsilon_n(u)$ where, for fixed $u$, $\epsilon_n(u) \to 0$ as $n \to \infty$. Putting these estimates together, we obtain that for large $n$ and for $1 \leqslant j \leqslant \alpha B_n$:

$$K_n(u,j) = \left| c^{1/\theta}\frac{j}{B'_n}\epsilon_n(u) + \frac{cu\sqrt{n}+\mathcal{O}(1)}{B'_{q_n(u)}} \right| \leqslant \alpha' c^{1/\theta}\epsilon_n(u) + \frac{cu}{n^{\epsilon'}} + \mathcal{O}\left(\frac{1}{n^{1/4}}\right),$$

which tends to 0 as $n \to \infty$.

We finally treat the case when $\sigma^2 = \infty$ and $\theta = 2$. Recall the definition of the slowly varying function $h'$ introduced in Section 4.3.1 and let $\epsilon_n(u)$ be as previously. By the remark following the proof of Theorem 4.1.10, $h'$ is increasing so that for $n$ large enough:

$$K_n(u,j) = \left| c^{1/\theta}\frac{j}{B'_n}\epsilon_n(u) + \frac{cu\sqrt{n}+\mathcal{O}(1)}{h'(q_n(u))\sqrt{q_n(u)}} \right| \leqslant \alpha' c^{1/\theta}\epsilon_n(u) + A\frac{u}{h'(n/(2c))} + \mathcal{O}\left(\frac{1}{n^{1/4}}\right)$$

for some $A > 0$. The latter quantity tends to 0 as $n \to \infty$ since $h'(n) \to \infty$ as $n \to \infty$ by the remark following the proof of Theorem 4.1.10. This completes the proof. □

*Proof of Lemma 4.5.4 (ii).* From Theorem 4.3.5 we have the bound $a_n(u) \leqslant C(1 \wedge u^{-2})$ and by (4.7), the functions $b_n$ are uniformly bounded. Since, for $j \leqslant \alpha B_n$,

$$\frac{j}{B'_n}\left| \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, \frac{n}{r_n(u)} a_n(u)b_n(u,j) - \mu_0 \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, a_n(u)b_n(u,j) \right| \leqslant C\alpha' \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du(1\wedge u^{-2}) \left| \frac{n}{r_n(u)} - \mu_0 \right|,$$

it follows from the dominated convergence theorem that:

$$\sup_{1\leqslant j\leqslant \alpha B_n} \left| \frac{j}{B'_n}\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, \frac{n}{r_n(u)} a_n(u)b_n(u,j) - \frac{j}{B'_n}\mu_0 \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, a_n(u)b_n(u,j) \right| \underset{n\to\infty}{\longrightarrow} 0. \quad (4.33)$$

Recall that $q_1(x) = xp_1(-x)$. By (4.30), (4.31) and (4.33), to prove Lemma 4.5.4 (ii), it is sufficient to establish that:

$$\sup_{1\leqslant j\leqslant \alpha B_n} \left| \mu_0 \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, a_n(u)b_n(u,j) - c^{1/\theta}p_1\left(-\frac{j}{B_{n/\mu_0}}\right) \right| \underset{n\to\infty}{\longrightarrow} 0 \quad (4.34)$$

Let us first show that:

$$\sup_{1\leqslant j\leqslant \alpha B_n}\left|\mu_0\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\, a_n(u)b_n(u,j)-\mu_0\int_{-\infty}^{+\infty} du\, F_0(u)G_0(u,j/B'_n)\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.35}$$

To this end, let us prove the following stronger convergence:

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} du\left(\sup_{1\leqslant j\leqslant \alpha B_n}|a_n(u)b_n(u,j)-F_0(u)G_0(u,j/B'_n)|\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.36}$$

It is clear that the function $G_0$ is uniformly bounded. Recall that the functions $b_n$ are uniformly bounded as well. Moreover, $F_0$ is an integrable function and we have the bound $a_n(u)\leqslant C(1\wedge u^{-2})$. The convergence (4.36) then follows from Lemma 4.5.13 and the dominated convergence theorem. This proves (4.35).

To conclude, we distinguish two cases. First assume that $\sigma^2<\infty$, so that $\theta=2$. Then $W'_1$ has finite variance $\sigma'^2$ as well. Recall that $\sigma'^2=(\sigma^2-\mu_0)/(1-\mu_0)-(\mu_0/(1-\mu_0))^2$ and $B'_n=\sigma'\sqrt{n/2}$. A straightforward calculation based on the fact that, for $\alpha,\beta,\gamma,\delta>0$,

$$\int_{-\infty}^{+\infty} du\, e^{-\alpha u^2}e^{-\beta(\gamma+\delta u)^2}=\frac{\sqrt{\pi}}{\sqrt{\alpha+\beta\delta^2}}e^{-\frac{\alpha\gamma^2}{\alpha/\beta+\delta^2}}$$

gives:

$$\begin{aligned}\mu_0\int_{-\infty}^{+\infty} du\, F_0(u)G_0(u,j/B'_n) &= \int_{-\infty}^{+\infty} du\, \mu_0F_0(u)c^{1/2}p_1\left(c^{1/2}\frac{j}{\sigma'\sqrt{n/2}}+\frac{\sqrt{2}c^{3/2}u}{\sigma'}\right)\\ &= c^{1/2}p_1\left(-\frac{j}{\sigma\sqrt{n/(2\mu_0)}}\right)=c^{1/2}p_1\left(-\frac{j}{B_{n/\mu_0}}\right).\end{aligned}$$

By combining this with (4.35), we get (4.34) as desired.

Now assume that $\sigma^2=\infty$. In this case:

$$\mu_0\int_{-\infty}^{+\infty} du\, F_0(u)G_0(u,j/B'_n)=\int_{-\infty}^{+\infty} du\, \mu_0F_0(u)c^{1/\theta}p_1\left(-c^{1/\theta}\frac{j}{B'_n}\right)=c^{1/\theta}p_1\left(-c^{1/\theta}\frac{j}{B'_n}\right). \tag{4.37}$$

By Lemma 4.3.4, $B'_n/B_{n/\mu_0}\to c^{1/\theta}$, which implies:

$$\sup_{1\leqslant j\leqslant \alpha B_n}\left|p_1\left(-c^{1/\theta}\frac{j}{B'_n}\right)-p_1\left(-\frac{j}{B_{n/\mu_0}}\right)\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.38}$$

By combining (4.37) and (4.38) with (4.35), we get (4.34) as desired. □

## 4.6 Convergence of rescaled Lukasiewicz paths when conditioning on having exactly $n$ leaves

We have previously established that the rescaled Lukasiewicz path, height function and contour process of a tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau)=n]$ converge in distribution on $[0,a]$

for every $a \in (0, 1)$. Recall that by means of a time-reversal argument, we were able to extend the convergence of the scaled height and contour functions to the whole segment $[0, 1]$. However, since the Lukasiewicz path $\mathcal{W}(\tau)$ of a tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$ is not invariant under time-reversal, another approach is needed to extend the convergence of $\mathcal{W}(\tau)$ (properly rescaled) to the whole segment $[0, 1]$. To this end, we will use a Vervaat transformation. Let us stress that the Lukasiewicz path of a tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$ does not have a deterministic length, so that special care is necessary to prove the following theorem.

Recall that $\mu$ is a probability distribution on $\mathbb{N}$ satisfying the hypothesis $(H_\theta)$ for some $\theta \in (1, 2]$. Recall also the definition of the sequence $(B_n)$, introduced just before Lemma 4.1.11.

**Theorem 4.6.1.** *For every $n \geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. Then:*

$$\left(\frac{1}{B_{\zeta(t_n)}}\mathcal{W}_{\lfloor \zeta(t_n) t \rfloor}(t_n);\ 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t; 0 \leqslant t \leqslant 1) \text{ under } \mathbf{N}(\,\cdot\,|\,\zeta = 1). \tag{4.39}$$

As previously, to avoid further technicalities, we prove Theorem 4.6.1 in the case where $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$ for all $n$ sufficiently large. Throughout this section, $(W_n)_{n \geqslant 0}$ will stand for the random walk of Proposition 4.1.4. Introduce the following notation for $n \geqslant 0$ and $u \geqslant 0$:

$$\Lambda(n) = \sum_{j=0}^{n-1} 1_{\{W_{j+1} - W_j = -1\}}, \;\; T_u = \inf\{k \geqslant 0;\ \Lambda(k) = \lfloor u \rfloor\}, \;\; \zeta = \inf\{k \geqslant 0;\ W_k = -1\}. \tag{4.40}$$

For technical reasons, we put $B_u = B_{\lfloor u \rfloor}$ for $u \geqslant 1$.

**Lemma 4.6.2.** *The following properties hold.*

*(i) We have* $\mathbb{P}\left[\left|\frac{T_n}{n} - \frac{1}{\mu_0}\right| > \frac{1}{n^{1/4}}\right] = oe_{1/2}(n)$.

*(ii) For every* $a > 0$, $\left(\frac{1}{B_{n/\mu_0}} W_{\lfloor \frac{t}{a} T_{an} \rfloor};\ 0 \leqslant t \leqslant a\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t;\ 0 \leqslant t \leqslant a)$ *under* $\mathbb{P}$.

*Proof.* The first assertion is an easy consequence of Lemma 4.2.5 (i). For (ii), we use a generalization of Donsker's invariance theorem to the stable case, which states that $\big(W_{\lfloor nt \rfloor}/B_n;\ t \geqslant 0\big)$ converges in distribution towards $(X_t;\ t \geqslant 0)$ as $n \to \infty$. See e.g. [57, Chapter VII]. By (i), $T_n/n$ converges almost surely towards $1/\mu_0$, and (ii) easily follows. □

### 4.6.1 The Vervaat transformation

We introduce the Vervaat transformation, which will allow us to deal with random paths with no positivity constraint. Recall the notation $\mathbf{x}^{(i)}$ introduced in Section 1.3.

**Definition 4.6.3.** Let $k \geqslant 1$ be an integer and let $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$. Set $w_j = x_1 + \cdots + x_j$ for $1 \leqslant j \leqslant k$ and let the integer $i_*(\mathbf{x})$ be defined by $i_*(\mathbf{x}) = \inf\{j \geqslant 1; w_j = \min_{1 \leqslant i \leqslant k} w_i\}$. The Vervaat transform of $\mathbf{x}$ is defined as $\mathbf{V}(\mathbf{x}) = \mathbf{x}^{(i_*(\mathbf{x}))}$.

Also introduce the following notation for positive integers $k \geqslant n$:

$$\mathcal{S}^k(n) = \{(x_1, \dots, x_k) \in \{-1, 0, 1, \dots\}; \sum_{i=1}^{k} x_i = -1 \text{ and } \mathrm{Card}\{1 \leqslant i \leqslant k; \, x_i = -1\} = n\},$$

as well as:

$$\overline{\mathcal{S}}^k(n) = \{(x_1, \dots, x_k) \in \mathcal{S}^k(n); \sum_{i=1}^{m} x_i > -1 \text{ for every } m \in \{1, 2, \dots, k-1\}\}.$$

Finally set $\overline{\mathcal{S}}(n) = \cup_{k \geqslant n} \overline{\mathcal{S}}^k(n)$.

**Lemma 4.6.4.** *Let $k \geqslant n$ be positive integers. Set $Z^k = (W_1, W_2 - W_1, \dots, W_k - W_{k-1})$.*

*(i) Conditionally on the event $\{W_k = -1\}$, the random variable $i_*(Z^k)$ is uniformly distributed on $\{1, 2, \dots, k\}$ and is independent of $\boldsymbol{V}(Z^k)$.*

*(ii) Let $\mathbf{x} \in \overline{\mathcal{S}}^k(n)$. Then:*

$$\mathbb{P}\left[\boldsymbol{V}(Z^k) = \mathbf{x}, Z^k_k = -1\right] = \frac{n}{k} \mathbb{P}\left[\boldsymbol{V}(Z^k) = \mathbf{x}\right]. \tag{4.41}$$

*Proof.* The first assertion is a well-known fact, but we give a proof for the sake of completeness. Let $\mathbf{x} \in \overline{\mathcal{S}}^k(n)$ with $k \geqslant n$. Then:

$$\mathbb{P}\left[Z^k = \mathbf{x}\right] = \frac{1}{k} \sum_{i=1}^{k} \mathbb{P}\left[\left(Z^k\right)^{(i)} = \mathbf{x}\right] = \frac{1}{k} \mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}\right]. \tag{4.42}$$

For the first equality, we have used the fact that $Z^k$ and $\left(Z^k\right)^{(i)}$ have the same law. The second equality follows from the fact that by the Cyclic Lemma, there exists a unique $1 \leqslant i_* \leqslant k$ such that $\left(Z^k\right)^{(i_*)} \in \cup_{n \geqslant 1} \overline{\mathcal{S}}^k(n)$, which entails $\mathbf{V}(Z^k) = \left(Z^k\right)^{(i_*)}$. Then, for $1 \leqslant i \leqslant k$:

$$\mathbb{P}\left[i_*(Z^k) = i, \mathbf{V}(Z^k) = \mathbf{x}\right] = \mathbb{P}\left[(Z^k)^{(i)} = \mathbf{x}\right] = \mathbb{P}\left[Z^k = \mathbf{x}\right] = \frac{1}{k} \mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}\right].$$

The conclusion follows.

For (ii), write $\mathbf{x} = (x_1, \dots, x_k)$ and observe that

$$\mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}, Z^k_k = -1\right] = \mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}, x_{k - i_*(Z^k)} = -1\right].$$

The conclusion follows from (i) since $\mathrm{Card}\{1 \leqslant i \leqslant k; \, x_i = -1\} = n$. □

**Proposition 4.6.5.** *For every integer $n \geqslant 1$, the law of the vector $\boldsymbol{V}(W_1, W_2 - W_1, \dots, W_{T_n} - W_{T_n - 1})$ under $\mathbb{P}[\,\cdot\,|\, W_{T_n} = -1]$ coincides with the law of the vector $(\mathcal{W}_1(\tau), \mathcal{W}_2(\tau) - \mathcal{W}_1(\tau), \dots,$ $\mathcal{W}_{\zeta(\tau)}(\tau) - \mathcal{W}_{\zeta(\tau)-1}(\tau))$ under $\mathbb{P}_\mu[\,\cdot\,|\, \lambda(\tau) = n]$.*

*Proof.* To simplify notation set $Z = (W_1, W_2 - W_1, \dots, W_{T_n} - W_{T_n - 1})$. Fix an integer $k \geqslant n$, and set $Z^k = (W_1, W_2 - W_1, \dots, W_k - W_{k-1})$. Let $\mathbf{x} = (x_1, \dots, x_k) \in \overline{\mathcal{S}}^k(n)$. We have:

$$\mathbb{P}\left[\mathbf{V}(Z) = \mathbf{x} \,|\, W_{T_n} = -1\right] = \mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x},\ T_n = k \,|\, W_{T_n} = -1\right]$$

simply because $Z = Z^k$ on the event $\{T_n = k\}$. Then write:

$$\begin{aligned}
\mathbb{P}\left[\mathbf{V}(Z) = \mathbf{x} \,|\, W_{T_n} = -1\right] &= \frac{\mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}, T_n = k\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} = \frac{\mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}, Z^k_k = -1\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} \\
&= \frac{n}{k} \frac{\mathbb{P}\left[\mathbf{V}(Z^k) = \mathbf{x}\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} = n \frac{\mathbb{P}\left[Z^k = \mathbf{x}\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} = n \frac{\mathbb{P}\left[Z = \mathbf{x}\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} \\
&= \frac{n \mathbb{P}\left[Z \in \overline{\mathcal{S}}(n)\right]}{\mathbb{P}\left[W_{T_n} = -1\right]} \mathbb{P}\left[Z = \mathbf{x} \,|\, Z \in \overline{\mathcal{S}}(n)\right],
\end{aligned}$$

where we have used (4.41) for the third equality and (4.42) for the fourth equality. Summing over all possible $\mathbf{x} \in \overline{\mathcal{S}}^k(n)$ and then over $k \geqslant n$, we get $\mathbb{P}\left[W_{T_n} = -1\right] = n \mathbb{P}\left[Z \in \overline{\mathcal{S}}(n)\right]$. As a consequence, we have $\mathbb{P}\left[\mathbf{V}(Z) = \mathbf{x} \,|\, W_{T_n} = -1\right] = \mathbb{P}\left[Z = \mathbf{x} \,|\, Z \in \overline{\mathcal{S}}(n)\right]$ for every $\mathbf{x} \in \overline{\mathcal{S}}(n)$.

On the other hand, by Proposition 4.1.4, for every $\mathbf{x} \in \overline{\mathcal{S}}(n)$,

$$\mathbb{P}_\mu\left[(\mathcal{W}_1(\tau), \dots, \mathcal{W}_{\zeta(\tau)}(\tau) - \mathcal{W}_{\zeta(\tau)-1}(\tau)) = \mathbf{x} \,|\, \lambda(\tau) = n\right] = \mathbb{P}\left[Z^\zeta = \mathbf{x} \,|\, \Lambda(\zeta) = n\right]$$

where we have used the notation $\zeta$ introduced in (4.40). The probability appearing in the right-hand side is equal to $\mathbb{P}\left[Z = \mathbf{x} \,|\, Z \in \overline{\mathcal{S}}(n)\right]$ because $\{\Lambda(\zeta) = n\} = \{Z \in \overline{\mathcal{S}}(n)\}$, and moreover we have $\zeta = T_n$ and $Z^\zeta = Z$ on this event. We conclude that:

$$\begin{aligned}
\mathbb{P}_\mu\left[(\mathcal{W}_1(\tau), \dots, \mathcal{W}_{\zeta(\tau)}(\tau) - \mathcal{W}_{\zeta(\tau)-1}(\tau)) = \mathbf{x} \,|\, \lambda(\tau) = n\right] &= \mathbb{P}\left[Z = \mathbf{x} \,|\, Z \in \overline{\mathcal{S}}(n)\right] \\
&= \mathbb{P}\left[\mathbf{V}(Z) = \mathbf{x} | W_{T_n} = -1\right].
\end{aligned}$$

This completes the proof. □

**Definition 4.6.6.** Set $\mathbb{D}_0([0,1], \mathbb{R}) = \{\omega \in \mathbb{D}([0,1], \mathbb{R});\ \omega(0) = 0\}$. The Vervaat transformation in continuous time, denoted by $\mathcal{V} : \mathbb{D}_0([0,1], \mathbb{R}) \to \mathbb{D}([0,1], \mathbb{R})$, is defined as follows. For $\omega \in \mathbb{D}_0([0,1], \mathbb{R})$, set $g_1(\omega) = \inf\{t \in [0,1]; \omega(t-) \wedge \omega(t) = \inf_{[0,1]} \omega\}$. Then define:

$$\mathcal{V}(\omega)(t) = \begin{cases} \omega(g_1(\omega) + t) - \inf_{[0,1]} \omega, & \text{if } g_1(\omega) + t \leqslant 1, \\ \omega(g_1(\omega) + t - 1) + \omega(1) - \inf_{[0,1]} \omega & \text{if } g_1(\omega) + t \geqslant 1. \end{cases}$$

**Corollary 4.6.7.** *The law of* $\left(\frac{1}{B_{\zeta(\tau)}} \mathcal{W}_{\lfloor \zeta(\tau) t \rfloor}(\tau);\ 0 \leqslant t \leqslant 1\right)$ *under* $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$ *coincides with the law of* $\mathcal{V}\left(\frac{1}{B_{T_n}} W_{\lfloor T_n t \rfloor};\ 0 \leqslant t \leqslant 1\right)$ *under* $\mathbb{P}[\,\cdot\,|\,W_{T_n} = -1]$.

This immediately follows from Proposition 4.6.5. In the next subsections, we first get a limiting result under $\mathbb{P}[\,\cdot\,|\,W_{T_n} = -1]$ and then apply the Vervaat transformation using the preceding remark. The advantage of dealing with $\mathbb{P}[\,\cdot\,|\,W_{T_n} = -1]$ is to avoid any positivity constraint.

### 4.6.2 Time Reversal

The probability measure $\mathbb{P}[\cdot \mid W_{T_n} = -1]$ enjoys a time-reversal invariance property that will be useful in our applications. Ultimately, as for the height and contour processes, this time-reversal property will allow us to get the convergence of rescaled Lukasiewicz paths over the whole segment $[0, 1]$.

**Proposition 4.6.8.** *Fix two integers $m \geqslant n \geqslant 1$ such that $\mathbb{P}[W_m = 0, \Lambda(m) = n] > 0$. For $0 \leqslant i \leqslant m$, set $\widehat{W}_i^{(m)} = W_m - W_{m-i}$. The law of the vector $(W_0, \dots, W_m)$ under $\mathbb{P}[\cdot \mid W_m = 0, \Lambda(m) = n]$ coincides with the law of the vector $\left(\widehat{W}_0^{(m)}, \dots, \widehat{W}_m^{(m)}\right)$ under the same probability measure.*

*Proof.* This is left as an exercise. □

### 4.6.3 The Lévy Bridge

The Lévy bridge $X^{br}$ can be seen informally as the path $(X_t; 0 \leqslant t \leqslant 1)$ conditioned to be at level zero at time one. See [15, Chapter VIII] for definitions.

**Proposition 4.6.9.** *The following two properties hold.*

*(i) The continuous Vervaat transformation $\mathcal{V}$ is almost surely continuous at $X^{br}$ and $\mathcal{V}(X^{br})$ has the same distribution as $X$ under $\mathbf{N}(\cdot \mid \zeta = 1)$.*

*(ii) Fix $a \in (0, 1)$. Let $F$ be a bounded continuous functional on $\mathbb{D}([0, a], \mathbb{R})$. We have:*

$$\mathbb{E}\left[F\left(X_t^{br}; 0 \leqslant t \leqslant a\right)\right] = \mathbb{E}\left[F\left(X_t; 0 \leqslant t \leqslant a\right) \frac{p_{1-a}(-X_a)}{p_1(0)}\right].$$

*Proof.* The continuity of $\mathcal{V}$ at $X^{br}$ follows from the fact that the absolute minimum of $X^{br}$ is almost surely attained at a unique time. See [25, Theorem 4] for a proof of the fact that $\mathcal{V}(X^{br})$ has the same distribution as $X$ under $\mathbf{N}(\cdot \mid \zeta = 1)$. For (ii), see [15, Formula (8), chapter VIII.3]. □

### 4.6.4 Absolute continuity and convergence of the Lukasiewicz path

By means of a discrete absolute continuity argument similar to the one used in Section 5, we shall show that for every $a \in (0, 1)$ the law of $\left(\frac{1}{B_{T_n}} W_{\lfloor T_n t \rfloor}; 0 \leqslant t \leqslant a\right)$ under $\mathbb{P}[\cdot \mid W_{T_n} = -1]$ converges to the law of $(X_t^{br}, 0 \leqslant t \leqslant a)$.

**Lemma 4.6.10.** *Fix $a \in (0, 1)$ and let $n$ be a positive integer. To simplify notation, set $W^{(u)} = (W_0, W_1, \dots, W_{T_{\lfloor u \rfloor}})$ for $u \geqslant 0$. For every function $f : \cup_{i \geqslant 1} \mathbb{Z}^i \to \mathbb{R}_+$ we have:*

$$\mathbb{E}\left[f(W^{(an)}) \mid W_{T_n} = -1\right] = \mathbb{E}\left[f(W^{(an)}) \frac{\chi_{n - \lfloor an \rfloor}(W_{T_{an}})}{\chi_n(0)}\right],$$

*where $\chi_k(j) = \mathbb{P}_j[W_{T_k} = -1]$ for every $j \in \mathbb{Z}$ and $k \geqslant 1$, and $W$ starts from $j$ under the probability measure $\mathbb{P}_j$.*

*Proof.* This follows from the strong Markov property for the random walk $W$. □

**Lemma 4.6.11.** *For every $\alpha > 0$, we have* $\lim_{n\to\infty} \sup_{|j|\leqslant \alpha B_n} \left| B_{n/\mu_0}\chi_n(j) - p_1\left(-\frac{j}{B_{n/\mu_0}}\right)\right| = 0.$

Note that j is allowed to take negative values. Note also that Lemma 4.6.11 implies that $\chi_n(0) \sim p_1(0)/B_{n/\mu_0}$ as $n \to \infty$.

*Proof.* Fix $\epsilon \in (0,1)$. Using Lemma, 4.6.2 (i), we have:

$$\begin{aligned}
\chi_n(j) = \mathbb{P}_j[W_{T_n} = -1] &= \mathbb{P}[W_{T_n} = -j-1] \\
&= \mathbb{P}\left[W_{T_n} = -j-1, \left|T_n - \frac{n}{\mu_0}\right| \leqslant \epsilon n\right] + oe_{1/2}(n) \\
&= \sum_{|k-n/\mu_0|\leqslant \epsilon n} \mathbb{P}[W_k = -j-1, T_n = k] + oe_{1/2}(n) \\
&= \mu_0 \sum_{|k-n/\mu_0|\leqslant \epsilon n} \mathbb{P}[W_{k-1} = -j, \Lambda(k-1) = n-1] + oe_{1/2}(n)
\end{aligned}$$

Recall that $S_n$ stands for the sum of $n$ iid Bernoulli random variables of parameter $\mu_0$ and that $W'$ is the random walk $W$ conditioned on having nonnegative jumps. By (4.5):

$$\begin{aligned}
&\mathbb{P}_j[W_{T_n} = -1] \\
&= \mu_0 \sum_{|k-n/\mu_0|\leqslant \epsilon n} \mathbb{P}[S_{k-1} = n-1]\mathbb{P}\left[W'_{k-n} = n-j-1\right] + oe_{1/2}(n) \\
&= \mu_0 \int_{n/\mu_0-\epsilon n+O(1)}^{n/\mu_0+\epsilon n+O(1)} dx\, \mathbb{P}\left[S_{\lfloor x-1\rfloor} = n-1\right]\mathbb{P}\left[W'_{\lfloor x\rfloor - n} = n-j-1\right] + oe_{1/2}(n). \\
&= \mu_0 \int_{-\epsilon\sqrt{n}+o(1)}^{\epsilon\sqrt{n}+o(1)} du\, \sqrt{n}\,\mathbb{P}\left[S_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}-1\rfloor} = n-1\right]\mathbb{P}\left[W'_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}\rfloor - n} = n-j-1\right] \\
&\qquad + \quad oe_{1/2}(n).
\end{aligned}$$

For $|u| \leqslant \epsilon\sqrt{n}$, set:

$$\widetilde{a}_n(u) = \sqrt{n}\,\mathbb{P}\left[S_{\lfloor \frac{n}{\mu_0}+u\sqrt{n}-1\rfloor} = n-1\right].$$

Using the notation of Section 4.5.5, we have then:

$$B'_n\mathbb{P}_j[W_{T_n} = -1] = \mu_0 \int_{-\epsilon\sqrt{n}+o(1)}^{\epsilon\sqrt{n}+o(1)} du\, \widetilde{a}_n(u) b_n(u, j+1) + oe_{1/2}(n).$$

The same argument that led us to (4.34) gives that:

$$\sup_{|j|\leqslant \alpha B_n} \left| B'_n\mathbb{P}_j[W_{T_n} = -1] - c^{1/\theta}p_1\left(-\frac{j}{B_{n/\mu_0}}\right)\right| \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

The conclusion follows from that fact that $B'_n/B_{n/\mu_0} \to c^{1/\theta}$. □

All the necessary ingredients have been gathered and we can now turn to the proof of the theorem.

*Proof of Theorem 4.6.1.* Let $F : \mathbb{D}([0,a],\mathbb{R}) \to \mathbb{R}_+$ be a bounded continuous function. Fix $a \in (0,1)$ and $\alpha > 0$. To simplify notation, we set $A_n^\alpha = \left\{|W_{T_{an}}| \leqslant \alpha B_{n/\mu_0}\right\}$ as well as $G^{(n)}(W) = F\left(\frac{1}{B_{n/\mu_0}} W_{\lfloor \frac{t}{a} T_{an} \rfloor}; 0 \leqslant t \leqslant a\right)$. We apply Lemma 4.6.10 with $f(W_0, W_1, \ldots, W_{T_{an}}) = G^{(n)}(W) 1_{A_n^\alpha}$ and get:

$$\mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \,|\, W_{T_n} = -1\right] = \mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \frac{\chi_{n-\lfloor an \rfloor}(W_{T_{an}})}{\chi_n(0)}\right].$$

Lemma 4.6.11 then entails:

$$\lim_{n\to\infty} \left| \mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \,|\, W_{T_n} = -1\right] - \mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \frac{(1-a)^{-1/\theta}}{p_1(0)} p_1\left(-\frac{W_{T_{an}}}{B_{(n-\lfloor an \rfloor)/\mu_0}}\right)\right]\right| = 0$$

From Lemma 4.6.2 (ii), we deduce that:

$$\lim_{n\to\infty} \mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \,|\, W_{T_n} = -1\right] = \mathbb{E}\left[F((X_t)_{0\leqslant t \leqslant a}) 1_{\{|X_a| < \alpha\}} \frac{(1-a)^{-1/\theta}}{p_1(0)} p_1\left(-\frac{X_a}{(1-a)^{1/\theta}}\right)\right].$$

By (4.16), we have $(1-a)^{-1/\theta} p_1\left(-\frac{X_a}{(1-a)^{1/\theta}}\right) = p_{1-a}(-X_a)$. Consequently, by Proposition 4.6.9 (ii), we conclude that:

$$\lim_{n\to\infty} \mathbb{E}\left[G^{(n)}(W) 1_{A_n^\alpha} \,|\, W_{T_n} = -1\right] = \mathbb{E}\left[F(X_t^{\text{br}};\ 0 \leqslant t \leqslant a) 1_{\{|X_a^{\text{br}}| \leqslant \alpha\}}\right]. \tag{4.43}$$

By taking $F \equiv 1$, we obtain:

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} \mathbb{P}\left[A_n^\alpha \,|\, W_{T_n} = -1\right] = 1. \tag{4.44}$$

By choosing $\alpha > 0$ sufficiently large, we easily deduce from the convergence (4.43) that:

$$\lim_{n\to\infty} \left| \mathbb{E}\left[F\left(\frac{1}{B_{n/\mu_0}} W_{\lfloor \frac{t}{a} T_{an} \rfloor}; 0 \leqslant t \leqslant a\right) | W_{T_n} = -1\right] - \mathbb{E}\left[F(X_t^{\text{br}};\ 0 \leqslant t \leqslant a)\right]\right| \quad = \quad 0.$$

Next write:

$$\frac{1}{B_{T_n}} W_{\lfloor T_n t \rfloor} = K_n \cdot \frac{1}{B_{n/\mu_0}} W_{\lfloor S_n \cdot \frac{t}{a} T_{an} \rfloor},$$

where $K_n = B_{n/\mu_0}/B_{T_n}$ and $S_n = aT_n/T_{an}$. Lemma 4.6.2 (i) entails that $K_n$ and $S_n$ both converge in probability towards 1. Lemma 4.5.7 then implies that the law of $\left(\frac{1}{B_{T_n}} W_{\lfloor T_n t \rfloor}; 0 \leqslant t \leqslant a\right)$ under $\mathbb{P}[\,\cdot\,|\, W_{T_n} = -1]$ converges to the law of $(X_t^{\text{br}}, 0 \leqslant t \leqslant a)$, and this holds for every $a \in (0,1)$.

We now show that the latter convergence holds also for $a = 1$ by using a time-reversal argument based on Proposition 4.6.8. By the usual tightness criterion (see e.g. [20, Formula (13.8)]), it is sufficient to show that, for every $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left[\sup_{s\in[1-\delta,1)} \left|\frac{1}{B_{T_n}} W_{\lfloor T_n s \rfloor}\right| > \eta \,\middle|\, W_{T_n} = -1\right] = 0. \tag{4.45}$$

Note that:

$$\sup_{s\in[1-\delta,1)}\left|\frac{1}{B_{T_n}}W_{\lfloor T_n s\rfloor}\right| = \sup_{\lfloor(1-\delta)T_n\rfloor\leqslant k\leqslant T_n-1}\left|\frac{1}{B_{T_n}}W_k\right|.$$

Using this remark, we write:

$$\begin{aligned}
&\mathbb{P}\left[\sup_{s\in[1-\delta,1)}\left|\frac{1}{B_{T_n}}W_{\lfloor T_n s\rfloor}\right|>\eta\;\middle|\;W_{T_n}=-1\right]\\
&\quad=\frac{1}{\mathbb{P}\left[W_{T_n}=-1\right]}\sum_{k=n}^{\infty}\mathbb{P}\left[T_n=k, W_k=-1,\;\sup_{s\in[1-\delta,1)}\left|\frac{1}{B_k}W_{\lfloor ks\rfloor}\right|>\eta\right]\\
&\quad=\frac{\mu_0}{\mathbb{P}\left[W_{T_n}=-1\right]}\sum_{k=n}^{\infty}\mathbb{P}\left[\Lambda(k-1)=n-1,\;W_{k-1}=0,\;\sup_{s\in[1-\delta,1)}\left|\frac{1}{B_k}W_{\lfloor ks\rfloor}\right|>\eta\right]\\
&\quad\leqslant\frac{\mu_0}{\mathbb{P}\left[W_{T_n}=-1\right]}\sum_{k=n}^{\infty}\mathbb{P}\left[\Lambda(k-1)=n-1,\;W_{k-1}=0,\;\sup_{s\in[0,\delta+1/k]}\left|-\frac{1}{B_k}W_{\lfloor ks\rfloor}\right|>\eta\right]\\
&\quad=\mathbb{P}\left[\sup_{s\in[0,\delta+1/T_n]}\left|\frac{1}{B_{T_n}}W_{\lfloor T_n s\rfloor}\right|>\eta\;\middle|\;W_{T_n}=-1\right],
\end{aligned}$$

using Proposition 4.6.8 in the upper bound of the last display. (4.45) then follows from the fact that that the law of $\left(\frac{1}{B_{T_n}}W_{\lfloor T_n t\rfloor}; 0\leqslant t\leqslant a\right)$ under $\mathbb{P}[\,\cdot\,|\,W_{T_n}=-1]$ converges to the law of $(X^{br}_t, 0\leqslant t\leqslant a)$ for every $a\in(0,1)$. We conclude that this convergence also holds for $a=1$.

We then combine the continuous Vervaat transformation $\mathcal{V}$ with the latter convergence. Since $\mathcal{V}$ is almost surely continuous at $X^{br}$ (Proposition 4.6.9 (i)), we get that the law of

$$\mathcal{V}\left(\frac{1}{B_{T_n}}W_{\lfloor T_n t\rfloor};\;0\leqslant t\leqslant 1\right)$$

under $\mathbb{P}[\,\cdot\,|\,W_{T_n}=-1]$ converges to the law of $\mathcal{V}(X^{br})$. Corollary 4.6.7 and Proposition 4.6.9 (i) entail:

$$\left(\frac{1}{B_{\zeta(t_n)}}\mathcal{W}_{\lfloor\zeta(t_n)t\rfloor}(t_n);\;0\leqslant t\leqslant 1\right)\quad\xrightarrow[n\to\infty]{(d)}\quad(X_t; 0\leqslant t\leqslant 1)\text{ under }\mathbf{N}(\,\cdot\,|\,\zeta=1).$$

This completes the proof. □

## 4.7 Application: maximum degree in a Galton-Watson tree conditioned on having many leaves

In this section, we study the asymptotic behavior of the distribution of the maximum degree in a Galton-Watson tree conditioned on having $n$ leaves. To this end, we use tools introduced in Section 6 such as the Vervaat transformation and absolute continuity arguments.

As earlier, we fix $\theta\in(1,2]$ and suppose that $\mu$ is a probability distribution satisfying the hypothesis $(H_\theta)$. For every $n\geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau)=n]>0$, let also $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. If $\tau \in \mathbb{T}$ is a tree, let $\Delta(\tau) = \max\{k_u;\ u \in \tau\}$ be the maximum number of children of individuals of $\tau$. We are interested in the asymptotic behavior of $\Delta(t_n)$.

The case $1 < \theta < 2$ easily follows from previous results. Indeed, let $(B_n)_{n\geqslant 1}$ be defined as before Lemma 4.1.11. Then, by Theorem 4.6.1 and Remark 4.5.10:

$$\left(\frac{\mu_0^{1/\theta}}{B_n}\mathcal{W}_{\lfloor \zeta(t_n)t\rfloor}(t_n);\ 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad X^{\text{exc}}. \tag{4.46}$$

If $Z \in D([0,1],\mathbb{R})$, let $\overline{\Delta}(Z)$ be the largest jump of $Z$. Note that by construction, $\overline{\Delta}(\mathcal{W}(t_n))) = \Delta(t_n) - 1$. Since $\overline{\Delta}$ is a continuous functional on $D([0,1],\mathbb{R})$, (4.46) immediately gives that $\mu_0^{1/\theta}\overline{\Delta}(X)/B_n$ converges in distribution towards $\overline{\Delta}(X^{\text{exc}})$, which is almost surely positive.

However, in the case $\sigma^2 < \infty$, $\overline{\Delta}(X^{\text{exc}}) = 0$ almost surely since $X^{\text{exc}}$ is continuous. It is natural to ask whether the suitably rescaled sequence $\Delta(t_n)$ converges to a non-degenerate limit. A similar question has been previously studied by Meir & Moon [84] when $t_n$ is distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\zeta(\tau) = n]$. We shall make the same assumptions on $\mu$ as Meir & Moon.

More precisely, let $\nu$ be a critical aperiodic probability distribution on $\mathbb{N}$ with finite variance. Let $R$ be the radius of convergence of $\sum \nu(i)z^i$. We say that $\nu$ satisfies hypothesis $\mathcal{H}$ if the following two conditions hold: $R > 1$ and if $R < \infty$, $\nu(n)^{1/n}$ converges towards $1/R$ as $n \to \infty$, if $R = \infty$ there exists $N \geqslant 0$ such that the sequence $(\nu(k)^{1/k})_{k\geqslant N}$ is decreasing.

**Theorem 4.7.1.**

*(i) If $1 < \theta < 2$, we have* $\mu_0^{1/\theta}\Delta(t_n)/B_n \quad \xrightarrow[n\to\infty]{(d)} \quad \overline{\Delta}(X^{\text{exc}})$.

*(ii) Set* $D(n) = \max\{k \geqslant 1;\ \mu([k,\infty)) \geqslant 1/n\}$. *If $\sigma^2 < \infty$, under the additional assumption that $\mu$ satisfies hypothesis $\mathcal{H}$, we have for every $\epsilon > 0$:*

$$\mathbb{P}[(1-\epsilon)D(n) \leqslant \Delta(t_n) \leqslant (1+\epsilon)D(n)] \quad \xrightarrow[n\to\infty]{} \quad 1.$$

Part (i) of the theorem follows from the preceding discussion. It remains to prove (ii). We suppose that $\mu$ satisfies the assumptions in (ii). The first step is to control the asymptotic behavior of $D(n)$.

**Lemma 4.7.2** (Meir & Moon)**.** *Let $\epsilon > 0$. For $n$ sufficiently large:*

$$\mu([(1-\epsilon)D(n),\infty)) \geqslant n^{-\frac{1}{1+\epsilon/3}}, \qquad \mu([(1+\epsilon)D(n),\infty)) \leqslant n^{-1-\epsilon/3}.$$

*Proof.* See the proof of Theorem 1 in [84], which uses the different assumptions made on $\mu$. $\square$

*Proof of Theorem 4.7.1 in the case $\sigma^2 < \infty$.* The idea of the proof consists in showing that if the Lukasiewicz path of a non-conditioned Galton-Watson tree satisfies asymptotically some property which is invariant under cyclic-shift (with some additional monotonicity condition), then the Lukasiewicz path of a conditioned Galton-Watson tree satisfies asymptotically the same property.

We first establish the lower bound. Recall the notation introduced in (4.40). For every $\mathbf{u} = (u_1,\dots,u_k) \in \mathbb{Z}^k$, set $\mathcal{M}(\mathbf{u}) = \max_{1\leqslant i\leqslant k} u_i$, so that $\Delta(t_n) = \mathcal{M}(\mathcal{W}_1(t_n) - \mathcal{W}_0(t_n),\dots,\mathcal{W}_{\zeta(t_n)}(t_n) -$

$\mathcal{W}_{\zeta(t_n)-1}(t_n)) + 1$. Note that $\mathcal{M}$ is invariant under cyclic shift. Set $p_n = (1-\epsilon)D(n)$. To simplify notation, for $u_1, \ldots, u_k \in \mathbb{Z}$ set $F^{(n)}(u_1, \ldots, u_k) = 1_{\{\mathcal{M}(u_1,\ldots,u_k)<p_n\}}$. We have:

$$\begin{aligned} \mathbb{P}\left[\Delta(t_n) < p_n + 1\right] &= \mathbb{E}_\mu\left[F^{(n)}(\mathcal{W}_1(\tau) - \mathcal{W}_0(\tau), \ldots, \ldots, \mathcal{W}_{\zeta(\tau)}(\tau) - \mathcal{W}_{\zeta(\tau)-1}(\tau))) \,|\, \lambda(\tau) = n\right] \\ &= \mathbb{E}\left[F^{(n)}(\mathbf{V}(W_1, W_2 - W_1, \ldots, W_{T_n} - W_{T_n-1})) \,|\, W_{T_n} = -1\right] \\ &= \mathbb{E}\left[F^{(n)}(W_1, W_2 - W_1, \ldots, W_{T_n} - W_{T_n-1}) \,|\, W_{T_n} = -1\right], \end{aligned} \tag{4.47}$$

where we have used Proposition 4.6.5 in the first equality, and the fact that $F^{(n)}(\mathbf{V}(\mathbf{u})) = F^{(n)}(\mathbf{u})$ for every $\mathbf{u} \in \mathbb{Z}^k$ ($k \geqslant 1$) in the second one. To simplify notation, we put $F_k^{(n)}(W) = F_n(W_1, W_2 - W_1, \ldots, W_k - W_{k-1})$. Note that $\mathbb{E}\left[F_{T_n}^{(n)}(W) \,|\, W_{T_n} = -1\right] \leqslant \mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) \,|\, W_{T_n} = -1\right]$. In order to establish the lower bound in Theorem 4.7.1 (ii), it then suffices to prove that the quantity $\mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) \,|\, W_{T_n} = -1\right]$ tends to $0$ as $n \to \infty$. Let $\alpha > 0$, and let the event $A_n^\alpha$ be defined by

$$A_n^\alpha = \left\{ |W_{T_{n/2}}| < \alpha\sigma\sqrt{n/(2\mu_0)} \right\},$$

where $\sigma^2$ is the variance of $\mu$. By Lemma 4.6.2 (i), we have:

$$\begin{aligned} &\mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) \,|\, W_{T_n} = -1\right] \\ &\qquad \leqslant \mathbb{E}\left[1_{\{A_n^\alpha\}^c} \,|\, W_{T_n} = -1\right] + \mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) 1_{\{A_n^\alpha, \frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}} \,|\, W_{T_n} = -1\right] + oe_{1/2}(n) \end{aligned}$$

By Lemma 4.6.10:

$$\mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) 1_{\{A_n^\alpha, \frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}} \,|\, W_{T_n} = -1\right] = \mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) 1_{\{A_n^\alpha, \frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}} \frac{\chi_{n-\lfloor n/2 \rfloor}(W_{T_{n/2}})}{\chi_n(0)}\right],$$

where $\chi_n(j) = \mathbb{P}_j[W_{T_n} = -1]$. By Lemma 4.6.11, there exists $C > 0$ such that for every $n$ large enough, $\chi_{n-\lfloor n/2 \rfloor}(W_{T_{n/2}})/\chi_n(0) \leqslant C$ on the event $A_n^\alpha$. By combining the previous observations, we get:

$$\begin{aligned} &\mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) \,|\, W_{T_n} = -1\right] \\ &\qquad \leqslant \mathbb{E}\left[1_{\{A_n^\alpha\}^c} \,|\, W_{T_n} = -1\right] + C\mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) 1_{\{\frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}}\right] + oe_{1/2}(n). \end{aligned} \tag{4.48}$$

By (4.44), we have:

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} \mathbb{E}\left[1_{\{A_n^\alpha\}^c} \,|\, W_{T_n} = -1\right] = 0.$$

Let us finally show that the second term in the right-hand side of (4.48) tends to zero as well. We have:

$$\begin{aligned} \mathbb{E}\left[F_{T_{n/2}}^{(n)}(W) 1_{\{\frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}}\right] &\leqslant \mathbb{P}\left[\mathcal{M}(W_1, W_2 - W_1, \ldots, W_{\lfloor n/4\mu_0 \rfloor} - W_{\lfloor n/4\mu_0 \rfloor - 1}) < p_n\right] \\ &= \mathbb{P}[W_1 < p_n]^{\lfloor n/4\mu_0 \rfloor} = (1 - P[W_1 \geqslant p_n])^{\lfloor n/4\mu_0 \rfloor}. \end{aligned}$$

The first part of Lemma 4.7.2, implies that the last quantity tends to $0$ as $n \to \infty$. By combining the previous estimates, we conclude that $\mathbb{P}[(1-\epsilon)D(n) \geqslant \Delta(t_n)] \to 0$ as $n \to \infty$.

Let us now establish the upper bound. Set $q_n = (1+\epsilon)D(n)$. By an argument similar to the one we used to establish (4.47), we get $\mathbb{P}[\Delta(t_n) > q_n + 1] = \mathbb{P}[\mathcal{M}(W_1, \dots, W_{T_n} - W_{T_n - 1}) > q_n \,|\, W_{T_n} = -1]$. It follows that:

$$\begin{aligned}\mathbb{P}[\Delta(t_n) > q_n + 1] \;\leqslant\;& \mathbb{P}[\mathcal{M}(W_1, W_2 - W_1, \dots, W_{T_{n/2}} - W_{T_{n/2}-1}) > q_n \,|\, W_{T_n} = -1] \\ &+ \mathbb{P}[\mathcal{M}(W_{T_{n/2}} - W_{T_{n/2}-1}, \dots, W_{T_n} - W_{T_n - 1}) > q_n \,|\, W_{T_n} = -1]\end{aligned}$$

By a time-reversal argument based on Proposition 4.6.8, it is sufficient to show that the first term of the last expression tends to 0. To this end, we use the same approach as for the proof of the lower bound, taking this time $F^{(n)}_k(W) = 1_{\{\mathcal{M}(W_1, \dots, W_k - W_{k-1}) > q_n\}}$. It is then sufficient to verify that:

$$\mathbb{E}\left[F^{(n)}_{T_{n/2}}(W) 1_{\{\frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}}\right] \quad \xrightarrow[n\to\infty]{} \quad 0.$$

To this end, write:

$$\begin{aligned}\mathbb{E}\left[F^{(n)}_{T_{n/2}}(W) 1_{\{\frac{n}{4\mu_0} \leqslant T_{n/2} \leqslant \frac{n}{\mu_0}\}}\right] \;&\leqslant\; \mathbb{P}[\mathcal{M}(W_1, W_2 - W_1, \dots, W_{\lfloor n/\mu_0 \rfloor} - W_{\lfloor n/\mu_0 \rfloor - 1})) > q_n] \\ &= \quad 1 - (1 - \mathbb{P}[W_1 > q_n])^{\lfloor n/\mu_0 \rfloor}\end{aligned}$$

which tends to 0 as $n \to \infty$ by Lemma 4.7.2. By combining the previous estimates, we conclude that $\mathbb{P}[(1+\epsilon)D(n) \leqslant \Delta(t_n)] \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

**Remark 4.7.3.** In particular cases, it is possible to obtain better bounds in the previous theorem. Let $\mu$ be defined by $\mu(0) = 2 - \sqrt{2}$, $\mu(1) = 0$ and $\mu(i) = \left((2-\sqrt{2})/2\right)^{i-1}$ for $i \geqslant 2$ (this probability distribution appears when we consider the tree associated with a uniform dissection of the $n$-gon, see [28]). One verifies that $\mu$ is a critical probability measure. For $n \geqslant 1$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. One easily checks that $\mu$ is the unique critical probability measure such that $t_n$ is distributed uniformly over the set of all rooted plane trees with $n$ leaves such that no vertex has exactly one child. In this particular case, Theorem 4.7.1 (ii) can be strengthened as follows:

$$\mathbb{P}[\log_b n - c\log_b \log_b n \leqslant \Delta(t_n) \leqslant \log_b n + c \log_b \log_b n] \quad \xrightarrow[n\to\infty]{} \quad 1, \qquad (4.49)$$

for every $c > 0$, where $b = 1/\mu(2) = \sqrt{2} + 2$. Indeed, the proof of Theorem 4.7.1 shows that it is sufficient to verify that for every $c > 0$:

$$(1 - P[W_1 \geqslant \log_b n - c\log_b\log_b n])^{n/4\mu_0} \xrightarrow[n\to\infty]{} 0, \; (1 - P[W_1 \geqslant \log_b n + c\log_b\log_b n])^{n/4\mu_0} \xrightarrow[n\to\infty]{} 1.$$

These asymptotics are easily obtained since the probabilities appearing in these two expressions can be calculated explicitly.

The convergence (4.49) yields an interesting application to the maximum face degree in a uniform dissection (see [28, Prop. 3.5]).

## 4.8 Extensions

Recall that if $\mathcal{A}$ is a non-empty subset of $\mathbb{N}$ and $\tau$ a tree, $\zeta_{\mathcal{A}}(\tau)$ is the total number of vertices $u \in \tau$ such that $k_u(\tau) \in \mathcal{A}$. For a forest $\mathbf{f}$, $\zeta_{\mathcal{A}}(\mathbf{f})$ is defined in a similar way. In this section, we extend the results (I) and (II) appearing in the Introduction to the case where $\mathcal{A} \neq \{0\}$. By slightly adapting the previous techniques, it is possible to obtain the following more general result.

Recall that $\mu$ is a probability distribution on $\mathbb{N}$ satisfying the hypothesis $(H_\theta)$ for some $\theta \in (1,2]$. We also consider the slowly varying function $h$ and the sequence $(B_n)_{n\geqslant 1}$ introduced just before Lemma 4.1.11.

**Theorem 4.8.1.** *Let $\mathcal{A}$ be a non-empty subset of $\mathbb{N}$. If $\mu$ has infinite variance, suppose in addition that either $\mathcal{A}$ is finite, or $\mathbb{N}\backslash\mathcal{A}$ is finite.*

*(I) Let $d \geqslant 1$ be the largest integer such that there exists $b \in \mathbb{N}$ such that supp$(\mu)\backslash\mathcal{A}$ is contained in $b + d\mathbb{Z}$, where supp$(\mu)$ is the support of $\mu$. Then :*

$$\mathbb{P}_\mu\left[\zeta_{\mathcal{A}}(\tau) = n\right] \underset{n\to\infty}{\sim} \mu(\mathcal{A})^{1/\theta} p_1(0) \frac{\gcd(b-1,d)}{h(n) n^{1/\theta+1}}$$

*for those values of $n$ such that $\mathbb{P}_\mu\left[\zeta_{\mathcal{A}}(\tau) = n\right] > 0$.*

*(II) For every $n \geqslant 1$ such that $\mathbb{P}_\mu\left[\zeta_{\mathcal{A}}(\tau) = n\right] > 0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot \,|\, \zeta_{\mathcal{A}}(\tau) = n]$. Then*

$$\left(\frac{1}{B_{\zeta(t_n)}} \mathcal{W}_{\lfloor \zeta(t_n) t \rfloor}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} C_{2\zeta(t_n)t}(t_n), \frac{B_{\zeta(t_n)}}{\zeta(t_n)} H_{\zeta(t_n)t}(t_n)\right)_{0\leqslant t\leqslant 1}$$

*converges in distribution to $(X^{\text{exc}}, H^{\text{exc}}, H^{\text{exc}})$ as $n \to \infty$.*

Theorem 4.8.1 can be established by the same arguments used to prove Theorems 4.3.1, 4.5.9 and 4.6.1. The main difference comes from the proof of the needed extension of Lemma 4.5.4 (ii), which is more technical. Let us explain the argument leading to the convergence

$$\lim_{n\to\infty} \sup_{1\leqslant j\leqslant \alpha B_n} \left| n\mathbb{P}_{\mu,j}\left[\zeta_{\mathcal{A}}(\mathbf{f}) = n\right] - q_1\left(\frac{j}{B_{n/\mu(\mathcal{A})}}\right)\right| = 0. \tag{4.50}$$

The first step is to generalize Proposition 4.1.6 and find the joint law of $(\zeta(\mathbf{f}), \zeta_{\mathcal{A}}(\mathbf{f}))$ under $\mathbb{P}_{\mu,j}$ (which is the contents of the latter proposition in the case $\mathcal{A} = \{0\}$). To this end, let $\rho$ and $\mu'$ be the two probability measures on $\mathbb{N} \cup \{-1\}$ defined by:

$$\rho(i) = \begin{cases} \frac{\mu(i+1)}{\mu(\mathcal{A})} & \text{if } i+1 \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}, \qquad \mu'(i) = \begin{cases} \frac{\mu(i+1)}{1-\mu(\mathcal{A})} & \text{if } i+1 \notin \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

It is then straightforward to adapt Proposition 4.1.6 and get that:

$$\mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = p, \zeta_{\mathcal{A}}(\mathbf{f}) = n] = \frac{j}{p}\, \mathbb{P}[S_p = n]\, \mathbb{P}[W'_{p-n} = -U_n - j].$$

where $S_p$ is the sum of $p$ independent Bernoulli random variables of parameter $\mu(\mathcal{A})$, $(W'_n)_{n\geqslant 1}$ is the random walk started from $0$ with jump distribution $\mu'$ and $(U_n)_{n\geqslant 1}$ is an independent random walk started from $0$ with jump distribution $\rho$. Note that $-U_n = n$ when $\mathcal{A} = \{0\}$.

First suppose that $\mu$ has finite variance. Then both $W'_1$ and $U_1$ have finite variance. As in the proof of Lemma 4.5.4, we have, for $0 < \epsilon < 1$:

$$n\mathbb{P}_{\mu,j}\left[\zeta_{\mathcal{A}}(\mathbf{f}) = n\right] = n\int_{n/\mu(\mathcal{A})-\epsilon n+O(1)}^{n/\mu(\mathcal{A})+\epsilon n+O(1)} dx\frac{j}{\lfloor x\rfloor}\mathbb{P}\left[S_{\lfloor x\rfloor} = n\right]\mathbb{P}\left[W'_{\lfloor x\rfloor-n} = -U_n - j\right] + oe_{1/2}(n). \tag{4.51}$$

By the law of large numbers, we can suppose that $\mathbb{P}\left[|U_n - n\mathbb{E}\left[U_1\right]| > \epsilon n\right] < \epsilon$ for $n$ sufficiently large. Set $t_n(\nu) = \left\lfloor n\mathbb{E}\left[U_1\right] + \nu\sqrt{n}\right\rfloor$ for $n \geqslant 1$ and $\nu \in \mathbb{R}$. It follows that:

$$\left|\mathbb{P}\left[W'_{\lfloor x\rfloor-n} = -U_n - j\right] - \int_{-\epsilon\sqrt{n}+o(1)}^{\epsilon\sqrt{n}+o(1)} d\nu\sqrt{n}\,\mathbb{P}\left[W'_{\lfloor x\rfloor-n} = -t_n(\nu) - j\right]\mathbb{P}\left[U_n = t_n(\nu)\right]\right| \leqslant \epsilon.$$

The local limit theorems give bounds and estimates for the quantities $\mathbb{P}\left[W'_{\lfloor x\rfloor-n} = -t_n(\nu) - j\right]$ and $\mathbb{P}\left[U_n = t_n(\nu)\right]$. As previously, we can then use the dominated convergence theorem to obtain an estimate of $\mathbb{P}\left[W'_{\lfloor x\rfloor-n} = -U_n - j\right]$ as $n \to \infty$. We substitute this estimate in (4.51) and using once again the dominated convergence theorem we obtain (4.50).

Now suppose that $\mu$ has infinite variance and that $\mathcal{A}$ is finite. Then $W'_1$ is in the domain of attraction of a stable law of index $\theta$ and $U_1$ has bounded support hence finite variance. The proof of (4.50) then goes along the same lines as in the finite variance case.

When $\mu$ has infinite variance and $\mathbb{N}\backslash\mathcal{A}$ is finite, $W'_1$ has finite variance and $U_1$ is in the domain of attraction of a stable law of index $\theta$. The proof of (4.50) goes along the same lines as when $\mu$ has finite variance by interchanging the roles of $W'$ and of $U$ (see [68] for details in the case $\mathcal{A} = \mathbb{N}$).

Chapitre 5

# Limit theorems for conditioned nongeneric Galton-Watson trees

*Les résultats de ce chapitre sont issus de l'article [71], soumis pour publication.*

## Contenu de ce chapitre



We study a particular type of subcritical Galton-Watson trees, which are called non-generic trees in the physics community. In contrast with the critical or supercritical case, it is known that condensation appears in certain large conditioned non-generic trees, meaning that with high probability there exists a unique vertex with macroscopic degree comparable to the total size of the tree. We investigate this phenomenon by studying scaling limits of such trees. Using recent results concerning subexponential distributions, we study the convergence of three functions coding these trees (the Lukasiewicz path, the contour function and the height function) and show that the situation is completely different from the critical case. In particular, the height of such trees grows logarithmically in their size. We also study fluctuations around the condensation vertex.

# Introduction

The behavior of large Galton-Watson trees whose offspring distribution $\mu = (\mu_i)_{i \geqslant 0}$ is *critical* and has *finite variance* has drawn a lot of attention. If $t_n$ is a Galton-Watson tree with offspring distribution $\mu$ (in short a $GW_\mu$ tree) conditioned on having total size $n$, Kesten [65] proved that $t_n$ converges locally in distribution as $n \to \infty$ to the so-called critical Galton-Watson tree conditioned to survive. Aldous [5] studied the scaled asymptotic behavior of $t_n$ by showing that the appropriately rescaled contour function of $t_n$ converges to the Brownian excursion.

These results have been extended in different directions. The "finite second moment" condition on $\mu$ has been relaxed by Duquesne [36], who showed that when $\mu$ belongs to the domain of attraction of a stable law of index $\theta \in (1, 2]$, the appropriately rescaled contour function of $t_n$ converges towards the normalized excursion of the $\theta$-stable height process, which codes the so-called $\theta$-stable tree (see also [68]). In a different direction, several authors have considered trees conditioned by other quantities than the total size, for example by the height [66, 72] or the number of leaves [94, 70].

Kennedy [63] noticed that, under certain conditions, the study of non-critical offspring distributions reduces to the study of critical ones. More precisely, if $\lambda > 0$ is a fixed parameter such that $Z_\lambda = \sum_{i \geqslant 0} \mu_i \lambda^i < \infty$, set $\mu_i^{(\lambda)} = \mu_i \lambda^i / Z_\lambda$ for $i \geqslant 0$. Then a $GW_\mu$ tree conditioned on having total size $n$ has the same distribution as a $GW_{\mu^{(\lambda)}}$ tree conditioned on having total size $n$. Thus, if one can find $\lambda > 0$ such that both $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical, then studying a conditioned non-critical Galton-Watson tree boils down to studying a critical one. This explains why the critical case has been extensively studied in the literature.

Let $\mu$ be a probability distribution such that $\mu_0 > 0$ and $\mu_k > 0$ for some $k \geqslant 2$. We are interested in the case where there exist no $\lambda > 0$ such that both $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical (see [59, Section 8] for a characterization of such probability distributions). An important example is when $\mu$ is subcritical (i.e. of mean strictly less than 1) and $\mu_i \sim c/i^\beta$ as $i \to \infty$ for a fixed parameter $\beta > 2$. The study of such $GW_\mu$ trees conditioned on having a large fixed size was initiated only recently by Jonsson & Stefánsson [62] who called such trees *non-generic* trees. They studied the above-mentioned case where $\mu_i \sim c/i^\beta$ as $i \to \infty$, with $\beta > 2$, and showed that if $t_n$ is a $GW_\mu$ tree conditioned on having total size $n$, then with probability tending to 1 as $n \to \infty$, there exists a unique vertex of $t_n$ with maximal degree, which is asymptotic to $(1 - m)n$ where $m < 1$ is the mean of $\mu$. This phenomenon is called *condensation* and appears in a variety of statistical mechanics models such as the Bose-Einstein condensation for bosons, the zero-range process [61, 52] or the Backgammon model [17] (see Fig. 5.1).

Jonsson and Stefánsson [62] have also constructed an infinite random tree $\widehat{\mathcal{T}}$ (with a unique vertex of infinite degree) such that $t_n$ converges locally in distribution towards $\widehat{\mathcal{T}}$ (meaning roughly that the degree of every vertex of $t_n$ converges towards the degree of the corresponding vertex of $\widehat{\mathcal{T}}$). See Proposition 5.2.4 below for the description of $\widehat{\mathcal{T}}$. In [59], Janson has extended this result to simply generated trees and has in particular given a very precise description of local properties of Galton-Watson trees conditioned on their size.

In this work, we are interested in the existence of scaling limits for the random trees $t_n$. When scaling limits exist, one often gets information concerning the global structure of the tree, which does not seem to follow from the existence of local limits.

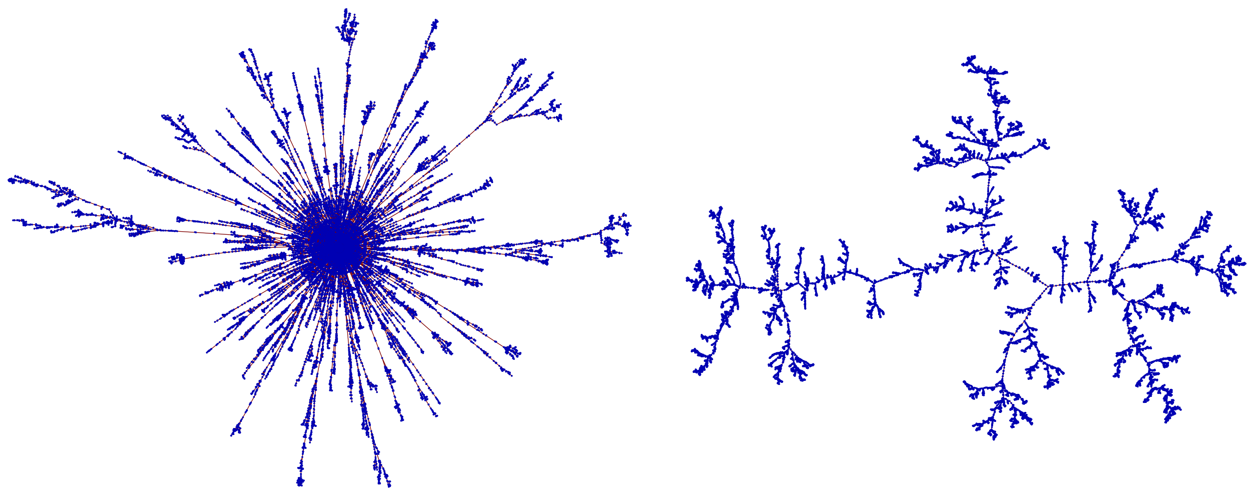

**Figure 5.1:** The first figure shows a non-generic Galton-Watson tree with 24045 vertices. The second figure shows a critical Galton-Watson tree with finite variance and with 55803 vertices.

**Notation and assumptions.** Throughout this work $\theta > 1$ will be a fixed parameter. We say that a probability distribution $(\mu_j)_{j \geqslant 0}$ on the nonnegative integers satisfies Assumption $(H_\theta)$ if the following two conditions hold:

(i) $\mu$ is subcritical, meaning that $0 < \sum_{j=0}^{\infty} j\mu_j < 1$.

(ii) There exists a function $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathcal{L}(x) > 0$ for $x$ large enough as well as $\lim_{x\to\infty} \mathcal{L}(tx)/\mathcal{L}(x) = 1$ for all $t > 0$ (such a function is called slowly varying) and $\mu_n = \mathcal{L}(n)/n^{1+\theta}$ for every $n \geqslant 1$.

Let $\zeta(\tau)$ be the total progeny or size of a tree $\tau$. Condition (ii) implies that $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ for sufficiently large $n$. Note that (ii) is more general than the analogous assumption in [62, 59], where only the case $\mathcal{L}(x) \to c$ as $x \to \infty$ was studied in detail. Throughout this text, $\theta > 1$ is a fixed parameter and $\mu$ is a probability distribution on $\mathbb{N}$ satisfying the Assumption $(H_\theta)$. In addition, for every $n \geqslant 1$ such that $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$, $t_n$ is a $GW_\mu$ tree conditioned on having $n$ vertices (note that $t_n$ is well defined for $n$ sufficiently large). The mean of $\mu$ will be denoted by $m$ and we set $\gamma = 1 - m$.

We use the standard formalism for (discrete) plane trees, see e.g. [73, Section 1]. Before stating our results, let us introduce three different coding functions which determine a finite tree $\tau$ (see Definition 5.1.3 for details). Recall that $\zeta(\tau)$ is the total progeny of $\tau$, and let $u(0), u(1), \ldots, u(\zeta(\tau) - 1)$ denote the vertices of $\tau$ in lexicographical order. By definition, the integer $i$ is called the index of $u(i)$. The Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ is defined by $\mathcal{W}_0(\tau) = 0$, and for $0 \leqslant n \leqslant \zeta(\tau) - 1$, $\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k(n) - 1$, where $k(n)$ is the number of children of $u(n)$. For $0 \leqslant i \leqslant \zeta(\tau) - 1$, define $H_i(\tau)$ as the generation of $u(i)$

and set $H_{\zeta(\tau)}(\tau) = 0$ by convention. The height function $H(\tau) = (H_t(\tau); 0 \leqslant t \leqslant \zeta(\tau))$ is then defined by linear interpolation on the real interval $[0, \zeta(\tau)]$. In order to define the contour function $(C_t(\tau), 0 \leqslant t \leqslant 2\zeta(\tau))$, consider a particle that explores the tree from the left to the right, moving at unit speed along the edges. Then, for every $0 \leqslant t \leqslant 2(\zeta(\tau) - 1)$, $C_t(\tau)$ is defined as the distance from the root of the position of the particle at time $t$, and we set $C_t(\tau) = 0$ for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$ for technical reasons. See Fig. 1.2 and 1.3 for an example.

We now state our main results after introducing some notation. If $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to $x$. If $I$ is an interval, we denote the space of all continuous functions $I \to \mathbb{R}$ equipped with the topology of uniform convergence on compact subsets of $I$ by $\mathcal{C}(I, \mathbb{R})$, and finally we let $\mathbb{D}(I, \mathbb{R})$ denote the space of all right-continuous with left limits (càdlàg) functions $I \to \mathbb{R}$, endowed with the Skorokhod $J_1$-topology (see [20, chap. 3] and [57, chap. VI] for background concerning the Skorokhod topology). We will denote by $(Y_t)_{t \geqslant 0}$ the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(t\lambda^{2 \wedge \theta})$.

If $\tau$ is a (finite) tree, we denote the maximal out-degree of a vertex of $\tau$ by $\Delta(\tau)$ (the out-degree of a vertex is by definition its number of children). Note that $\Delta(\tau) - 1$ is equal to the maximal jump of $\mathcal{W}(\tau)$. Recall that $t_n$ is a $GW_\mu$ tree conditioned on having $n$ vertices, and that $\mu$ satisfies Assumption $(H_\theta)$.

**Theorem 1.** *Let $U(t_n) = \min\{j \geqslant 0;\ \mathcal{W}_{j+1}(t_n) - \mathcal{W}_j(t_n) = \Delta(t_n) - 1\}$ be the index of the first vertex of $t_n$ with maximal out-degree. Then:*

*(i)* $U(t_n)/n$ *converges in probability to* $0$ *as* $n \to \infty$.

*(ii) We have:*

$$\sup_{0 \leqslant i \leqslant U(t_n)} \frac{\mathcal{W}_i(t_n)}{n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0.$$

*(iii) The following convergence holds in distribution in* $\mathbb{D}([0,1], \mathbb{R})$*:*

$$\left( \frac{\mathcal{W}_{\lfloor nt \rfloor \vee (U(t_n)+1)}(t_n)}{n}, 0 \leqslant t \leqslant 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\gamma(1-t), 0 \leqslant t \leqslant 1).$$

*(iv) We have:*

$$\frac{\Delta(t_n)}{\gamma n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 1.$$

Using completely different techniques, a similar result has been proved by Durrett [40, Theorem 3.2] when $t_n$ is a $GW_\mu$ conditioned on having *at least* $n$ vertices and in addition $\mu$ has finite variance. Property (ii) shows that $\left(\mathcal{W}_{\lfloor nt \rfloor}(t_n)/n, 0 \leqslant t \leqslant 1\right)$ does not converge in distribution in $\mathbb{D}([0,1], \mathbb{R})$ towards $(\gamma(1-t), 0 \leqslant t \leqslant 1)$ and this explains why we look at the Lukasiewicz path only after time $U(t_n)$ in (iii). We will establish later that $U(t_n)$ actually converges in distribution as $n \to \infty$, thus improving (i). Theorem 1 shows that, with probability tending to 1 as $n \to \infty$, there exists a vertex of $t_n$ with out-degree roughly $\gamma n$ whose index is $o(n)$, and the out-degree of all other vertices of $t_n$ is $o(n)$. In particular the vertex with maximal out-degree is unique. This is consistent with the previously mentioned results of Jonsson & Stefánsson and Janson concerning the local convergence of $t_n$. The proof of Theorem 1 is based on a connection between the Lukasiewicz path of a $GW_\mu$ tree and a certain conditioned random walk (see Proposition 5.1.4), and combines the Vervaat transformation (see Proposition 5.1.6) with recent results of Armendáriz & Loulakis [9] concerning random walks whose jump distribution is subexponential.

Let us also mention that there exist probability distributions which do not satisfy Assumption $(H_\theta)$ but such that Theorem 1 holds, and that there also exist subcritical probability distributions such that Theorem 1 does not hold (see Remark 5.2.2).

We will need the following useful consequence of Theorem 1:

**Corollary 1.** *Fix* $c \in (0, \gamma)$. *We have* $U(t_n) = \min\{j \geqslant 0; \ \mathcal{W}_{j+1}(t_n) - \mathcal{W}_j(t_n) \geqslant cn\}$ *with probability tending to one as* $n \to \infty$.

This means that for every $c \in (0, \gamma)$, $U(t_n)$ coincides with the index of the first vertex of $t_n$ with at least $cn$ children, with probability tending to one as $n \to \infty$.

We then study the fluctuations of $\Delta(t_n)$ around $\gamma n$. The following statement, which is a consequence of results of Armendáriz & Loulakis [9], establishes a central limit theorem for $\Delta(t_n)$. Recall that $(Y_t)_{t \geqslant 0}$ is the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(t\lambda^{2\wedge\theta})$.

**Theorem 2.** *There exists a slowly varying function* $L$ *such that if* $B_n = L(n) n^{1/(2\wedge\theta)}$, *one has:*

$$\frac{\Delta(t_n) - \gamma n}{B_n} \quad \xrightarrow[n\to\infty]{(d)} \quad -Y_1.$$

When $\mu$ has finite variance $\sigma^2$, one may take $B_n = \sigma\sqrt{n/2}$. Theorem 2 has also been obtained independently of the present work by Janson [59] in the case $\mu_n \sim c/n^{1+\theta}$ as $n \to \infty$ (that is when $\mathcal{L} = c + o(1)$). In the latter case, one may choose $L$ to be a constant function.

Our next result concerns the location of the vertex with maximal out-degree in $t_n$. In order to state this result, we first need to introduce some notation. If $\tau$ is a finite tree, let $u_\star(\tau)$ be the smallest vertex of $\tau$ (in the lexicographical sense) with maximal out-degree. Note that the index of $u_\star(\tau)$ is $U(\tau)$ and that the number of children of $u_\star(\tau)$ is $\Delta(\tau)$. Denote the generation of $u_\star(\tau)$ by $|u_\star(\tau)|$.

**Theorem 3.** *The following three convergences hold:*

*(i)* *For* $i \geqslant 0$, $\mathbb{P}[U(t_n) = i] \quad \xrightarrow[n\to\infty]{} \quad \gamma \cdot \mathbb{P}_\mu[\zeta(\tau) \geqslant i+1]$.

*(ii)* *For* $i \geqslant 0$, $\mathbb{P}[|u_\star(t_n)| = i] \quad \xrightarrow[n\to\infty]{} \quad (1-m)m^i$.

*(iii)* *The total number of vertices of* $t_n$ *which are not strict descendants of* $u_\star(t_n)$ *converges in distribution as* $n \to \infty$ *to a finite random variable.*

Again a result similar to assertion (i) has been proved by Durrett [40, Theorem 3.2] when $t_n$ is a $GW_\mu$ conditioned on having *at least* $n$ vertices and in addition $\mu$ has finite variance. Note that $\sum_{i\geqslant 1} \mathbb{P}_\mu[\zeta(\tau) \geqslant i] = \mathbb{E}_\mu[\zeta(\tau)] = 1/\gamma$, so that the limit in (i) is a probability distribution. The proof of (i) combines the coding of $t_n$ by the Lukasiewicz path with the use of local limit theorems. The proof of the other two assertions uses (i) together with the local convergence of

$t_n$ towards an infinite random tree, which has been obtained by Janson [59] and was already mentioned above.

We are also interested in the size of the subtrees grafted on $u_\star(t_n)$. If $\tau$ is a tree, for $1 \leqslant j \leqslant \Delta(\tau)$, let $\zeta_j(\tau)$ be the number of descendants of the j-th child of $u_\star(\tau)$ and set $Z_j(\tau) = \zeta_1(\tau) + \zeta_2(\tau) + \cdots + \zeta_j(\tau)$. Recall the sequence $(B_n)_{n \geqslant 1}$ from Theorem 2.

**Theorem 4.** *The following convergence holds in distribution in* $\mathbb{D}([0,1],\mathbb{R})$*:*

$$\left( \frac{Z_{\lfloor \Delta(t_n) t \rfloor}(t_n) - \Delta(t_n)t/\gamma}{B_n}, 0 \leqslant t \leqslant 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad \left( \frac{1}{\gamma} Y_t, 0 \leqslant t \leqslant 1 \right).$$

Note that in the case when $\mu$ has finite variance, we have $\theta \geqslant 2$ and $Y$ is just a constant times standard Brownian motion.

Theorem 4 yields an interesting consequence concerning the maximum size of the subtrees grafted on $u_\star(t_n)$.

**Corollary 2.** *If* $\theta \geqslant 2$, $\max_{1 \leqslant i \leqslant \Delta(t_n)} \zeta_i(t_n)/B_n$ *converges in probability towards* $0$ *as* $n \to \infty$*. If* $\theta < 2$*, for every* $u > 0$ *we have:*

$$\mathbb{P}\left[ \frac{1}{B_n} \max_{1 \leqslant i \leqslant \Delta(t_n)} \zeta_i(t_n) \leqslant u \right] \quad \xrightarrow[n\to\infty]{} \quad \exp\left( \frac{1}{\gamma^\theta \Gamma(1-\theta)} u^{-\theta} \right),$$

*where* $\Gamma$ *is Euler's Gamma function.*

We then turn our attention to the height $\mathcal{H}(t_n)$ of $t_n$, which is by definition the maximal generation in $t_n$. We establish that $\mathcal{H}(t_n)$ grows logarithmically in $n$:

**Theorem 5.** *For every sequence* $(\lambda_n)_{n \geqslant 1}$ *of positive real numbers tending to infinity:*

$$\mathbb{P}\left[ \left| \mathcal{H}(t_n) - \frac{\ln(n)}{\ln(1/\mathfrak{m})} \right| \leqslant \lambda_n \right] \quad \xrightarrow[n\to\infty]{} \quad 1.$$

Note that the situation is different from the critical case, where $\mathcal{H}(t_n)$ grows like a power of $n$. Theorem 5 stems from the fact that $\mathcal{H}(t_n)$ is close to the maximum of the height of $\Delta(t_n)$ independent $GW_\mu$ trees. Note that Theorem 5 implies that $\mathcal{H}(t_n)/\ln(n) \to 1/\ln(1/\mathfrak{m})$ in probability as $n \to \infty$, thus partially answering Problem 20.7 in [59].

As an application of Theorem 5, we investigate the scaled asymptotic behavior of the contour and height functions of $t_n$.

**Theorem 6.** *Let* $(r_n)_{n \geqslant 1}$ *be a sequence of positive real numbers. For* $n \geqslant 1$*, set either* $Y^{(n)} = (C_{2nt}(t_n)/r_n, 0 \leqslant t \leqslant 1)$ *or* $Y^{(n)} = (H_{nt}(t_n)/r_n, 0 \leqslant t \leqslant 1)$*.*

*(i) If* $r_n/\ln(n) \to \infty$*, then* $Y^{(n)}$ *converges in distribution in the space* $\mathcal{C}([0,1],\mathbb{R})$ *towards the constant function equal to* $0$ *on* $[0,1]$ *as* $n \to \infty$*.*

*(ii) Otherwise, the sequence* $(Y^{(n)})_{n \geqslant 1}$ *is not tight in the space* $\mathcal{C}([0,1],\mathbb{R})$*.*

Theorem 6 implies that we cannot hope to obtain a nontrivial scaling limit from the contour and height functions of $t_n$, in contrast to the critical case (see [36]). This partially answers a question of Janson [59, Problem 20.11].

This text is organized as follows. We first recall the definition and basic properties of Galton-Watson trees. In Section 2, we establish limit theorems for large conditioned non-generic Galton-Watson trees. We conclude by giving possible extensions and formulating some open problems.

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# 5.1 Galton-Watson trees

## 5.1.1 Basic definitions

We briefly recall the formalism of plane trees (also known in the literature as rooted ordered trees) which can be found in [74] for example.

**Definition 5.1.1.** Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers and let $\mathbb{N}^*$ be the set of all positive integers. Let also $U$ be the set of all labels defined by:

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. An element of $U$ is a sequence $u = u_1 \cdots u_k$ of positive integers and we set $|u| = k$, which represents the "generation" of $u$. If $u = u_1 \cdots u_i$ and $v = v_1 \cdots v_j$ belong to $U$, we write $uv = u_1 \cdots u_i v_1 \cdots v_j$ for the concatenation of $u$ and $v$. In particular, we have $u\emptyset = \emptyset u = u$. Finally, a *plane tree* $\tau$ is a finite or infinite subset of $U$ such that:

(i) $\emptyset \in \tau$,

(ii) if $v \in \tau$ and $v = ui$ for some $i \in \mathbb{N}^*$, then $u \in \tau$,

(iii) for every $u \in \tau$, there exists $k_u(\tau) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ (the number of children of $u$) such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

Note that in contrast with [73, 74] we allow the possibility $k_u(\tau) = \infty$ in (iii). In the following, by *tree* we will always mean plane tree, and we denote the set of all trees by $\mathbb{T}$. We will often view each vertex of a tree $\tau$ as an individual of a population whose $\tau$ is the genealogical tree. The total progeny or size of $\tau$ will be denoted by $\zeta(\tau) = \text{Card}(\tau)$. If $\tau$ is a tree and $u \in \tau$, we define the shift of $\tau$ at $u$ by $T_u\tau = \{v \in U;\ uv \in \tau\}$, which is itself a tree.

**Definition 5.1.2.** Let $\rho$ be a probability measure on $\mathbb{N}$. The law of the Galton-Watson tree with offspring distribution $\rho$ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbb{T}$ such that:

(i) $\mathbb{P}_\rho[k_\emptyset = j] = \rho(j)$ for $j \geqslant 0$,

(ii) for every $j \geqslant 1$ with $\rho(j) > 0$, conditionally on $\{k_\varnothing = j\}$, the subtrees $T_1\tau, \dots, T_j\tau$ are independent and identically distributed with distribution $\mathbb{P}_\rho$.

A random tree whose distribution is $\mathbb{P}_\rho$ will be called a Galton-Watson tree with offspring distribution $\rho$, or in short a $GW_\rho$ tree.

In the sequel, for every integer $j \geqslant 1$, $\mathbb{P}_{\rho,j}$ will denote the probability measure on $\mathbb{T}^j$ which is the distribution of j independent $GW_\rho$ trees. The canonical element of $\mathbb{T}^j$ is denoted by $\mathbf{f}$. For $\mathbf{f} = (\tau_1, \dots, \tau_j) \in \mathbb{T}^j$, let $\zeta(\mathbf{f}) = \zeta(\tau_1) + \cdots + \zeta(\tau_j)$ be the total progeny of $\mathbf{f}$.

### 5.1.2 Coding Galton-Watson trees

We now explain how trees can be coded by three different functions. These codings are important in the understanding of large Galton-Watson trees.

**Definition 5.1.3.** We write $u < v$ for the lexicographical order on the labels $\mathcal{U}$ (for example $\emptyset < 1 < 21 < 22$). Let $\tau$ be a finite tree and order the individuals of $\tau$ in lexicographical order: $\emptyset = u(0) < u(1) < \cdots < u(\zeta(\tau) - 1)$. The height process $H(\tau) = (H_n(\tau), 0 \leqslant n < \zeta(\tau))$ is defined, for $0 \leqslant n < \zeta(\tau)$, by:

$$H_n(\tau) = |u(n)|.$$

We set $H_{\zeta(\tau)}(\tau) = 0$ for technical reasons. The height $\mathcal{H}(\tau)$ of $\tau$ is by definition $\max_{0 \leqslant n < \zeta(\tau)} H_n(\tau)$.

Consider a particle that starts from the root and visits continuously all the edges of $\tau$ at unit speed, assuming that every edge has unit length. When the particle leaves a vertex, it moves towards the first non visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all the edges are crossed twice, the total time needed to explore the tree is $2(\zeta(\tau) - 1)$. For $0 \leqslant t \leqslant 2(\zeta(\tau) - 1)$, $C_\tau(t)$ is defined as the distance to the root of the position of the particle at time t. For technical reasons, we set $C_\tau(t) = 0$ for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. The function $C(\tau)$ is called the contour function of the tree $\tau$. See Figure 1.3 for an example, and [36, Section 2] for a rigorous definition.

Finally, the Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ of $\tau$ is defined by $\mathcal{W}_0(\tau) = 0$ and for $0 \leqslant n \leqslant \zeta(\tau) - 1$:

$$\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1.$$

Note that necessarily $\mathcal{W}_{\zeta(\tau)}(\tau) = -1$.

The following proposition explains the importance of the Lukasiewicz path. Let $\rho$ be a critical or subcritical probability distribution on $\mathbb{N}$ with $\rho(1) < 1$.

**Proposition 5.1.4.** *Let $(W_n)_{n \geqslant 0}$ be a random walk with starting point $W_0 = 0$ and jump distribution $\nu(k) = \rho(k+1)$ for $k \geqslant -1$. Set $\zeta = \inf\{n \geqslant 0;\ W_n = -1\}$. Then $(W_0, W_1, \dots, W_\zeta)$ has the same distribution as the Lukasiewicz path of a $GW_\rho$ tree. In particular, the total progeny of a $GW_\rho$ tree has the same law as $\zeta$.*

*Proof.* See [73, Proposition 1.5]. □

### 5.1.3 The Vervaat transformation

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, denote by $\mathbf{x}^{(i)}$ the $i$-th cyclic shift of $\mathbf{x}$ defined by $x_k^{(i)} = x_{i+k \mod n}$ for $1 \leqslant k \leqslant n$.

**Definition 5.1.5.** Let $n \geqslant 1$ be an integer and let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. Set $w_j = x_1 + \cdots + x_j$ for $1 \leqslant j \leqslant n$ and let the integer $i_*(\mathbf{x})$ be defined by $i_*(\mathbf{x}) = \inf\{j \geqslant 1; w_j = \min_{1 \leqslant i \leqslant n} w_i\}$. The Vervaat transform of $\mathbf{x}$, denoted by $\mathbf{V}(\mathbf{x})$, is defined to be $\mathbf{x}^{(i_*(\mathbf{x}))}$.

The following fact is well known (see e.g. [90, Section 5]):

**Proposition 5.1.6.** *Let $(W_n, n \geqslant 0)$ be as in Proposition 5.1.4 and $X_k = W_k - W_{k-1}$ for $k \geqslant 1$. Fix an integer $n \geqslant 1$ such that $\mathbb{P}[W_n = -1] > 0$. The law of $\mathbf{V}(X_1, \ldots, X_n)$ under $\mathbb{P}[\,\cdot\,| W_n = -1]$ coincides with the law of $(X_1, \ldots, X_n)$ under $\mathbb{P}[\,\cdot\,| \zeta = n]$.*

From Proposition 5.1.4, it follows that the law of $\mathbf{V}(X_1, \ldots, X_n)$ under $\mathbb{P}[\,\cdot\,| W_n = -1]$ coincides with the law of $(\mathcal{W}_1(t_n), \mathcal{W}_2(t_n) - \mathcal{W}_1(t_n), \ldots, \mathcal{W}_n(t_n) - \mathcal{W}_{n-1}(t_n))$ where $t_n$ is a $\mathrm{GW}_\rho$ tree conditioned on having total progeny equal to $n$.

We now introduce the Vervaat transformation in continuous time.

**Definition 5.1.7.** Set $\mathbb{D}_0([0,1], \mathbb{R}) = \{\omega \in \mathbb{D}([0,1], \mathbb{R});\ \omega(0) = 0\}$. The Vervaat transformation in continuous time, denoted by $\mathcal{V} : \mathbb{D}_0([0,1], \mathbb{R}) \to \mathbb{D}([0,1], \mathbb{R})$, is defined as follows. For $\omega \in \mathbb{D}_0([0,1], \mathbb{R})$, set $g_1(\omega) = \inf\{t \in [0,1]; \omega(t-) \wedge \omega(t) = \inf_{[0,1]} \omega\}$. Then define:

$$\mathcal{V}(\omega)(t) = \begin{cases} \omega(g_1(\omega) + t) - \inf_{[0,1]} \omega, & \text{if } g_1(\omega) + t \leqslant 1, \\ \omega(g_1(\omega) + t - 1) + \omega(1) - \inf_{[0,1]} \omega & \text{if } g_1(\omega) + t \geqslant 1. \end{cases}$$

By combining the Vervaat transformation with limit theorems under the conditional probability distribution $\mathbb{P}[\,\cdot\,| W_n = -1]$ and using Proposition 5.1.4 we will obtain information about conditioned Galton-Watson trees. The advantage of dealing with $\mathbb{P}[\,\cdot\,| W_n = -1]$ is to avoid any positivity constraint.

### 5.1.4 Slowly varying functions

Recall that a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be slowly varying if $L(x) > 0$ for $x$ large enough and $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$. Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying function. Without further notice, we will use the following standard facts:

(i) The convergence $\lim_{x\to\infty} L(tx)/L(x) = 1$ holds uniformly for $t$ in a compact subset of $(0, \infty)$.

(ii) Fix $\epsilon > 0$. There exists a constant $C > 1$ such that $\frac{1}{C} x^{-\epsilon} \leqslant L(nx)/L(n) \leqslant C x^{\epsilon}$ for every integer $n$ sufficiently large and $x \geqslant 1$.

These results are immediate consequences of the so-called representation theorem for slowly varying functions (see e.g. [21, Theorem 1.3.1]).

## 5.2 Limit theorems for conditioned non-generic Galton-Watson trees

In the sequel, $(W_n; n \geqslant 0)$ denotes the random walk introduced in Proposition 5.1.4 with $\rho = \mu$. Note that $\mathbb{E}[W_1] = -\gamma < 0$. Set $X_0 = 0$ and $X_k = W_k - W_{k-1}$ for $k \geqslant 1$. It will be convenient to work with centered random walks, so we also set $\overline{W}_n = W_n + \gamma n$ and $\overline{X}_n = X_n + \gamma$ for $n \geqslant 0$ so that $\overline{W}_n = \overline{X}_1 + \cdots + \overline{X}_n$. Obviously, $W_n = -1$ if and only if $\overline{W}_n = \gamma n - 1$.

### 5.2.1 Invariance principle for the Lukasiewicz path

In this section, we study the scaling limit of the Lukasiewicz path of $t_n$ and first introduce some notation. Denote by $T : \cup_{n\geqslant 1}\mathbb{R}^n \to \cup_{n\geqslant 1}\mathbb{R}^n$ the operator that interchanges the last and the (first) maximal component of a finite sequence of real numbers:

$$T(x_1, \ldots, x_n)_k = \begin{cases} \max\limits_{1\leqslant i\leqslant n} x_i & \text{if } k = n \\ x_n & \text{if } x_k > \max\limits_{1\leqslant i<k} x_i \text{ and } x_k = \max\limits_{k\leqslant i\leqslant n} x_i \\ x_k & \text{otherwise.} \end{cases}$$

Since $\mu$ satisfies Assumption $(H_\theta)$, we have $\mathbb{P}\left[\overline{W}_1 \in (x, x+1]\right] \sim \mathcal{L}(x)/x^{1+\theta}$ as $x \to \infty$. Then, by [32, Theorem 9.1], we have:

$$\mathbb{P}\left[\overline{W}_n \in (x, x+1]\right] \underset{n\to\infty}{\sim} n\mathbb{P}\left[\overline{W}_1 \in (x, x+1]\right], \tag{5.1}$$

uniformly in $x \geqslant \epsilon n$ for every fixed $\epsilon > 0$. In other words, the distribution of $\overline{W}_1$ is $(0, 1]$–subexponential, so that we can apply a recent result of Armendáriz & Loulakis [9] concerning conditioned random walks with subexponential jump distribution. In our particular case, this result can be stated as follows:

**Proposition 5.2.1** (Armendáriz & Loulakis, Theorem 1 in [9])**.** *For $n \geqslant 1$ and $x > 0$, let $\mu_{n,x}$ be the probability measure on $\mathbb{R}^n$ which is the distribution of $(\overline{X}_1, \ldots, \overline{X}_n)$ under the conditional probability distribution $\mathbb{P}\left[\,\cdot\,|\,\overline{W}_n \in (x, x+1]\right]$.*

*Then for every $\epsilon > 0$, we have:*

$$\lim_{n\to\infty} \sup_{x\geqslant \epsilon n} \sup_{A\in\mathfrak{B}(\mathbb{R}^{n-1})} \left|\mu_{n,x} \circ T^{-1}[A \times \mathbb{R}] - \mu^{\otimes(n-1)}[A]\right| = 0.$$

As explained in [9], this means that under $\mathbb{P}\left[\,\cdot\,|\,\overline{W}_n \in (x, x+1]\right]$, asymptotically one gets $n-1$ independent random variables after forgetting the largest jump.

*Proof of Theorem 1.* We first prove that the following convergence holds in $\mathbb{D}([0,1], \mathbb{R})$:

$$\left(\frac{\overline{W}_{\lfloor nt\rfloor}}{n}, 0 \leqslant t \leqslant 1 \,\middle|\, \overline{W}_n = \gamma n - 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\gamma \mathbb{1}_{U\leqslant t}, 0 \leqslant t \leqslant 1), \tag{5.2}$$

where $U$ is a uniformly distributed random variable on $[0,1]$. Denote by $V_n$ the coordinate of the first maximal component of $(\overline{X}_1, \ldots, \overline{X}_n)$. Set $\widetilde{W}_0 = 0$ and for $1 \leqslant i \leqslant n-1$ set:

$$\widetilde{W}_i = \begin{cases} \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_i & \text{if } i < V_n \\ \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_{V_n - 1} + \overline{X}_{V_n+1} + \cdots + \overline{X}_{i+1} & \text{otherwise.} \end{cases}$$

By Proposition 5.2.1, for every $\epsilon > 0$:

$$\lim_{n \to \infty} \left| \mathbb{P}\left[ \forall t \in [0,1], \left| \frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leqslant \epsilon \middle| \overline{W}_n = \gamma n - 1 \right] - \mathbb{P}\left[ \forall t \in [0,1], \left| \frac{\overline{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leqslant \epsilon \right] \right| = 0. \tag{5.3}$$

We now claim that for every $\epsilon > 0$:

$$\mathbb{P}\left[ \forall t \in [0,1], \left| \frac{\overline{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leqslant \epsilon \right] \quad \underset{n \to \infty}{\longrightarrow} \quad 1. \tag{5.4}$$

To establish (5.4), note that $(|\overline{W}_i|)_{i \geqslant 0}$ is a submartingale, so that Doob's maximal inequality entails:

$$\mathbb{P}\left[ \max_{1 \leqslant i \leqslant n-1} |\overline{W}_i| \geqslant (n-1)\epsilon \right] \leqslant \frac{\mathbb{E}\left[ |\overline{W}_{n-1}| \right]}{\epsilon(n-1)}.$$

By [92], $\mathbb{E}\left[ |\overline{W}_{n-1}| \right] / (n-1) \to 0$ as $n \to \infty$. This shows (5.4).

Combining (5.4) with (5.3), we get the following convergence in $\mathbb{D}([0,1], \mathbb{R})$:

$$\left( \frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1}; 0 \leqslant t \leqslant 1 \middle| \overline{W}_n = \gamma n - 1 \right) \quad \overset{(\mathbb{P})}{\underset{n \to \infty}{\longrightarrow}} \quad \mathbf{0}, \tag{5.5}$$

where $\mathbf{0}$ stands for the constant function equal to $0$ on $[0,1]$. In addition, note that on the event $\{\overline{W}_n = \gamma n - 1\}$, we have $\overline{X}_{V_n} = \gamma n - 1 - \widetilde{W}_{\lfloor n-1 \rfloor}$. The following joint convergence in distribution thus holds in $\mathbb{D}([0,1], \mathbb{R}) \otimes \mathbb{R}$:

$$\left( \left( \frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1}; 0 \leqslant t \leqslant 1 \right), \frac{\overline{X}_{V_n}}{n} \middle| \overline{W}_n = \gamma n - 1 \right) \quad \overset{(\mathbb{P})}{\underset{n \to \infty}{\longrightarrow}} \quad (\mathbf{0}, \gamma). \tag{5.6}$$

Standard properties of the Skorokhod topology then show that the following convergence holds in $\mathbb{D}([0,1], \mathbb{R})$:

$$\left( \frac{\overline{W}_{\lfloor nt \rfloor}}{n} - \frac{\overline{X}_{V_n}}{n} \mathbb{1}_{\{t \geqslant \frac{V_n}{n}\}}; 0 \leqslant t \leqslant 1 \middle| \overline{W}_n = \gamma n - 1 \right) \quad \overset{(\mathbb{P})}{\underset{n \to \infty}{\longrightarrow}} \quad \mathbf{0}. \tag{5.7}$$

Next, note that the convergence (5.6) implies that under $\mathbb{P}\left[ \cdot \,|\, \overline{W}_n = \gamma n - 1 \right]$, $(\overline{X}_1, \ldots, \overline{X}_n)$ has a unique maximal component with probability tending to one as $n \to \infty$. Since the distribution of $(\overline{X}_1, \ldots, \overline{X}_n)$ under $\mathbb{P}\left[ \cdot \,|\, \overline{W}_n = \gamma n - 1 \right]$ is cyclically exchangeable, one easily gets that the law of $V_n/n$ under $\mathbb{P}\left[ \cdot \,|\, \overline{W}_n = \gamma n - 1 \right]$ converges to the uniform distribution on $[0,1]$. Also from

(5.6) we know that $\overline{X}_{V_n}/n$ under $\mathbb{P}\left[\,\cdot\,|\,\overline{W}_n = \gamma n - 1\right]$ converges in probability to $\gamma$. It follows that

$$\left( \frac{\overline{X}_{V_n}}{n} \mathbb{1}_{\{\frac{V_n}{n} \leqslant t\}}, 0 \leqslant t \leqslant 1 \,\middle|\, \overline{W}_n = \gamma n - 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\gamma \mathbb{1}_{U \leqslant t}, 0 \leqslant t \leqslant 1), \tag{5.8}$$

where $U$ is uniformly over $[0,1]$. Since (5.7) holds in probability, we can combine (5.7) and (5.8) to get (5.2).

We now prove the assertions of Theorem 1. From (5.2) and the definition of $\overline{W}$, we have:

$$\left( \frac{W_{\lfloor nt \rfloor}}{n}, 0 \leqslant t \leqslant 1 \,\middle|\, W_n = -1 \right) \qquad \xrightarrow[n\to\infty]{(d)} \qquad (-\gamma t + \gamma \mathbb{1}_{U \leqslant t}, 0 \leqslant t \leqslant 1). \tag{5.9}$$

Let $(W_0^{br,n}, W_1^{br,n}, \ldots, W_n^{br,n})$ be a random vector distributed as $(W_0, W_1, \ldots, W_n)$ under the conditional probability distribution $\mathbb{P}[\,\cdot\,|\,W_n = -1]$. By (5.9) and Skorokhod's representation theorem we can suppose assume that we have a.s.

$$\left( \frac{W_{\lfloor nt \rfloor}^{br,n}}{n}, 0 \leqslant t \leqslant 1 \right) \qquad \xrightarrow[n\to\infty]{} \qquad (-\gamma t + \gamma \mathbb{1}_{U \leqslant t}, 0 \leqslant t \leqslant 1). \tag{5.10}$$

By the remark following Proposition 5.1.6, we can also assume that

$$\left( \frac{\mathcal{W}_{\lfloor nt \rfloor}(t_n)}{n}, 0 \leqslant t \leqslant 1 \right) \quad = \quad \mathcal{V} \left( \frac{W_{\lfloor nt \rfloor}^{br,n}}{n}, 0 \leqslant t \leqslant 1 \right) \tag{5.11}$$

where we recall that $\mathcal{V}$ denotes the Vervaat transformaton in continuous time. Now set $i_0^{(n)} = \inf\{0 \leqslant i \leqslant n; W_i^{br,n} = \min_{0 \leqslant j \leqslant n} W_j^{br,n}\}$ and

$$j_0^{(n)} = \inf \left\{ 0 \leqslant j \leqslant n-1; W_{j+1}^{br,n} - W_j^{br,n} = \max_{0 \leqslant i \leqslant n-1} (W_{i+1}^{br,n} - W_i^{br,n}) \right\}$$

Recall the notation $U(t_n)$ defined in the statement of Theorem 1. By (5.11) and the definition of the Vervaat transformation, we have $U(t_n) = j_0^{(n)} - i_0^{(n)}$. From (5.10), it follows that a.s. for $n$ sufficiently large $i_0^{(n)} \leqslant j_0^{(n)}$, and $(j_0^{(n)} - i_0^{(n)})/n \to 0$. Assertion (i) follows.

For assertion (ii), note that, by (5.11) and the definition of the Vervaat transformation:

$$\sup_{0 \leqslant i \leqslant U(t_n)} \frac{\mathcal{W}_i(t_n)}{n} = \sup_{i_0^{(n)} \leqslant j \leqslant j_0^{(n)}} W_j^{br,n} - W_{i_0^{(n)}}^{br,n}.$$

From (5.10) and properties of the Skorokhod topology, we get that the right-hand side of the previous expression tends to $0$ as $n \to \infty$. This shows (ii).

The proof of assertion (iii) is similar and left to the reader. For assertion (iv), it is sufficient to note that $\Delta(t_n) = W_{j_0+1}^{br,n} - W_{j_0}^{br,n} + 1$ by (5.11), so that $\Delta(t_n)/n \to \gamma$ as $n \to \infty$ by (5.10). This completes the proof of Theorem 1. □

**Remark 5.2.2.** The preceding proof shows that Theorem 1 remains true when $\mu$ is subcritical and both (5.1) and Proposition 5.2.1 hold. These conditions are more general than those of Assumption $(H_\theta)$: see e.g. [32, Section 9] for examples of probability distributions that do not satisfy Assumption $(H_\theta)$ but such that (5.1) holds. Note also that there exist subcritical probability distributions such that Theorem 1 does not hold (see [59, Example 19.37] for an example).

Corollary 1 is an immediate consequence of Theorem 1. Indeed, assertions (ii) and (iii) of the latter theorem imply that for every $c \in (0, \gamma)$, there is a unique vertex of $t_n$ with out-degree at least $cn$, with probability tending to one as $n \to \infty$.

### 5.2.2 Asymptotic behavior of the maximal out-degree

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, set $\mathcal{M}(\mathbf{x}) = \max_{1 \leqslant i \leqslant n} x_i$. Recall the notation $\mathbf{V}(\mathbf{x})$ for the Vervaat transform of $\mathbf{x}$. Note that $\mathcal{M}(\mathbf{x}) = \mathcal{M}(\mathbf{V}(\mathbf{x}))$. Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Recall that $\Delta(t_n)$ denotes the maximal out-degree of a vertex of $t_n$. Since the maximal jump of $\mathcal{W}(t_n)$ is equal to $\Delta(t_n) - 1$, it follows from the remark following Proposition 5.1.6 that:

$$\begin{aligned} \mathbb{E}\left[F(\Delta(t_n))\right] &= \mathbb{E}\left[F\left(\mathcal{M}\left(\mathbf{V}(X_1, X_2, \ldots, X_n)\right) + 1\right) \mid W_n = -1\right] \\ &= \mathbb{E}\left[F\left(\mathcal{M}\left(X_1, X_2, \ldots, X_n\right) + 1\right) \mid W_n = -1\right] \end{aligned} \tag{5.12}$$

Recall that since $\mu$ satisfies Assumption $(H_\theta)$, $\overline{W}_1$ belongs to the domain of attraction of a spectrally positive strictly stable law of index $2 \wedge \theta$. Hence there exists a slowly varying function $L$ such that $\overline{W}_n / \left(L(n) n^{1/(2\wedge\theta)}\right)$ converges in distribution towards $Y_1$. We set $B_n = L(n) n^{1/(2\wedge\theta)}$ and prove that Theorem 2 holds with this choice of $B_n$. The function $L$ is not unique, but if $\widetilde{L}$ is another slowly function with the same property we have $L(n)/\widetilde{L}(n) \to 1$ as $n \to \infty$. So our results do not depend on the choice of $L$. Note that when $\mu$ has finite variance $\sigma^2$, one may take $B_n = \sigma\sqrt{n/2}$, and when $\mathcal{L} = c + o(1)$ one may choose $L$ to be a constant function.

*Proof of Theorem 2.* If $V_n$ is as in the proof of Theorem 1, we have

$$\mathcal{M}(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) = \overline{X}_{V_n} = \gamma n - 1 - \sum_{i \neq V_n} \overline{X}_i$$

on the event $\left\{\overline{W}_n = \gamma n - 1\right\}$. As noted in [9, Formula (2.7)], it follows from Proposition 5.2.1 that

$$\left( \frac{\mathcal{M}(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) - \gamma n}{B_n} \,\middle|\, \overline{W}_n = \gamma n - 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad -Y_1.$$

Since $\mathcal{M}(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) = \mathcal{M}(X_1, X_2, \ldots, X_n) + \gamma$, we thus get that

$$\left( \frac{\mathcal{M}\left(X_1, X_2, \ldots, X_n\right) + 2 - \gamma n}{B_n} \,\middle|\, W_n = -1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad -Y_1.$$

Theorem 2 then immediately follows from (5.12). □

### 5.2.3 Location of the vertex with maximal out-degree

Recall the notation $\mathbb{P}_{\mu,j}$ for the law of a forest of $j$ independent $GW_\mu$ trees.

Our goal is to prove Theorem 4. We first need to introduce some notation. It is well known that the mean number of vertices of a $GW_\mu$ tree at generation $n$ is $m^n$. As a consequence, we have $\mathbb{E}_\mu[\zeta(\tau)] = 1 + m + m^2 + \cdots = 1/(1-m) = 1/\gamma$. Moreover, for $n \geqslant 1$, by Kemperman's formula (see e.g. [90, Section 5]):

$$\mathbb{P}_\mu[\zeta(\tau) = n] = \frac{1}{n}\mathbb{P}[W_n = -1] = \frac{1}{n}\mathbb{P}\left[\overline{W}_n = \gamma n - 1\right] \underset{n\to\infty}{\sim} \frac{\mathcal{L}(n)}{(\gamma n)^{1+\theta}}, \tag{5.13}$$

where we have used (5.1) for the last estimate. It follows that the total progeny of a $GW_\mu$ tree belongs to the domain of attraction of a spectrally positive strictly stable law of index $2\wedge\theta$. Hence we can find a slowly varying function $L'$ such that the law of $(\zeta(\mathbf{f}) - n/\gamma)/\left(L'(n)n^{1/(2\wedge\theta)}\right)$ under $\mathbb{P}_{\mu,n}$ converges as $n \to \infty$ to the law of $Y_1$. We set $B'_n = L'(n)n^{1/(2\wedge\theta)}$. The local limit theorem (see [56, Theorem 4.2.1]) implies that if we set $\varphi_j(k) = \mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = k]$ for $j \geqslant 1$ and $k \geqslant 0$ (with the convention $\varphi_j(k) = 0$ for $k < 0$), then

$$\lim_{m\to\infty} \sup_{k\in\mathbb{Z}} \left| B'_m \varphi_m(k) - p_1\left(\frac{k - m/\gamma}{B'_m}\right)\right| = 0, \tag{5.14}$$

where $p_1$ is the density of $Y_1$. It is well known that $p_1$ is a bounded continuous function (see e.g. [99, I. 4]).

The following technical result establishes a useful link between $B_n$ and $B'_n$.

**Lemma 5.2.3.** *We have* $B'_n/B_n \to 1/\gamma^{1+1/(2\wedge\theta)}$ *as* $n \to \infty$.

The proof of Lemma 5.2.3 is postponed to the end of this section.

*Proof of Theorem 3 (i).* Fix $\epsilon > 0$ and an integer $i_0 \geqslant 0$. By Theorem 2 and Lemma 5.2.3, we may choose $A > 0$ such that:

$$|\mathbb{P}[U(t_n) = i_0] - \mathbb{P}[U(t_n) = i_0, |k_{i_0}(t_n) - \gamma n| \leqslant AB'_n]| \leqslant \epsilon \tag{5.15}$$

By Theorem 1, for $n$ large enough:

$$|\mathbb{P}[U(t_n) = i_0, |k_{i_0}(t_n) - \gamma n| \leqslant AB'_n] - \mathbb{P}[|k_{i_0}(t_n) - \gamma n| \leqslant AB'_n]| \leqslant \epsilon. \tag{5.16}$$

By Proposition 5.1.4, we have for $n$ large enough:

$$\begin{aligned}
&\mathbb{P}[|k_{i_0}(t_n) - \gamma n| \leqslant AB'_n] \\
&\quad = \frac{1}{\mathbb{P}_\mu[\zeta(\tau) = n]} \sum_{|j-\gamma n|\leqslant AB'_n} \mathbb{P}[X_{i_0+1} = j-1, \zeta = n] \\
&\quad = \sum_{|j-\gamma n|\leqslant AB'_n} \frac{\mathbb{P}[X_{i_0+1} = j-1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}\left[\mathbb{1}_{\{\forall m\leqslant i_0, W_m\geqslant 0\}} \varphi_{W_{i_0}+j}(n - i_0 - 1)\right],
\end{aligned} \tag{5.17}$$

where we have used the Markov property of the random walk $W$ at time $i_0 + 1$ for the last equality. To simplify notation, set $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then write:

$$\begin{aligned}
&\sum_{|j-\gamma n|\leqslant AB'_n} \frac{\mathbb{P}[X_{i_0+1} = j-1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}\left[\mathbb{1}_{\{\forall m\leqslant i_0, W_m\geqslant 0\}} \varphi_{W_{i_0}+j}(n - i_0 - 1)\right] \\
&\quad = \mathbb{E}\left[\mathbb{1}_{\{\forall m\leqslant i_0, W_m\geqslant 0\}} \int_{-AB'_n-\mathcal{O}(1)}^{AB'_n+\mathcal{O}(1)} du \frac{\mathbb{P}[X_1 = \lfloor \gamma n + u\rfloor - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{W_{i_0}+\lfloor \gamma n+u\rfloor}(n - i_0 - 1)\right] \\
&\quad = B'_n\, \mathbb{E}\left[\mathbb{1}_{\{\forall m\leqslant i_0, W_m\geqslant 0\}} \int_{-A-o(1)}^{A+o(1)} du \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{W_{i_0}+j_n(u)}(n - i_0 - 1)\right]
\end{aligned} \tag{5.18}$$

By using the dominated convergence theorem, we will now show that for a fixed integer $k \geqslant 0$

$$B'_n \int_{-A-o(1)}^{A+o(1)} du \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{k+j_n(u)}(n - i_0 - 1) \quad \xrightarrow[n\to\infty]{} \quad \int_{-A}^{A} \frac{du}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right) \tag{5.19}$$

We will then apply (5.19) with $k = W_{i_0}$. Since $\mu$ satisfies assumption (ii) in $(H_\theta)$, we have $\mathbb{P}[X_1 = j_n(u) - 1] = \mathcal{L}(j_n(u))/(j_n(u))^{1+\theta}$. From (5.13), it follows that

$$\sup_{-A-1 \leqslant u \leqslant A+1} \left| \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} - 1 \right| \quad \xrightarrow[n\to\infty]{} \quad 0 \tag{5.20}$$

Next, note that $B'_n/B'_{k+j_n(u)} \to 1/\gamma^{1/(2\wedge\theta)}$ as $n \to \infty$, uniformly in $u \in (-A-1, A+1)$. From (5.14), it follows that

$$B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) - \frac{1}{\gamma^{1/(2\wedge\theta)}} p_1\left(\frac{n - i_0 - 1 - \frac{1}{\gamma}(k + \lfloor \gamma n + uB'_n \rfloor)}{B'_{k+j_n(u)}}\right) \quad \xrightarrow[n\to\infty]{} \quad 0.$$

Hence

$$B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) \quad \xrightarrow[n\to\infty]{} \quad \frac{1}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right).$$

In addition, by (5.14), there exists a constant $C > 0$ which is independent of $k$ and such that $0 \leqslant B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) \leqslant C$ for every $n$ sufficiently large and $u \in (-A-1, A+1)$. The convergence (5.19) then follows from an application of the dominated convergence theorem.

Another application of the dominated convergence theorem gives that the expression appearing in (5.18) converges towards

$$\mathbb{P}[\forall m \leqslant i_0, W_m \geqslant 0] \int_{-A}^{A} \frac{du}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right).$$

as $n \to \infty$. It follows from (5.15), (5.16) and (5.17) that for $n$ sufficiently large:

$$\left| \mathbb{P}[U(t_n) = i_0] - \mathbb{P}[\forall m \leqslant i_0, W_m \geqslant 0] \int_{-A}^{A} \frac{du}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right) \right| \leqslant 3\epsilon.$$

Finally, since $p_1$ is the density of a probability distribution, we have:

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{du}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right) = \int_{-\infty}^{\infty} \frac{du}{\gamma^{1/(2\wedge\theta)}} p_1\left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2\wedge\theta)}}\right) = \gamma.$$

Thus, by choosing $A$ sufficiently large, we get that for $n$ sufficiently large

$$|\mathbb{P}[U(t_n) = i_0] - \gamma \mathbb{P}[\forall m \leqslant i_0, W_m \geqslant 0]| \leqslant 4\epsilon.$$

Since this holds for every $\epsilon > 0$, we get $\lim_{n\to\infty} \mathbb{P}[U(t_n) = i_0] = \gamma \mathbb{P}[\forall m \leqslant i_0 W_m \geqslant 0]$. By Proposition 5.1.4, we have $\mathbb{P}[\forall m \leqslant i_0, W_m \geqslant 0] = \mathbb{P}_\mu[\zeta(\tau) \geqslant i_0 + 1]$. This completes the proof of assertion (i) in Theorem 3. □

To prove the second assertion of Theorem 3, we will need the size-biased distribution associated to $\mu$, which is the distribution of the random variable $\zeta^*$ such that:

$$\mathbb{P}[\zeta^* = k] := \frac{k\mu_k}{m} \qquad k = 0, 1, \ldots.$$

The following result concerning the local convergence of $t_n$ as $n \to \infty$ will be useful. We refer the reader to [59, Section 6] for definitions and background concerning local convergence of trees (note that we need to consider trees that are not locally finite, so that this is slightly different from the usual setting).

**Proposition 5.2.4** (Jonsson & Stefánsson [62], Janson [59]). *Let $\widehat{\mathcal{T}}$ be the infinite random tree constructed as follows. Start with a spine composed of a random number S of vertices, where S is defined by:*

$$\mathbb{P}[S = i] = (1 - m)m^{i-1}, \qquad i = 1, 2, \ldots. \tag{5.21}$$

*Then attach further branches as follows (see also Figure 5.2 below). At the top of the spine, attach an infinite number of branches, each branch being a $\mathrm{GW}_\mu$ tree. At all the other vertices of the spine, a random number of branches distributed as $\zeta^* - 1$ is attached to either to the left or to the right of the spine, each branch being a $\mathrm{GW}_\mu$ tree. At a vertex of the spine where $k$ new branches are attached, the number of new branches attached to the left of the spine is uniformly distributed on $\{0, \ldots, k\}$. Moreover all random choices are independent.*

*Then $t_n$ converges locally in distribution towards $\widehat{\mathcal{T}}$ as $n \to \infty$.*

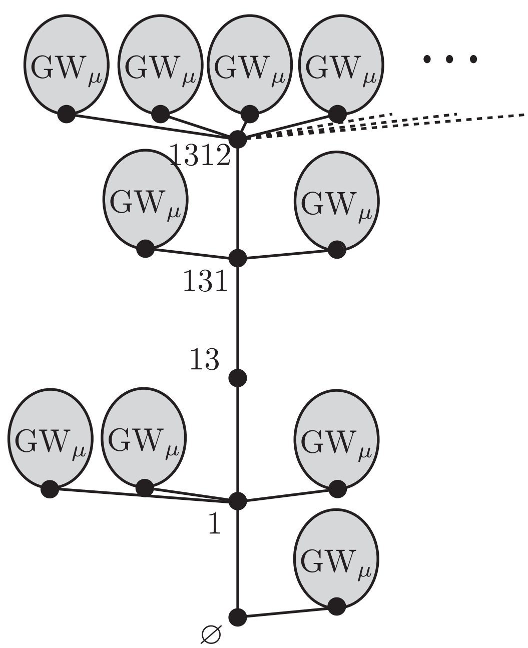


**Figure 5.2:** An illustration of $\widehat{\mathcal{T}}$. Here, the spine is composed of the vertices $\emptyset, 1, 13, 13, 131, 1312$.

*Proof of Theorem 3 (ii) and (iii).* By Skorokhod's representation theorem (see e.g. [20, Theorem 6.7]) we can suppose that the convergence $t_n \to \widehat{\mathcal{T}}$ as $n \to \infty$ holds almost surely for the local topology. Let $u_\star \in \widehat{\mathcal{T}}$ be the vertex of the spine with largest generation. By (5.21), we have for $i \geqslant 0$:

$$\mathbb{P}\left[|u_\star| = i\right] = (1-m)m^i. \tag{5.22}$$

Recall the notation $U(t_n)$ for the index of $u_\star(t_n)$. Let $\epsilon > 0$. By assertion (i) of Theorem 3, which was proved at the beginning of this section, we can fix an integer $K$ such that, for every $n$, $\mathbb{P}\left[U(t_n) \leqslant K\right] > 1 - \epsilon$. From the local convergence of $t_n$ to $\widehat{\mathcal{T}}$ (and the properties of local convergence, see in particular Lemma 6.3 in [59]) we can easily verify that

$$\mathbb{P}\left[\{u_\star(t_n) \neq u_\star\} \cap \{U(t_n) \leqslant K\}\right] \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

We conclude that $\mathbb{P}\left[u_\star(t_n) \neq u_\star\right] \to 0$ as $n \to \infty$. Assertion (ii) of Theorem 3 now follows from (5.22). Similarly, the local convergence of $t_n$ to $\widehat{\mathcal{T}}$ implies that the number of vertices of $t_n$ which are not strict descendants of $u_\star(t_n)$ converges in distribution towards the number of vertices of $\widehat{\mathcal{T}}$ which are not strict descendants of $u_\star$, giving assertion (iii). □

Note that assertion (i) in Theorem 3 was needed to prove assertions (ii) and (iii). Indeed, the local convergence of $t_n$ towards $\widehat{\mathcal{T}}$ would not have been sufficient to get that $\mathbb{P}\left[u_\star(t_n) \neq u_\star\right] \to 0$.

We conclude this section by proving Lemma 5.2.3.

*Proof of Lemma 5.2.3.* Let $\sigma^2$ be the variance of $\mu$. Note that $\sigma^2 = \infty$ if $\theta \in (1,2)$, $\sigma^2 < \infty$ if $\theta > 2$ and that we can have either $\sigma^2 = \infty$ or $\sigma^2 < \infty$ for $\theta = 2$. When $\sigma^2 = \infty$, the desired result follows from classical results expressing $B_n$ in terms of $\mu$. Indeed, in the case $\theta < 2$, we may choose $B_n$ and $B'_n$ such that (see e.g. [70, Theorem 1.10]):

$$\frac{B'_n}{B_n} = \frac{\inf\left\{x \geqslant 0;\ \mathbb{P}_\mu\left[\zeta(\tau) \geqslant x\right] \leqslant \frac{1}{n}\right\}}{\inf\left\{x \geqslant 0;\ \mu([x,\infty)) \leqslant \frac{1}{n}\right\}}.$$

Property (ii) in Assumption $(H_\theta)$ and (5.13) entail that $\mathbb{P}_\mu\left[\zeta(\tau) \geqslant x\right]/\mu([x,\infty)) \to 1/\gamma^{1+\theta}$ as $x \to \infty$. The result easily follows. The case when $\sigma^2 = \infty$ and $\theta = 2$ is treated by using similar arguments. We leave details to the reader.

We now concentrate on the case $\sigma^2 < \infty$. Note that necessarily $\theta \geqslant 2$. Let $\sigma'^2$ be the variance of $\zeta(\tau)$ under $\mathbb{P}_\mu$ (from (5.13) this variance is finite when $\sigma^2 < \infty$). We shall show that $\sigma' = \sigma/\gamma^{3/2}$. The desired result will then follow by the classical central limit theorem since we may take $B_n = \sigma\sqrt{n/2}$ and $B'_n = \sigma'\sqrt{n/2}$. In order to calculate $\sigma'^2$, we introduce the Galton-Watson process $(\mathcal{Z}_i)_{i\geqslant 0}$ with offspring distribution $\mu$ such that $\mathcal{Z}_0 = 1$. Recall that $\mathbb{E}\left[\mathcal{Z}_i\right] = m^i$. Then note that:

$$\sigma'^2 = \mathbb{E}_\mu\left[\zeta(\tau)^2\right] - \mathbb{E}_\mu\left[\zeta(\tau)\right]^2 = \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \mathcal{Z}_i\right)^2\right] - \frac{1}{\gamma^2}.$$

Since $(\mathcal{Z}_i/m^i)_{i\geqslant 0}$ is a martingale with respect to the filtration generated by $(\mathcal{Z}_i)_{i\geqslant 0}$, we have $\mathbb{E}\left[Z_i Z_j\right] = m^{j-i}\mathbb{E}\left[Z_i^2\right]$. Also using the well-known fact that for $i \geqslant 1$ the variance of $\mathcal{Z}_i$ is

$\sigma^2 m^{i-1}(m^i - 1)/(m-1)$ (see e.g. [11, Section 1.2]), write:

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=0}^{\infty}\mathfrak{Z}_i\right)^2\right] &= \sum_{i=0}^{\infty}\mathbb{E}\left[\mathfrak{Z}_i^2\right] + 2\sum_{0\leqslant i<j} m^{j-i}\mathbb{E}\left[\mathfrak{Z}_i^2\right] = \sum_{i=0}^{\infty}\mathbb{E}\left[\mathfrak{Z}_i^2\right]\left(1+\frac{2m}{1-m}\right)\\ &= \left(1+\sum_{i=1}^{\infty}\left(\frac{\sigma^2 m^{i-1}(m^i-1)}{m-1}+m^{2i}\right)\right)\left(1+\frac{2m}{1-m}\right)\\ &= \frac{\sigma^2}{\gamma^3}+\frac{1}{\gamma^2}.\end{aligned}$$

This entails $\sigma' = \sigma/\gamma^{3/2}$ and the conclusion follows. □

### 5.2.4 Subtrees branching off the vertex with maximum out-degree

In order to prove Theorem 4, we need to introduce some notation. If $\tau$ is a tree, let $u_\star^{(n)}(\tau)$ be the first vertex of $\tau$ in lexicographical order with strictly more than $\gamma n/2$ children (with the convention $u_\star^{(n)}(\tau) = \partial$ if there is no such vertex, where $\partial$ is a cemetery vertex). Let $\kappa_n(\tau)$ be the number of children of $u_\star^{(n)}(\tau)$ (with the convention $\kappa_n(\tau) = 0$ if $u_\star^{(n)}(\tau) = \partial$). Recall the notation $T_u\tau$ for the tree shifted at $u$. For $1 \leqslant j \leqslant \kappa_n(\tau)$, let $u_j^{(n)}(\tau)$ denote the j-th child of $u_\star^{(n)}(\tau)$ and set $\tau_j^{(n)} = T_{u_j^{(n)}}(\tau)$. For $1 \leqslant i \leqslant j \leqslant \kappa_n(\tau)$, we set $\mathcal{F}_{i,j}^{(n)} = (\tau_i^{(n)}, \dots, \tau_j^{(n)})$. Finally, let $N_0^{(n)}(\tau) = \zeta(\tau) + 1 - \zeta\left(T_{u_\star^{(n)}}\tau\right)$ denote the number of vertices of $\tau$ which are not strict descendants of $u_\star^{(n)}(\tau)$.

Recall the notation $\mathbb{P}_{\mu,j}$ for the law of a forest of j independent $GW_\mu$ trees. The following result easily follows from the properties of Galton-Watson trees:

**Lemma 5.2.5.** *For every $j \geqslant 1$ such that $\mathbb{P}_\mu[\kappa_n(\tau) = j] > 0$, under the conditional probability measure $\mathbb{P}_\mu[\,\cdot\,|\,\kappa_n = j]$, the random variables $N_0^{(n)}(\tau)$ and $\mathcal{F}_{1,j}^{(n)}$ are independent and, in addition, $\mathcal{F}_{1,j}^{(n)}$ is distributed according to $\mathbb{P}_{\mu,j}$.*

By Corollary 1, we have $u_\star^{(n)}(t_n) = u_\star(t_n)$ with probability tending to one as $n \to \infty$. It is thus sufficient to establish Theorem 4 when the process $Z$ is replaced by the process $Z^{(n)}$ defined as follows. For every tree $\tau$ and for every $1 \leqslant j \leqslant \kappa_n(\tau)$, let $\zeta_j^{(n)}(\tau) = \zeta(\tau_j^{(n)})$ be the number of descendants of $u_j^{(n)}(\tau)$, and set $Z_j^{(n)}(\tau) = \zeta_1^{(n)}(\tau) + \zeta_2^{(n)}(\tau) + \cdots + \zeta_j^{(n)}(\tau)$. We need to prove that:

$$\left(\frac{Z^{(n)}_{\lfloor \kappa_n(t_n)t\rfloor}(t_n) - \kappa_n(t_n)t/\gamma}{B_n}, 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad \left(\frac{1}{\gamma}Y_t, 0 \leqslant t \leqslant 1\right). \tag{5.23}$$

Let $(\mathcal{Z}_i)_{i\geqslant 0}$ be the random walk which starts at 0 and whose jump distribution has the same law as the total progeny of a $GW_\mu$ tree. Note that $\mathbb{P}[\mathcal{Z}_j = k] = \mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = k]$. Recall from the beginning of Section 5.2.3 that the distribution of $\mathcal{Z}_1$ belongs to the domain of attraction of a spectrally positive strictly stable law of index $2 \wedge \theta$. In particular, if $B'_n$ is defined as in the beginning of Section 5.2.3, the following convergence holds in distribution in the space $\mathbb{D}([0,1],\mathbb{R})$:

$$\left(\frac{\mathcal{Z}_{\lfloor nt \rfloor} - nt/\gamma}{B'_n}; 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (Y_t; 0 \leqslant t \leqslant 1). \tag{5.24}$$

Before we proceed to the proof of Theorem 4, we state a technical lemma.

**Lemma 5.2.6.** *Set $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then for every $A > 0$:*

$$\lim_{n\to\infty} \sup_{u\in[-A-1,A+1]} \left| \frac{\mathbb{P}_\mu\left[\kappa_n(\tau) = j_n(u)\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]} - \frac{1}{\gamma} \right| = 0.$$

*Proof.* By Proposition 5.1.4, for $n$ sufficiently large, we have for every $u \in [-A-1, A+1]$:

$$\begin{aligned}
&\mathbb{P}_\mu\left[\kappa_n(\tau) = j_n(u)\right] \\
&\qquad = \sum_{i=1}^{\infty} \mathbb{P}\left[X_1 < \frac{\gamma n}{2}, \dots, X_{i-1} < \frac{\gamma n}{2}, X_i = j_n(u) - 1, W_m \geqslant 0 \text{ for } 1 \leqslant m \leqslant i-1\right] \\
&\qquad = \mathbb{P}\left[X_1 = j_n(u) - 1\right] \sum_{i=1}^{\infty} \mathbb{P}\left[X_1 < \frac{\gamma n}{2}, \dots, X_{i-1} < \frac{\gamma n}{2}, W_m \geqslant 0 \text{ for } 1 \leqslant m \leqslant i-1\right].
\end{aligned}$$

The monotone convergence theorem entails:

$$\frac{\mathbb{P}_\mu\left[\kappa_n(\tau) = j_n(u)\right]}{\mathbb{P}\left[X_1 = j_n(u) - 1\right]} \quad \xrightarrow[n\to\infty]{} \quad \sum_{i=1}^{\infty} \mathbb{P}\left[W_m \geqslant 0 \text{ for } 1 \leqslant m \leqslant i-1\right],$$

uniformly in $u \in [-A-1, A+1]$. By Proposition 5.1.4, the last sum is equal to :

$$\sum_{i=1}^{\infty} \mathbb{P}_\mu\left[\zeta(\tau) \geqslant i\right] = \mathbb{E}_\mu\left[\zeta(\tau)\right] = \frac{1}{\gamma}$$

In addition, (5.20) gives that $\mathbb{P}\left[X_1 = j_n(u) - 1\right] / \mathbb{P}_\mu\left[\zeta(\tau) = n\right]$ converges towards 1 as $n \to \infty$, uniformly in $u \in [-A-1, A+1]$. The conclusion immediately follows. □

*Proof of Theorem 4.* We shall show that for every fixed $\eta \in (0,1)$:

$$\left(\frac{Z^{(n)}_{\lfloor \kappa_n(t_n)t \rfloor}(t_n) - \kappa_n(t_n)t/\gamma}{B'_{\kappa_n(t_n)}}, 0 \leqslant t \leqslant \eta\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (Y_t, 0 \leqslant t \leqslant \eta). \tag{5.25}$$

Recalling Lemma 5.2.3, our claim (5.23) will follow from a time-reversal argument since the vectors $(\zeta^{(n)}_1(t_n), \zeta^{(n)}_2(t_n), , \dots, \zeta^{(n)}_{\kappa_n(t_n)}(t_n))$ and $(\zeta^{(n)}_{\kappa_n(t_n)}(t_n), \zeta^{(n)}_{\kappa_n(t_n)-1}(t_n), \dots, \zeta^{(n)}_1(t_n))$ have the same distribution. Tightness follows from the time-reversal argument and also continuity at $t = 1$.

Let $F : \mathbb{D}([0,\eta],\mathbb{R}) \to \mathbb{R}_+$ be a bounded continuous function and to simplify notation set, for every tree $\tau$:

$$\widetilde{Z}^{(\kappa_n)}(\tau) = \left(\frac{Z^{(n)}_{\lfloor \kappa_n(\tau)t \rfloor}(\tau) - \kappa_n(\tau)t/\gamma}{B'_{\kappa_n(\tau)}}, 0 \leqslant t \leqslant \eta\right).$$

Fix $\epsilon > 0$. Recall that by Corollary 1, $u_\star^{(n)}(t_n) = u_\star(t_n)$ with probability tending to one as $n \to \infty$. It follows from Theorem 2 and Lemma 5.2.3 that $(\kappa_n(t_n) - \gamma n)/B'_n$ converges in distribution as $n \to \infty$. We can thus choose $A > 0$ such that for every $n$ sufficiently large:

$$\left| \mathbb{E}_\mu \left[ F\left(\widetilde{Z}^{(\kappa_n)}(\tau)\right) \mid \zeta(\tau) = n \right] - \mathbb{E}_\mu \left[ F\left(\widetilde{Z}^{(\kappa_n)}(\tau)\right) \mathbb{1}_{\{|\kappa_n(\tau) - \gamma n| \leqslant AB'_n\}} \mid \zeta(\tau) = n \right] \right| \leqslant \epsilon. \tag{5.26}$$

Without risk of confusion, in the sequel we write $\kappa_n$ instead of $\kappa_n(\tau)$, $N_0^{(n)}$ instead of $N_0^{(n)}(\tau)$ and so on. Obviously $\{\zeta(\tau) = n\} = \{Z_{\kappa_n}^{(n)} + N_0^{(n)} = n\}$. To simplify notation, let $(\mathcal{Z}_j)_{j\geqslant 0}$ and $\mathcal{N}_0^{(n)}$ be independent under $\mathbb{P}$ and such that $(\mathcal{Z}_j)_{j\geqslant 0}$ is distributed as explained before (5.24), and $\mathcal{N}_0^{(n)}$ has the same law as $N_0^{(n)}(\tau)$ under $\mathbb{P}_\mu$. Finally, set for $k \geqslant 1$:

$$\mathcal{Z}^{(k)} = \left( \frac{\mathcal{Z}_{\lfloor kt \rfloor} - kt/\gamma}{B'_k}, 0 \leqslant t \leqslant \eta \right).$$

By Lemma 5.2.5:

$$\begin{aligned}
&\mathbb{E}_\mu \left[ F\left(\widetilde{Z}^{(\kappa_n)}\right) \mathbb{1}_{\{|\kappa_n - \gamma n| \leqslant AB'_n\}} \mid \zeta(\tau) = n \right] \\
&\qquad = \sum_{|j - \gamma n| \leqslant AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}\left[ F\left(\widetilde{Z}^{(j)}\right) \mathbb{1}_{\zeta(\mathcal{F}_{1,j}) + N_0^{(n)} = n} \mid \kappa_n = j \right] \\
&\qquad = \sum_{|j - \gamma n| \leqslant AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}\left[ F\left(\mathcal{Z}^{(j)}\right) \mathbb{1}_{\{\mathcal{Z}_j + \mathcal{N}_0^{(n)} = n\}} \right].
\end{aligned} \tag{5.27}$$

Recall the notation $\varphi_j(k) = \mathbb{P}[\mathcal{Z}_j = k]$. To simplify notation, for integers $j, n \geqslant 0$ we set $\mathcal{R}_j^{(n)} = \mathbb{E}\left[ F\left(\mathcal{Z}^{(j)}\right) \mathbb{1}_{\{\mathcal{Z}_j + \mathcal{N}_0^{(n)} = n\}} \right]$. It follows from the Markov property of the random walk $(\mathcal{Z}_j)_{j\geqslant 0}$ applied at time $\lfloor \eta j \rfloor$ that:

$$\mathcal{R}_j^{(n)} = \mathbb{E}\left[ F\left(\mathcal{Z}^{(j)}\right) \varphi_{j - \lfloor \eta j \rfloor}(n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j \rfloor}) \right].$$

Recall that $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then the sum appearing in (5.27) is equal to:

$$\int_{-AB'_n + \mathcal{O}(1)}^{AB'_n + \mathcal{O}(1)} du \frac{\mathbb{P}_\mu[\kappa_n = \lfloor \gamma n + u \rfloor]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathcal{R}_{\lfloor \gamma n + u \rfloor}^{(n)} = \int_{-A + o(1)}^{A + o(1)} du B'_n \frac{\mathbb{P}_\mu[\kappa_n = j_n(u)]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathcal{R}_{j_n(u)}^{(n)} \tag{5.28}$$

We shall now show that there exist $\alpha, \beta > 0$ such that :

$$\int_{-A + o(1)}^{A + o(1)} du B'_n \frac{\mathbb{P}_\mu[\kappa_n = j_n(u)]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathcal{R}_{j_n(u)}^{(n)} \quad \underset{n \to \infty}{\longrightarrow} \quad \frac{1}{\beta\gamma} \int_{-A}^{A} du \mathbb{E}\left[ F((Y_t)_{0 \leqslant t \leqslant \eta}) p_1 \left( -\frac{u}{\beta\gamma} + \frac{Y_\eta}{\alpha} \right) \right], \tag{5.29}$$

where $p_1$ is the density of $Y_1$. Recall that $p_1$ is a bounded continuous function.

We first show that there exist $\alpha, \beta > 0$ such that for fixed $u \in \mathbb{R}$:

$$B'_n \mathcal{R}_{j_n(u)}^{(n)} \quad \underset{n \to \infty}{\longrightarrow} \quad \frac{1}{\beta} \mathbb{E}\left[ F((Y_t)_{0 \leqslant t \leqslant \eta}) p_1 \left( -\frac{u}{\beta\gamma} + \frac{Y_\eta}{\alpha} \right) \right]. \tag{5.30}$$

To this end, we start by establishing a few useful convergences. Set $\alpha = (1-\eta)^{1/(2\wedge\theta)}$. By (5.24) and Theorem 3 (iii), we have the following joint convergence in distribution:

$$\left(\mathcal{Z}^{(j_n(u))}, \frac{\mathcal{Z}_{\lfloor j_n(u)\eta\rfloor} - j_n(u)\eta/\gamma}{B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}}, \mathcal{N}_0^{(n)}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad \left((Y_t)_{0\leqslant t\leqslant \eta}, \frac{Y_\eta}{\alpha}, \mathcal{N}\right), \tag{5.31}$$

where $\mathcal{N}$ is a finite random variable independent of $Y$. Note that for every sequence of real numbers $(r_n)_{n\geqslant 1}$ such that $r_n \to \infty$ as $n \to \infty$ we have $\mathcal{N}_0^{(n)}/r_n \to 0$ as $n \to \infty$, in probability.

Set $\beta = (\gamma(1-\eta))^{1/(2\wedge\theta)}$. Since $B'_n/B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor} \to 1/\beta$ as $n\to\infty$, it immediately follows that:

$$\frac{n - \mathcal{N}_0^{(n)} - j_n(u)/\gamma}{B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad -\frac{u}{\beta\gamma} \tag{5.32}$$

We now establish (5.30). By the definition of $\mathcal{R}^{(n)}_{j_n(u)}$, we have:

$$B'_n\mathcal{R}^{(n)}_{j_n(u)} = \frac{B'_n}{B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}} \mathbb{E}\left[F\left(\mathcal{Z}^{(j_n(u))}\right) B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}\varphi_{j_n(u)-\lfloor \eta j_n(u)\rfloor}(n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u)\rfloor})\right].$$

To simplify notation, set $G_n = B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}\varphi_{j_n(u)-\lfloor \eta j_n(u)\rfloor}(n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u)\rfloor})$. By (5.14), we have:

$$\lim_{n\to\infty}\left|G_n - p_1\left(\frac{n-\mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u)\rfloor} - \frac{1}{\gamma}(j_n(u)-\lfloor \eta j_n(u)\rfloor)}{B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor}}\right)\right| = 0.$$

It follows from (5.31) and (5.32) that:

$$F\left(\mathcal{Z}^{(j_n(u))}\right) G_n \quad \xrightarrow[n\to\infty]{(d)} \quad F((Y_t)_{0\leqslant t\leqslant\eta})p_1\left(-\frac{u}{\beta\gamma} + \frac{Y_\eta}{\alpha}\right).$$

In addition, by (5.14), there exists a (deterministic) constant $C > 0$ such that $0 \leqslant G_n \leqslant C$ for every $n \geqslant 1$. Using the fact that $B'_n/B'_{j_n(u)-\lfloor \eta j_n(u)\rfloor} \to 1/\beta$, the convergence (5.30) follows from the dominated convergence theorem.

Let us now establish the convergence (5.29). By Lemma 5.2.6, the convergence

$$\frac{\mathbb{P}_\mu\left[\kappa_n = j_n(u)\right]}{\mathbb{P}_\mu\left[\zeta(\tau)=n\right]} \quad \xrightarrow[n\to\infty]{} \quad \frac{1}{\gamma} \tag{5.33}$$

holds uniformly in $u \in [-A-1, A+1]$. Moreover, it is clear from (5.14) that $B'_n\mathcal{R}^{(n)}_{j_n(u)}$ is bounded uniformly in $n \geqslant 1$ and $u \in [-A-1, A+1]$. The convergence (5.29) then follows from an application of the dominated convergence theorem after taking into account (5.33) and (5.30).

By Fubini's theorem:

$$\begin{aligned}\frac{1}{\beta\gamma}\int_{-\infty}^{\infty} du\mathbb{E}\left[F((Y_t)_{0\leqslant t\leqslant\eta})p_1\left(-\frac{u}{\beta\gamma}+\frac{Y_\eta}{\alpha}\right)\right] &= \frac{1}{\beta\gamma}\mathbb{E}\left[F((Y_t)_{0\leqslant t\leqslant\eta})\int_{-\infty}^{\infty} dup_1\left(-\frac{u}{\beta\gamma}+\frac{Y_\eta}{\alpha}\right)\right] \\ &= \mathbb{E}\left[F((Y_t)_{0\leqslant t\leqslant\eta})\right]\end{aligned}$$

where we have used the fact that $p_1$ is a probability distribution in the second equality. Hence we can choose $A > 0$ sufficiently large so that in addition to (5.26) we also have:

$$\left| \frac{1}{\beta\gamma} \int_{-A}^{A} du \mathbb{E}\left[ F((Y_t)_{0\leqslant t\leqslant \eta}) p_1 \left( -\frac{u}{\beta\gamma} + \frac{Y_\eta}{\alpha} \right) \right] - \mathbb{E}\left[ F((Y_t)_{0\leqslant t\leqslant \eta}) \right] \right| \leqslant \epsilon.$$

By putting (5.27), (5.28) and (5.29) together, we get that for $n$ sufficiently large:

$$\left| \mathbb{E}_\mu \left[ F\left( \widetilde{Z}^{(\kappa_n)} \right) \mid \zeta(\tau) = n \right] - \mathbb{E}\left[ F((Y_t)_{0\leqslant t\leqslant \eta}) \right] \right| \leqslant 2\epsilon.$$

This establishes (5.25) and completes the proof. □

*Proof of Corollary 2.* By Theorem 1 (iv), $\Delta(t_n)/n$ converges in probability towards $\gamma$ as $n \to \infty$ so that $B_{\Delta(t_n)}/B_n$ converges in probability towards $\gamma^{1/(2\wedge\theta)}$. In addition, the map $Z \mapsto \sup_{s\in[0,1]}(Z_s - Z_{s-})$ is continuous on $\mathbb{D}([0,1],\mathbb{R})$. It follows from Theorem 4 that:

$$\frac{1}{B_n} \max_{1\leqslant i\leqslant \Delta(t_n)} \zeta_i(t_n) \quad \xrightarrow[n\to\infty]{(d)} \quad \frac{1}{\gamma} \sup_{s\in[0,1]} \Delta Y_s.$$

If $\theta \geqslant 2$, $Y$ is continuous and the first assumption of Corollary 2 follows. If $\theta < 2$, the result easily follows from the fact that the Lévy measure of $Y$ is $\nu(dx) = \mathbb{1}_{\{x>0\}} dx/(\Gamma(-\theta)x^{1+\theta})$. □

### 5.2.5 Height of large conditioned non-generic trees

We now prove Theorem 5. We keep the notation $u_\star(\tau)$ of the Introduction as well as the notation $u_\star^{(n)}, \kappa_n, \mathcal{F}_{i,j}^{(n)}$ and $\varphi_j(k)$ introduced in the previous section. Recall also the notation $\mathbb{P}_{\mu,j}$ for the law of a forest of $j$ independent $GW_\mu$ trees. If $\mathbf{f} = (\tau_1, \ldots, \tau_k)$ is a forest, its height $\mathcal{H}(\mathbf{f})$ is by definition $\max(\mathcal{H}(\tau_1), \ldots, \mathcal{H}(\tau_k))$.

*Proof of Theorem 5.* If $\tau$ is a tree, let $\mathcal{H}_\star^{(n)}(\tau) = \mathcal{H}\left( T_{u_\star^{(n)}} \tau \right)$ be the height of the subtree of descendants of $u_\star^{(n)}$ in $\tau$. Recall that by Corollary 1, $u_\star^{(n)}(t_n) = u_\star(t_n)$ with probability tending to one as $n \to \infty$ and that by Theorem 3 (ii), the generation of $u_\star^{(n)}(t_n)$ converges in distribution. It is thus sufficient to establish that, if $(\lambda_n)_{n\geqslant 1}$ of positive real numbers tending to infinity:

$$\mathbb{P}\left[ \left| \mathcal{H}_\star^{(n)}(t_n) - \frac{\ln(n)}{\ln(1/m)} \right| \leqslant \lambda_n \right] \quad \xrightarrow[n\to\infty]{} \quad 1. \tag{5.34}$$

To simplify notation, set $\mathcal{H}_{i,j}^{(n)}(\tau) = \mathcal{H}(\mathcal{F}_{i,j}^{(n)})$ and $\zeta_{i,j}^{(n)} = \zeta(\mathcal{F}_{i,j}^{(n)})$. Set $p_n = \ln(n)/\ln(1/m) - \lambda_n$. Let us first prove the lower bound, that is $\mathbb{P}\left[ \mathcal{H}_\star^{(n)}(t_n) \leqslant p_n \right] \to 0$ as $n \to \infty$. Fix $0 < \epsilon < \gamma$. As in the proof of Theorem 4, we can choose $A > 0$ such that for $n$ sufficiently large:

$$\mathbb{P}_\mu \left[ \left| \frac{\kappa_n(\tau) - \gamma n}{B'_n} \right| \geqslant A \;\middle|\; \zeta(\tau) = n \right] < \epsilon.$$

Then for $n$ large enough:

$$
\begin{aligned}
\mathbb{P}\left[\mathcal{H}_\star^{(n)}(t_n) \leqslant p_n\right] &= \mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor \kappa_n \rfloor} \leqslant p_n - 1 \,|\, \zeta(\tau) = n\right] \\
&\leqslant \mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor \kappa_n/2 \rfloor} \leqslant p_n \,|\, \zeta(\tau) = n\right] \\
&\leqslant \sum_{|j-\gamma n| \leqslant AB'_n} \frac{\mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor \kappa_n/2 \rfloor} \leqslant p_n, \kappa_n = j, \zeta(\tau) = n\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]} + \epsilon \\
&= \sum_{|j-\gamma n| \leqslant AB'_n} \frac{\mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor j/2 \rfloor} \leqslant p_n, \kappa_n = j, \zeta^{(n)}_{1,\lfloor j/2 \rfloor} + \zeta^{(n)}_{\lfloor j/2 \rfloor+1,j} + N_0^{(n)} = n\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]} + \epsilon
\end{aligned}
$$

To simplify notation, let $\mathfrak{f}_{\lfloor j/2 \rfloor}$ and $\mathcal{N}_0^{(n)}$ be two independent random variables defined under $\mathbb{P}$ such that $\mathfrak{f}_{\lfloor j/2 \rfloor}$ is distributed according to $\mathbb{P}_{\mu,\lfloor j/2 \rfloor}$ and $\mathcal{N}_0^{(n)}$ has the same law as $N_0^{(n)}(\tau)$ under $\mathbb{P}_\mu$. Then, by Lemma 5.2.5:

$$
\begin{aligned}
&\frac{1}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]}\mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor j/2 \rfloor} \leqslant p_n, \kappa_n = j, \zeta^{(n)}_{1,\lfloor j/2 \rfloor} + \zeta^{(n)}_{\lfloor j/2 \rfloor+1,j} + N_0^{(n)} = n\right] \\
&\qquad = \frac{\mathbb{P}_\mu\left[\kappa_n = j\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]}\mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor j/2 \rfloor} \leqslant p_n, \zeta^{(n)}_{1,\lfloor j/2 \rfloor} + \zeta^{(n)}_{\lfloor j/2 \rfloor+1,j} + N_0^{(n)} = n \,\middle|\, \kappa_n = j\right] \\
&\qquad = \frac{\mathbb{P}_\mu\left[\kappa_n = j\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]}\mathbb{E}\left[\mathbb{1}_{\{\mathcal{H}(\mathfrak{f}_{\lfloor j/2 \rfloor}) \leqslant p_n\}} \cdot \varphi_{j-\lfloor j/2 \rfloor}\left(n - \mathcal{N}_0^{(n)} - \zeta(\mathfrak{f}_{\lfloor j/2 \rfloor})\right)\right]
\end{aligned}
$$

where we use the notation $\varphi_j(k) = \mathbb{P}_{\mu,j}\left[\zeta(\mathbf{f}) = k\right]$ as in the proof of Theorem 4. It follows that:

$$
\begin{aligned}
&\mathbb{P}\left[\mathcal{H}_\star^{(n)}(t_n) \leqslant p_n\right] \\
&\qquad \leqslant \sum_{|j-\gamma n| \leqslant AB'_n} \frac{\mathbb{P}_\mu\left[\kappa_n = j\right]}{\mathbb{P}_\mu\left[\zeta(\tau) = n\right]}\mathbb{E}\left[\mathbb{1}_{\{\mathcal{H}(\mathfrak{f}_{\lfloor j/2 \rfloor}) \leqslant p_n\}} \cdot \varphi_{j-\lfloor j/2 \rfloor}\left(n - \mathcal{N}_0^{(n)} - \zeta(\mathfrak{f}_{\lfloor j/2 \rfloor})\right)\right] \\
&\qquad\qquad + \quad \epsilon. \qquad (5.35)
\end{aligned}
$$

We now claim that it is sufficient to establish that for $n$ sufficiently large, for every $j \in [\gamma n - AB'_n, \gamma n + AB'_n]$ and $k \in \mathbb{Z}$ the following three estimates hold with some constants $C_1$ and $C_2$:

$$
\begin{cases}
\varphi_{j-\lfloor j/2 \rfloor}(k) \leqslant C_1/B'_n. & (5.36) \\
\mathbb{P}_\mu\left[\kappa_n = j\right] \leqslant C_2 \mathbb{P}_\mu\left[\zeta(\tau) = n\right] & (5.37) \\
\mathbb{P}\left[\mathcal{H}(\mathfrak{f}_{\lfloor j/2 \rfloor}) \leqslant p_n\right] \leqslant \epsilon/A & (5.38)
\end{cases}
$$

Indeed, from (5.35), we will then get that

$$\mathbb{P}\left[\mathcal{H}_\star^{(n)}(t_n) \leqslant p_n\right] \leqslant ((2A+1)C_1C_2/A + 1)\epsilon,$$

and the lower bound will follow.

The bound (5.36) is an immediate consequence of the local limit theorem (5.14). The estimate (5.37) follows from Lemma 5.2.6. Let us finally establish (5.38). Note that for $n$ sufficiently large all the indices j appearing in the sum (5.35) satisfy $j \geqslant (\gamma-\epsilon)n$. Consequently, for n sufficiently large:

$$\mathbb{P}\left[\mathcal{H}(\mathfrak{f}_{\lfloor j/2\rfloor}) \leqslant p_n\right] \leqslant \left(1-\mathbb{P}_\mu\left[\mathcal{H}(\tau) > p_n\right]\right)^{\lfloor(\gamma-\epsilon)n/2\rfloor}. \tag{5.39}$$

Since $\mu$ satisfies Assumption $(H_\theta)$, we have $\sum_{i\geqslant 1} i\ln(i)\mu_i < \infty$ . It follows from [55, Theorem 2] that there exists a constant $c > 0$ such that:

$$\mathbb{P}_\mu\left[\mathcal{H}(\tau) > k\right] \quad \underset{k\to\infty}{\sim} \quad c \cdot \mathfrak{m}^k. \tag{5.40}$$

Hence the right-hand side of (5.39) tends to 0 as $n \to \infty$. The estimate (5.38) follows, and the proof of the lower bound is complete.

Set $q_n = \ln(n)/\ln(1/\mathfrak{m}) + \lambda_n$. The proof of the fact that $\mathbb{P}\left[\mathcal{H}_\star^{(n)}(t_n) \geqslant q_n\right] \to 0$ as $n \to \infty$ is similar and we only sketch the argument. Write:

$$\mathbb{P}\left[\mathcal{H}_\star^{(n)}(t_n) \geqslant q_n\right] \leqslant \mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{1,\lfloor\kappa_n/2\rfloor} \geqslant q_n \,|\, \zeta(\tau)=n\right] + \mathbb{P}_\mu\left[\mathcal{H}^{(n)}_{\lfloor\kappa_n/2\rfloor+1,\kappa_n} \geqslant q_n \,|\, \zeta(\tau)=n\right]$$

Under $\mathbb{P}_\mu$, $\mathcal{H}^{(n)}_{\lfloor\kappa_n/2\rfloor+1,\kappa_n}$ has the same distribution as $\mathcal{H}^{(n)}_{1,\kappa_n-\lfloor\kappa_n/2\rfloor}$. It thus sufficies to show that the first term of the last sum tends to 0 as $n \to \infty$. By similar arguments as in the proof of the lower bound, it is enough to verify that

$$\mathbb{P}_{\mu,\lfloor(\gamma+\epsilon)n/2\rfloor}\left[\mathcal{H}(\mathbf{f}) \geqslant q_n\right] \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

This follows from the fact that $\mathbb{P}_{\mu,\lfloor(\gamma+\epsilon)n/2\rfloor}\left[\mathcal{H}(\mathbf{f}) \geqslant q_n\right] = 1-\left(1-\mathbb{P}_\mu\left[\mathcal{H}(\tau) \geqslant q_n\right]\right)^{\lfloor(\gamma+\epsilon)n/2\rfloor}$ and (5.40). This completes the proof of the upper bound and establishes (5.34). □

### 5.2.6 The contour and height functions of subcritical trees

We now turn our attention to the contour and height functions of $t_n$ and prove Theorem 6.

*Proof of Theorem 6.* The first assertion is an immediate consequence of the convergence in probability:

$$\frac{\mathcal{H}(t_n)}{\ln(n)} \quad \overset{(\mathbb{P})}{\underset{n\to\infty}{\longrightarrow}} \quad 1/\ln(1/\mathfrak{m}) \tag{5.41}$$

which follows from Theorem 5.

We only prove the second assertion when $Y^{(n)} = (H_{nt}(t_n)/r_n, 0 \leqslant t \leqslant 1)$ (the case of the contour function is similar and left to the reader). First note that if $r_n$ does not tend to infinity as $n \to \infty$ then it follows from (5.41) that $\left(Y^{(n)}\right)_{n\geqslant 1}$ is not tight. In the sequel, we thus suppose that $r_n \to \infty$ as $n \to \infty$. Since $r_n/\ln(n)$ does not tend to infinity, there exist $\eta > 0$ and a subsequence $(r_{a_n}, n \geqslant 1)$ such that $\ln(a_n) \geqslant \eta r_{a_n}$ for every $n \geqslant 1$. In particular:

$$\sup Y^{(a_n)} = \frac{\mathcal{H}(t_{a_n})}{r_{a_n}} \quad \overset{(\mathbb{P})}{\underset{n\to\infty}{\not\longrightarrow}} \quad 0, \tag{5.42}$$

by (5.41). We argue by contradiction and suppose that $(Y^{(n)})_{n\geqslant 1}$ is tight. This implies that there exists a subsequence of $(Y^{(a_n)})_{n\geqslant 1}$ that converges in distribution towards a continuous random process denoted by $\mathcal{Y}$. To simplify notation, in the sequel, all values of $n$ will belong to this subsequence. Fix $\epsilon > 0$. We shall show that with probability tending to one as $n \to \infty$, we can find a (random) sequence of integers $1 \leqslant U_0 < U_1 < \cdots < U_N \leqslant n$ such that the following two conditions are satisfied:

(i) $U_0 \leqslant \epsilon n, U_N \geqslant (1-\epsilon)n$ and $U_{k+1} - U_k \leqslant \epsilon n$ for every $0 \leqslant k \leqslant N-1$,

(ii) $H_{U_k}(t_n)/r_n \leqslant 2\epsilon$ for every $0 \leqslant k \leqslant N$.

This will indeed imply that $\mathcal{Y} = 0$, in contradiction with (5.42).

Recall the notation $u_\star(t_n)$ for the smallest vertex of $t_n$ of maximal out-degree and the notation $U(t_n)$ for its index. By Theorem 3 (ii), the generation of $u_\star(t_n)$ converges in distribution as $n \to \infty$. Then, since $r_n \to \infty$ as $n \to \infty$:

$$\frac{H_{U(t_n)}(t_n)}{r_n} = \frac{|u_\star(t_n)|}{r_n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0. \tag{5.43}$$

Now recall the notation $\Delta(t_n)$ for the number of children of $u_\star(t_n)$. Set $U_0 = U(t_n)$ and for $1 \leqslant k \leqslant \Delta(t_n)$ set:

$$U_k = \inf\{i \geqslant U_0 + 1;\ \mathcal{W}_i(t_n) = \mathcal{W}_{U_0+1}(t_n) - (k-1)\}.$$

First note that by the convergence in Theorem 1 (iii) and standard properties of the Skorokhod topology, property (i) holds with probability tending to one as $n \to \infty$. Next, the integers $U_1, U_2, \ldots, U_{\Delta(t_n)}$ are the indices of the children of $u_\star(t_n)$ (see e.g. [73]). As a consequence, $H_{U_k}(t_n) = H_{U_0}(t_n) + 1$ for $1 \leqslant k \leqslant \Delta(t_n)$. By (5.43), it follows that:

$$\frac{1}{r_n} \sup_{0 \leqslant k \leqslant \Delta(t_n)} H_{U_k}(t_n) \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0.$$

Hence property (ii) holds with probability tending to one as $n \to \infty$, completing the proof. □

## 5.3 Extensions and comments

We conclude by proposing possible extensions and stating a few open questions.

**Other types of conditioning.** Throughout this text, we have only considered the case of Galton-Watson trees conditioned on having a fixed total progeny. It is natural to consider different types of conditioning. For instance, for $n \geqslant 1$, let $t_n^h$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\mathcal{H}(\tau) \geqslant n]$. In [59, Section 22], Janson has in particular proved that when $\mu$ is critical or subcritical, as $n \to \infty$, $t_n^h$ converges locally to a random infinite tree $\mathcal{T}^*$, which is different from $\mathcal{T}$. It would be interesting to know whether the theorems of the present work apply in this case.

Another type of conditioning involving the number of leaves has been introduced in [28, 70, 94]. If $\tau$ is a tree, denote by $\lambda(\tau)$ the number of leaves of $\tau$ (that is the number of individuals with no child). For $n \geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$, let $t_n^l$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\lambda(\tau) = n]$. Do results similar to those we have obtained hold when $t_n$ is replaced by $t_n^l$? We expect the answer to be positive, since a $GW_\mu$ tree with $n$ leaves is very close to a $GW_\mu$ with total progeny $n/\mu_0$ (see [70] for details), and we believe that the techniques of the present work can be adapted to solve this problem.

**Concentration of $\mathcal{H}(t_n)$ around $\ln(n)/\ln(1/m)$.** By Theorem 5, the sequence of random variable $(\mathcal{H}(t_n) - \ln(n)/\ln(1/m))_{n\geqslant 1}$ is tight. It is therefore natural to ask the following question, due to Nicolas Broutin. Does there exist a random variable $\mathscr{H}$ such that:

$$\mathcal{H}(t_n) - \frac{\ln(n)}{\ln(1/m)} \quad \xrightarrow[n\to\infty]{(d)} \quad \mathscr{H} \quad ?$$

We expect the answer to be negative. Let us give a heuristic argument to support this prediction. In the proof of Theorem 5, we have seen that the height of $\mathcal{H}(t_n)$ is close to the height of $\lfloor \gamma n \rfloor$ independent $\mathrm{GW}_\mu$ trees and the height of each of these trees satisfies the estimate (5.40). However, if $(Q_i)_{i\geqslant 1}$ is an i.i.d. sequence of random variables such that $\mathbb{P}[Q_1 \geqslant k] = c \cdot m^k$, then it is known (see e.g. [58, Example 4.3]) that the random variables

$$\max(Q_1, Q_2, \dots, Q_n) - \frac{\ln(n)}{\ln(1/m)}$$

do not converge in distribution.

**Behavior of $\mathbb{E}[\mathcal{H}(t_n)]$.** We conjecture that:

$$\mathbb{E}[\mathcal{H}(t_n)] \quad \underset{n\to\infty}{\sim} \quad \frac{\ln(n)}{\ln(1/m)}.$$

However, it seems that more precise estimates that the ones we have used are needed to prove this statement.

**Other types of trees**. Janson [59] gives a very general limit theorem concerning the local asymptotic behavior of simply generated trees conditioned on having a fixed large number of vertices. Let us briefly recall the definition of simply generated trees. Fix a sequence $\mathbf{w} = (w_k)_{k\geqslant 0}$ of nonnegative real numbers such that $w_0 > 0$ and such that there exists $k > 1$ with $w_k > 0$ ($\mathbf{w}$ is called a weight sequence). Let $\mathbb{T}_f \subset \mathbb{T}$ be the set of all finite plane trees and, for every $n \geqslant 1$, let $\mathbb{T}_n$ be the set of all plane trees with $n$ vertices. For every $\tau \in \mathbb{T}_f$, define the weight $w(\tau)$ of $\tau$ by:

$$w(\tau) = \prod_{u\in\tau} w_{k_u(\tau)}.$$

Then for $n \geqslant 1$ set

$$Z_n = \sum_{\tau\in\mathbb{T}_n} w(\tau).$$

For every $n \geqslant 1$ such that $Z_n \neq 0$, let $\mathcal{T}_n$ be a random tree taking values in $\mathbb{T}_n$ such that for every $\tau \in \mathbb{T}_n$:

$$\mathbb{P}[\mathcal{T}_n = \tau] = \frac{w(\tau)}{Z_n}.$$

The random tree $\mathcal{T}_n$ is said to be finitely generated. Galton-Watson trees conditioned on their total progeny are particular instances of simply generated trees. Conversely, if $\mathcal{T}_n$ is as above, there exists an offspring distribution $\mu$ such that $\mathcal{T}_n$ has the same distribution as a $\mathrm{GW}_\mu$ tree conditioned on having $n$ vertices if, and only if, the radius of convergence of $\sum w_i z^i$ is positive (see [59, Section 8]).

It would thus be interesting to find out if the theorems obtained in the present work for Galton-Watson trees can be extended to the setting of simply generated trees whose associated radius of convergence is $0$. In the latter case, Janson [59] proved that $\mathcal{T}_n$ converges locally as $n \to \infty$ towards a deterministic tree consisting of a root vertex with an infinite number of leaves attached to it. We thus expect that the asymptotic properties derived in the present work will take a different form in this case. We hope to investigate this in future work.

# Troisième Partie

# *Laminations et dissections aléatoires du disque*

CHAPITRE 6

# Random stable laminations of the disk

*Les résultats de ce chapitre sont issus de l'article [69], accepté pour publication dans* Ann. Probab.

## Contenu de ce chapitre




We study large random dissections of polygons. We consider random dissections of a regular polygon with $n$ sides, which are chosen according to Boltzmann weights in the domain of attraction of a stable law of index $\theta \in (1, 2]$. As $n$ goes to infinity, we prove that these random dissections converge in distribution towards a random compact set, called the random stable lamination. If $\theta = 2$, we recover Aldous' Brownian triangulation. However, if $\theta \in (1, 2)$, large faces remain in the limit and a different random compact set appears. We show that the random stable lamination can be coded by the continuous-time height function associated to the normalized excursion of a strictly stable spectrally positive Lévy process of index $\theta$. Using this coding, we establish that the Hausdorff dimension of the stable random lamination is almost surely $2 - 1/\theta$.

# Introduction

In this article, we study large random dissections of polygons. A *dissection* of a polygon is the union of the sides of the polygon and of a collection of diagonals that may intersect only at their endpoints. The faces are the connected components of the complement of the dissection in the polygon. The particular case of triangulations (when all faces are triangles) has been extensively studied in the literature. For every integer $n \geqslant 3$, let $P_n$ be the regular polygon with $n$ sides whose vertices are the $n$-th roots of unity. It is well known that the number of triangulations of $P_n$ is the Catalan number of order $n-2$. In the general case, where faces of degree greater than three are allowed, there is no known explicit formula for the number of dissections of $P_n$, although an asymptotic estimate is known (see [45, 28]). Probabilistic aspects of uniformly distributed random triangulations have been investigated, see e.g. the articles [47, 48] which study graph-theoretical properties of uniform triangulations (such as the maximal vertex degree or the number of vertices of degree $k$). Graph-theoretical properties of uniform dissections of $P_n$ have also been studied, extending the previously mentioned results for triangulations (see [14, 28]).

From a more geometrical point of view, Aldous [6, 7] studied the shape of a large uniform triangulation viewed as a random compact subset of the closed unit disk. See also the work of Curien and Le Gall [27], who discuss a random continuous triangulation (different from Aldous' one) obtained as a limit of random dissections constructed recursively. Our goal is to generalize Aldous' result by studying the shape of large random dissections of $P_n$, viewed as random variables with values in the space of all compact subsets of the disk, which is equipped with the usual Hausdorff metric.

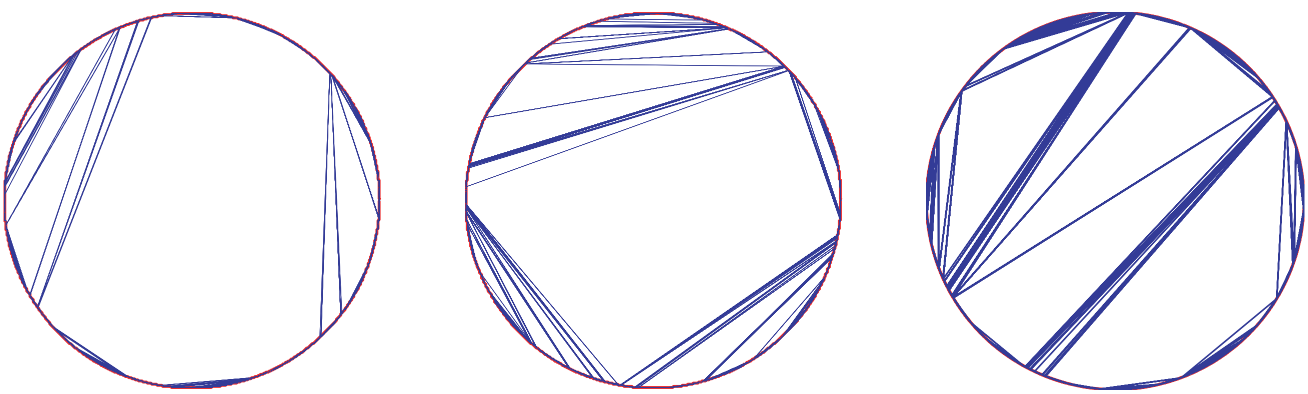

**Figure 6.1:** Random dissections of $P_{27183}$ for $\theta = 1.1$, $P_{11655}$ for $\theta = 1.5$ and of $P_{20999}$ for $\theta = 1.9$.

Let us state more precisely Aldous' results. Denote by $t_n$ a uniformly distributed random triangulation of $P_n$. There exists a random compact subset $t$ of the closed unit disk $\overline{\mathbb{D}}$ such that the sequence $(t_n)$ converges in distribution towards $t$. The random compact set $t$ is a continuous triangulation, in the sense that $\overline{\mathbb{D}} \backslash t$ is a disjoint union of open triangles whose vertices belong to the unit circle. Aldous also explains how $t$ can be explicitly constructed using the Brownian excursion and computes the Hausdorff dimension of $t$, which is equal almost surely to $3/2$ (see also [77]).

In this work, we propose to study the following generalization of this model. Consider a probability distribution $(\mu_j)_{j\geqslant 0}$ on the nonnegative integers such that $\mu_1 = 0$ and the mean of $\mu$ is equal to 1. We suppose that $\mu$ is in the domain of attraction of a stable law of index $\theta \in (1, 2]$. For every integer $n \geqslant 2$, let $\mathcal{D}_n$ be the set of all dissections of $P_{n+1}$, and consider the following Boltzmann probability measure on $\mathcal{D}_n$ associated to the weights $(\mu_j)$:

$$\mathbb{P}_n^\mu(\omega) = \frac{1}{Z_n} \prod_{f \text{ face of } \omega} \mu_{\deg(f)-1}, \qquad \omega \in \mathcal{D}_n,$$

where $\deg(f)$ is the degree of the face $f$, that is the number of edges in the boundary of $f$, and $Z_n$ is a normalizing constant. Note that the definition of $\mathbb{P}_n^\mu$ involves only $\mu_2, \mu_3, \ldots$, and $\mu_0$ is the missing constant to obtain a probability measure. Under appropriate conditions on $\mu$, this definition makes sense for all sufficiently large integers $n$. Let us mention two important special cases. If $\mu_0 = 2 - \sqrt{2}$ and $\mu_i = ((2-\sqrt{2})/2)^{i-1}$ for every $i \geqslant 2$, one easily checks that $\mathbb{P}_n^\mu$ is uniform over $\mathcal{D}_n$. If $p \geqslant 3$ is an integer and if $\mu_0 = 1 - 1/(p-1)$, $\mu_{p-1} = 1/(p-1)$ and $\mu_i = 0$ otherwise, $\mathbb{P}_n^\mu$ is uniform over dissections of $\mathcal{D}_n$ with all faces of degree $p$ (in that case, we must restrict our attention to values of $n$ such that $n-1$ is a multiple of $p-2$, but our results carry over to this setting).

We are interested in the following problem. Let $\mathcal{D}_n$ be a random dissection distributed according to $\mathbb{P}_n$. Does the sequence $(\mathcal{D}_n)$ converge in distribution to a random compact subset of $\overline{\mathbb{D}}$ ? Let us mention that this setting is inspired by [75], where Le Gall and Miermont consider random planar maps chosen according to a Boltzmann probability measure, and show that if the Boltzmann weights do not decrease sufficiently fast, large faces remain in the scaling limit. We will see that this phenomenon occurs in our case as well.

In our main result Theorem 6.3.1, we first consider the case where the variance of $\mu$ is finite and then show that $\mathcal{D}_n$ converges in distribution to Aldous' Brownian triangulation as $n \to \infty$. This extends Aldous' theorem to random dissections which are not necessarily triangulations. For instance, we may let $\mathcal{D}_n$ be uniformly distributed over the set of all dissections whose faces are all quadrangles (or pentagons, or hexagons, etc.). As noted above, this requires that we restrict our attention to a subset of values of $n$, but the convergence of $\mathcal{D}_n$ towards the Brownian triangulation still holds. This maybe surprising result comes from the fact that certain sides of the squares (or of the pentagons, or of the hexagons, etc.) degenerate in the limit. See also the recent paper [28] for other classes of non-crossing configurations of the polygon that converge to the Brownian triangulation.

On the other hand, if $\mu$ is in the domain of attraction of a stable law of index $\theta \in (1, 2)$, Theorem 6.3.1 shows that $(\mathcal{D}_n)$ converges in distribution to another random compact subset $\mathfrak{l}$ of $\overline{\mathbb{D}}$, which we call the $\theta$-stable random lamination of the disk. The random compact subset $\mathfrak{l}$ is the union of the unit circle and of infinitely many non-crossing chords, which can be constructed as follows. Let $X^{\text{exc}} = (X^{\text{exc}}_t)_{0\leqslant t\leqslant 1}$ be the normalized excursion of the strictly stable spectrally positive Lévy process of index $\theta$ (see Section 2.1 for a precise definition). For $0 \leqslant s < t \leqslant 1$, we set $s \simeq^{X^{\text{exc}}} t$ if $t = \inf\{u > s;\ X^{\text{exc}}_u \leqslant X^{\text{exc}}_{s-}\}$, and $s \simeq^{X^{\text{exc}}} s$ by convention. Then:

$$\mathfrak{l} = \bigcup_{s \simeq^{X^{\text{exc}}} t} [e^{-2i\pi s}, e^{-2i\pi t}], \tag{6.1}$$

where $[u, v]$ stands for the line segment between the two complex numbers $u$ and $v$. In particular, the latter set is compact, which is not obvious *a priori*.

In order to study fine properties of the set $\mathfrak{l}$, we derive an alternative representation in terms of the so-called height process $H^{exc} = (H^{exc}_t)_{0\leqslant t\leqslant 1}$ associated with $X^{exc}$ (see [36, 37] for the definition and properties of $H^{exc}$). Note that $H^{exc}$ is a random continuous function on $[0,1]$ that vanishes at $0$ and at $1$ and takes positive values on $(0,1)$. Then Theorem 6.4.5 states that:

$$\mathfrak{l} = \bigcup_{s \approx^{H^{exc}} t} [e^{-2i\pi s}, e^{-2i\pi t}], \tag{6.2}$$

where, for $s,t \in [0,1]$, $s \approx^{H^{exc}} t$ if $H^{exc}_s = H^{exc}_t$ and $H^{exc}_r > H^{exc}_s$ for every $r \in (s\wedge t, s\vee t)$, or if $(s,t)$ is a limit of pairs satisfying these properties. This is very closely related to the equivalence relation used to define the so-called stable tree, which is coded by $H^{exc}$ (see [36]). The representation (6.2) thus shows that the $\theta$-stable random lamination is connected to the $\theta$-stable tree in the same way as the Brownian triangulation is connected to the Brownian CRT (see [7] for applications of the latter connection). The representation (6.2) also allows us to establish that the Hausdorff dimension of $\mathfrak{l}$ is almost surely equal to $2-1/\theta$. Note that for $\theta = 2$, we obtain a Hausdorff dimension equal to $3/2$, which is consistent with Aldous' result. Additionally, we verify that the Hausdorff dimension of the set of endpoints of all chords in $\mathfrak{l}$ is equal to $1-1/\theta$.

Finally, we derive precise information about the faces of $\mathfrak{l}$, which are the connected components of the complement of $\mathfrak{l}$ in the closed unit disk. When $\theta = 2$, we already noted that all faces are triangles. On the other hand, when $\theta \in (1,2)$ each face is bounded by infinitely many chords. We prove more precisely that the set of extreme points of the closure of a face (or equivalently the set of points of the closure that lie on the circle) has Hausdorff dimension $1/\theta$.

Let us now sketch the main techniques and arguments used to establish the previous assertions. A key ingredient is the fact that the dual graph of $\mathcal{D}_n$ is a Galton-Watson tree conditioned on having $n$ leaves. In our previous work [70], we establish limit theorems for Galton-Watson trees conditioned on their number of leaves and, in particular, we prove an invariance principle stating that the rescaled Lukasiewicz path of a Galton-Watson tree conditioned on having $n$ leaves converges in distribution to $X^{exc}$ (see Theorem 6.3.3 below). Using this result, we are able to show that $\mathcal{D}_n$ converges towards the random compact set $\mathfrak{l}$ described by (6.1). The representation (6.2) then follows from relations between $X^{exc}$ and $H^{exc}$. Finally, we use (6.2) to verify that the Hausdorff dimension of $\mathfrak{l}$ is almost surely equal to $2-1/\theta$. This calculation relies in part on the time-reversibility of the process $H^{exc}$. It seems more difficult to derive the Hausdorff dimension of $\mathfrak{l}$ from the representation (6.1).

The paper [28] develops a number of applications of the present work to enumeration problems and asymptotic properties of uniformly distributed random dissections.

The paper is organized as follows. In Section 1, we present the discrete framework. In particular, we introduce Galton-Watson trees and their coding functions. In Section 2, we discuss the normalized excursion of the strictly stable spectrally positive Lévy process of index $\theta$ and its associated lamination $L(X^{exc})$. In Section 3, we prove that $(\mathcal{D}_n)$ converges in distribution towards $L(X^{exc})$. In Section 4, we start by introducing the continuous-time height process $H^{exc}$ associated to $X^{exc}$ and we then show that $L(X^{exc})$ can be coded by $H^{exc}$. In Section 5, we use the time-reversibility of $H^{exc}$ to calculate the Hausdorff dimension of the stable lamination.

Throughout this work, the notation $\overline{A}$ stands for the closure of a subset $A$ of the plane.

**Acknowledgments.** I am deeply indebted to Jean-François Le Gall for suggesting me to study this model, for insightful discussions and for carefully reading the manuscript and making many useful suggestions.

## 6.1 The discrete setting: dissections and trees

### 6.1.1 Boltzmann dissections

**Definition 6.1.1.** A *dissection* of a polygon is the union of the sides of the polygon and of a collection of diagonals that may intersect only at their endpoints. A *face* $f$ of a dissection $\omega$ of a polygon $P$ is a connected component of the complement of $\omega$ inside $P$; its degree, denoted by $\deg(f)$, is the number of sides surrounding $f$. See Figure 6.1 for an example.

Let $(\mu_i)_{i \geqslant 2}$ be a sequence of nonnegative real numbers. For every integer $n \geqslant 3$, let $P_n$ be the regular polygon of the plane whose vertices are the $n$-th roots of unity. For every $n \geqslant 2$, let $\mathcal{D}_n$ be the set of all dissections of $P_{n+1}$. Note that $\mathcal{D}_n$ is a finite set. Let $\mathbb{L} = \cup_{n \geqslant 2} \mathcal{D}_n$ be the set of all dissections. A weight $\pi(\omega)$ is associated to each dissection $\omega \in \mathcal{D}_n$ by setting:

$$\pi(\omega) = \prod_{f \text{ face of } \omega} \mu_{\deg(f)-1}.$$

We define a probability measure on $\mathcal{D}_n$ by normalizing these weights. More precisely, we set:

$$Z_n = \sum_{w \in \mathcal{D}_n} \pi(w), \tag{6.3}$$

and for every $n \geqslant 2$ such that $Z_n > 0$:

$$\mathbb{P}_n^{\mu}(\omega) = \frac{1}{Z_n} \pi(\omega)$$

for $\omega \in \mathcal{D}_n$.

We are interested in the asymptotic behavior of random dissections sampled according to $\mathbb{P}_n^{\mu}$. Let $\overline{\mathbb{D}}$ be the closed unit disk of the complex plane and let $\mathcal{C}$ be the set of all compact subsets of $\overline{\mathbb{D}}$. We equip $\mathcal{C}$ with the Hausdorff distance $d_H$, so that $(\mathcal{C}, d_H)$ is a compact metric space. In the following, we will always view a dissection as an element of this metric space.

We are interested in the following question. For every $n \geqslant 2$ such that $Z_n > 0$, let $\mathcal{D}_n$ be a random dissection distributing according to $\mathbb{P}_n^{\mu}$. Does there exist a limiting random compact set $\mathfrak{l}$ such that $\mathcal{D}_n$ converges in distribution towards $\mathfrak{l}$ ?

We shall answer this question for some specific families of sequences $(\mu_i)_{i \geqslant 2}$ defined as follows. Let $\theta \in (1, 2]$. We say that a sequence of nonnegative real numbers $(\mu_j)_{j \geqslant 2}$ satisfies the condition $(H_\theta)$ if:

- $\mu$ is critical, meaning that $\sum_{i=2}^{\infty} i\mu_i = 1$. Note that this condition implies $\sum_{i=2}^{\infty} \mu_i < 1$.
- Set $\mu_1 = 0$ and $\mu_0 = 1 - \sum_{i=2}^{\infty} \mu_i$. Then $(\mu_j)_{j \geqslant 0}$ is a probability measure in the domain of attraction of a stable law of index $\theta$.

Recall that the second condition is equivalent to saying that if $X$ is random variable such that $\mathbb{P}[X = j] = \mu_j$ for $j \geqslant 0$, then either $X$ has finite variance, or $\mathbb{P}[X \geqslant j] = j^{-\theta} L(j)$, where $L$ is a function such that $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$ (such a function is called slowly varying at infinity). We refer to [21] or [41, chapter 3.7] for details.

### 6.1.2 Random dissections and Galton-Watson trees

In this subsection, we explain how to associate a dual object to a dissection. This dual object is a finite rooted ordered tree. The study of large random dissections will then boil down to the study of large Galton-Watson trees, which is a more familiar realm.

**Definition 6.1.2.** Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of all nonnegative integers, $\mathbb{N}^* = \{1, 2, \ldots\}$ and let $U$ be the set of labels:

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\varnothing\}$. An element of $U$ is a sequence $u = u_1 \cdots u_m$ of positive integers, and we set $|u| = m$, which represents the "generation" of $u$. If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ belong to $U$, we write $uv = u_1 \cdots u_m v_1 \cdots v_n$ for the concatenation of $u$ and $v$. In particular, note that $u\varnothing = \varnothing u = u$. Finally, a *rooted ordered tree* $\tau$ is a finite subset of $U$ such that:

1. $\varnothing \in \tau$;
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$;
3. for every $u \in \tau$, there exists an integer $k_u(\tau) \geqslant 0$ such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

In the following, by *tree* we will always mean rooted ordered tree. We denote the set of all trees by $\mathbb{T}$. We will often view each vertex of a tree $\tau$ as an individual of a population whose $\tau$ is the genealogical tree. The total progeny of $\tau$, Card$(\tau)$, will be denoted by $\zeta(\tau)$. A leaf of a tree $\tau$ is a vertex $u \in \tau$ such that $k_u(\tau) = 0$. The total number of leaves of $\tau$ will be denoted by $\lambda(\tau)$. If $\tau$ is a tree and $u \in \tau$, we define the shift of $\tau$ at $u$ by $T_u\tau = \{v \in U;\ uv \in \tau\}$, which is itself a tree.

Given a dissection $\omega \in \mathcal{D}_n$, we construct a (rooted ordered) tree $\phi(\omega)$ as follows: consider the dual graph of $\omega$, obtained by placing a vertex inside each face of $\omega$ and outside each side of the polygon $P_{n+1}$ and by joining two vertices if the corresponding faces share a common edge, thus giving a connected graph without cycles. Then remove the dual edge intersecting the side $\left[1, e^{\frac{2i\pi}{n+1}}\right]$ of $P_n$. Finally root the graph at the dual vertex corresponding to the face adjacent to the side $\left[1, e^{\frac{2i\pi}{n+1}}\right]$ (see Fig. 6.2). The planar structure now allows us to associate a tree $\phi(\omega)$ to this graph, in a way that should be obvious from Fig 6.2. Note that $k_u(\phi(\omega)) \neq 1$ for every $u \in \phi(\omega)$.

For every integer $n \geqslant 2$, let $\mathbb{T}_{(n)}$ stand for the set of all trees $\tau \in \mathbb{T}$ with exactly $n$ leaves and such that $k_u(\tau) \neq 1$ for every $u \in \tau$. The preceding construction provides a bijection $\phi$ from $\mathcal{D}_n$ onto $\mathbb{T}_{(n)}$. Furthermore, if $\tau = \phi(\omega)$ for $\omega \in \mathcal{D}_n$, there is a one-to-one correspondence between internal vertices of $\tau$ and faces of $\omega$, such that if $u$ is an internal vertex of $\tau$ and $f$ is the associated face of $\omega$, we have $\deg f = k_u(\tau) + 1$. The latter property should be clear from our construction.

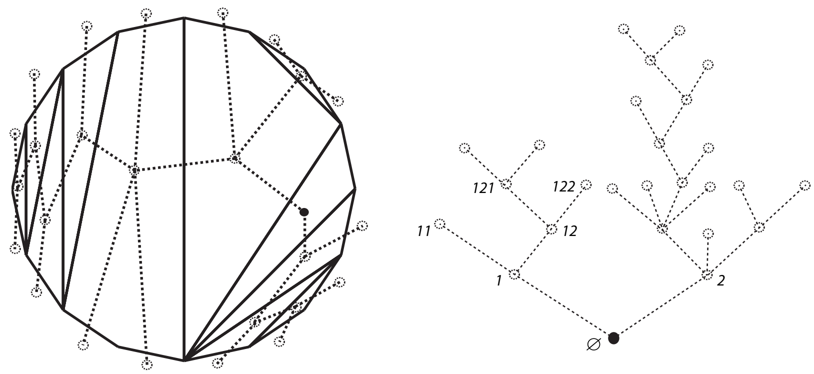


**Figure 6.2:** The dual tree of a dissection, rooted at the bold vertex.

**Definition 6.1.3.** Let $\rho$ be a probability measure on $\mathbb{N}$ with mean less than or equal to $1$ and such that $\rho(1) < 1$. The law of the Galton-Watson tree with offspring distribution $\rho$ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbb{T}$ such that:

1. $\mathbb{P}_\rho[k_\varnothing = j] = \rho(j)$ for $j \geqslant 0$;
2. for every $j \geqslant 1$ with $\rho(j) > 0$, the shifted trees $T_1\tau, \ldots, T_j\tau$ are independent under the conditional probability $\mathbb{P}_\rho[\,\cdot\,|k_\varnothing = j]$ and their conditional distribution is $\mathbb{P}_\rho$.

A random tree with distribution $\mathbb{P}_\rho$ will sometimes be called a $GW_\rho$ tree.

**Proposition 6.1.4.** *Let $(\mu_j)_{j\geqslant 2}$ be a sequence of nonnegative real numbers such that $\sum_{j=2}^{\infty} j\mu_j = 1$. Put $\mu_1 = 0$ and $\mu_0 = 1 - \sum_{j=2}^{\infty} \mu_j$ so that $\mu = (\mu_j)_{j\geqslant 0}$ defines a probability measure on $\mathbb{N}$, which satisfies the assumptions of Definition 6.1.3. Let $n \geqslant 2$ and let $Z_n$ be defined as in (6.3). Then $Z_n > 0$ if, and only if, $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$. Assume that this condition holds. Then if $\mathcal{D}_n$ is a random dissection distributed according to $\mathbb{P}_n^\mu$, the tree $\phi(\mathcal{D}_n)$ is distributed according to $\mathbb{P}_\mu[\,\cdot\,|\lambda(\tau) = n]$.*

*Proof.* Let $\tau \in \mathbb{T}_{(n)}$ and $\omega = \phi^{-1}(\tau)$. Then:

$$\mathbb{P}_\mu(\tau) = \prod_{u\in\tau} \mu_{k_u(\tau)} = \mu_0^n \prod_{\text{f face of } \omega} \mu_{\deg(f)-1} = \mu_0^n \pi(\omega). \tag{6.4}$$

The first equality is a well-known property of Galton-Watson trees (see e.g. Proposition 1.4 in [73]). The second one follows from the observations preceding Definition 6.1.3, and the last one is the definition of $\pi(\omega)$. From (6.4), we now get that $\mathbb{P}_\mu(\mathbb{T}_{(n)}) = \mu_0^n Z_n$, and then (if these quantities are positive) that $\mathbb{P}_\mu(\tau \mid \mathbb{T}_{(n)}) = \mathbb{P}_n^\mu(\omega)$, giving the last assertion of the proposition. □

**Remark 6.1.5.** The preceding proposition will be a major ingredient of our study. We will derive information about the random dissection $\mathcal{D}_n$ (when $n \to \infty$) from asymptotic results for the random trees $\phi(\mathcal{D}_n)$. To this end, we will assume that $(\mu_j)_{j\geqslant 2}$ satisfies condition $(H_\theta)$ for some $\theta \in (1, 2]$, which will allow us to use the limit theorems of [70] for Galton-Watson trees conditioned to have a (fixed) large number of leaves.

### 6.1.3 Coding trees and dissections

In the previous subsection, we have seen that certain random dissections are coded by conditioned Galton-Watson trees . We now explain how trees themselves can be coded by two functions, called respectively the Lukasiewicz path and the height function (see Figure 6.4 for an example). These codings are crucial in the understanding of large Galton-Watson trees and thus of large random dissections.

We write $u < v$ for the lexicographical order on the set $U$ (for example $\varnothing < 1 < 21 < 22$). In the following, we will denote the children of a tree $\tau$ listed in lexicographical order by $\varnothing = u(0) < u(1) < \cdots < u(\zeta(\tau) - 1)$.

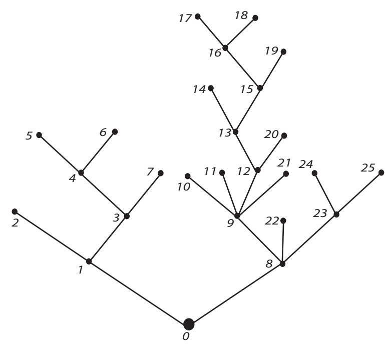


**Figure 6.3:** The dual tree $\tau$ associated to the dissection of figure 6.2 with its vertices indexed in lexicographical order. Here, $\zeta(\tau) = 26$.

**Figure 6.4:** The Lukasiewicz path $(W_u(\tau), 0 \leqslant u \leqslant \zeta(\tau))$ and the height function $(H_u(\tau), 0 \leqslant u < \zeta(\tau)$ of $\tau$.

**Definition 6.1.6.** Let $\tau \in \mathbb{T}$. The height process $H(\tau) = (H_n(\tau), 0 \leqslant n < \zeta(\tau))$ is defined, for $0 \leqslant n < \zeta(\tau)$, by $H_n(\tau) = |u(n)|$ . The Lukasiewicz path $W(\tau) = (W_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ is defined by $W_0(\tau) = 0$ and $W_{n+1}(\tau) = W_n(\tau) + k_{u(n)}(\tau) - 1$ for $0 \leqslant n \leqslant \zeta(\tau) - 1$.

It is easy to see that $W_n(\tau) \geqslant 0$ for $0 \leqslant n < \zeta(\tau)$ but $W_\zeta(\tau) = -1$ (see e.g. [73]).

Consider a dissection $\omega$, its dual tree $\tau = \phi(\omega)$ and the associated Lukasiewicz path $W(\tau)$. We now explain how to reconstruct $\omega$ from $W(\tau)$. As a first step, recall that an internal vertex $u$ of $\tau$ is associated to a face $f$ of $\omega$, and that the chords bounding $f$ are in bijection with the dual edges linking $u$ to its children and to its parent. The following proposition explains how to find all the children of a given vertex of $\tau$ using only $W$ or $H$, and will be useful to construct the edges linking the vertex $u \in \tau$ to its children.

**Proposition 6.1.7.** *Let $\tau \in \mathbb{T}$, and let $u(0), \dots, u(\zeta(\tau)-1)$ be as above the vertices of $\tau$ listed in lexicographical order. Fix $n \in \{0, 1, \dots, \zeta(\tau)-1\}$ such that $k_{u(n)}(\tau) > 0$ and set $k = k_{u(n)}(\tau)$.*

*(i) Let $s_1, \dots, s_k \in \{0, 1, \dots, \zeta(\tau)-1\}$ be defined by setting $s_i = \inf\{l \geqslant n+1;\ W_l(\tau) = W_{n+1}(\tau) - (i-1)\}$ for $1 \leqslant i \leqslant k$ (in particular $s_1 = n+1$). Then $u(s_1), u(s_2), \dots, u(s_k)$ are the children of $u(n)$ listed in lexicographical order.*

*(ii) We have $H_{s_1}(\tau) = H_{s_2}(\tau) = \cdots = H_{s_k}(\tau) = H_n(\tau) + 1$. Furthermore, for $1 \leqslant i \leqslant k-1$,*

$$H_j(\tau) > H_{s_i}(\tau) = H_{s_{i+1}}(\tau), \quad \forall j \in (s_r, s_{r+1}) \cap \mathbb{N}.$$

*Proof.* We leave this as an exercise (or see the proof of Proposition 1.2 in [73]) and encourage the reader to visualize what this means on Figure 6.4. □

In a second step, we explain how to reconstruct the dissection from the Lukasiewicz path of its dual tree.

**Proposition 6.1.8.** *Let $\zeta \geqslant 2$ be an integer and let $Z = (Z_n, 0 \leqslant n \leqslant \zeta)$ be a sequence of integers such that $Z_0 = 0$, $Z_\zeta = -1$, $Z_k \geqslant 0$ for $0 \leqslant k < \zeta$ and $Z_{i+1} - Z_i \in \{-1, 1, 2, 3, \dots\}$ for $0 \leqslant i < \zeta$. For $0 \leqslant i < \zeta$, set $X_i = Z_{i+1} - Z_i$ and, for $1 \leqslant i \leqslant \zeta$:*

$$\Lambda(i) = \mathrm{Card}\{0 \leqslant j < i; X_j = -1\}.$$

*For every integer $i \in \{0, 1, \dots, \zeta(\tau)-1\}$ such that $X_i \geqslant 1$, set $k_i = X_i + 1$ and let $s^i_1, \dots, s^i_{k_i+2}$ be defined by $s^i_1 = s^i_{k_i+2} = i+1$ and $s^i_{m+1} = \inf\{l \geqslant i+1;\ Z_l = Z_{i+1} - m\}$ for $1 \leqslant m \leqslant k_i$. Then the set $D(Z)$ defined by*

$$D(Z) = \bigcup_{i; X_i \geqslant 1} \bigcup_{j=1}^{k_i+1} \left[ \exp\left(-2i\pi \frac{\Lambda(s^i_j)}{\Lambda(\zeta)+1}\right), \exp\left(-2i\pi \frac{\Lambda(s^i_{j+1})}{\Lambda(\zeta)+1}\right) \right] \tag{6.5}$$

*is a dissection of the polygon $P_{\Lambda(\zeta)+1}$ called the dissection coded by $Z$.*

Note that if $\tau$ is a tree (different from the trivial tree $\{\varnothing\}$), if $u(0), \dots, u(\zeta(\tau)-1)$ are its vertices listed in lexicographical order and $Z = W(\tau)$, then $\Lambda(i)$ is the number of leaves among $u(0), u(1), \dots u(i-1)$ (in particular $\Lambda(\zeta)$ is the number of leaves of $\tau$), $k_i$ is the number of children of $u(i)$, $s^i_m$ is the index of the $m$-th child of $u(i)$ for $1 \leqslant m \leqslant k_i$.

*Proof.* First notice that, for all pairs $(i, j)$ occurring in the union of (6.5), we have $\Lambda(s^i_j) \neq \Lambda(s^i_{j+1})$. We then check that all edges of the polygon $P_{\Lambda(\zeta)+1}$ appear in the right-hand side of (6.5). To this end, fix $\ell \in \{0, 1, \dots, \Lambda(\zeta)-1\}$. Then there is a unique integer $k \in \{1, 2, \dots, \zeta-1\}$ such that $X_k = -1$ and $\Lambda(k) = \ell$. Set

$$i = \sup\{j \in \{0, 1, \dots, k-1\} : Z_j \leqslant Z_k\}$$

and $m = Z_{i+1} - Z_k + 1$. Notice that $1 \leqslant m \leqslant k_i$ since $Z_k \geqslant Z_i$ by construction. It is now a simple matter to verify that $s^i_m = k$ and $s^i_{m+1} = k+1$. Recalling that $\Lambda(k) = \ell$ and $\Lambda(k+1) = \ell + 1$, we get that the line segment

$$\left[\exp\left(-2i\pi\frac{\ell}{\Lambda(\zeta)+1}\right), \exp\left(-2i\pi\frac{\ell+1}{\Lambda(\zeta)+1}\right)\right]$$

appears in the right-hand side of (6.5). We therefore get that $D(Z)$ contains all edges of $P_{\Lambda(\zeta)+1}$ with the possible exception of the edge $[1, \exp(-2i\pi\frac{\Lambda(\zeta)}{\Lambda(\zeta)+1})]$. However, the latter edge also appears in the union of (6.5), taking $i = 0$ and $j = k_0 + 1$ and noting that $s^0_{k_0+1} = \zeta$ and $s^0_{k_0+2} = 1$.

Next suppose that $0 \leqslant i < \zeta, 0 \leqslant i' < \zeta$ are such that $k_i \geqslant 1, k_{i'} \geqslant 1$. Let $j \in \{1, \ldots, k_i + 1\}$, $j' \in \{1, \ldots, k_{i'} + 1\}$. If $(i,j) \neq (i',j')$, one easily checks that either the intervals $(s^i_j, s^i_{j+1})$ are disjoint, or one of them is contained in the either one. It follows that the chords corresponding respectively to $(i,j)$ and to $(i',j')$ in the union of (6.5) are non-crossing. Hence $D(Z)$ is a dissection. □

**Lemma 6.1.9.** *For every dissection $\omega \in \mathbb{L}$, we have $D(W(\phi(\omega))) = \omega$. In other words, a dissection is equal to the dissection coded by the Lukasiewicz path of its dual tree.*

*Proof.* This is a consequence of our construction. Suppose that $\omega \in \mathcal{D}_n$, for some $n \geqslant 2$, and set $\tau = \phi(\omega)$. Fix a face $f$ of $\omega$ and the corresponding dual vertex $u(i) \in \phi(\omega)$ (recall that the faces of $f$ are in one-to-one correspondence with the internal vertices of $\tau$). Denote the Lukasiewicz path of $\tau$ by $Z = W(\tau)$. First notice that the degree of $f$ is equal to $1 + k_{u(i)} = Z_{i+1} - Z_i + 2$, where $k_{u(i)}$ is the number of children of $u(i)$. To simplify notation, set $k_i = k_{u(i)}$. Let $s^i_1, \ldots, s^i_{k_i+2}$ be defined as in Proposition 6.1.8. By Proposition 6.1.7, $u(s^i_1), u(s^i_2), \ldots, u(s^i_{k_i})$ are the children of $u(i)$.

As in Proposition 6.1.8, we set, for every $1 \leqslant i \leqslant \zeta$, $\Lambda(i) = \mathrm{Card}\{0 \leqslant j < i;\ Z_{j+1} - Z_j = -1\}$, which represents the number of leaves among the first $i$ vertices of $\tau$. Note that $\Lambda(\zeta(\tau)) = n$. Then, assuming that $k_i \geqslant 2$:

- For every $1 \leqslant j \leqslant k_i$ the chord of $\omega$ which intersects the dual edge linking $u(i)$ to its $j$-th child is
$$\left[\exp\left(-2i\pi\frac{\Lambda(s^i_j)}{n+1}\right), \exp\left(-2i\pi\frac{\Lambda(s^i_{j+1})}{n+1}\right)\right].$$
- The chord of $\omega$ intersecting the dual edge linking $u(i)$ to its parent is
$$\left[\exp\left(-2i\pi\frac{\Lambda(s^i_{k_i+1})}{n+1}\right), \exp\left(-2i\pi\frac{\Lambda(s^i_1)}{n+1}\right)\right].$$

Indeed, a look at Fig. 6.2 should convince the reader that the vertices

$$\exp\left(-2i\pi\frac{\Lambda(s^i_j)}{n+1}\right), \qquad 1 \leqslant j \leqslant k_i + 1$$

are exactly the vertices belonging to the boundary of the face associated with $u(i)$ listed in clockwise order. Consequently the preceding chords are exactly the ones that bound this face. Since this holds for every face $f$ of $\omega$, the conclusion follows. □

## 6.2 The continuous setting: construction of the stable lamination

In this section, we present the continuous background by first introducing the normalized excursion $X^{exc}$ of the $\theta$-stable Lévy process. This process is important for our purposes because $X^{exc}$ will appear as the limit in the Skorokhod sense of the rescaled Lukasiewicz paths of large $GW_\mu$ trees coding discrete dissections. We then use $X^{exc}$ to construct a random compact subset of the closed unit disk, which will be our candidate for the limit in distribution of the random dissections we are considering. Two cases will be distinguished: the case $\theta = 2$, where $X^{exc}$ is continuous, and the case $\theta \in (1, 2)$, where the set of discontinuities of $X^{exc}$ is dense.

### 6.2.1 The normalized excursion of the Lévy process

We follow the presentation of [36] and refer to [15] for the proof of the results recalled in this subsection. The underlying probability space will be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a process with paths in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, the space of right-continuous with left limits (càdlàg) real-valued functions, endowed with the Skorokhod topology. We refer the reader to [20, chap. 3] and [57, chap. VI] for background concerning the Skorokhod topology. We denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the canonical filtration of X augmented with the $\mathbb{P}$-negligible sets. We assume that X is a strictly stable spectrally positive Lévy process of index $\theta$ normalized so that for every $\lambda > 0$:

$$\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\theta).$$

In the following, by $\theta$-*stable Lévy process* we will always mean such a Lévy process. In particular, for $\theta = 2$ the process X is $\sqrt{2}$ times the standard Brownian motion on the line. Recall that X enjoys the following scaling property: For every $c > 0$, the process $(c^{-1/\theta} X_{ct}, t \geqslant 0)$ has the same law as X. Also recall that when $1 < \theta < 2$, the Lévy measure $\pi$ of X is:

$$\pi(dr) = \frac{\theta(\theta-1)}{\Gamma(2-\theta)} r^{-\theta-1} 1_{(0,\infty)} dr.$$

For $s > 0$, we set $\Delta X_s = X_s - X_{s-}$. The following notation will be useful: for $0 \leqslant s < t$,

$$I_t^s = \inf_{[s,t]} X, \qquad I_t = \inf_{[0,t]} X, \qquad S_t = \sup_{[0,t]} X.$$

Notice that the process I is continuous since X has no negative jumps.

We have $X_0 = 0$ and $I_t < 0 < S_t$ for every $t > 0$ almost surely (meaning that the point $0$ is regular both for $(0, \infty)$ and for $(-\infty, 0)$ with respect to X). The process $X - I$ is a strong Markov process and $0$ is regular for itself with respect to $X - I$. We may and will choose $-I$ as the local time of $X - I$ at level $0$. Let $(g_i, d_i), i \in \mathcal{I}$ be the excursion intervals of $X - I$ away from $0$. For every $i \in \mathcal{I}$ and $s \geqslant 0$, set $\omega_s^i = X_{(g_i+s)\wedge d_i} - X_{g_i}$. We view $\omega^i$ as an element of the excursion space $\mathcal{E}$, which is defined by:

$$\mathcal{E} = \{\omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+);\ \omega(0) = 0 \text{ and } \zeta(\omega) := \sup\{s > 0; \omega(s) > 0\} \in (0, \infty)\}.$$

If $\omega \in \mathcal{E}$, we call $\zeta(\omega)$ the lifetime of the excursion $\omega$. From Itô's excursion theory, the point measure

$$\mathcal{N}(dt d\omega) = \sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, \omega^i)}$$

is a Poisson measure with intensity $dtN(d\omega)$, where $N(d\omega)$ is a $\sigma$-finite measure on the set $\mathcal{E}$.

Let us define the normalized excursion of the $\theta$-stable Lévy process. Define, for every $\lambda > 0$, the re-scaling operator $S^{(\lambda)}$ on the set of excursions by $S^{(\lambda)}(\omega) = \left(\lambda^{1/\theta}\omega(s/\lambda),\, s \geqslant 0\right)$. The scaling property of X shows that the image of $N(\cdot \mid \zeta > t)$ under $S^{(1/\zeta)}$ does not depend on $t > 0$. This common law, which is supported on the càdlàg paths with unit lifetime, is called the law of the normalized excursion of X and denoted by $\mathbb{P}^{\text{exc}}$. Informally, $\mathbb{P}^{\text{exc}}$ is the law of an excursion under the Itô measure conditioned to have unit lifetime. In the following, $(X^{\text{exc}}_t;\, 0 \leqslant t \leqslant 1)$ will stand for a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with paths in $\mathbb{D}([0,1], \mathbb{R}_+)$ and whose distribution under $\mathbb{P}$ is $\mathbb{P}^{\text{exc}}$. Note that $X^{\text{exc}}_0 = X^{\text{exc}}_1 = 0$.

As for the Brownian excursion, the normalized excursion can be constructed directly from the Lévy process X. We state Chaumont's result [25] without proof. Let $(\underline{g}_1, \underline{d}_1)$ be the excursion interval of $X - I$ straddling 1. More precisely, $\underline{g}_1 = \sup\{s \leqslant 1;\, X_s = I_s\}$ and $\underline{d}_1 = \inf\{s > 1;\, X_s = I_s\}$. Let $\zeta_1 = \underline{d}_1 - \underline{g}_1$ be the length of this excursion.

**Proposition 6.2.1** (Chaumont). *For every* $s \in [0,1]$, *set* $X^*_s = \zeta_1^{-1/\theta}(X_{\underline{g}_1 + \zeta_1 s} - X_{\underline{g}_1})$. *Then* $X^*$ *is distributed according to* $\mathbb{P}^{\text{exc}}$.

## 6.2.2 The $\theta$-stable lamination of the disk

The open unit disk of the complex plane $\mathbb{C}$ is denoted by $\mathbb{D} = \{z \in \mathbb{C};\, |z| < 1\}$ and $\mathbb{S}_1$ is the unit circle. If $x, y$ are distinct points of $\mathbb{S}_1$, we recall that $[x, y]$ stands for the line segment between $x$ and $y$. By convention, $[x, x]$ is equal to the singleton $\{x\}$.

**Definition 6.2.2.** A geodesic lamination L of $\overline{\mathbb{D}}$ is a closed subset L of $\overline{\mathbb{D}}$ which can be written as the union of a collection of non-crossing chords. The lamination L is maximal if it is maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$. In the sequel, by lamination we will always mean geodesic lamination of $\overline{\mathbb{D}}$.

**Remark 6.2.3.** In hyperbolic geometry, geodesic laminations of the disk are defined as closed subsets of the open hyperbolic disk [22]. As in [27], we prefer to see these laminations as compact subsets of $\overline{\mathbb{D}}$ because this will allow us to study the convergence of laminations in the sense of the Hausdorff distance on compact subsets of $\overline{\mathbb{D}}$.

It is not hard to check that the set of all geodesic laminations is closed with respect to the Hausdorff distance.

#### The Brownian triangulation

**Definition 6.2.4.** The Brownian excursion $\mathbb{e}$ is defined as $X^{\text{exc}}$ for $\theta = 2$. For $u, v \in [0,1]$ we set $u \overset{\mathbb{e}}{\sim} v$ if $\mathbb{e}_{u\wedge v} = \mathbb{e}_{u \vee v} = \min_{t \in [u\wedge v, u \vee v]} \mathbb{e}_t$.

Note that, with our normalization of $X^{\text{exc}}$, $\mathbb{e}/\sqrt{2}$ is the standard Brownian excursion. It is well known that the local minima of $\mathbb{e}$ are distinct almost surely. In the following, we always discard the set of probability zero where this property fails.

**Proposition 6.2.5** (Aldous [6] - Le Gall & Paulin [77]). *Define $\boldsymbol{L}(\mathbb{e})$ by:*

$$\boldsymbol{L}(\mathbb{e}) = \bigcup_{s \overset{\mathbb{e}}{\sim} t} [e^{-2i\pi s}, e^{-2i\pi t}].$$

*Then $\boldsymbol{L}(\mathbb{e})$ is a maximal geodesic lamination of $\overline{\mathbb{D}}$ (see Fig. 6.5 for a simulation of $\boldsymbol{L}(\mathbb{e})$).*

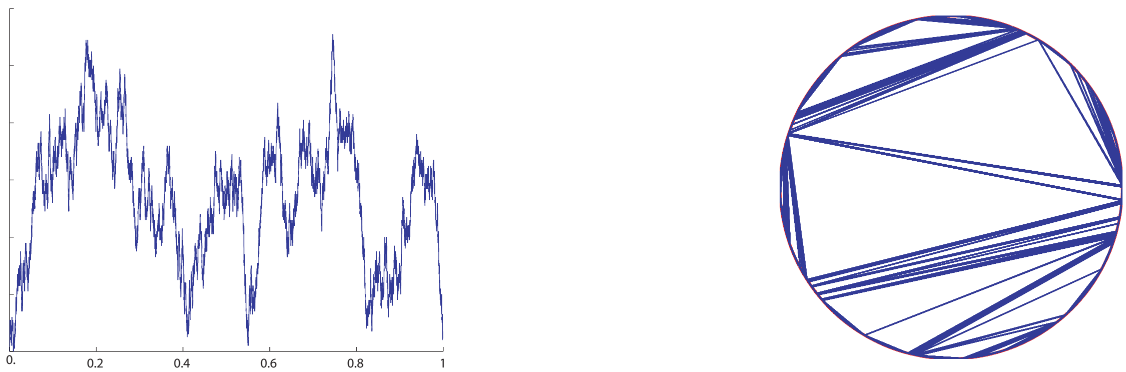


**Figure 6.5:** A Brownian excursion $\mathbb{e}$ and the associated triangulation $\mathbf{L}(\mathbb{e})$.

**Remark 6.2.6.** Both the property that $\mathbf{L}(\mathbb{e})$ is a lamination and its maximality property are related to the fact that local minima of $\mathbb{e}$ are distinct. The connected components of $\overline{\mathbb{D}} \backslash \mathbf{L}(\mathbb{e})$ are open triangles whose vertices belong to $\mathbb{S}_1$. For this reason we call $\mathbf{L}(\mathbb{e})$ the Brownian triangulation. Notice also that $\mathbb{S}_1 \subset \mathbf{L}(\mathbb{e})$.

### The $\theta$-stable lamination

Here, $\theta \in (1, 2)$ so that the $\theta$-stable Lévy process $X$ is not continuous. In the beginning of this section we fix $Z \in \mathbb{D}([0, 1], \mathbb{R})$ such that $Z_0 = Z_1 = 0$, $\Delta Z_s \geqslant 0$ for $s \in (0, 1]$ and $Z_s > 0$ for $s \in (0, 1)$. We then consider the case when $Z = X^{\text{exc}}$ is the normalized excursion of the $\theta$-stable Lévy process $X$.

**Definition 6.2.7.** For $0 \leqslant s < t \leqslant 1$, we set $s \simeq^Z t$ if and only if $t = \inf\{u > s;\, Z_u \leqslant Z_{s-}\}$ (where $Z_{0-} = 0$ by definition). For $0 \leqslant t < s \leqslant 1$, we set $s \simeq^Z t$ if and only if $t \simeq^Z s$. Finally, we set $s \simeq^Z s$ for every $s \in [0, 1]$.

Note that $\simeq^Z$ is not necessarily an equivalence relation. For example, if $0 < r < s < t < 1$ are such that $\Delta Z_r = 0$, $Z_r = Z_s = Z_t$ and $Z_u > Z_r$ for $u \in (r, s) \cup (s, t)$, then $r \simeq^Z s$ and $s \simeq^Z t$, but we do not have $r \simeq^Z t$.

**Remark 6.2.8.** If $s \simeq^Z t$ and $s < t$, then $Z_{s-} = Z_t$ and $Z_r > Z_{s-}$ for $r \in (s, t)$.

**Proposition 6.2.9.** *We say that $Z$ attains a local minimum at $t \in (0, 1)$ if there exists $\eta > 0$ such that $\inf_{[t-\eta, t+\eta]} Z = Z_t$. Suppose that $Z$ satisfies the following four assumptions:*

*(H1) If* $0 \leqslant s < t \leqslant 1$ *, there exists at most one value* $r \in (s,t)$ *such that* $Z_r = \inf_{[s,t]} Z$ *(we say that local minima of* $Z$ *are distinct);*

*(H2) If* $t \in (0,1)$ *is such that* $\Delta Z_t > 0$ *then* $\inf_{[t,t+\epsilon]} Z < Z_t$ *for all* $0 < \epsilon \leqslant 1-t$*;*

*(H3) If* $t \in (0,1)$ *is such that* $\Delta Z_t > 0$ *then* $\inf_{[t-\epsilon,t]} Z < Z_{t-}$ *for all* $0 < \epsilon \leqslant t$*;*

*(H4) Suppose that* $Z$ *attains a local minimum at* $t \in (0,1)$ *(in particular* $\Delta Z_t = 0$ *by (H3)). Let* $s = \sup\{r \in [0,t];\ Z_r < Z_t\}$*. Then* $\Delta Z_s > 0$ *and* $Z_{s-} < Z_t$*. Note that then* $Z_s > Z_t$ *by (H2).*

*Then the set*

$$L(Z) := \bigcup_{s \simeq^Z t} [e^{-2i\pi s}, e^{-2i\pi t}]$$

*is a geodesic lamination of* $\overline{\mathbb{D}}$*, called the lamination coded by the càdlàg function* $Z$*.*

Notice that $\mathbb{S}_1 \subset L(Z)$ since $s \simeq^Z s$ for every $s \in [0,1]$.

*Proof.* It easily follows from Remark 6.2.8 that the chords appearing in the definition of $L(Z)$ are non-crossing. We have to prove that $L(Z)$ is closed. To this end, it is enough to verify that the relation $\simeq^Z$ is closed, in the sense that its graph is a closed subset of $[0,1]^2$. Consider two sequences $(s_n), (t_n)$ of reals such that $0 \leqslant s_n < t_n \leqslant 1$, $s_n \simeq^Z t_n$ and the pairs $(s_n, t_n)$ converge to $(s,t)$. We need to verify that $s \simeq^Z t$. Clearly $s \leqslant t$ and we can assume that $s < t$ since $\mathbb{S}_1 \subset L(Z)$.

The property $s_n \simeq^Z t_n$ implies that $Z_r \geqslant Z_{t_n}$ for every $r \in (s_n, t_n)$. By passing to the limit $n \to \infty$, we get $Z_r \geqslant Z_{t-}$ for every $r \in (s,t)$. If $\Delta Z_t > 0$, this contradicts (H3). So we can assume that $\Delta Z_t = 0$, implying that the sequence $(Z_{t_n})$ converges to $Z_t$ as $n \to \infty$.

*Case 1.* Assume that $\Delta Z_s > 0$ and thus $s > 0$. By (H2) and right-continuity, we can find $\eta > 0$ such that $\eta < (t-s)/2$ and :

$$\inf_{[s,s+\eta)} Z > \inf_{[s+\eta,(s+t)/2]} Z.$$

It follows from (H3) that the infimum of $Z$ over a compact interval is achieved at some point of this interval. Hence we may take $r_0 \in [s+\eta, (s+t)/2]$ such that $Z_{r_0} = \inf_{[s+\eta,(s+t)/2]} Z$. If $s < s_n$ for infinitely many $n$, we can find infinitely many values of $n$ for which $s < s_n < s+\eta \leqslant r_0 < t_n$. For those values of $n$, $r_0 \in (s_n, t_n)$ and $Z_{r_0} < Z_{s_n-}$, which contradicts Remark 6.2.8. We can thus suppose that $s_n \leqslant s$ for every sufficiently large $n$. Consequently, $(Z_{s_n-})$ converges to $Z_{s-}$ as $n$ tends to infinity. Since $Z_{s_n-} = Z_{t_n}$ for all $n$, it follows that $Z_t = Z_{s-}$. Recall that $Z_r \geqslant Z_t$ for $r \in (s,t)$. We now demonstrate by contradiction that, in fact, $Z_r > Z_t$ for all $r \in (s,t)$. Suppose that there exists $r_1 \in (s,t)$ such that $Z_{r_1} = Z_t$. Notice that $Z$ then attains a local minimum at $r_1$. Property (H3) ensures that

$$s = \sup\{u \in [0,r_1];\ Z_u < Z_{r_1}\},$$

and the fact that $Z_{s-} = Z_t = Z_{r_1}$ contradicts (H4). We conclude that $Z_r > Z_{s-}$ for every $r \in (s,t)$. Therefore $t = \inf\{u > s;\ Z_u \leqslant Z_{s-}\}$. This implies that $s \simeq^Z t$, as desired.

*Case 2.* Assume that $\Delta Z_s = 0$. In this case, $(Z_{s_n})$ converges to $Z_s$ as $n$ tends to infinity. Since $Z_{s_n-} = Z_{t_n}$ for all $n$, it follows that $Z_s = Z_t$. We also know that $Z_r \geqslant Z_s$ for $r \in (s,t)$. If $s = 0$, we necessarily have $t = 1$ and the fact that $Z$ is positive on $(0,1)$ implies $0 \simeq^Z 1$. We thus suppose that $s > 0$. Argue by contradiction and suppose that there exists $r_1 \in (s,t)$ such that $Z_{r_1} = Z_t$. Then $r_1$ is a local minimum of $Z$. If $\inf_{[s-\epsilon,s]} Z < Z_s$ for every $\epsilon \in (0,s]$, then $s = \sup\{u \in [0,r_1];\ Z_u < Z_{r_1}\}$. By (H4), $s$ must be a jump time of $Z$, which is a contradiction. If

$\inf_{[s-\epsilon,s]} Z \geqslant Z_s$ for some $\epsilon \in (0, s]$, this means that s is a local minimum of Z. Since $Z_s = Z_{r_1}$, this contradicts (H1). We conclude that $Z_r > Z_t$ for $r \in (s,t)$. This implies that $s \simeq^Z t$. □

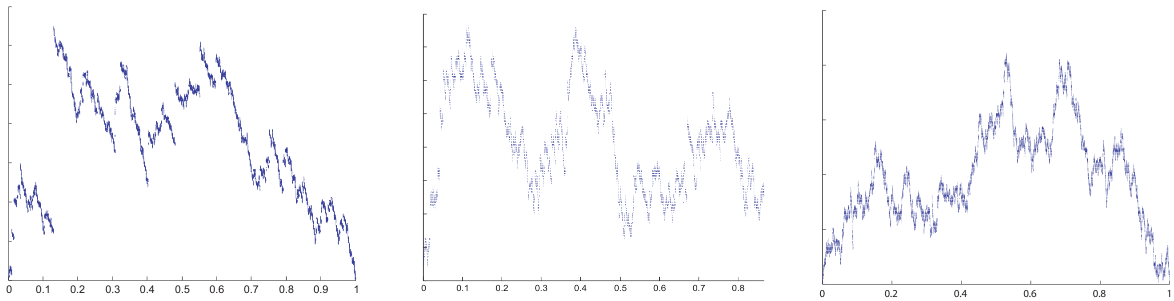

**Figure 6.6:** Simulations of $X^{exc}$ for respectively $\theta = 1.1, 1.5, 1.9$.

Let (H0) be the property: $\{s \in [0,1];\ \Delta Z_s \neq 0\}$ is dense in $[0,1]$.

**Proposition 6.2.10.** *Let $1 < \theta < 2$. With probability one, the normalized excursion $X^{exc}$ of the $\theta$-stable Lévy process satisfies the assumptions (H0), (H1), (H2), (H3) and (H4).*

*Proof.* It is sufficient to prove that properties analogous to (H0) – (H4) hold for the Lévy process X. The case of (H0) is clear. (H1) and (H2) are consequences of the (strong) Markov property of X and the fact that 0 is regular for $(-\infty, 0)$ with respect to X.

For the remaining properties, we will use the time-reversal property of X, which states that if $t > 0$ and $\widehat{X}^{(t)}$ is the process defined by $\widehat{X}^{(t)}_s = X_t - X_{(t-s)-}$ for $0 \leqslant s < t$ and $\widehat{X}^{(t)}_t = X_t$, then the two processes $(X_s, 0 \leqslant s \leqslant t)$ and $(\widehat{X}^{(t)}_s, 0 \leqslant s \leqslant t)$ have the same law. For (H3), the time-reversal property of X and the regularity of 0 for $(0,\infty)$ shows that a.s. for every jump time s of X and every $v \in [0, s)$:

$$\inf_{r \in [v,s]} X_r < X_{s-}.$$

We finally prove the analog of (H4) for X. By the time-reversal property of X, it is sufficient to prove that if $q > 0$ is rational and $T = \inf\{t \geqslant q;\ X_t \geqslant S_q\}$ then $X_T > S_q \geqslant X_{T-}$ almost surely. This follows from the Markov property at time q and the fact that for any $a > 0$, X jumps a.s. across a at its first passage time above a (see [15, Proposition VIII.8 (ii)]). □

In the following, we always discard the set of zero probability where one of the properties (H0) – (H4) does not hold.

**Definition 6.2.11.** The $\theta$-stable lamination is defined as the geodesic lamination $L(X^{exc})$, where $X^{exc}$ is the normalized excursion of the $\theta$-stable Lévy process.

See Figure 6.1 for some examples. The following proposition is immediate from the definition of the relation $\simeq^{X^{exc}}$ and Remark 6.2.8.

**Proposition 6.2.12.** *Almost surely, for every choice of $0 \leqslant \alpha < \beta \leqslant 1$ with $(\alpha, \beta) \neq (0,1)$, we have $\alpha \simeq^{X^{exc}} \beta$ if and only if one of the following two mutually exclusive cases holds:*

*(i)* $\Delta X^{exc}_\alpha > 0$ *and* $\beta = \inf\{u \geqslant \alpha;\ X^{exc}_u = X^{exc}_{\alpha-}\}$*;*

*(ii)* $\Delta X^{exc}_\alpha = 0$*,* $X^{exc}_\alpha = X^{exc}_\beta$*, and* $X^{exc}_r > X^{exc}_\alpha$ *for every* $r \in (\alpha, \beta)$*.*

**Definition 6.2.13.** Let $\mathcal{E}_1$ be the set of all pairs $(\alpha, \beta)$ where $0 \leqslant \alpha < \beta \leqslant 1$ satisfy condition (i) in Proposition 6.2.12.

**Proposition 6.2.14.** *The following holds almost surely for any pair* $(s, t)$ *such that* $0 \leqslant s < t \leqslant 1$ *and* $X^{exc}_s = X^{exc}_t$ *and* $X^{exc}_r > X^{exc}_s$ *for every* $r \in (s, t)$*. For every* $\epsilon \in (0, (t-s)/2)$*, there exist* $s' \in [s, s+\epsilon]$ *and* $t' \in [t-\epsilon, t]$ *such that* $\Delta X^{exc}_{s'} > 0$ *and* $t' = \inf\{u \geqslant s';\ X^{exc}_u = X^{exc}_{s'-}\}$*, so that in particular* $(s', t') \in \mathcal{E}_1$*.*

*Proof.* Let $0 \leqslant s < t \leqslant 1$ be such that the assumptions in the proposition hold. Take $\epsilon < (t-s)/4$, then set $m = \inf_{[s+\epsilon, t-\epsilon]} X^{exc}$ and note that $m > X^{exc}_s$ as an easy consequence of (H3). By right-continuity, there exists $\epsilon'$ with $0 < \epsilon' < \epsilon$ such that $\sup_{[s,s+\epsilon']} X^{exc} < m$. Let $w \in (s, s+\epsilon')$ be a jump time of $X^{exc}$, so that, by (H2),

$$\inf_{r \in [w, s+\epsilon']} X^{exc}_r < X^{exc}_w.$$

We already noticed that the property (H3) implies that the minimum of $X^{exc}$ over a compact interval is achieved at a point of this interval. Hence there exists $u \in [w, s+\epsilon]$ such that $X^{exc}_u = \inf_{[w,s+\epsilon]} X^{exc}$. Finally, let $s' = \sup\{r \in [s, u];\ X^{exc}_r < X^{exc}_u\}$. By (H4), we see that $s'$ is a jump time. Set $t' = \inf\{u > s';\ X^{exc}_u = X^{exc}_{s'-}\}$. By construction, $s \leqslant s' \leqslant w \leqslant u \leqslant s+\epsilon < t-\epsilon \leqslant t' \leqslant t$ and the desired result follows. □

**Proposition 6.2.15.** *We have a.s.*

$$L(X^{exc}) = \overline{\bigcup_{(s,t) \in \mathcal{E}_1} [e^{-2i\pi s}, e^{-2i\pi t}]}.$$

*Proof.* Denote the compact subset of $\overline{\mathbb{D}}$ in the right-hand side by $K$. The fact that $L(X^{exc})$ is closed implies that $K \subset L(X^{exc})$. We have to show the reverse inclusion. To this end, let $0 \leqslant u < v \leqslant 1$ such that $u \overset{X^{exc}}{\simeq} v$ but $(u, v) \notin \mathcal{E}_1$. Then condition (ii) in Proposition 6.2.12 holds for $(\alpha, \beta) = (u, v)$, and it follows from Proposition 6.2.14 that $(u, v)$ is the limit of a sequence of pairs $(u_n, v_n)$ belonging to $\mathcal{E}_1$. Since $K$ is closed, we get that $[e^{-2i\pi u}, e^{-2i\pi v}] \subset K$. Finally, from the fact that $X^{exc}$ satisfies properties (H0) and (H2), it is easy to verify that in any nontrivial open subinterval of $[0, 1]$ we can find a pair $(u, v)$ such that $(u, v) \in \mathcal{E}_1$, and it follows that $\mathbb{S}_1 \subset K$. This completes the proof. □

## 6.3 Convergence to the stable lamination

In this section, we show that the Boltzmann dissections of $P_{n+1}$ considered in Section 6.1.1 converge in distribution to the stable laminations introduced in the previous section. To this end, we use limit theorems for rescaled Lukasiewicz paths of critical Galton-Watson trees conditioned on their number of leaves, which we obtained in [70]. We combine these limit theorems with Proposition 6.1.4 (which states that the dual tree of a Boltzmann dissection is a Galton-Watson tree conditioned on having a given number of leaves) to deduce that the underlying tree structures of large dissections converge. As before, we will deal separately with the case $\theta = 2$ and the case $\theta \in (1,2)$. Our goal is to prove the following:

**Theorem 6.3.1.** *Let $(\mu_j)_{j\geqslant 2}$ be a sequence satisfying Assumption $(H_\theta)$ for some $\theta \in (1,2]$. For every integer $n \geqslant 2$ such that the definition of $\mathbb{P}_n^\mu$ makes sense, let $\mathcal{D}_n$ be a random dissection distributed according to $\mathbb{P}_n^\mu$. Then:*

$$\mathcal{D}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \begin{cases} \boldsymbol{L}(\mathbb{e}) & \text{if } \theta = 2, \\ \mathrm{L}(X^{\mathrm{exc}}) & \text{if } \theta \in (1,2), \end{cases}$$

*where the convergence holds in distribution for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.*

**Remarks 6.3.2.** (i) This theorem generalizes Aldous' result [6, 7], stating that uniformly distributed triangulations of $P_n$ converge to $\mathbf{L}(\mathbb{e})$ as $n \to \infty$. Indeed, in our setting, uniform triangulations of $P_n$ are obtained by taking $\mu_0 = 1/2, \mu_2 = 1/2$ and $\mu_j = 0$ otherwise.

(ii) In [28], it is shown that Theorem 6.3.1 can be used to study uniformly distributed dissections. More precisely, if one sets $\mu_0 = 2-\sqrt{2}$ and $\mu_i = ((2-\sqrt{2})/2)^{i-1}$ for every $i \geqslant 2$ then the Boltzmann probability measure $\mathbb{P}_n^\mu$ associated to $\mu$ is the uniform probability measure on dissections of $P_{n+1}$.

(iii) It would be interesting to understand what happens when the sequence $(\mu_i)_{i\geqslant 2}$ does not satisfy $(H_\theta)$, for instance if $\sum_{i=2}^\infty i\mu_i = \infty$. We hope to investigate this in future work.

### 6.3.1 Galton-Watson trees conditioned on their number of leaves

Let $\tau \in \mathbb{T}$. Recall our notation $(u(i), 0 \leqslant i \leqslant \zeta(\tau)-1)$ for the vertices of $\tau$ listed in lexicographical order and denote the number of children of $u(j)$ by $k_j$. Define $\Lambda_\tau(l)$ for every $\ell \in \{0, 1, \ldots, \zeta(\tau)\}$ by:

$$\Lambda_\tau(\ell) = \sum_{0\leqslant j<\ell} 1_{\{k_j=0\}}.$$

Note that if $Z = W(\tau)$ is the Lukasiewicz path of $\tau$, $\Lambda_\tau$ coincides with $\Lambda$ as defined in Proposition 6.1.8. Also note that $\Lambda_\tau(\zeta(\tau)) = \lambda(\tau)$ is the total number of leaves of $\tau$.

**Theorem 6.3.3 ([70]).** *Let $(\mu_j)_{j\geqslant 2}$ be a sequence of nonnegative real numbers satisfying the assumption $(H_\theta)$ for some $\theta \in (1,2]$. Put $\mu_1 = 0$ and $\mu_0 = 1-\sum_{j=2}^\infty \mu_j$, so that $\mu = (\mu_j)_{j\geqslant 0}$ is a critical probability measure on $\mathbb{N}$. For every $n \geqslant 1$ such that $\mathbb{P}_\mu[\lambda(\tau) = n] > 0$, let $t_n$ be a random tree distributed according to $\mathbb{P}_\mu[\,\cdot\,|\,\lambda(\tau) = n]$. The following two properties hold.*

*(i) We have:*

$$\sup_{0\leqslant t\leqslant 1}\left|\frac{\Lambda_{t_n}(\lfloor \zeta(t_n)t\rfloor)}{n} - t\right| \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0.$$

*(ii) There exists a sequence $(B_k)_{k\geqslant 1}$ of positive constants converging to $\infty$ such that:*

$$\left(\frac{1}{B_{\zeta(t_n)}} W_{\lfloor \zeta(t_n)t\rfloor}(t_n);\, 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_t^{\mathrm{exc}}; 0 \leqslant t \leqslant 1). \tag{6.6}$$

*Proof.* Note that $\Lambda_{t_n}(\zeta(t_n)) = \lambda(t_n) = n$. In [70, Corollary 3.3], it is shown that, for every $0 < \eta < 1$:

$$\sup_{\eta \leqslant t \leqslant 1} \left| \frac{\Lambda_{t_n}(\lfloor \zeta(t_n)t \rfloor)}{\zeta(t_n)t} - \mu_0 \right| \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0.$$

In particular, this implies that $\zeta(t_n)/n$ converges in probability to $1/\mu_0$. Assertion (i) follows from the preceding convergences, noting that, for every $t \in (0,1]$,

$$\frac{\Lambda_{t_n}(\lfloor \zeta(t_n)t \rfloor)}{n} - t = t\,\frac{\zeta(t_n)}{n} \left( \frac{\Lambda_{t_n}(\lfloor \zeta(t_n)t \rfloor)}{\zeta(t_n)t} - \mu_0 \right) + t \left( \frac{\mu_0 \zeta(t_n)}{n} - 1 \right).$$

The second assertion is a particular case of [70, Theorem 6.1]. □

### 6.3.2 Convergence to the stable lamination

We fix a sequence of nonnegative real numbers $(\mu_j)_{j\geqslant 2}$ satisfying Assumption $(H_\theta)$ for some $\theta \in (1,2]$ and we define $\mu_0$ and $\mu_1$ as previously. Throughout this section, for every $n \geqslant 1$ such that $Z_n$ defined by (6.3) is positive (so that $\mathbb{P}^\mu_n$ is well defined), $\mathcal{D}_n$ stands for a random dissection distributed according to the Boltzmann probability measure $\mathbb{P}^\mu_n$, and $t_n$ stands for its dual tree $\phi(\mathcal{D}_n)$, which is distributed according to $\mathbb{P}_\mu[\,\cdot\,|\lambda(\tau) = n]$ by Proposition 6.1.4. The total progeny of $t_n$ is denoted by $\zeta_n$. The Lukasiewicz path of $t_n$ is denoted by $W^n$ and $u^n_0, u^n_1, \ldots, u^n_{\zeta_n - 1}$ are the vertices of $t_n$ listed in lexicographical order. Let $(B_n)_{n\geqslant 1}$ be a sequence of positive real numbers such that (6.6) holds. Define the rescaled Lukasiewicz path $X^n$ of $t_n$ by $X^n_t = \frac{1}{B_{\zeta_n}} W^n_{\lfloor \zeta_n t \rfloor}$ for $0 \leqslant t \leqslant 1$. By Theorem 6.3.3 and Skorokhod's representation theorem (see e.g. [20, Theorem 6.7]) we may and will assume that the following convergence holds almost surely in the space $\mathbb{R} \otimes \mathbb{D}([0,1],\mathbb{R})$:

$$\left( \sup_{0\leqslant t \leqslant 1} \left| \frac{\Lambda_{t_n}(\lfloor \zeta_n t \rfloor)}{n+1} - t \right|, X^n \right) \quad \xrightarrow[n\to\infty]{a.s.} \quad (0, X^{exc}). \tag{6.7}$$

**Convergence to the Brownian triangulation**

Here, we suppose that $\theta = 2$.

**Proposition 6.3.4.** *When $n$ tends to infinity, $D(W^n) \overset{a.s.}{\to} \mathbf{L}(\mathbb{e})$ in the sense of the Hausdorff distance $d_H$ between compact subsets of $\overline{\mathbb{D}}$.*

*Proof.* We fix $\omega$ in the underlying probability space so that the convergence (6.7) holds for this value of $\omega$ and we will verify that for this particular value of $\omega$ we have also $D(W^n) \to \mathbf{L}(\mathbb{e})$. Since the space $(\mathcal{C}, d_H)$ is compact, we may find a random subsequence $(n_k(\omega))$ (depending on $\omega$) such that $D(W^{n_k})$ converges to a compact subset $K$ of $\overline{\mathbb{D}}$, and we need to verify that $K = \mathbf{L}(\mathbb{e})$. Since $D(W^{n_k})$ is a dissection for every $k$, one easily checks that $K$ must be a geodesic lamination of $\overline{\mathbb{D}}$. Since $\mathbf{L}(\mathbb{e})$ is a maximal lamination of $\overline{\mathbb{D}}$, the proof will be complete if we can verify that $\mathbf{L}(\mathbb{e}) \subset K$.

So we let $0 \leqslant s < t \leqslant 1$ be such that $s \overset{\mathbb{e}}{\sim} t$ and we aim at proving that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset K$. Let $\epsilon > 0$. Simple arguments using the convergence (6.7) (and the fact that local minima of the Brownian excursion are distinct) show that for every $n$ large enough, we can find integers $i_n, j_n \in \{1, \ldots, \zeta_n - 1\}$ such that $|i_n/\zeta_n - s| < \epsilon$, $|j_n/\zeta_n - t| < \epsilon$ and:

$$W^n_{i_n} > W^n_{i_n - 1}, \qquad j_n = \min\{k > i_n;\, W^n_k < W^n_{i_n}\}.$$

By Proposition 6.1.7, $u^n_{i_n}$ and $u^n_{j_n}$ are consecutive children of $u^n_{i_n-1}$. Recalling that $\Lambda_{t_n}(\zeta(t_n)) = n$, we get from Lemma 6.1.9 that:

$$\left[\exp\left(-2i\pi\frac{\Lambda_{t_n}(i_n)}{n+1}\right), \exp\left(-2i\pi\frac{\Lambda_{t_n}(j_n)}{n+1}\right)\right] \subset D(W^n).$$

To simplify notation, set $s_n = \Lambda_{t_n}(i_n)/(n+1)$ and $t_n = \Lambda_{t_n}(j_n)/(n+1)$. From the convergence (6.7), we get $|s_n - s| < \epsilon$ and $|t_n - t| < \epsilon$ for every large enough $n$. In particular, we see that the chord $[e^{-2i\pi s}, e^{-2i\pi t}]$ lies within distance $2\epsilon$ from $D(W^n)$ for every large enough $n$. It follows that the chord $[e^{-2i\pi s}, e^{-2i\pi t}]$ is within distance $2\epsilon$ from $K$. Since $\epsilon > 0$ was arbitrary, we get that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset K$, which completes the proof. □

**Convergence to the stable lamination when $\theta \neq 2$**

We now assume that $\theta \in (1, 2)$. Recall that the convergence (6.7) is assumed to hold a.s.

**Proposition 6.3.5.** *We have $D(W^n) \overset{a.s.}{\to} L(X^{exc})$ as $n \to \infty$ in the sense of the Hausdorff distance $d_H$ between compact subsets of $\overline{\mathbb{D}}$.*

We fix $\omega$ in the underlying probability space so that both the conclusion of Proposition 6.2.15 and the convergence (6.7) hold for this value of $\omega$, and furthermore the path $X^{exc}(\omega)$ satisfies properties (H0) – (H4). We then consider a subsequence $(n_k(\omega))$ such that $D(W^{n_k})$ converges to a compact subset $K$ of $\overline{\mathbb{D}}$, and we need to verify that $K = L(X^{exc})$. We will first prove that $L(X^{exc}) \subset K$ before proving the reverse inclusion. In both cases, the precise description of $L(X^{exc})$ as a union of chords will be crucial. Note that $K$ must contain the circle $\mathbb{S}^1$ because the dissection $D(W^n)$ contains the polygon $P_{n+1}$. We stress that the lamination $L(X^{exc})$ is not maximal, in contrast to the case $\theta = 2$. As a consequence, we will have to prove the non-trivial reverse inclusion.

**Lemma 6.3.6.** *Let $s$ be a jump time of $X^{exc}$ and $t = \inf\{u > s;\, X^{exc}_u = X^{exc}_{s-}\}$. For $\epsilon \in (0, (t-s)/2)$ small enough, we can choose an integer $n_0(\epsilon)$ such that, for every $n \geqslant n_0(\epsilon)$, there exists $s_n \in (s - \epsilon, s + \epsilon) \cap \zeta_n^{-1}\mathbb{N}$ such that the following inequalities hold:*

$$\inf_{[t-\epsilon, t+\epsilon]} X^n < X^n_{s_n -} < \inf_{[s_n, t-\epsilon]} X^n. \tag{6.8}$$

Lemma 6.3.6 follows from the convergence of $X^n$ to $X^{exc}$ and well-known properties of the Skorokhod topology. We give only the main ideas of the proof and leave the details to the reader. The time $s_n$ can be chosen (arbitrarily close to $s$ when $n$ is large) so that $X^n_{s_n-}$ is close to $X^{exc}_{s-}$ and $\Delta X^n_{s_n}$ is close to $\Delta X^{exc}_s$. Then (6.8) is derived by observing that, for $\epsilon > 0$ small enough

$$\inf_{[t, t+\epsilon]} X^{exc} < X^{exc}_t = X^{exc}_{s-} < \inf_{[s, t-\epsilon]} X^{exc}.$$

Notice that the bound $\inf_{[t,t+\epsilon]} X^{exc} < X^{exc}_t$ holds because otherwise $t$ would be a time of local minimum of $X$ and this would contradict (H4).

**Lemma 6.3.7.** *We have* $L(X^{exc}) \subset K$.

*Proof.* Since $K$ is closed, the property of Proposition 6.2.15 shows that it is enough to verify that $\left[e^{-2i\pi\alpha}, e^{-2i\pi\beta}\right] \subset K$ for every $(\alpha,\beta) \in \mathcal{E}_1$. So let $(\alpha,\beta) \in \mathcal{E}_1$. Then $\alpha$ is a jump time of $X^{exc}$ and $\beta = \inf\{u > \alpha; X^{exc}_u = X^{exc}_{\alpha-}\}$. To show that $\left[e^{-2i\pi\alpha}, e^{-2i\pi\beta}\right] \subset K$ it is sufficient to show that for every $\epsilon > 0$ and every $n$ sufficiently large we can find $\alpha_n, \beta_n \in [0,1]$ such that $|\alpha_n - \alpha| \leqslant 2\epsilon, |\beta_n - \beta| \leqslant 2\epsilon$ and $\left[e^{-2i\pi\alpha_n}, e^{-2i\pi\beta_n}\right] \subset D(W^n)$. We fix $\epsilon > 0$. Using Lemma 6.3.6 with $(s,t) = (\alpha,\beta)$, we can, for every large enough $n$, find $\alpha'_n \in (\alpha-\epsilon, \alpha+\epsilon) \cap \zeta_n^{-1}\mathbb{N}$ such that:

$$\inf_{[\beta-\epsilon,\beta+\epsilon]} X^n < X^n_{\alpha'_n-} < \inf_{[\alpha'_n,\beta-\epsilon]} X^n.$$

Then put $\beta'_n = \inf\{u \geqslant \alpha'_n; X^n_u < X^n_{\alpha'_n-}\}$ and note that $|\alpha-\alpha'_n| \leqslant \epsilon$, $|\beta-\beta'_n| \leqslant \epsilon$. The time $\zeta_n\alpha'_n$ must correspond to a positive jump of $W^n$, and we have also

$$\zeta_n\beta'_n = \inf\{l \geqslant \zeta_n\alpha'_n\,; W^n_l = W^n_{\zeta_n\alpha'_n} - (W^n_{\zeta_n\alpha'_n} - W^n_{\zeta_n\alpha'_n-1} + 1)\}.$$

Using formula (6.5) and recalling that $\Lambda_{t_n}$ coincides with the process $\Lambda$ of Proposition 6.1.8 if $Z = W^n$, we get from Lemma 6.1.9 that

$$\left[\exp\left(-2i\pi\frac{\Lambda_{t_n}(\zeta_n\alpha'_n)}{n+1}\right), \exp\left(-2i\pi\frac{\Lambda_{t_n}(\zeta_n\beta'_n)}{n+1}\right)\right] \subset D(W^n).$$

If we set $\alpha_n = (n+1)^{-1}\Lambda_{t_n}(\zeta_n\alpha'_n)$ and $\beta_n = (n+1)^{-1}\Lambda_{t_n}(\zeta_n\beta'_n)$, the convergence (6.7) shows that $\alpha_n$ and $\beta_n$ satisfy $|\alpha_n-\alpha| \leqslant 2\epsilon$ and $|\beta_n-\beta| \leqslant 2\epsilon$ for all sufficiently large $n$, thus giving the desired result. □

We now prove the reverse inclusion.

**Lemma 6.3.8.** *We have* $K \subset L(X^{exc})$.

*Proof.* Recall that $D(W^{n_k})$ converges to $K$ in the Hausdorff sense. By the formula of Proposition 6.1.8, we can write:

$$D(W^{n_k}) = \bigcup_{(u,v)\in\mathcal{E}_{(n_k)}} \left[e^{-2i\pi u}, e^{-2i\pi v}\right],$$

where $\mathcal{E}_{(n_k)}$ is a (finite) symmetric subset of $[0,1]^2$. By extracting a subsequence if necessary, we may assume that $\mathcal{E}_{(n_k)} \to \mathcal{E}_\infty$ in the Hausdorff sense as $k \to \infty$, where $\mathcal{E}_\infty$ is a symmetric closed subset of $[0,1]^2$. It is easy to verify that:

$$K = \bigcup_{(u,v)\in\mathcal{E}_\infty} \left[e^{-2i\pi u}, e^{-2i\pi v}\right].$$

The proof of the inclusion $K \subset L(X^{exc})$ then reduces to checking that if $u,v \in \mathcal{E}_\infty$ with $u < v$, we have $u \overset{X^{exc}}{\simeq} v$.

So fix $u,v \in \mathcal{E}_\infty$ such that $u < v$. Then the pair $(u,v)$ is the limit of a sequence $(u_k,v_k)$ with $(u_k,v_k) \in \mathcal{E}_{(n_k)}$ for every $k$. From Proposition 6.1.8, we can find integers $l_{n_k} < m_{n_k}$ in $\{0,1,\ldots,\zeta_{n_k}\}$ such that:

$$u = \lim_{k\to\infty}\frac{\Lambda_{t_{n_k}}(l_{n_k})}{n_k+1}, \qquad v = \lim_{k\to\infty}\frac{\Lambda_{t_{n_k}}(m_{n_k})}{n_k+1}$$

and

$$m_{n_k} = \inf\left\{i \geqslant l_{n_k};\, W_i^{n_k} = W_{l_{n_k}}^{n_k} - 1\right\}. \tag{6.9}$$

By (6.7), we have also:

$$u = \lim_{k\to\infty} \frac{l_{n_k}}{\zeta_{n_k}}, \qquad v = \lim_{k\to\infty} \frac{m_{n_k}}{\zeta_{n_k}}. \tag{6.10}$$

>From (6.9), we have $W_i^{n_k} \geqslant W_{m_{n_k}}^{n_k}$ for every $i \in [l_{n_k}, m_{n_k}]$. Thus, using the convergence of $X^n$ to $X^{\mathrm{exc}}$ and (6.10):

$$X_s^{\mathrm{exc}} \geqslant X_{v-}^{\mathrm{exc}}, \qquad \text{for every } s \in (u, v). \tag{6.11}$$

>From property (H3) this implies that $X_v^{\mathrm{exc}} = X_{v-}^{\mathrm{exc}}$, and then $(B_{\zeta_{n_k}})^{-1} W_{m_{n_k}}^{n_k} = X_{m_{n_k}/\zeta_{n_k}}^{n_k}$ must converge to $X_v^{\mathrm{exc}}$. Note that $X_{u-}^{\mathrm{exc}}$ and $X_u^{\mathrm{exc}}$ are the only possible accumulation points for the sequence $(B_{\zeta_{n_k}})^{-1} W_{l_{n_k}}^{n_k} = X_{l_{n_k}/\zeta_{n_k}}^{n_k}$. Now consider two cases:

If $X_u^{\mathrm{exc}} = X_{u-}^{\mathrm{exc}}$, then $(B_{\zeta_{n_k}})^{-1} W_{l_{n_k}}^{n_k} = X_{l_{n_k}/\zeta_{n_k}}^{n_k}$ converges to $X_u^{\mathrm{exc}}$ and, using (6.9), we get that $X_u^{\mathrm{exc}} = X_v^{\mathrm{exc}}$. It follows that $X_s^{\mathrm{exc}} > X_v^{\mathrm{exc}}$ for every $s \in (u, v)$, because otherwise this would contradict (H1) or (H4). Clearly we obtain $u \overset{X^{\mathrm{exc}}}{\simeq} v$.

If $X_u^{\mathrm{exc}} > X_{u-}^{\mathrm{exc}}$ then we must have $X_{l_{n_k}/\zeta_{n_k}}^{n_k} \to X_{u-}^{\mathrm{exc}}$ (otherwise (6.9) would give $X_u^{\mathrm{exc}} = X_v^{\mathrm{exc}}$, and (6.11) would contradict (H2)). Then (6.9) gives $X_v^{\mathrm{exc}} = X_{u-}^{\mathrm{exc}}$. The inequality (6.11) can then be reinforced in $X_s^{\mathrm{exc}} > X_v^{\mathrm{exc}} = X_{u-}^{\mathrm{exc}}$ for every $s \in (u, v)$, since otherwise $X^{\mathrm{exc}}$ would have a local minimum equal to $X_v^{\mathrm{exc}} = X_{u-}^{\mathrm{exc}}$ in $(u, v)$, which would contradict (H4). Hence we also get $u \overset{X^{\mathrm{exc}}}{\simeq} v$ in that case.

This completes the proof. □

Together with Lemma 6.3.7, Lemma 6.3.8 completes the proof of Theorem 6.3.1 in the case $\theta \neq 2$.

### 6.3.3 Description of the faces of $L(X^{\mathrm{exc}})$ for $\theta \neq 2$

We still consider the case $1 < \theta < 2$. By definition, the faces of $L(X^{\mathrm{exc}})$ are the connected components of $\overline{\mathbb{D}} \backslash L(X^{\mathrm{exc}})$. In this section, we study the faces of $L(X^{\mathrm{exc}})$ and we show in particular that, almost surely, every face of $L(X^{\mathrm{exc}})$ is bounded by infinitely many chords (in contrast to the case $\theta = 2$ where all faces are triangles).

**Lemma 6.3.9.** *Almost surely, for every face $U$ of $L(X^{\mathrm{exc}})$, if $\Gamma = \mathbb{S}_1 \cap \overline{U}$ denotes the part of the boundary of $U$ lying on the circle, then:*

*(i) $U$ is a convex open set;*

*(ii) $\Gamma$ is not a singleton;*

*(iii) $1 \notin \Gamma$.*

*Proof.* Assertions (i) and (ii) hold for any geodesic lamination of $\overline{\mathbb{D}}$, and we leave the proof to the reader. To get (iii), fix $\epsilon > 0$ and note that by Proposition 6.2.14 we can find $s \in (0, \epsilon]$ and $t \in [1-\epsilon, 1)$ such that the chord $[e^{-2i\pi s}, e^{-2i\pi t}]$ is contained in $L(X^{\mathrm{exc}})$. It follows that 1 cannot belong to the boundary of a connected component of $\overline{\mathbb{D}} \backslash L(X^{\mathrm{exc}})$. □

For distinct $s, t \in (0,1)$, we denote by $\mathbb{H}_t^s$ the open half-plane bounded by the line containing $e^{-2i\pi s}$ and $e^{-2i\pi t}$ and such that $1 \notin \mathbb{H}_t^s$. We write $\widetilde{\mathbb{H}}_t^s$ for the other open half-plane bounded by the same line.

**Proposition 6.3.10.** *Let* $s$ *be a jump time of* $X^{\text{exc}}$ *and* $t = \inf\{u > s;\ X_u^{\text{exc}} = X_{s-}^{\text{exc}}\}$. *There exists a unique face* $U$ *of* $L(X^{\text{exc}})$ *contained in* $\mathbb{H}_t^s$ *and whose closure* $\overline{U}$ *contains the chord* $[e^{-2i\pi s}, e^{-2i\pi t}]$. *The face* $U$ *is called the face associated to* $s$. *The mapping* $s \mapsto U$ *is a one-to-one correspondence between jump times of* $X^{\text{exc}}$ *and faces of* $L(X^{\text{exc}})$.

*Proof.* We start by giving a description of the face associated to $s$. Let $(\alpha_i, \beta_i)_{i \geqslant 1}$ be defined by:

$$\{(\alpha_i, \beta_i);\ i \geqslant 1\} = \{(\alpha, \beta);\ s \leqslant \alpha < \beta \leqslant t, X_\alpha = X_\beta = \inf_{[s,\alpha]} X \text{ and } X_r^{\text{exc}} > X_\alpha^{\text{exc}} \text{ for } r \in (\alpha, \beta)\},$$

where the pairs $(\alpha_i, \beta_i)$ are listed in such a way that $\beta_i - \alpha_i > \beta_j - \alpha_j$ for $i < j$. The intervals $(\alpha_i, \beta_i)$ are exactly the excursion intervals of $(X_r - I_r^s)_{s \leqslant r \leqslant t}$ away from $0$. Note that $\alpha_i \simeq^{X^{\text{exc}}} \beta_i$ by Proposition 6.2.12, and that the intervals $(\alpha_i, \beta_i)$, $i \geqslant 1$ are disjoint. Furthermore, the fact that (H3) holds for $X^{\text{exc}}$ shows that the times $\alpha_i$, $i \geqslant 1$ are not jump times of $X^{\text{exc}}$.

For every $n \geqslant 1$, let $V_n$ be the convex open polygon whose vertices are

$$\{e^{-2i\pi s}, e^{-2i\pi t}\} \cup \bigcup_{i=1}^{n} \{e^{-2i\pi\alpha_i}, e^{-2i\pi\beta_i}\}.$$

Observe that $V_n \subset V_{n+1}$. We finally set:

$$V = \bigcup_{n \geqslant 1} V_n,$$

which is a convex open set. It is clear that $V$ is contained in the open half-plane $\mathbb{H}_t^s$ and that $\overline{V}$ contains $[e^{-2i\pi s}, e^{-2i\pi t}]$. To prove that $V$ is a connected component of $\overline{\mathbb{D}} \backslash L(X^{\text{exc}})$, we proceed in two steps. We first prove that $V \subset \overline{\mathbb{D}} \backslash L(X^{\text{exc}})$ and then that $V$ is a maximal connected open subset of $\overline{\mathbb{D}} \backslash L(X^{\text{exc}})$.

Let us prove that $V \subset \overline{\mathbb{D}} \backslash L(X^{\text{exc}})$. Argue by contradiction and suppose that there exist $P \in L(X^{\text{exc}})$ and $N \geqslant 1$ such that $P \in V_N$. By the definition of $L(X^{\text{exc}})$, there exist $0 \leqslant u \leqslant v < 1$ such that $u \simeq^{X^{\text{exc}}} v$ and $P \in [e^{-2i\pi u}, e^{-2i\pi v}]$. Since $V$ is contained in the open half-plane $\mathbb{H}_t^s$, we must have $s \leqslant u < v \leqslant t$. Let us first show that $s < u$. If $s = u$, the definition of $\simeq^{X^{\text{exc}}}$ implies that $v = \inf\{r > s;\ X_r^{\text{exc}} = X_{s-}^{\text{exc}}\} = t$. Consequently, $P \in [e^{-2i\pi s}, e^{-2i\pi t}]$, contradicting the fact that $P \in V_N$. We thus have $s < u$. Since $P \in V_N$ and since for every $j \in \{1, \ldots, N\}$ the chord $\left[e^{-2i\pi\alpha_j}, e^{-2i\pi\beta_j}\right]$ does not cross the chord $\left[e^{-2i\pi u}, e^{-2i\pi v}\right]$ a simple argument shows that there exists $1 \leqslant i \leqslant N$ such that $u \leqslant \alpha_i < \beta_i \leqslant v$, the case $(u, v) = (\alpha_i, \beta_i)$ being excluded. We examine two cases:

- If $u < \alpha_i$, then $X_{u-}^{\text{exc}} > X_{\alpha_i}^{\text{exc}}$ because $\inf_{[s,\alpha_i]} X^{\text{exc}} = X_{\alpha_i}^{\text{exc}}$, $\alpha_i$ is a local minimum time for $X^{\text{exc}}$ and local minima are almost surely distinct. Since $\alpha_i \in [u, v]$ and $u \simeq^{X^{\text{exc}}} v$, this contradicts Remark 6.2.8.
- If $u = \alpha_i$, we know that $u$ is not a jump time of $X^{\text{exc}}$ and the property $u \simeq^{X^{\text{exc}}} v$ implies $v = \inf\{r > u;\ X_r^{\text{exc}} \leqslant X_{\alpha_i}^{\text{exc}}\} = \beta_i$, which is excluded.

In each case, a contradiction occurs. This completes the first step.

Let us then prove that V is a maximal connected open subset of $\overline{\mathbb{D}}\backslash L(X^{\text{exc}})$. To this end, we observe that we have

$$V = \mathbb{H}_t^s \cap \left( \bigcap_{i=1}^{\infty} \widetilde{\mathbb{H}}_{\beta_i}^{\alpha_i} \right) \cap \mathbb{D}.$$

The fact that V is contained in the set in the right-hand side is immediate from our construction, and the reverse inclusion is also easy. Set $R = (\mathbb{H}_t^s)^c \cap \overline{\mathbb{D}}$ and $R_i = (\widetilde{\mathbb{H}}_{\beta_i}^{\alpha_i})^c \cap \overline{\mathbb{D}}$ for $i \geqslant 1$. It follows that

$$\overline{\mathbb{D}}\backslash V = \mathbb{S}_1 \cup R \cup \left( \bigcup_{i=1}^{\infty} R_i \right). \tag{6.12}$$

This implies that the boundary of V is contained in $L(X^{\text{exc}})$, and it follows that V is a maximal connected open subset of $\overline{\mathbb{D}}\backslash L(X^{\text{exc}})$. From the preceding formula for $\overline{\mathbb{D}}\backslash V$, it is also clear that the boundary of V contains the chord $[e^{-2i\pi s}, e^{-2i\pi t}]$, as well as all chords $[e^{-2i\pi \alpha_i}, e^{-2i\pi b_i}]$, and we have obtained the existence of the face associated to s. The uniqueness of this face is obvious for geometric reasons.

We still have to prove the last assertion of the proposition. Let U be a face of $L(X^{\text{exc}})$. We need to verify that U is the face associated to a certain jump time of $X^{\text{exc}}$. To this end, let $\Gamma = \mathbb{S}_1 \cap \overline{U}$ be the part of the boundary of U lying on the circle and set:

$$s = \inf\{u \geqslant 0;\ e^{-2i\pi u} \in \Gamma\}, \qquad t = \sup\{0 \leqslant u \leqslant 1;\ e^{-2i\pi u} \in \Gamma\}.$$

By Lemma 6.3.9 (iii), we have $0 < s < t < 1$. By the compactness of $L(X^{\text{exc}})$ and a convexity argument, it is easy to verify that $\left[e^{-2i\pi s}, e^{-2i\pi t}\right] \subset L(X^{\text{exc}})$. We then claim that s is a jump time of $X^{\text{exc}}$. If not, by Proposition 6.2.12, this means that $X_s^{\text{exc}} = X_t^{\text{exc}}$ and $X_u^{\text{exc}} > X_s^{\text{exc}}$ for $u \in (s,t)$. But then Proposition 6.2.14 could be used to produce a chord of $L(X^{\text{exc}})$ partitioning U into two disjoint open sets, which is impossible. So s is a jump time of $X^{\text{exc}}$ and we then know that $t = \inf\{u > s;\ X_u^{\text{exc}} = X_{s-}^{\text{exc}}\}$. Let V be the face associated to s. To prove that $U = V$, it is sufficient to show that $U \cap V \neq \varnothing$. This follows from simple geometric considerations. This completes the proof. □

## 6.4 The stable lamination coded by a continuous function

The definitions of the limiting random laminations $\mathbf{L}(\mathbb{e})$ and $L(X^{\text{exc}})$ that appear in our main result Theorem 6.3.1 for $\theta = 2$ and $\theta \neq 2$ were somewhat different. The goal of this section is to unify these two cases by explaining how $L(X^{\text{exc}})$ (for $\theta \neq 2$) can also be constructed from a random continuous function. This will allow us to make the connection between our stable laminations and the so-called stable trees, which were studied in particular in [37, 38], in the same way as the Brownian triangulation is connected to the Brownian CRT [7], and this will also be useful when we calculate the Hausdorff dimension of $L(X^{\text{exc}})$. The relevant random function, called the height process in continuous time, was introduced in [76] and studied in great detail in [37].

In this section, X is the strictly stable spectrally positive Lévy of index $\theta$, as defined in Section 2.1 and $1 < \theta < 2$.

### 6.4.1 The height process

The continuous-time height process associated with X can be defined by the following approximation formula. For every $t \geqslant 0$,

$$H_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t ds \mathbb{1}_{\{X_s \leqslant I_t^s + \epsilon\}}$$

where the convergence holds in probability. The process $(H_t)_{t \geqslant 0}$ has a continuous modification, which we consider from now on.

A very useful ingredient in the study of the height process is the so-called exploration process $(\rho_t)_{t \geqslant 0}$, which is a strong Markov process taking values in the space $M_f(\mathbb{R}_+)$ of all finite measures on $\mathbb{R}_+$. For every $t \geqslant 0$, $\rho_t$ is defined by:

$$\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_t^s f(H_s), \tag{6.13}$$

for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Here the notation $d_s I_t^s$ refers to the integration with respect to the nondecreasing function $s \to I_t^s$ (recall the definition of $I_t^s$ in subsection 2.1). Note in particular that $\langle \rho_t, 1 \rangle = X_t - I_t$. The process $(\rho_t)_{t \geqslant 0}$ enjoys the following two important properties [37, Lemma 1.2.2]:

(i) Almost surely for every $t \geqslant 0$, $\rho_t(\{0\}) = 0$ and $\text{supp}(\rho_t) = [0, H_t]$ (here and later $\text{supp}(\mu)$ denotes the topological support of $\mu \in M_f(\mathbb{R}_+)$, with the convention that $\text{supp}(0) = \{0\}$).

(ii) Almost surely $\{t \geqslant 0;\ H_t = 0\} = \{t \geqslant 0;\ \rho_t = 0\} = \{t \geqslant 0;\ X_t = I_t\}$.

In addition to (i), one can prove that, for every fixed $t \geqslant 0$, $\rho_t(\{H_t\}) = 0$ almost surely. This follows from formula (17) in [37]. Moreover, almost surely for every jump time s of X, $\rho_s(\{H_s\}) = \Delta X_s$ (see formula (19) in [37]).

We will need another important property of the exploration process. To state this property we need to introduce some notation. If $\mu \in M_f(\mathbb{R}_+)$ and $\alpha \geqslant 0$, the "killed" measure $k_\alpha \mu$ is the unique element of $M_f(\mathbb{R}_+)$ such that, for every $t \geqslant 0$:

$$k_\alpha \mu([0,t]) = \mu([0,t]) \wedge (\mu(\mathbb{R}_+) - \alpha)^+.$$

Suppose that $\mu \in M_f(\mathbb{R}_+)$ has compact support and set $h(\mu) = \sup(\text{supp}(\mu))$. Then if $\nu \in M_f(\mathbb{R}_+)$ the concatenation $[\mu, \nu] \in M_f(\mathbb{R}_+)$ is defined by:

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \int \nu(dt) f(h(\mu) + t).$$

Let T be a stopping time of the filtration of X and let $X_t^{(T)} = X_{T+t} - X_T$ for every $t \geqslant 0$. Recall that $(X_t^{(T)})_{t \geqslant 0}$ has the same distribution as $(X_t)_{t \geqslant 0}$ by the strong Markov property of X. Set $I_t^{(T)} = \inf_{s \leqslant t} X_s^{(T)}$ for every $t \geqslant 0$, and let $(H_t^{(T)})_{t \geqslant 0}$ and $(\rho_t^{(T)})_{t \geqslant 0}$ be respectively the height process and the exploration process associated with $X^{(T)}$. According to formula (20) in [37] we have almost surely for every $t \geqslant 0$,

$$\rho_{T+t} = \left[ k_{-I_t^{(T)}} \rho_T, \rho_t^{(T)} \right]. \tag{6.14}$$

It follows that almost surely for every $t \geqslant 0$,

$$H_{T+t} - \inf_{s \in [T, T+t]} H_s = H_t^{(T)} \tag{6.15}$$

(see [37, Lemma 1.4.5] for the case where $T$ is deterministic, but the derivation is the same in the general case).

The following result is a continuous analog of Proposition 6.1.7.

**Proposition 6.4.1.** *The following holds almost surely. Let $s \geqslant 0$ be a jump time of $X$ and $t = \inf\{u > s;\ X_u = X_{s-}\}$. Then:*

*(i) for every $u \in [s, t]$, $H_u \geqslant H_s$ and $H_u = H_s$ if and only if $X_u = \inf_{[s,u]} X$;*

*(ii) for every $\alpha \in [0, s)$, $\inf_{[\alpha,s]} H < H_s$;*

*(iii) for every $u \in (t, \infty)$, $\inf_{[s,u]} H < H_s$.*

*Proof.* Since the set of all jump times can be written as a countable collection of stopping times, it is sufficient to consider the case when $s = S$ is a stopping time that is also a jump time of $X$, and $t = T = \inf\{r \geqslant S; X_r = X_{S-}\}$. By preceding observations, we know that $\rho_S(\{H_S\}) = \Delta X_S$.

Let us prove (i). From (6.14) applied to the stopping time $S$ we have $\rho_{S+r} \geqslant k_{\Delta X_S} \rho_S$ for every $r \in [0, T - S]$ and thus:

$$H_{S+r} = \sup(\text{supp}\ \rho_{S+r}) \geqslant \sup(\text{supp}\ k_{\Delta X_S} \rho_S) = H_S.$$

Furthermore, for the same values of $r$, (6.14) shows that $H_{S+r} = H_S$ can only hold if $\rho_r^{(S)} = 0$, which is equivalent (by (6.13)) to $X_r^{(S)} = I_r^{(S)}$. This completes the proof of (i).

To get (ii), we observe that we can always pick a rational $\beta \in (\alpha, S)$ such that $X_\beta < X_S$. By (6.15) applied to $T = \beta$,

$$H_S - \inf_{r \in [\alpha, S]} H_r \geqslant H_S - \inf_{r \in [\beta, S]} H_r = H_{S-\beta}^{(\beta)}.$$

Since $X_S > X_\beta$ we have $\langle \rho_{S-\beta}^{(\beta)}, 1 \rangle \geqslant X_{S-\beta}^{(\beta)} > 0$ and thus $H_{S-\beta}^{(\beta)} > 0$ completing the proof of (ii).

Finally for every $\epsilon > 0$ set $T_\epsilon = \inf\{r \geqslant S;\ X_r \leqslant X_{S-} - \epsilon\}$. By (6.14) we have $\rho_{T_\epsilon} = k_{\Delta X_s + \epsilon} \rho_S$ and $H_{T_\epsilon} = \sup(\text{supp}\ k_{\Delta X_s + \epsilon} \rho_S) < H_S$ because $\rho_S(\{H_S\}) = \Delta X_S$. This completes the proof. □

The following result will also be useful.

**Proposition 6.4.2.** *The following holds almost surely for every choice of $0 \leqslant s < t$ such that $H_s = H_t$ and $H_u > H_s$ for all $u \in (s, t)$. For every $\epsilon \in (0, (t-s)/2)$, there exist $s' \in (s, s+\epsilon)$ and $t' \in (t-\epsilon, t)$ such that $s' < t'$ and:*

*(i) $H$ does not attain a local minimum at $s'$ nor at $t'$;*

*(ii) $H_{s'} = H_{t'} = \inf_{[s',t']} H$ and there exists $v \in (s', t')$ such that $H_v = H_{s'}$.*

*Proof.* We can assume that $\epsilon < (t-s)/4$. Set $m = \inf_{[s+\epsilon, t-\epsilon]} H$. By the continuity of $H$, there exists $\epsilon' \in (0, \epsilon)$ such that $\sup_{[s,s+\epsilon']} H < m$. Let $u \in (s, s+\epsilon') \cap \mathbb{Q}$. We have:

$$\inf_{[u, s+\epsilon']} H < H_u$$

because it easily follows from formula (6.14) that $\inf_{[q,q+\delta]} H < H_q$ for every rational $q > 0$ and every $\delta > 0$, almost surely (the point is that the measure $\rho_q$ gives no mass to $\{H_q\}$, so that the supremum of the support of $k_a \rho_q$ will be strictly smaller than $H_q$, for every $a > 0$).

Then let $v \in (u, s+\epsilon']$ be such that $H_v = \inf_{[u,s+\epsilon']} H$. Finally set $s' = \inf\{r \in [s, s+\epsilon']; H_r = H_v\}$ and $t' = \sup\{r \in [s+\epsilon', t]; H_r = H_v\}$ so that H does not attain a local minimum at $s'$ nor at $t'$. By construction and using the continuity of H, we have

$$s < s' \leqslant u < v \leqslant s + \epsilon < t - \epsilon < t' < t.$$

Since $H_{s'} = H_v = H_{t'}$, the proposition is proved. □

### 6.4.2 The normalized excursion of the height process

Recall the notation of Section 2.1, where we have constructed the normalized excursion $X^{\text{exc}}$ from the excursion of X straddling 1.

The normalized excursion of the height process, which is denoted by $H^{\text{exc}}$, is defined as follows. Set $\beta_\epsilon = \theta/(\Gamma(2-\theta)\epsilon^{\theta-1})$. Using Proposition 6.2.1, one shows that there exists a continuous process $(H^{\text{exc}}_t)_{0 \leqslant t \leqslant 1}$, such that, for every t belonging to a subset of $[0, 1]$ of full Lebesgue measure,

$$H^{\text{exc}}_t = \lim_{\epsilon \to 0} \frac{1}{\beta_\epsilon} \text{Card}\{u \in [0,t];\ X^{\text{exc}}_{u-} < \inf_{[u,t]} X^{\text{exc}},\ \Delta X^{\text{exc}}_u > \epsilon\}, \quad \text{a.s.}$$

See [36, Section 3] for details of the argument. This process $H^{\text{exc}}$ is called the normalized excursion of the height process. The pair $(X^{\text{exc}}, H^{\text{exc}})$ can be constructed explicitly from the process X via the formula

$$(X^{\text{exc}}_t, H^{\text{exc}}_t)_{0 \leqslant t \leqslant 1} = \left(\zeta_1^{-\frac{1}{\theta}}(X_{\underline{g}_1 + \zeta_1 t} - X_{\underline{g}_1}), \zeta_1^{\frac{1}{\theta}-1} H_{\underline{g}_1 + \zeta_1 t}\right)_{0 \leqslant t \leqslant 1} \tag{6.16}$$

where we recall the notation $\underline{g}_1 = \sup\{s \leqslant 1; X_s = I_s\}$ and $\zeta_1 = \underline{g}_1 - \inf\{s > 1; X_s = I_s\}$.

**Remark 6.4.3.** From formula (6.16), we see that the results of Propositions 6.4.1 and 6.4.2 remain valid if we replace X with $X^{\text{exc}}$ and H with $H^{\text{exc}}$. More precisely, we will use these results in the following form. Almost surely:

1. Let $0 \leqslant s \leqslant 1$ be a jump time of $X^{\text{exc}}$ and $t = \inf\{u > s; X^{\text{exc}}_u = X^{\text{exc}}_{s-}\}$. Then for $u \in [s, t]$, $H^{\text{exc}}_u \geqslant H^{\text{exc}}_s$, and $H^{\text{exc}}_u = H^{\text{exc}}_s$ if and only if $X^{\text{exc}}_u = \inf_{[s,u]} X^{\text{exc}}$. Moreover, if $0 \leqslant \alpha < s$ then $\inf_{[\alpha,s]} H^{\text{exc}} < H^{\text{exc}}_s$, and if $t < u \leqslant 1$ then $\inf_{[s,u]} H^{\text{exc}} < H^{\text{exc}}_s$;
2. For every choice of $0 \leqslant s < t \leqslant 1$, the conditions $H^{\text{exc}}_s = H^{\text{exc}}_t$ and $H^{\text{exc}}_u > H^{\text{exc}}_s$ for all $u \in (s, t)$ imply that for every $\epsilon > 0$ sufficiently small, there exist $s' \in (s, s+\epsilon)$ and $t' \in (t-\epsilon, t)$ such that:
   (i) $H^{\text{exc}}$ does not attain a local minimum at $s'$ nor at $t'$,
   (ii) $\inf_{[s',t']} H^{\text{exc}} = H^{\text{exc}}_{s'} = H^{\text{exc}}_{t'}$ and there exists $u \in (s', t')$ such that $H^{\text{exc}}_u = H^{\text{exc}}_{s'} = H^{\text{exc}}_{t'}$.

The main result of [36] states that if $t_n$ is a $GW_\mu$ tree conditioned on having total progeny n, the discrete height process $(H_k(t_n))_{0 \leqslant k \leqslant n}$, appropriately rescaled, converges in distribution to $H^{\text{exc}}$. However, we will not use this fact.

### 6.4.3 Laminations coded by continuous functions

Let $g : [0,1] \to \mathbb{R}_+$ be a continuous function such that $g(0) = g(1) = 0$. We define a pseudo-distance on $[0,1]$ by:

$$d_g(s,t) = g(s) - g(t) - 2 \min_{r \in [s \wedge t, s \vee t]} g(r)$$

for $s,t \in [0,1]$. The associated equivalence relation on $[0,1]$ is defined by setting $s \overset{g}{\sim} t$ if and only if $d_g(s,t) = 0$, or equivalently $g(s) = g(t) = \min_{r \in [s\wedge t, s\vee t]} g(r)$ (in the special case $g = \mathbb{e}$, this equivalence relation was already used in Section 2).

The quotient set $T_g := [0,1]/ \overset{g}{\sim}$ equipped with the distance $d_g$ is an $\mathbb{R}$-tree, called the tree coded by the function $g$. We refer to [38, 42] for more information about $\mathbb{R}$-trees, which are natural generalizations of discrete trees, and their coding by functions.

For $s \in [0,1]$, we let $\mathrm{cl}_g(s)$ be the equivalence class of $s$ with respect to the equivalence relation $\overset{g}{\sim}$. Then, for $s,t \in [0,1]$, we set $s \overset{g}{\approx} t$ if at least one of the following two conditions holds:

- $s \overset{g}{\sim} t$ and $g(r) > g(s)$ for every $r \in (s \wedge t, s \vee t)$;
- $s \overset{g}{\sim} t$ and $s \wedge t = \min \mathrm{cl}_g(s)$, $s \vee t = \max \mathrm{cl}_g(s)$.

By [27, Proposition 2.5], the set

$$\mathbf{L}(g) := \bigcup_{s \overset{g}{\approx} t} [e^{-2i\pi s}, e^{-2i\pi t}]$$

is a geodesic lamination of $\overline{\mathbb{D}}$. Note that if $g = \mathbb{e}$, this coincides with the definition in Section 2, thanks to the fact that local minima of $\mathbb{e}$ are distinct.

In what follows we take $g = H^{\mathrm{exc}}$ and write $\approx^{H^{\mathrm{exc}}}$ rather than $\overset{H^{\mathrm{exc}}}{\approx}$ for notational reasons.

**Proposition 6.4.4.** *Almost surely, for every* $u \in [0,1]$ *such that* $\mathrm{Card}(\mathrm{cl}_{H^{\mathrm{exc}}}(u)) \geqslant 3$, *there exists a jump time* $\alpha$ *of* $X^{\mathrm{exc}}$ *such that* $\alpha \in \mathrm{cl}_{H^{\mathrm{exc}}}(u)$. *Conversely, let* $\alpha$ *be a jump time of* $X^{\mathrm{exc}}$ *and* $\beta = \inf\{r > \alpha;\ X^{\mathrm{exc}}_r = X^{\mathrm{exc}}_{\alpha-}\}$. *Then* $\mathrm{Card}(\mathrm{cl}_{H^{\mathrm{exc}}}(\alpha)) = \infty$, *and furthermore* $\min \mathrm{cl}_{H^{\mathrm{exc}}}(\alpha) = \alpha$ *and* $\max \mathrm{cl}_{H^{\mathrm{exc}}}(\alpha) = \beta$, *so that, in particular,* $\alpha \approx^{H^{\mathrm{exc}}} \beta$.

*Proof.* The first assertion is a consequence of Theorem 4.7 in [38] and the discussion following this statement. The fact that $\mathrm{Card}(\mathrm{cl}_{H^{\mathrm{exc}}}(\alpha)) = \infty$ if $\alpha$ is a jump time of $X^{\mathrm{exc}}$ follows from [38, Theorem 4.6]. Finally, let $\alpha$ be a jump time of $X^{\mathrm{exc}}$ and let $\beta = \inf\{r \geqslant \alpha;\ X^{\mathrm{exc}}_r = X^{\mathrm{exc}}_{\alpha-}\}$. By the first part of Remark 6.4.3, we know that $H^{\mathrm{exc}}_\alpha = \inf_{[\alpha,\beta]} H^{\mathrm{exc}} = H^{\mathrm{exc}}_\beta$ and that for any $\epsilon > 0$ :

$$\inf_{[\alpha-\epsilon,\alpha]} H^{\mathrm{exc}} < H^{\mathrm{exc}}_\alpha, \qquad \inf_{[\beta,\beta+\epsilon]} H^{\mathrm{exc}} < H^{\mathrm{exc}}_\beta.$$

The desired result follows. ☐

**Theorem 6.4.5.** *Almost surely, the relations* $\simeq^{X^{\mathrm{exc}}}$ *and* $\approx^{H^{\mathrm{exc}}}$ *coincide. In particular,*

$$L(X^{\mathrm{exc}}) = \boldsymbol{L}(H^{\mathrm{exc}}) \quad a.s.$$

*Proof.* We first observe that both relations $\simeq^{X^{exc}}$ and $\approx^{H^{exc}}$ are closed, in the sense that their graphs are closed subsets of $[0,1]^2$. In the case of $\simeq^{X^{exc}}$, this was already observed in the proof of Proposition 6.2.9. In the case of $\approx^{H^{exc}}$, this is elementary (see [27, Section 2.3]).

Let $s,t \in [0,1]$ such that $s < t$ and $s \simeq^{X^{exc}} t$. From Proposition 6.2.14, we can write the pair $(s,t)$ as the limit of a sequence $(s_n, t_n)$ in $\mathcal{E}_1$ (of course if $s$ is a jump time of $X^{exc}$ we take $s_n = s$ and $t_n = t$ for every $n$). However Proposition 6.4.4 then implies that $s_n \approx^{H^{exc}} t_n$, for every $n$, and it follows that $s \approx^{H^{exc}} t$.

Let us prove the converse. Let $(s,t)$ be such that $0 \leqslant s < t \leqslant 1$ and $s \approx^{H^{exc}} t$. If $\mathrm{Card}(\mathrm{cl}_{H^{exc}}(s)) \geqslant 3$, we must have $s = \min \mathrm{Card}(\mathrm{cl}_{H^{exc}}(s))$ and $t = \max \mathrm{Card}(\mathrm{cl}_{H^{exc}}(s))$, so that Proposition 6.4.4 implies that the pair $(s,t)$ belongs to $\mathcal{E}_1$, and in particular $s \simeq^{X^{exc}} t$. If $\mathrm{Card}(\mathrm{cl}_{H^{exc}}(s)) = 2$, then the second part of Remark 6.4.3 shows that $(s,t)$ is the limit of a sequence of pairs $s_n, t_n$ such that $s_n \approx^{H^{exc}} t_n$ and $\mathrm{Card}(\mathrm{cl}_{H^{exc}}(s_n)) \geqslant 3$. We have then $s_n \simeq^{X^{exc}} t_n$ for every $n$ and $s \simeq^{X^{exc}} t$ since the relation $\simeq^{X^{exc}}$ is closed. □

**Remark 6.4.6.** In the discrete setting, the definition of the dissection $D(W(\tau))$ via formula (6.5) uses the times $s^i_1, \dots, s^i_{k_i}$, which can be defined either from the Lukasiewicz path of $\tau$ as in Proposition 6.1.7 (i), or from the discrete height process of $\tau$ as in part (ii) of the same proposition. In the continuous setting, we recover these two different points of view in the definition of the $\theta$-stable lamination as $L(X^{exc})$ or $\mathbf{L}(H^{exc})$.

## 6.5 The Hausdorff dimension of the stable lamination

In this section, we determine the Hausdorff dimension of $L(X^{exc})$ and of some other random sets related to $L(X^{exc})$. We refer the reader to [82] for background concerning Hausdorff and Minkowski dimensions.

**Theorem 6.5.1.** *Fix $\theta \in (1,2]$. Let $L(X^{exc})$ be the random lamination coded by the normalized excursion $X^{exc}$ of the $\theta$-stable Lévy process and let $A$ stand for the set of all endpoints of chords in $L(X^{exc})$. Then:*

$$\dim(A) = 1 - \frac{1}{\theta}, \quad \dim(L(X^{exc})) = 2 - \frac{1}{\theta},$$

*where $\dim(K)$ stands for the Hausdorff dimension of a subset $K$ of $\mathbb{C}$. Furthermore, if $1 < \theta < 2$, then a.s. for every face $V$ of $L(X^{exc})$,*

$$\dim(\overline{V} \cap \mathbb{S}^1) = \frac{1}{\theta}.$$

**Remark 6.5.2.** In the case $\theta = 2$, the results of the theorem are already known: See [6] for a sketch of the argument and [77] for a detailed proof. We thus restrict our attention to $\theta \in (1,2)$. We follow the idea of the proof of [77] but a different argument is needed because of the existence of jump times.

It will be convenient to identify the interval $[0,1)$ with $\mathbb{S}_1$ via the mapping $x \mapsto e^{-2i\pi x}$. The set $A$ of the theorem is the set of all $x \in \mathbb{S}^1$ such that there exists $y \in \mathbb{S}^1$ with $y \neq x$ and $x \simeq^{X^{exc}} y$. We also let $\mathcal{I}$ be the set of all (ordered) pairs $(I,J)$ where $I$ and $J$ are two disjoint closed subarcs of $\mathbb{S}_1$ with nonempty interior and rational endpoints. If $(I, J) \in \mathcal{J}$, we denote by $A^{(I,J)}$ the set of all $x \in I$ such that $x \simeq^{X^{exc}} y$ for some $y \in J$. In particular:

$$A = \bigcup_{(I,J)\in\mathcal{J}} A^{(I,J)}.$$

In the following, $\underline{\dim}_M(B)$ and $\overline{\dim}_M(B)$ will denote respectively the lower and the upper Minkowski dimensions of a set B (see [82] for definitions). In order to compute Hausdorff and Minkowski dimensions, the following proposition will be useful.

**Proposition 6.5.3.** *Almost surely, for every $t > 0$, the set $\{0 \leqslant s \leqslant t; S_s = X_s\}$ has Hausdorff dimension and upper Minkowski dimension equal to $1 - 1/\theta$, and the set $\{0 \leqslant s \leqslant t; I_s = X_s\}$ has Hausdorff dimension and upper Minkowski dimension equal to $1/\theta$.*

*Proof.* Recall that if $(\tau_t, t \geqslant 0)$ is a stable subordinator of parameter $\rho \in (0, 1)$ then, almost surely, for all $t > 0$, the Hausdorff dimension and the upper Minkowski dimension of $\{\tau_s; 0 \leqslant s \leqslant t\}$, or of the closure of this set, is equal to $\rho$ (see e.g. [16, Theorem 5.1, Corollary 5.3]). Let $L = (L_t, t \geqslant 0)$ stand for a local time of $S - X$ at $0$, and let $L^{-1}$ be the right-continuous inverse of L. Since X has only positive jumps the set $\{0 \leqslant s < t; S_s = X_s\}$ is closed. By [15, Lemma VIII.1], $L^{-1}$ is a subordinator of index $1 - 1/\theta$ and by [15, Proposition IV.7 ], $\{0 \leqslant s < t; S_s = X_s\}$ coincides with the closure of $\{L_s^{-1}; 0 \leqslant s < L_t\}$. As $L_t > 0$ almost surely, the first assertion of the proposition follows. The proof of the second assertion is similar, noting that $-I$ is a local time at $0$ for $X - I$ and that the right-continuous inverse of $-I$ is a stable subordinator of index $1/\theta$, again by [15, Lemma VIII.1]. □

**Lemma 6.5.4.** *For $a \in (0, 1]$, set $\widehat{F}_a := \{u \in (0, a); X^{exc}_{u-} \leqslant \inf_{[u,a]} X^{exc}\}$. Almost surely, for every jump time $a$ of $X^{exc}$ in $(0, 1)$ we have:*

$$\dim(\widehat{F}_a) = \overline{\dim}_M(\widehat{F}_a) = 1 - \frac{1}{\theta}. \tag{6.17}$$

Informally, if one identifies the interval $[0, 1]$ with the circle $\mathbb{S}_1$ by using the map $x \to e^{-2i\pi x}$, the set $\widehat{F}_a$ corresponds to endpoints in $(0, a)$ of chords that connect a point of $(0, a)$ to a point of $(a, 1)$.

*Proof of Lemma 6.5.4.* We first consider an analog of $\widehat{F}_a$ where $X^{exc}$ is replaced by the Lévy process X. Precisely, for every $a > 0$, we set

$$\widetilde{F}_a := \{u \in (0, a); X_{u-} \leqslant \inf_{[u,a]} X\}.$$

Note that, under the condition $X_a > I_a$, $\widetilde{F}_a$ is contained in the (closure of the) excursion interval of $X - I$ that straddles $a$. Thanks to this observation and to the connection between $X^{exc}$ and X given by Proposition 6.2.1, the result of the lemma will follow if we can verify that

$$\dim(\widetilde{F}_a) = \overline{\dim}_M(\widetilde{F}_a) = 1 - \frac{1}{\theta}, \tag{6.18}$$

for every jump time $a$ of $X$ (note that if $X^{exc}$ is given by the formula of Proposition 6.2.1, the jump times of $X^{exc}$ exactly correspond to jump times of $X$ over $(\underline{g}_1, \underline{d}_1)$). Let $K > 0$ and consider only jump times that are bounded above by $K$. The desired result for such jump times follows by considering the process $X$ time-reversed at time $K$ and using the strong Markov property together with Proposition 6.5.3. $\square$

*Proof of Theorem 6.5.1.* We first prove the last assertion of the theorem. By Proposition 6.3.10, a face $V$ of $L(X^{exc})$ is associated to a jump time $s$ of $X^{exc}$, and we set $t = \inf\{r > s : X^{exc}_r = X^{exc}_{s-}\}$. Let the intervals $(\alpha_i, \beta_i)$, $i \geqslant 1$ be defined as in the proof of Proposition 6.3.10. Then, it easily follows from (6.12) that

$$\overline{V} \cap \mathbb{S}^1 = [s,t] \backslash \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i) = \{r \in [s,t]; X^{exc}_r = \inf_{[s,r]} X^{exc}\},$$

where we recall that $\mathbb{S}^1$ is identified with $[0,1)$. The calculation of $\dim(\overline{V} \cap \mathbb{S}^1)$ now follows from the second assertion of Proposition 6.5.3, using also Proposition 6.2.1.

Let us turn to the first part of the theorem. We follow the ideas of the proof of the analogous result in [77]. We will prove that:

$$\dim(A) = 1 - 1/\theta, \qquad \overline{\dim}_M(A^{(I,J)}) \leqslant 1 - 1/\theta \tag{6.19}$$

for every $(I,J) \in \mathcal{I}$, a.s. If (6.19) holds, then:

$$\underline{\dim}_M(A^{(I,J)} \cup A^{(J,I)}) \leqslant \overline{\dim}_M(A^{(I,J)} \cup A^{(J,I)}) = \max(\overline{\dim}_M(A^{(I,J)}), \overline{\dim}_M(A^{(J,I)})) \leqslant \dim(A),$$

and then the same argument as in Proposition 2.3 of [77] entails that $\dim(L(X^{exc})) = 1 + \dim(A) = 2 - 1/\theta$. It remains to establish (6.19). In order to verify that

$$\overline{\dim}_M(A^{(I,J)}) \leqslant 1 - 1/\theta$$

for every $(I,J) \in \mathcal{I}$, we need only consider the case $I = [u,v]$, $J = [u',v']$ with $0 \leqslant u' < v' \leqslant 1$, $0 \leqslant u < v \leqslant 1$ (if one of the subarcs $I$ or $J$ contains $0$ as an interior point, partition it into two subarcs whose interior does not contain $0$). Since the relations $\simeq^{X^{exc}}$ and $\approx^{H^{exc}}$ coincide, the time-reversal invariance property of $H^{exc}$ (see [37, Corollary 3.1.6]) allows us to restrict to the case $0 \leqslant u < v < u' < v' \leqslant 1$. Choose a jump time $a$ of $X^{exc}$ such that $v < a < u'$ and observe that $\widehat{F}_a \subset A$ and $A^{(I,J)} \subset \widehat{F}_a$, with the notation of Lemma 6.5.4. Hence, by the latter lemma, $\overline{\dim}_M(A^{(I,J)}) \leqslant \overline{\dim}_M(\widehat{F}_a) = 1 - 1/\theta$. Lemma 6.5.4 and the property $\widehat{F}_a \subset A$ also give $1 - 1/\theta \leqslant \dim A$. We have then:

$$1 - \frac{1}{\theta} \leqslant \dim A \leqslant \overline{\dim}_M(A) \leqslant \max_{(I,J)\in\mathcal{I}} \overline{\dim}_M(A^{(I,J)}) \leqslant 1 - \frac{1}{\theta}.$$

In particular $\dim A = 1 - 1/\theta$ and (6.19) holds. This completes the proof. $\square$

Chapitre 7

# Random non-crossing plane configurations : A conditioned Galton-Watson tree approach

*Les résultats de ce chapitre sont issus de l'article [28] écrit en collaboration avec Nicolas Curien, accepté pour publication dans* Random Struct. Alg.

## Contenu de ce chapitre




We study various models of random non-crossing configurations consisting of diagonals of convex polygons, and focus in particular on uniform dissections and non-crossing trees. For both these models, we prove convergence in distribution towards Aldous' Brownian triangulation of the disk. In the case of dissections, we also refine the study of the maximal vertex degree and validate a conjecture of Bernasconi, Panagiotou and Steger. Our main tool is the use of an underlying Galton-Watson tree structure.


## Introduction

Various models of non-crossing geometric configurations involving diagonals of a convex polygon in the plane have been studied both in geometry, probability theory and especially in enumerative combinatorics (see e.g. [45]). Three specific models of non-crossing configurations – triangulations, dissections and non-crossing trees – have drawn particular attention. Let us first recall the definition of these models.

Let $P_n$ be the convex polygon inscribed in the unit disk of the complex plane whose vertices are the $n$-th roots of unity. By definition, a *dissection* of $P_n$ is the union of the sides of $P_n$ and of a collection of diagonals that may intersect only at their endpoints. A *triangulation* is a dissection whose inner faces are all triangles. Finally, a *non-crossing tree* of $P_n$ is a tree drawn on the plane whose vertices are all vertices of $P_n$ and whose edges are non-crossing line segments. See Fig. 7.1 for examples.

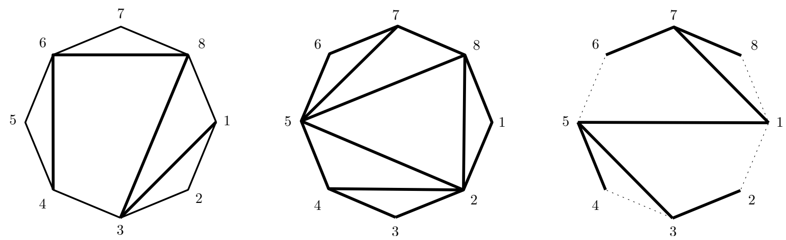


**Figure 7.1:** A dissection, a triangulation and a non-crossing tree of the octogon.

Graph theoretical properties of uniformly distributed triangulations have been recently investigated in combinatorics. For instance, the study of the asymptotic behavior of the maximal vertex degree has been initiated in [34] and pursued in [47]. Afterwards, the same random variable has been studied in the case of dissections [14].

We shall continue the study of graph-theoretical properties of large uniform dissections and in particular focus on the maximal vertex degree. Our method is based on finding and exploiting an underlying Galton-Watson tree structure. More precisely, we show that the *dual tree* associated with a uniformly distributed dissection of $P_n$ is a critical Galton-Watson tree conditioned on having exactly $n-1$ *leaves*. This new conditioning of Galton-Watson trees has been studied recently in [70] (see also [94]) and is well adapted to the study of dissections, see [69]. In particular we are able to validate a conjecture contained in [14] concerning the asymptotic behavior of the maximal vertex degree in a uniform dissection (Theorem 7.3.7). Using the critical Galton-Watson tree conditioned to survive introduced in [65], we also give a simple probabilistic explanation of the fact that the inner degree of a given vertex in a large uniform dissection converges in distribution to the sum of two independent geometric variables (Proposition 7.3.6). We finally obtain new results about the asymptotic behavior of the maximal face degree in a uniformly distributed dissection.

As a by-product of our techniques, we give a very simple probabilistic approach to the following enumeration problem. Let $\mathcal{A}$ be a non-empty subset of $\{3,4,5,\ldots\}$ and $\mathbf{D}_n^{(\mathcal{A})}$ the set of all dissections of $P_{n+1}$ whose face degrees all belong to the set $\mathcal{A}$. Theorem 7.2.7 gives an explicit asymptotic formula for $\#\mathbf{D}_n^{(\mathcal{A})}$ as $n \to \infty$ (for those values of $n$ for which $\mathbf{D}_n^{(\mathcal{A})} \neq \varnothing$). In particular when $\mathcal{A}_0 = \{3,4,5,\ldots\}$, then $\mathbf{D}_{n-1} := \mathbf{D}_{n-1}^{(\mathcal{A}_0)}$ is the set of all dissections of $P_n$ and

$$\#\mathbf{D}_{n-1} \quad \underset{n\to\infty}{\sim} \quad \frac{1}{4}\sqrt{\frac{99\sqrt{2}-140}{\pi}}\, n^{-3/2}(3+2\sqrt{2})^n.$$

This formula (Corollary 7.1.5) was originally derived by Flajolet & Noy [45] using very different techniques.

From a geometrical perspective, Aldous [7, 6] proposed to consider triangulations of $P_n$ as closed subsets of the unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ rather than viewing them as graphs. He proved that large uniform triangulations of $P_n$ converge in distribution (for the Hausdorff distance on compact subsets of the unit disk) towards a random compact subset. This limiting object is called *the Brownian triangulation* (see Fig. 7.2). This name comes from the fact that the Brownian triangulation can be constructed from the Brownian excursion as follows: Let $\mathbb{e} : [0,1] \to \mathbb{R}$ be a normalized excursion of linear Brownian motion. For every $s, t \in [0,1]$, we set $s \sim t$ if we have $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s \wedge t, s \vee t]} \mathbb{e}$. The Brownian triangulation is then defined as:

$$\mathcal{B} \ := \ \bigcup_{s \sim t} \left[ e^{-2i\pi s}, e^{-2i\pi t} \right], \tag{7.1}$$

where $[x, y]$ stands for the Euclidean line segment joining two complex numbers $x$ and $y$. It

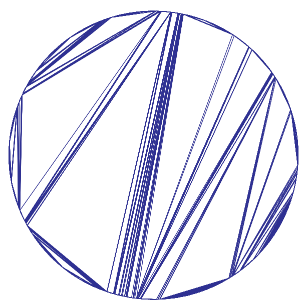

**Figure 7.2:** A sample of the Brownian triangulation $\mathcal{B}$.

turns out that $\mathcal{B}$ is almost surely a closed set which is a continuous triangulation of the unit disk (in the sense that the complement of $\mathcal{B}$ in $\overline{\mathbb{D}}$ is a disjoint union of open Euclidean triangles whose vertices belong to the unit circle). Aldous also observed that the Hausdorff dimension of $\mathcal{B}$ is almost surely equal to $3/2$ (see [77]). Later, in the context of random maps, the Brownian triangulation has been studied by Le Gall & Paulin in [77] where it serves as a tool in the proof of the homeomorphism theorem for the Brownian map. See also [27, 69] for analogs of the Brownian triangulation.

However, neither large random uniform dissections, nor large uniform non-crossing trees have yet been studied from this geometrical point of view.

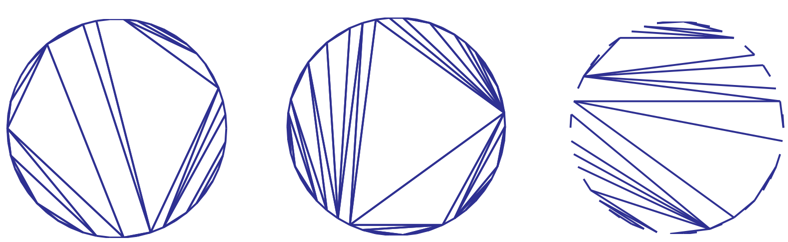

**Figure 7.3:** Uniform dissection, triangulation and non-crossing tree of size $50$. The same continuous model?

In this work, we extend Aldous' theorem by showing that both large uniform dissections and large uniform non-crossing trees converge in distribution towards the Brownian triangulation (Theorem 7.2.1). The maybe surprising fact that large uniform dissections (which may have non-triangular faces) converge to a continuous triangulation stems from the fact that many diagonals degenerate in the limit. For both models, the key is to use a Galton-Watson tree structure, which was already described above in the case of dissections. In the case of non-crossing trees this structure has been identified by Marckert & Panholzer [81] who established that the *shape* of a uniform non-crossing tree of $P_n$ is *almost* a Galton-Watson tree conditioned on having $n$ vertices (see Theorem 7.1.7 below for a precise statement).

We also consider other random configurations of non-crossing diagonals of $P_n$ such as non-crossing graphs, non-crossing partitions and non-crossing pair partitions, and prove the convergence towards the Brownian triangulation, again by using an appropriate underlying tree structure. We also show that a uniformly distributed dissection over $\mathbf{D}_n^{(\mathcal{A})}$ converges towards $\mathcal{B}$ as $n \to \infty$. The Brownian triangulation thus appears as a universal limit for random non-crossing configurations. This has interesting applications: For instance, let $\chi_n$ be a random non-crossing configuration on the vertices of $P_n$ that converges in distribution towards $\mathcal{B}$ in the sense of the Hausdorff metric. Then the corresponding angle of the longest diagonal of $\chi_n$, suitably normalized, converges in distribution towards the length of the longest chord of the Brownian triangulation with law

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\leqslant x\leqslant\frac{1}{2}}dx.$$

This has been shown in the particular case of triangulations in [6] (see also [34]).

The paper is organized as follows. In the first section we introduce the discrete models and explain the underlying Galton-Watson tree structures. The second section is devoted to the convergence of different random non-crossing configurations towards the Brownian triangulation and to applications. The final section contains the analysis of graph-theoretical properties of large uniform dissections, such as the maximal vertex or face degree.

**Acknowledgments.** We are indebted to Jean-François Le Gall for a very careful reading of a first version of this article and for useful suggestions. We thank the participants of the ANR A3 meeting in Nancy (March 2011) for stimulating discussions and especially Jean-François Marckert for telling us about non-crossing trees.

## 7.1 Dissections, non-crossing trees and Galton-Watson trees

### 7.1.1 Dissections and plane trees

Throughout this work, for every integer $n \geqslant 3$, $P_n$ stands for the regular polygon of the plane with $n$ sides whose vertices are the $n$-th roots of unity.

**Definition 7.1.1.** A *dissection* $\mathcal{D}$ of the polygon $P_n$ is the union of the sides of the polygon and of a collection of diagonals that may intersect only at their endpoints. A *face* $f$ of $\mathcal{D}$ is a connected component of the complement of $\mathcal{D}$ inside $P_n$; its degree, denoted by $\deg(f)$, is the number of sides surrounding $f$. See Fig. 7.4 for an example.

Let $\mathbb{L}_n$ be the set of all dissections of $P_{n+1}$. Given a dissection $\mathcal{D} \in \mathbb{L}_n$, we construct a (rooted ordered) tree $\phi(\mathcal{D})$ as follows: Consider the dual graph of $\mathcal{D}$, obtained by placing a vertex inside each face of $\mathcal{D}$ and outside each side of the polygon $P_{n+1}$ and by joining two vertices if the corresponding faces share a common edge, thus giving a connected graph without cycles. Then remove the dual edge intersecting the side of $P_{n+1}$ which connects $1$ to $e^{\frac{2i\pi}{n+1}}$. Finally, root the tree at the corner adjacent to the latter side (see Fig. 7.4).

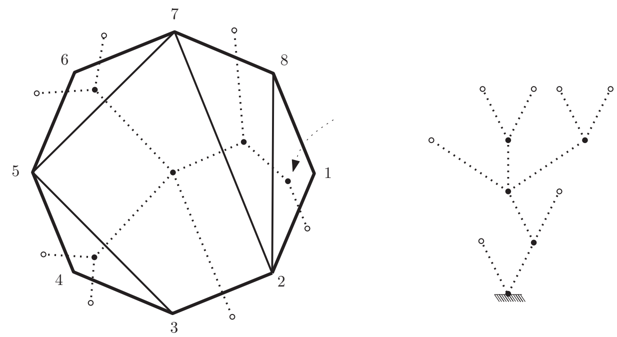


**Figure 7.4:** The dual tree of a dissection of $P_8$, note that the tree has $7$ leaves.

The dual tree of a dissection is a plane tree (also known as rooted ordered tree in the literature). We briefly recall the formalism of plane trees which can be found in [73] for example. Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of nonnegative integers, $\mathbb{N}^* = \{1, \ldots\}$ and let $\mathcal{U}$ be the set of labels

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\varnothing\}$. An element of $\mathcal{U}$ is a sequence $u = u_1 \cdots u_m$ of positive integers, and we set $|u| = m$, which represents the "generation" of $u$. If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ belong to $\mathcal{U}$, we write $uv = u_1 \cdots u_m v_1 \cdots v_m$ for the concatenation of $u$ and $v$. Finally, a *plane tree* $\tau$ is a finite or infinite subset of $\mathcal{U}$ such that:

1. $\varnothing \in \tau$,
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,
3. for every $u \in \tau$, there exists an integer $k_u(\tau) \geqslant 0$ (the number of children of $u$) such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

In the following, by *tree* we will always mean plane tree. We denote the set of all trees by $\mathbb{T}$. We will often view each vertex of a tree $\tau$ as an individual of a population whose $\tau$ is the genealogical tree. If $\tau$ is a tree and $u \in \tau$, we define the shift of $\tau$ at $u$ by $\sigma_u \tau = \{v \in U : uv \in \tau\}$, which is itself a tree. If $u, v \in \tau$ we denote by $[\![u, v]\!]$ the discrete geodesic path between $u$ and $v$ in $\tau$. The total progeny of $\tau$, which is the total number of vertices of $\tau$, will be denoted by $\zeta(\tau)$. The number of leaves (vertices $u$ of $\tau$ such that $k_u(\tau) = 0$) of the tree $\tau$ is denoted by $\lambda(\tau)$. Finally, we let $\mathbb{T}_n^{(\ell)}$ denote the set of all plane trees with $n$ leaves such that there is no vertex with exactly one child.

The following proposition is an easy combinatorial property, whose proof is omitted.

**Proposition 7.1.2.** *The duality application $\phi$ is a bijection between $\mathbf{D}_n$ and $\mathbb{T}_n^{(\ell)}$.*

The dissection is conversely easily obtained from the tree as shown in Fig. 7.4.

Finally, we briefly recall the standard definition of Galton-Watson trees. Let $\rho$ be a probability measure on $\mathbb{N}$ such that $\rho(1) < 1$. The law of the Galton-Watson tree with offspring distribution $\rho$ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbb{T}$ such that:

1. $\mathbb{P}_\rho(k_\varnothing = j) = \rho(j)$ for $j \geqslant 0$,
2. for every $j \geqslant 1$ with $\rho(j) > 0$, conditionally on $\{k_\varnothing = j\}$, the subtrees $\sigma_1\tau, \dots, \sigma_j\tau$ are i.i.d. with distribution $\mathbb{P}_\rho$.

It is well-known that if $\rho$ has mean less than or equal to 1 then a $\rho$-Galton-Watson tree is almost surely finite. In the sequel, for every integer $j \geqslant 1$, $\mathbb{P}_{\rho,j}$ will stand for the probability measure on $\mathbb{T}^j$ which is the distribution of $j$ independent trees of law $\mathbb{P}_\rho$. The canonical element of $\mathbb{T}^j$ will be denoted by $\mathfrak{f}$. For $\mathbf{f} = (\tau_1, \dots, \tau_j) \in \mathbb{T}^j$, let $\lambda(\mathbf{f}) = \lambda(\tau_1) + \cdots + \lambda(\tau_j)$ be the total number of leaves of $\mathbf{f}$.

### 7.1.2 Uniform dissections are conditioned Galton-Watson trees

In the rest of this work, $\mathcal{D}_n$ is a random dissection uniformly distributed over $\mathbf{D}_n$. We also set $t_n = \phi(\mathcal{D}_n)$ to simplify notation. Remark that $t_n$ is a random tree which belongs to $\mathbb{T}_n^{(\ell)}$.

Fix $c \in (0, 1/2)$ and define a probability distribution $\mu^{(c)}$ on $\mathbb{N}$ as follows:

$$\mu_0^{(c)} = \frac{1-2c}{1-c}, \qquad \mu_1^{(c)} = 0, \qquad \mu_i^{(c)} = c^{i-1} \text{ for } i \geqslant 2.$$

It is straightforward to check that $\mu^{(c)}$ is a probability measure, which moreover has mean equal to 1 when $c = 1 - 2^{-1/2}$. In the latter case, we drop the exponent $(c)$ in the notation, so that $\mu := \mu^{(1-1/\sqrt{2})}$. The following theorem gives a connection between uniform dissections of $P_n$ and Galton-Watson trees conditioned on their number of leaves. This connection has been obtained independently of the present work in [91].

**Proposition 7.1.3.** *The conditional probability distribution $\mathbb{P}_{\mu^{(c)}}(\,\cdot \mid \lambda(\tau) = n)$ does not depend on the choice of $c \in (0, 1/2)$ and coincides with the distribution of the dual tree $t_n$ of a uniformly distributed dissection of $P_{n+1}$.*

*Proof.* We adapt the proof of [69, Proposition 1.8] in our context. By Proposition 7.1.2, it is sufficient to show that for every $c \in (0, 1/2)$ the probability distribution $\mathbb{P}_{\mu^{(c)}}(\cdot \mid \lambda(\tau) = n)$ is the uniform probability distribution over $\mathbb{T}_n^{(\ell)}$. If $\tau$ is a tree, we denote by $u_0, \dots, u_{\zeta(\tau)-1}$ the vertices of $\tau$ listed in lexicographical order and recall that $k_{u_i}$ stands for the number of children of $u_i$. Let $\tau_0 \in \mathbb{T}_n^{(\ell)}$. By the definition of $\mathbb{P}_{\mu^{(c)}}$, we have

$$\mathbb{P}_{\mu^{(c)}}(\tau = \tau_0 \mid \lambda(\tau) = n) \quad = \quad \frac{1}{\mathbb{P}_{\mu^{(c)}}(\lambda(\tau) = n)} \prod_{i=0}^{\zeta(\tau_0)-1} \mu_{k_{u_i}}^{(c)}.$$

Using the definition of $\mu^{(c)}$, the product appearing in the last expression can be written as

$$\prod_{i=0}^{\zeta(\tau_0)-1} \mu_{k_{u_i}}^{(c)} = \left(\frac{1-2c}{1-c}\right)^{\lambda(\tau_0)} c^{\zeta(\tau_0)-1-(\zeta(\tau_0)-\lambda(\tau_0))} = c^{-1}\left(\frac{c(1-2c)}{1-c}\right)^{\lambda(\tau_0)}.$$

Thus $\mathbb{P}_{\mu^{(c)}}(\tau = \tau_0 \mid \lambda(\tau) = n)$ depends only on $\lambda(\tau_0)$. We conclude that $\mathbb{P}_{\mu^{(c)}}(\,\cdot \mid \lambda(\tau) = n)$ is the uniform distribution over $\mathbb{T}_n^{(\ell)}$. □

In the following, we will always choose $c = 1 - 2^{-1/2}$ for $\mu^{(c)} = \mu$ to be critical. Hence, the random tree $t_n$ has law $\mathbb{P}_\mu(\cdot \mid \lambda(\tau) = n)$. A general study of Galton-Watson trees conditioned by their number of leaves is made in [70]. In particular, we will make an extensive use of the following asymptotic estimate which is a particular case of [70, Theorem 3.1]:

**Lemma 7.1.4.** *We have*

$$\mathbb{P}_\mu\left[\lambda(\tau) = n\right] \quad \underset{n\to\infty}{\sim} \quad \frac{n^{-3/2}}{2\sqrt{\pi\sqrt{2}}}. \tag{7.2}$$

Let us give an application of Proposition 7.1.3 and Lemma 7.1.4 to the enumeration of dissections. There exists no easy closed formula for the number $\#\mathbf{D}_n$ of dissections of $P_{n+1}$. However, a recursive decomposition easily shows that the generating function

$$D(z) \quad := \quad \sum_{n\geqslant 3} z^n \#\mathbf{D}_{n-1},$$

is equal to $\frac{z}{4}(1+z-\sqrt{z^2-6z+1})$, see e.g. [14, Section 3] and [45]. Using classical techniques of analytic combinatorics [46], it is then possible to get the asymptotic behavior of $\#\mathbf{D}_n$, see [45]. Here, we present a very short "probabilistic" proof of this result.

**Corollary 7.1.5** (Flajolet & Noy, [45])**.** *We have*

$$\#\mathbb{L}_{n-1} \quad \underset{n\to\infty}{\sim} \quad \frac{1}{4}\sqrt{\frac{99\sqrt{2}-140}{\pi}}\, n^{-3/2}(3+2\sqrt{2})^n.$$

*Proof.* Let $n \geqslant 3$ and let $\tau_0 = \{\varnothing, 1, 2, \dots, n-1\}$ be the tree consisting of the root and its $n-1$ children. By Proposition 7.1.3, we have

$$\frac{1}{\#\mathbb{L}_{n-1}} = \mathbb{P}_\mu(\tau = \tau_0 \,|\, \lambda(\tau) = n-1) = \frac{\mathbb{P}_\mu(\tau = \tau_0)}{\mathbb{P}_\mu(\lambda(\tau) = n-1)} = \frac{\mu_{n-1}\mu_0^{n-1}}{\mathbb{P}_\mu(\lambda(\tau) = n-1)}.$$

Thus

$$\#\mathbb{L}_{n-1} = \frac{\mathbb{P}_\mu(\lambda(\tau) = n-1)}{(2-\sqrt{2})^{n-1}\left(\frac{2-\sqrt{2}}{2}\right)^{n-2}} = \frac{(2-\sqrt{2})^3}{4}\frac{\mathbb{P}_\mu(\lambda(\tau) = n-1)}{(3-2\sqrt{2})^n}. \tag{7.3}$$

The statement of the corollary now follows from (7.2) and (7.3). □

### 7.1.3 Non-crossing trees are *almost* conditioned Galton-Watson trees

**Definition 7.1.6.** A *non-crossing tree* $\mathscr{C}$ of $P_n$ is a tree drawn in the plane whose vertices are all the vertices of $P_n$ and whose edges are Euclidean line segments that do not intersect except possibly at their endpoints.

Every non-crossing tree $\mathscr{C}$ inherits a plane tree structure by rooting $\mathscr{C}$ at the vertex 1 of $P_n$ and keeping the planar ordering induced on $\mathscr{C}$. The children of the root vertex are ordered by going in clockwise order around the point 1 of $P_n$, starting from the edge connecting 1 to $e^{-2i\pi/n}$, which may or may not be in $\mathscr{C}$. As in [81], we call this plane tree *the shape* of $\mathscr{C}$ and denote it by $S(\mathscr{C})$. Obviously $\zeta(S(\mathscr{C})) = n$. Note that the mapping $\mathscr{C} \mapsto S(\mathscr{C})$ is not one-to-one. However, we will later see that large scale properties of uniform non-crossing trees are governed by their shapes.

In the following, we let $\mathscr{C}_n$ be uniformly distributed over the set of all non-crossing trees of $P_n$. We also set $\mathscr{T}_n = S(\mathscr{C}_n)$ to simplify notation. We start by recalling a result of Marckert and Panholzer stating that $\mathscr{T}_n$ is *almost* a Galton-Watson tree. Consider the two offspring distributions:

$$\begin{aligned} \nu_\varnothing(k) &= 2 \cdot 3^{-k}, && \text{for } k = 1, 2, 3, \ldots \\ \nu(k) &= 4(k+1)3^{-k-2}, && \text{for } k = 0, 1, 2, \ldots \end{aligned}$$

Following [81], we introduce a *modified* version of the $\nu$-Galton-Watson tree where the root vertex has a number of children distributed according to $\nu_\varnothing$ and all other individuals have offspring distribution $\nu$. We denote the resulting probability measure on plane trees obtained by $\widetilde{\mathbb{P}}_\nu$. The following theorem is the main result of [81] and will be useful for our purposes:

**Theorem 7.1.7** (Marckert & Panholzer, [81]). *The random plane tree $\mathscr{T}_n$ is distributed according to $\widetilde{\mathbb{P}}_\nu(\cdot \mid \zeta(\tau) = n)$.*

## 7.2 The Brownian triangulation: A universal limit for random non-crossing configurations

Recall that $\mathcal{D}_n$ is a uniform dissection of $P_{n+1}$ and that $\mathcal{T}_n$ stands for its dual plane tree. Recall also that $\mathscr{C}_n$ is a uniform non-crossing tree of $P_n$ and that $\mathscr{T}_n$ stands for its shape. In the following, we will view both $\mathcal{D}_n$ and $\mathscr{C}_n$ as random closed subsets of $\overline{\mathbb{D}}$ as suggested by Fig. 7.1. Recall that the *Hausdorff distance* between two closed subsets of $A, B \subset \overline{\mathbb{D}}$ is

$$d_{\mathrm{Haus}}(A, B) \;=\; \inf\left\{\varepsilon > 0 : A \subset B^{(\varepsilon)} \text{ and } B \subset A^{(\varepsilon)}\right\},$$

where $X^{(\varepsilon)}$ is the $\varepsilon$-enlargement of a set $X \subset \overline{\mathbb{D}}$. The set of all closed subsets of $\overline{\mathbb{D}}$ endowed with the Hausdorff distance is a compact metric space. Recall that the Brownian triangulation $\mathcal{B}$ is defined by (7.1). The main result of this section is:

**Theorem 7.2.1.** *The following two convergences in distribution hold for the Hausdorff metric on closed subsets of $\overline{\mathbb{D}}$:*

$$\text{(i)} \quad \mathcal{D}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad\qquad \text{(ii)} \quad \mathscr{C}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}.$$

The main ingredient in the proof of Theorem 7.2.1 is a scaling limit theorem for functions coding the trees $\mathcal{T}_n$ and $\mathscr{T}_n$. In order to state this result, let us introduce the contour function associated to a plane tree.

Fix a tree $\tau$ and consider a particle that starts from the root and visits continuously all the edges of $\tau$ at unit speed (assuming that every edge has unit length). When leaving a vertex, the particle moves towards the first non visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all the edges will be crossed twice, the total time needed to explore the tree is $2(\zeta(\tau)-1)$. For $0 \leqslant t \leqslant 2(\zeta(\tau)-1)$, $C_\tau(t)$ is defined as the distance to the root of the position of the particle at time $t$. For technical reasons, we set $C_\tau(t) = 0$ for $t \in [2(\zeta(\tau)-1), 2\zeta(\tau)]$. The function $C_\tau(\cdot)$ is called the contour function of the tree $\tau$. See [73] for a rigorous definition. For $t \in [0, 2(\zeta(\tau)-1)]$ and $u \in \tau$, we say that the contour process visits the vertex $u$ at time $t$ if the particle is at $u$ at time $t$. Similarly, if we say that the contour process visits an edge $\mathfrak{e}$ if the particle belongs to $\mathfrak{e}$ at time $t$.

Let $\mathbb{e}$ bet the normalized excursion of linear Brownian motion. The following convergences in distribution will be useful for our purposes:

$$\left(\frac{C_{\mathcal{T}_n}(2\zeta(\mathcal{T}_n)t)}{\sqrt{n}}\right)_{0\leqslant t\leqslant 1} \xrightarrow[n\to\infty]{(d)} \left((3\sqrt{2}-4)^{-1/2}\mathbb{e}(t)\right)_{0\leqslant t\leqslant 1}, \tag{7.4}$$

$$\left(\frac{C_{\mathscr{T}_n}(2nt)}{\sqrt{n}}\right)_{0\leqslant t\leqslant 1} \xrightarrow[n\to\infty]{(d)} \left(2\sqrt{\frac{2}{3}}\mathbb{e}(t)\right)_{0\leqslant t\leqslant 1}. \tag{7.5}$$

The convergence (7.4) has been proved by Kortchemski [70, Theorem 5.9, Remark 5.10], and (7.5) has been obtained by Marckert and Panholzer [81, Proposition 4]. Although not relevant for our purposes, let us mention that it follows from [70, Corollary 3.3] that $\zeta(\mathcal{T}_n)$ is concentrated around $\mu_0^{-1}n$ as $n \to \infty$ and thus (7.4) means that the global shape of $\mathcal{T}_n$ is the same as that of a $\mu$-Galton-Watson tree conditioned on having $\lfloor \mu_0^{-1}n \rfloor$ vertices in total.

The proof of Theorem 7.2.1 will be different for dissections and non-crossing trees, although the main ideas are the same in both cases. Notice for example that in the case of non-crossing trees, there is no need to consider a dual structure since the shape of the non-crossing tree already furnishes a plane tree.

### 7.2.1 Large uniform dissections

#### The Brownian triangulation is the limit of large uniform dissections

In [69], a general convergence result is proved for dissections whose dual tree is a conditioned Galton-Watson tree whose offspring distribution belongs to the domain of attraction of a stable law. Since our approach to the convergence of large uniform non-crossing trees towards the Brownian triangulation will be similar in spirit, we reproduce the main steps of the proof in our particular case.

The following lemma, which is an easy consequence of [70, Corollary 3.3], roughly says that leaves are distributed uniformly in a conditioned Galton-Watson tree. Formally, if $\tau$ is a plane tree, for $0 \leqslant t \leqslant 2\zeta(\tau)-2$, we let $\Lambda_\tau(t)$ be the number of leaves among the vertices of $\tau$ visited by the contour process up to time $t$, and we set $\Lambda_\tau(t) = \lambda(\tau)$ for $2\zeta(\tau)-2 \leqslant t \leqslant 2\zeta(\tau)$.

**Lemma 7.2.2.** *We have*

$$\sup_{0\leqslant t\leqslant 1}\left|\frac{\Lambda_{t_n}\ (2\zeta(t_n)t)}{n}-t\right| \xrightarrow[n\to\infty]{(\mathbb{P})} 0, \tag{7.6}$$

*where* $(\mathbb{P})$ *stands for the convergence in probability.*

*Proof of Theorem 7.2.1 part (i).* We can apply Skorokhod's representation theorem (see e.g. [20, Theorem 6.7]) and assume, without loss of generality, that the convergences (7.4) and (7.6) hold almost surely and we aim at showing that $\mathcal{D}_n$ converges almost surely towards the Brownian triangulation $\mathcal{B}$ defined by (7.1). Since the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff metric is compact, it is sufficient to show that the sequence $(\mathcal{D}_n)_{n\geqslant 1}$ has a unique accumulation point which is $\mathcal{B}$. We fix $\omega$ such that both convergences (7.4) and (7.6) hold for this value of $\omega$. Up to extraction, we thus suppose that $(\mathcal{D}_n)_{n\geqslant 1}$ converges towards a certain compact subset $\mathcal{D}_\infty$ of $\overline{\mathbb{D}}$ and we aim at showing that $\mathcal{D}_\infty = \mathcal{B}$.

We first show that $\mathcal{B} \subset \mathcal{D}_\infty$. Fix $0 < s < t < 1$ such that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$. We first consider the case when we have also $\mathbb{e}(r) > \mathbb{e}(s)$ for every $r \in (s,t)$. Let us prove that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{D}_\infty$. Using the convergence (7.4), one can find an edge of $t_n$ such that if $s_n$ is the time of the first visit of this edge by the contour process and if $t_n$ is the time of its last visit, then $s_n/(2\zeta(\mathcal{T}_n)) \to s$ and $t_n/(2\zeta(\mathcal{T}_n)) \to t$ as $n \to \infty$, see Fig. 7.5.

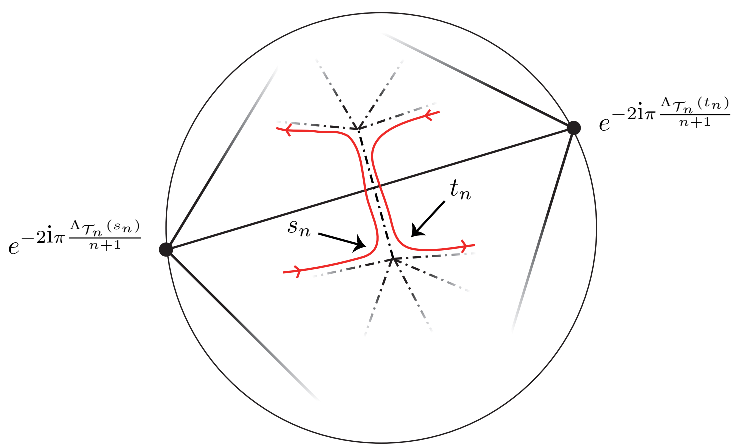


**Figure 7.5:** The arrows show the first visit time $s_n$ and last visit time $t_n$ of an edge of $\mathcal{T}_n$.

Since the sides of $P_{n+1}$, excepting the side connecting 1 to $e^{2i\pi/(n+1)}$, are in one-to-one correspondence with the leaves of $\mathcal{T}_n$ we have (see Fig. 7.5):

$$\left[e^{-2i\pi\frac{\Lambda_{t_n}(s_n)}{n+1}}, e^{-2i\pi\frac{\Lambda_{t_n}(t_n)}{n+1}}\right] \in \mathcal{D}_n.$$

We refer to [69] for a more complete proof. From Lemma 7.2.2, we can pass to the limit and obtain $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{D}_\infty$.

Let us now suppose that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$ and, moreover, there exists $r \in (s,t)$ such that $\mathbb{e}(r) = \mathbb{e}(s)$. Since local minima of Brownian motion are distinct, there exist two sequences of real numbers $(\alpha_n)_{n\geqslant 1}$ and $(\beta_n)_{n\geqslant 1}$ taking values in $[0,1]$ such that $\alpha_n \to s$, $\beta_n \to t$ as $n \to \infty$ and such that for every $n \geqslant 1$ and $r \in (\alpha_n, \beta_n)$ we have $\mathbb{e}(r) > \mathbb{e}(\alpha_n) = \mathbb{e}(\beta_n)$. The preceding argument yields $[e^{-2i\pi\alpha_n}, e^{-2i\pi\beta_n}] \subset \mathcal{D}_\infty$ for every $n \geqslant 1$. Since $\mathcal{D}_\infty$ is closed, we conclude that $\mathcal{B} \subset \mathcal{D}_\infty$.

The reverse inclusion is obtained by making use of a maximality argument. More precisely, it is easy to show that $\mathcal{D}_\infty$ is a *lamination*, that is a closed subset of $\overline{\mathbb{D}}$ which can be written as a union of chords that do not intersect each other inside $\mathbb{D}$. However, the Brownian triangulation $\mathcal{B}$, which is also a lamination, is almost surely maximal for the inclusion relation among the set of all laminations of $\overline{\mathbb{D}}$, see [77]. It follows that $\mathcal{D}_\infty = \mathcal{B}$. This completes the proof of the theorem. □

**Application to the study of the number of intersections with a given chord**

We now explain how the ingredients of the previous proof can be used to study the number of intersections of a large dissection with a given chord. For $\alpha, \beta \in [0,1]$, we denote by $I_n^{\alpha,\beta}$ the number of intersections of $\mathcal{D}_n$ with the chord $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}]$, with the convention $I_n^{\alpha,\beta} = 0$ if $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}] \subset \mathcal{D}_n$.

**Proposition 7.2.3.** *For* $0 < \alpha < \beta < 1$ *we have*

$$\frac{I_n^{\alpha,\beta}}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{\mathbb{e}(\beta-\alpha)}{\sqrt{3\sqrt{2}-4}},$$

*where* $\mathbb{e}$ *is the normalized excursion of linear Brownian motion.*

*Proof.* For $1 \leqslant i \leqslant n$, denote by $l^n(i)$ the $i$-th leaf of $t_n$ in the lexicographical order. Then, for $1 \leqslant i < j \leqslant n$, the construction of the dual tree shows that for every $s \in \left(\frac{i-1}{n+1}, \frac{i}{n+1}\right)$ and $t \in \left(\frac{j-1}{n+1}, \frac{j}{n+1}\right)$, $I_n^{s,t}$ is equal to the graph distance in the tree $t_n$ between the leaves $l^n(i)$ and $l^n(j)$. Indeed, the edges of $\mathcal{D}_n$ that intersect the chord $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}]$ correspond exactly to the edges composing the shortest path between $l^n(i)$ and $l^n(j)$ in $\mathcal{T}_n$. However, the situation is more complicated when $e^{-2i\pi s}$ or $e^{-2i\pi t}$ coincides with a vertex of $P_{n+1}$. To avoid these particular cases, we note that for $\varepsilon, \varepsilon' \in (-\frac{1}{n+1}, \frac{1}{n+1})$ we have

$$\left| I_n^{s+\varepsilon, t+\varepsilon'} - I_n^{s,t} \right| \leqslant 2\Delta^{(n)},$$

where $\Delta^{(n)}$ is the maximal number of diagonals of $\mathcal{D}_n$ adjacent to a vertex of $P_{n+1}$. We claim that

$$\frac{\Delta^{(n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{(\mathbb{P})} 0. \tag{7.7}$$

We leave the proof of the claim to the reader since a (much) stronger result will be given in Theorem 7.3.7. Let $0 < \alpha < \beta < 1$. Set $i_n = \lfloor (n+1)\alpha \rfloor + 1$ and $j_n = \lfloor (n+1)\beta \rfloor + 1$. Choose $n$ sufficiently large so that $j_n < n$. The preceding discussion shows that

$$\left| I_n^{\alpha,\beta} - d_{gr}(l^n(i_n), l^n(j_n)) \right| \leqslant 2\Delta^{(n)}, \tag{7.8}$$

where $d_{gr}$ stands for the graph distance between two vertices in $t_n$. Now note that the graph metric of the tree $t_n$ can be recovered from the contour function of $t_n$, see [37]: If $u_n, v_n$ are two vertices of $t_n$ such that the contour process reaches $u_n$ (resp. $v_n$) at the instant $s_n$ (resp. $t_n$), then

$$d_{gr}(u_n, v_n) \quad = \quad C_{t_n}(s_n) + C_{t_n}(t_n) - 2 \inf_{u \in [s_n \wedge t_n, s_n \vee t_n]} C_{t_n}(u). \tag{7.9}$$

If we choose $u_n = l^n(i_n)$ and $v_n = l^n(j_n)$ with respective first visit times $s_n$ and $t_n$, Lemma 7.2.2 shows that $s_n/2\zeta(t_n) \to \alpha$ and $t_n/2\zeta(t_n) \to \beta$ in probability as $n \to \infty$. Consequently, using (7.7),(7.8), together with (7.4), and (7.9) we finally obtain

$$\frac{I_n^{\alpha,\beta}}{\sqrt{n}} \quad \xrightarrow[n\to\infty]{(d)} \quad (3\sqrt{2}-4)^{-1/2} \left( \mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2 \inf_{[\alpha\wedge\beta, \alpha\vee\beta]} \mathbb{e} \right).$$

To conclude, observe that by the re-rooting property of the Brownian excursion (see [80, Proposition 4.9]), the variable $\mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2\inf_{[\alpha\wedge\beta,\alpha\vee\beta]} \mathbb{e}$ has the same distribution as $\mathbb{e}(\beta - \alpha)$. □

**Remark 7.2.4.** The preceding proof can be adapted easily to show the following functional convergence in distribution

$$\left( \frac{I_n^{\alpha,\beta}}{\sqrt{n}} \right)_{\substack{0\leqslant\alpha\leqslant 1\\ 0\leqslant\beta\leqslant 1}} \quad \xrightarrow[n\to\infty]{(d)} \quad (3\sqrt{2}-4)^{-1/2} \cdot \left( \mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2 \inf_{[\alpha\wedge\beta, \alpha\vee\beta]} \mathbb{e} \right)_{\substack{0\leqslant\alpha\leqslant 1\\ 0\leqslant\beta\leqslant 1}}.$$

### 7.2.2 Large uniform non-crossing trees

In order to study large uniform non-crossing trees, the following lemma will be useful. It roughly implies that the location of a leaf in a non-crossing tree $\mathscr{C}$ can be deduced from its location in the shape of $\mathscr{C}$ up to an error that is bounded by its height. Recall that if $\tau$ is a tree and $u \in \tau$, $k_u$ denotes the number of children of $u$.

**Lemma 7.2.5.** *Let $\mathscr{C}$ be a non-crossing tree with $n$ vertices and shape $S(\mathscr{C}) = \tau$. Fix a vertex $u \in \tau$ and let $a \in \{0, 1, \dots, n-1\}$ be such that the vertex in $\mathscr{C}$ corresponding to $u$ is $\exp(-2i\pi a/n)$. Then there exists $i_0 \in \{1, ..., k_u + 1\}$ such that*

$$|a - \#\{v \in \tau : v \prec ui_0\}| \quad \leqslant \quad |u|,$$

*where $\prec$ stands for the strict lexicographical order on $\mathcal{U}$.*

*Proof.* Let $u \in \tau \backslash \{\varnothing\}$. Consider the discrete geodesic path from $\varnothing$ to $u$ in $\tau$ and its image $\mathcal{L}$ in $\mathscr{C}$. There exists $1 \leqslant i_0 \leqslant k_u + 1$ such that, in $\mathscr{C}$, the first $i_0 - 1$ chidren of $u$ as well as their descendants are folded on the left of $\mathcal{L}$ (oriented from the root) and the rest of the descendants of $u$ are folded on the right of $\mathcal{L}$, see Fig. 7.6. Now, consider the set $E = \{1, \exp(-2i\pi/n), ..., \exp(-2i\pi a/n)\}$ of all the vertices of $P_n$ that are between $1$ and $\exp(-2i\pi a/n)$ in clockwise order. A geometric argument (see Fig. 7.6) shows that if a vertex $x$ of $\mathscr{C}$ belongs to $E$, then its corresponding vertex in the tree $\tau$ must belong to the set $\{v \in \tau : v \prec ui_0\}$. On the other hand, if $w \in \{v \in \tau : v \prec ui_0\}$ and if, moreover, $w$ is not a strict ancestor of $u$ in $\tau$ then its corresponding vertex in $\mathscr{C}$ belongs to $E$. Consequently, we have

$$\#\{v \in \tau : v \prec ui_0\} - |u| + 1 \quad \leqslant \quad \#E = a + 1 \quad \leqslant \quad \#\{v \in \tau : v \prec ui_0\}.$$

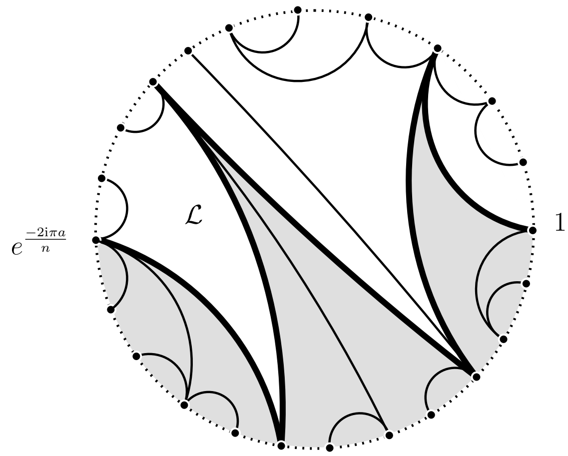


**Figure 7.6:** Illustration of the proof of Lemma 7.2.5. We represent the non-crossing tree $\mathscr{C}$ with curved chords for better visibility. The left-hand side of $\mathcal{L}$ is in gray whereas its right-hand side is in white.

The lemma follows (the case $u = \varnothing$ being trivial).

□

For our purpose, it will be convenient to reinterpret this lemma using the contour function. Fix a tree $\tau$, and define $\mathcal{Z}_\tau(j)$ as the number of distinct vertices of $\tau$ visited by the contour process of $\tau$ up to time $j$, for $0 \leqslant j \leqslant 2\zeta(\tau)-2$. For technical reasons, we set $\mathcal{Z}_\tau(j) = \zeta(\tau)$ for $j = 2\zeta(\tau)-1$ and $j = 2\zeta(\tau)$, and then extend $\mathcal{Z}_\tau(\cdot)$ to the whole segment $[0, 2\zeta(\tau)]$ by linear interpolation. Note that a vertex $u \in \tau$ with $k_u$ children is visited exactly $k_u+1$ times by the contour function of $\tau$, and that if $t^{(1)}, \ldots, t^{(k_u+1)}$ are these times, then for every $i_0 \in \{1, \ldots, k_u+1\}$ we have

$$\#\{v \in \tau : v \prec ui_0\} = \mathcal{Z}_\tau\left(t^{(i_0)}\right). \tag{7.10}$$

The idea of the proof of Theorem 7.2.1 part (ii) is the following. Let $\mathscr{C}$ be a large non-crossing tree with shape $S(\mathscr{C})$. Pick a vertex $u \in S(\mathscr{C})$ corresponding to the point $\exp(-2ia/n)$ in $\mathscr{C}$. The goal is to recover $a$ (with error at most $o(n)$) from the knowledge of $S(\mathscr{C})$ and $u$. Assume that $u$ is a leaf of $S(\mathscr{C})$. Then $u$ is visited only once by the contour process, say at time $t_u$. By Lemma 7.2.5 the quantity $|a - \mathcal{Z}_{S(\mathscr{C})}(t_u)|$ is less than the height of the tree $S(\mathscr{C})$ which is small in comparison with $n$ by (7.5). Hence $a$ in is known up to an error $o(n)$. We will see that the control by the leaves of $S(\mathscr{C})$ is sufficient for proving the convergence towards the Brownian triangulation.

The next lemma is an analogous to Lemma 7.2.2.

**Lemma 7.2.6.** *We have*

$$\sup_{0\leqslant t\leqslant 1}\left|\frac{\mathcal{Z}_{\mathscr{T}_n}(2nt)}{n} - t\right| \xrightarrow[n\to\infty]{(\mathbb{P})} 0. \tag{7.11}$$

*Proof.* This is a consequence of (7.5). Indeed, by [73, Eq. (13) in Section 1.6]:

$$\sup_{0\leqslant t\leqslant 1}\left|\frac{\mathcal{Z}_{\mathscr{T}_n}(2nt)}{n} - t\right| \leqslant \frac{1}{2n}\sup_{0\leqslant t\leqslant 1} C_{\mathscr{T}_n}(2nt) + \frac{1}{n} = \frac{1}{2\sqrt{n}}\sup_{0\leqslant t\leqslant 1}\frac{C_{\mathscr{T}_n}(2nt)}{\sqrt{n}} + \frac{1}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} 0.$$

□

*Proof of Theorem 7.2.1 part (ii).* Similarly to the proof of part $(i)$ of the theorem, we can apply Skorokhod's theorem and assume that the convergences (7.5) and (7.11) hold almost surely. We fix $\omega$ such that both convergences (7.5) and (7.11) hold for this value of $\omega$. Up to extraction, we thus suppose that $(\mathscr{C}_n)_{n\geqslant 1}$ converges towards a compact subset $\mathscr{C}_\infty$ of $\overline{\mathbb{D}}$ and we aim at showing that $\mathscr{C}_\infty = \mathcal{B}$.

We first show that $\mathcal{B} \subset \mathscr{C}_\infty$. Fix $0 < s < t < 1$ such that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$ and assume furthermore that $\mathbb{e}(r) > \mathbb{e}(s)$ for $r \in (s,t)$. Let us show that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathscr{C}_\infty$. To this end, we fix $\varepsilon > 0$ and show that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathscr{C}_n^{(6\varepsilon)}$ for $n$ sufficiently large (recall that $X^{(\varepsilon)}$ is the $\varepsilon$-enlargement of a closed subset $X \subset \overline{\mathbb{D}}$). Using the convergence (7.5), for every $n$ large enough, one can find integers $0 \leqslant s_n < t_n \leqslant 2n-2$ such that if $u_n$ (resp. $v_n$) denotes the vertex of $\mathscr{T}_n$ visited at time $s_n$ (resp. $t_n$) by the contour process, the following three properties are satisfied:

- $s_n/2n \to s$, $t_n/2n \to t$,
- $u_n$ and $v_n$ are leaves in $\mathscr{T}_n$,
- for every vertex $w_n$ in $[\![u_n, v_n]\!]$ (the discrete geodesic path between $u_n$ and $v_n$ in $\mathscr{T}_n$) and every visit time $r_n$ of $w_n$ by the contour process, we have

$$\min\left(\left|\frac{r_n}{2n} - s\right|, \left|\frac{r_n}{2n} - t\right|\right) \leqslant \varepsilon. \tag{7.12}$$

For the second property, we can for instance use the fact that local maxima of $\mathbb{e}$ are dense in $[0,1]$. We now claim that under these assumptions, for $n$ large enough, the image $\mathcal{L}_n$ in $\mathscr{C}_n$ of the discrete geodesic path $[\![u_n, v_n]\!]$ in $\mathscr{T}_n$ lies within Hausdorff distance $6\varepsilon$ from the line segment $[e^{-2i\pi s}, e^{-2i\pi t}]$.

Indeed, let $w_n \in [\![u_n, v_n]\!]$ and let $a_n \in \{0,1,\ldots,n-1\}$ such that the vertex of $\mathscr{C}_n$ corresponding to $w_n$ is $z_n = \exp(-2i\pi a_n/n)$. Applying Lemma 7.2.5 to $u = w_n$ and using (7.10) we can find a time $r_n$ at which the contour process is at $w_n$ and such that

$$|a_n - \mathcal{Z}_{\mathscr{T}_n}(r_n)| \leqslant |w_n|.$$

By the convergence (7.11) and the bound (7.12), there exists an integer $N \geqslant 1$, independent of the choice of $w_n$, such that for $n \geqslant N$

$$\min\left(\left|\frac{\mathcal{Z}_{\mathscr{T}_n}(r_n)}{n} - s\right|, \left|\frac{\mathcal{Z}_{\mathscr{T}_n}(r_n)}{n} - t\right|\right) \leqslant 2\varepsilon.$$

On the other hand, thanks to the convergence (7.5) we have

$$\frac{1}{n}\sup_{u\in\mathscr{T}_n}|u| = \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sup_{0\leqslant t\leqslant 1} C_{\mathscr{T}_n}(2nt) \quad \xrightarrow[n\to\infty]{} \quad 0.$$

We conclude that there exists an integer $N' \geqslant 1$, independent of the choice of $w_n$, such that for $n \geqslant N'$, we have $\min(|\frac{a_n}{n} - s|, |\frac{a_n}{n} - t|) \leqslant 3\varepsilon$ and thus that $\min\left(|z_n - e^{-2i\pi s}|, |z_n - e^{-2i\pi t}|\right) \leqslant 6\varepsilon$. It follows that for large $n$, we have

$$\mathcal{L}_n \subset [e^{-2i\pi s}, e^{-2i\pi t}]^{(6\varepsilon)}. \tag{7.13}$$

On the other hand, $u_n$ and $v_n$ are leaves, so they are visited at a unique time by the contour process. By the same arguments, for every $n$ sufficiently large, we deduce that their images $\alpha_n$ and $\beta_n$ in $\mathscr{C}_n$ satisfy $|\alpha_n - e^{-2i\pi s}| \leqslant 6\varepsilon$ and $|\beta_n - e^{-2i\pi t}| \leqslant 6\varepsilon$. Consequently, since $\mathcal{L}_n$ is a finite union of line segments connecting $\alpha_n$ to $\beta_n$, we deduce from (7.13) that for every $n$ sufficiently large enough

$$[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{L}_n^{(6\varepsilon)} \subset \mathscr{C}_n^{(6\varepsilon)}. \tag{7.14}$$

The case when there exists $r \in (s,t)$ such that $\mathbb{e}(s) = \mathbb{e}(r) = \mathbb{e}(t)$ (this $r$ is then a.s. unique by standard properties of the Brownian excursion) is treated exactly as in the proof of the first assertion of this theorem. We conclude that $\mathcal{B} \subset \mathscr{C}_\infty$. The reverse inclusion is obtained by making use of a maximality argument, see part $(i)$. □

### 7.2.3 Universality of the Brownian triangulation and applications

The convergence in distribution of random compact subsets towards the Brownian triangulation yields information on their asymptotic geometrical properties that are preserved under the Hausdorff convergence. Let us give an example of application of this fact.

Let $\chi_n$ be a random configuration on the vertices of $P_n$, that is a random closed subset made of line segments connecting some of the vertices of the polygon. Assume that $\chi_n$ converges in distribution towards $\mathcal{B}$ in the sense of the Hausdorff metric. Let $\mathrm{diag}(\chi_n)$ be the angle of the longest diagonal of $\chi_n$ (by definition, the angle of $[e^{-2i\pi s}, e^{-2i\pi t}]$ with $0 \leqslant s \leqslant t \leqslant 1$ is $\min(t-s, 1-t+s)$). Then, as $n \to \infty$, the law of $\mathrm{diag}(\chi_n)$ converges in distribution towards the angle of the longest diagonal of $\chi_n$, given by:

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\leqslant x\leqslant\frac{1}{2}}dx.$$

This distribution has been computed in [6] (see also [34]). The preceding convergence follows from the fact that the length of the longest chord is a continuous function of configurations for the Hausdorff metric. Similar limit theorems hold for a large variety of other functionals, such as the area of the face with largest area, etc.

It is plausible that many other uniformly distributed non-crossing configurations (see [45]) converge towards the Brownian triangulation in the Hausdorff sense. We give here a few instances of this phenomenon.

**Dissections with constrained face degrees.**

**Theorem 7.2.7.** *Let $\mathcal{A}$ be a non-empty subset of $\{3,4,5,\ldots\}$. Let $\mathbf{D}_n^{(\mathcal{A})}$ be the set of all dissections of $P_{n+1}$ whose face degrees all belong to the set $\mathcal{A}$. We restrict our attention to the values of $n$ for which $\mathbf{D}_n^{(\mathcal{A})} \neq \varnothing$.*

*(i) There exists a probability distribution $\nu_{\mathcal{A}}$ on $\mathbb{N}$ such that if $\sigma_{\mathcal{A}}^2$ denotes the variance of $\nu_{\mathcal{A}}$, we have*

$$\#\mathbf{D}_{n-1}^{(\mathcal{A})} \quad \underset{n\to\infty}{\sim} \quad \sqrt{\frac{\nu_{\mathcal{A}}(2)^4\nu_{\mathcal{A}}(0)^3}{2\pi\sigma_{\mathcal{A}}^2}} \cdot \frac{n^{-3/2}}{(\nu_{\mathcal{A}}(2)\nu_{\mathcal{A}}(0))^n}.$$

*(ii) Let $\mathcal{D}_n^{(\mathcal{A})}$ be uniformly distributed over $\mathbf{D}_n^{(\mathcal{A})}$. Then $\mathcal{D}_n^{(\mathcal{A})}$ converges towards the Brownian triangulation.*

Note that the case $\mathcal{A} = \{3\}$ corresponds to uniform triangulations and the case $\mathcal{A} = \{3, 4, 5, \ldots\}$ corresponds to uniform dissections.

*Proof.* The proof of this statement goes along the very same lines as the proofs of Corollary 7.1.5 and Theorem 7.2.1 (i) by noticing that the dual tree $\phi(\mathcal{D}_n^{(\mathcal{A})})$ is a Galton-Watson tree conditioned on having $n$ leaves for a certain finite variance offspring distribution $\nu_{\mathcal{A}}$. More precisely, if we denote the set $\{a-1 : a \in \mathcal{A}\}$ by $\mathcal{A}-1$, let $c_{\mathcal{A}} \in (0,1)$ be the unique real number in $(0,1)$ such that

$$\sum_{i \in \mathcal{A}-1} i c_{\mathcal{A}}^{i-1} = 1.$$

Then $\nu_{\mathcal{A}}$ is defined by

$$\nu_{\mathcal{A}}(0) = 1 - \sum_{i \in \mathcal{A}-1} c_{\mathcal{A}}^{i-1}, \qquad \nu_{\mathcal{A}}(i) = c_{\mathcal{A}}^{i-1} \text{ for } i \in \mathcal{A}-1.$$

Note that $\nu_{\mathcal{A}}(2) = c_{\mathcal{A}}$ and that $\nu_{\mathcal{A}}$ automatically has a finite variance $\sigma_{\mathcal{A}}^2 > 0$. □

**Non-crossing graphs.** A *non-crossing graph* of $P_n$ is a graph drawn on the plane, whose vertices are the vertices of $P_n$ and whose edges are non-crossing line segments. Let $\mathcal{G}_n$ be uniformly distributed over the set of all non-crossing graphs of $P_n$. Note that $\mathcal{G}_n$ can be seen as a compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{G}_n$ converges in distribution towards the Brownian triangulation.

This fact easily follows from the convergence of uniform dissections towards the Brownian triangulation. Indeed, if $\mathcal{G}$ is a non-crossing graph of $P_n$, let $\psi(\mathcal{G})$ be the compact subset of $\overline{\mathbb{D}}$ obtained from $\mathcal{G}$ by adding the sides of $P_n$. As noticed at the end of Section 3.1 in [45], $\psi(\mathcal{G})$ is a dissection, and every dissection has $2^n$ pre-images by $\psi$. It follows that the random dissection $\psi(\mathcal{G}_n)$ is a uniform dissection of $P_n$. The conclusion follows, since the Hausdorff distance between $\psi(\mathcal{G}_n)$ and $\mathcal{G}_n$ tends to $0$.

**Non-crossing partitions and non-crossing pair partitions.** A non-crossing partition of $P_n$ is a partition of the vertices of $P_n$ (labeled by the set $\{1, 2, \ldots, n\}$) such that the convex hulls of its blocks are pairwise disjoint (see Fig. 7.7 where the partition $\{\{1,2,4,8\},\{3\},\{5\},\{6,7\}\}$ is represented). A non-crossing pair-partition of $P_n$ is a non-crossing partition of $P_n$ whose blocks are all of size $2$ (see Fig. 7.7 where the pair-partition $\{\{1,16\},\{2,3\},\{4,7\},\{5,6\},\{8,15\},\{9,10\},\{11,14\},\{12,13\}\}$ is represented).

Let $\mathcal{P}_n^{(2)}$ be a uniformly distributed random variable on the set of all non-crossing pair-partitions of $P_{2n}$, seen as a compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{P}_n^{(2)}$ converges towards the Brownian triangulation.

To establish this fact, we rely once again on a coding of $\mathcal{P}_n^{(2)}$ by a critical Galton-Watson tree. One easily sees that the dual tree of $\mathcal{P}_n^{(2)}$ (see Fig. 7.7) is a uniform tree with $n$ edges, which is also well-known to be a Galton-Watson tree with geometric offspring distribution, conditioned on having $n$ edges. One can then show the convergence of $\mathcal{P}_n$ towards the Brownian triangulation using the same methods as in the case of uniform dissections. Details are left to the reader.

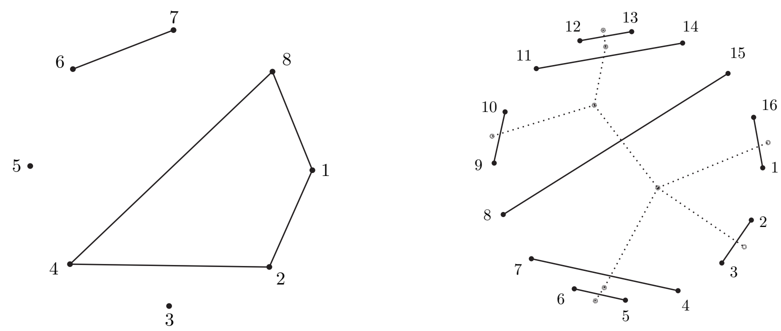


**Figure 7.7:** A non-crossing partition of $P_8$ and a non-crossing pair-partition of $P_{16}$ together with its dual tree.

Let us now discuss non-crossing partitions. Let $\mathcal{P}_n$ be a uniformly distributed random variable on the set of all non-crossing partitions of $P_n$, and view $\mathcal{P}_n$ as a random compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{P}_n$ converges towards the Brownian triangulation.

This follows from the convergence of non-crossing pair-partitions. Indeed, given a non-crossing pair-partition of $P_{2n}$, we get a non-crossing partition of $P_n$ by identifying the $n$ pairs of vertices of the form $(2i-1, 2i)$ for $1 \leqslant i \leqslant n$ (see Fig. 7.7 where the non-crossing partition of $P_8$ is obtained by contraction of vertices from the non-crossing pair-partition of $P_{16}$). This identification gives a bijection between non-crossing partitions of $P_n$ and non-crossing pair partitions of $P_{2n}$ and the desired result easily follows.

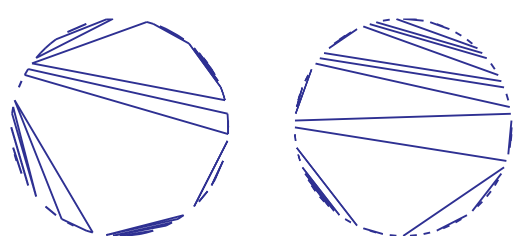

**Figure 7.8:** Uniform non-crossing partition and pair-partition of $P_{100}$.

Let us state the previously discussed results in a global statement:

**Theorem 7.2.8.** *The following convergences hold in distribution for the Hausdorff metric on closed subsets of $\overline{\mathbb{D}}$:*

$$\mathcal{G}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad \mathcal{P}_n^{(2)} \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad \mathcal{P}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}.$$

At first sight, it may seem mysterious that Galton-Watson trees appear behind many different models of uniform non-crossing configurations. In [45], using suitable parameterizations, Flajolet and Noy manage to find a Lagrange inversion-type implicit equation for the generating functions of these configurations. Generating functions verifying a Lagrange inversion-type implicit equation are those of simply-generated trees, which are very closely related to Galton-Watson trees (see [4, Section 2.1]). This explains why Galton-Watson trees are hidden behind various models of uniformly distributed non-crossing configurations.

## 7.3 Graph-theoretical properties of large uniform dissections

In this section, we study graph-theoretical properties of large dissections using the Galton-Watson tree structure identified in Proposition 7.1.3. Let us stress that as in Proposition 7.2.7, all the results contained in this section can be adapted easily to uniform dissections constrained on having all face degrees in a fixed non-empty subset of $\{3, 4, \ldots\}$ (and in particular to uniform triangulations).

As previously, $\mathcal{D}_n$ is a uniformly distributed dissection of $P_{n+1}$ and $\mathcal{T}_n$ denotes its dual tree with $n$ leaves. We start by recalling the definition and the construction of the so-called critical Galton-Watson tree conditioned to survive.

### 7.3.1 The critical Galton-Watson tree conditioned to survive

If $\tau$ is a tree and $k$ is a nonnegative integer, we let $[\tau]_k = \{u \in \tau : |u| \leqslant k\}$ denote the tree obtained from $\tau$ by keeping the vertices in the first $k$ generations. Let $= (\xi_i)_{i \geqslant 0}$ be an offspring distribution with $\xi_1 \neq 1$ and $\sum i\xi_i = 1$. We denote by $T_n$ a Galton-Watson tree with offspring distribution $\xi$ conditioned on having height at least $n \geqslant 0$. Kesten [65, Lemma 1.14] showed that for every $k \geqslant 0$, we have the following convergence in distribution

$$[T_n]_k \quad \xrightarrow[n\to\infty]{(d)} \quad [T_\infty]_k ,$$

where $T_\infty$ is a random infinite plane tree called the critical $\xi$-Galton-Watson tree conditioned to survive.

We denote the law of the $\xi$-Galton-Watson tree conditioned to survive by $\widehat{\mathbb{P}}_\xi$ and by $T_\infty$ a random tree distributed according to $\widehat{\mathbb{P}}_\xi$. Let us describe this tree (see [65, 78]). We let $\overline{\xi}$ be the size-biased distribution of $\xi$ defined by $\overline{\xi}_k = k\xi_k$ for $k \geqslant 0$. Let $(D_i)_{i\geqslant 0}$ be a sequence of i.i.d. random variables distributed according to $\overline{\xi}$. Let also $(U_i)_{i\geqslant 1}$ be a sequence of random variables such that, conditionally on $(D_i)_{i\geqslant 0}$, $(U_i)_{i\geqslant 1}$ are independent and $U_{k+1}$ is uniformly distributed over $\{1, 2, ..., D_k\}$ for every $k \geqslant 0$. The tree $T_\infty$ has a unique spine, that is a unique infinite path $(\varnothing, U_1, U_1U_2, U_1U_2U_3, ...) \in \mathbb{N}^{*\mathbb{N}^*}$ and the degree of $U_1U_2...U_k$ is $D_k$. Finally, conditionally on $(U_i)_{i\geqslant 1}$ and $(D_i)_{i\geqslant 0}$ all the remaining subtrees are independent $\xi$-Galton-Watson trees, see Fig. 7.9.

The critical Galton-Watson tree conditioned to survive also arises in other conditionings of Galton-Watson trees. Recall that $t_n$ is a $\mu$-Galton-Watson tree conditioned on having $n$ leaves.

**Theorem 7.3.1.** *For every $k \geqslant 0$, we have the convergence in distribution*

$$[t_n]_k \quad \xrightarrow[n\to\infty]{(d)} \quad [\mathcal{T}_\infty]_k ,$$

*where $\mathcal{T}_\infty$ is the critical Galton-Watson tree with offspring distribution $\mu$ conditioned to survive.*

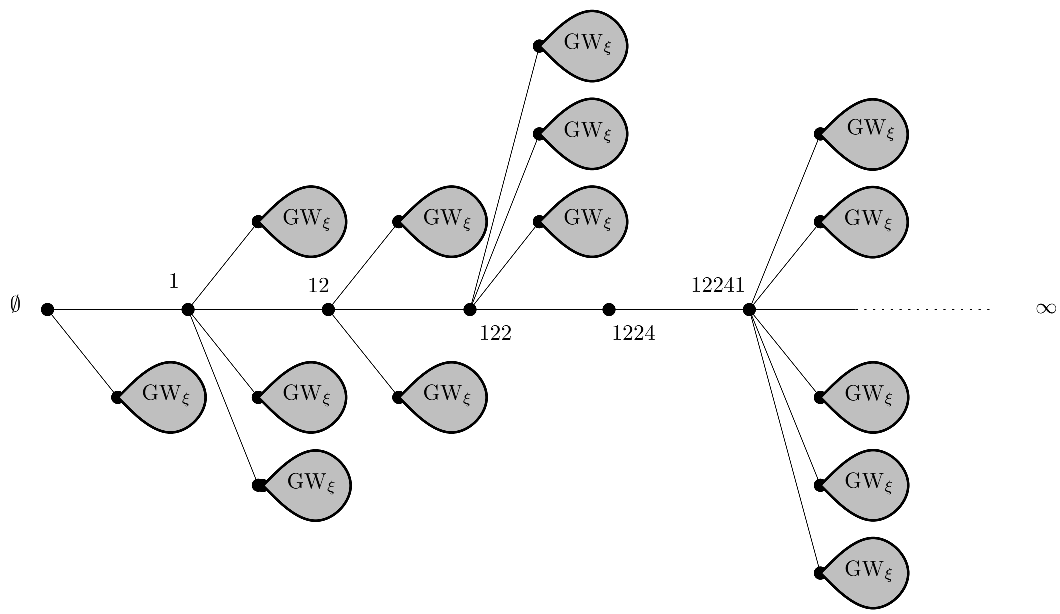


**Figure 7.9:** An illustration of $T_\infty$ under $\widehat{\mathbb{P}}_\xi$.

**Remark 7.3.2.** Theorem 7.3.1 is true when $\mu$ is replaced by any finite variance offspring distribution $\nu$ such that $\mathbb{P}_\nu\left(\lambda(\tau)=n\right)>0$ for every $n$ large enough.

*Proof.* This follows from another description of the law $\widehat{\mathbb{P}}_\mu$: If $\tau_0$ is a plane tree and $k \geqslant 0$ is an integer, we denote by $L_k(\tau_0)$ the number of individuals of $\tau_0$ at height exactly $k$. Then, if $\tau_0$ has height $k$, we have from [65, Lemma 1.14]:

$$\widehat{\mathbb{P}}_\mu([\tau]_k=\tau_0) \quad = \quad L_k(\tau_0)\mathbb{P}_\mu([\tau]_k=\tau_0).$$

Fix an integer $k \geqslant 1$ as well as a tree $\tau_0 \in \mathbb{T}$ of height $k$. In order to prove the theorem, it is thus sufficient to show that

$$\mathbb{P}_\mu([\tau]_k=\tau_0 \,|\, \lambda(\tau)=n) \quad \underset{n\to\infty}{\longrightarrow} \quad L_k(\tau_0)\mathbb{P}_\mu([\tau]_k=\tau_0).$$

Denote by $q$ the number of leaves of $\tau_0$ that have a height strictly smaller than $k$. By the branching property of Galton-Watson trees, we have

$$\mathbb{P}_\mu([\tau]_k=\tau_0 \,|\, \lambda(\tau)=n)=\mathbb{P}_\mu([\tau]_k=\tau_0)\frac{\mathbb{P}_{\mu,L_k(\tau_0)}(\lambda(\mathbf{f})=n-q)}{\mathbb{P}_\mu(\lambda(\tau)=n)},$$

where we recall that $\mathbb{P}_{\mu,i}$ denotes the probability distribution of a forest of $i$ independent Galton-Watson trees with law $\mathbb{P}_\mu$. Since $q$ and $L_k(\tau_0)$ are fixed, it is thus sufficient to show that for a fixed integer $i \geqslant 1$, as $n \to \infty$

$$\mathbb{P}_{\mu,i}(\lambda(\mathbf{f})=n) \quad \underset{n\to\infty}{\sim} \quad i \times \mathbb{P}_\mu(\lambda(\tau)=n).$$

This follows from the next lemma, which we state in a more general form than needed here in view of further applications. □

**Lemma 7.3.3.** *There exists $\varepsilon > 0$ such that if $(i_n)_{n\geqslant 1}$ is a sequence of positive integers such that $i_n \leqslant n^\varepsilon$ for every $n \geqslant 1$, we have*

$$\mathbb{P}_{\mu,i_n}(\lambda(\mathfrak{f}) = n) \underset{n\to\infty}{\sim} i_n \cdot \mathbb{P}_\mu(\lambda(\tau) = n).$$

*Proof.* We show that the conclusion of the lemma holds for any $\varepsilon \in (0, 1/9)$. To simplify notation, we set $p_k := \mathbb{P}_\mu(\lambda(\tau) = k)$ for every integer $k \geqslant 1$ and write $i = i_n$ in the proof. By the definition of $\mathbb{P}_{\mu,i}$, we have

$$\mathbb{P}_{\mu,i}(\lambda(\mathfrak{f}) = n) = \sum_{k_1+\ldots+k_i=n} \prod_{j=1}^{i} p_{k_j}.$$

We will show that when $n$ is large, the main contribution in the previous sum is obtained when the indices $k_1, \ldots, k_i$ are such that only one of them is of order $n$ and the others are small in comparison. Let $A \geqslant 1$. Firstly, notice that at least one of the indices $k_1, \ldots, k_i$ is larger than $n/i$. Secondly, let us evaluate the contribution of the sum when $k_1 \geqslant n/i$ and $k_2$ is larger than $A$. By Lemma 7.1.4,

$$\sum_{\substack{k_1+\cdots+k_i=n\\ k_1\geqslant n/i\\ k_2\geqslant A}} \prod_{j=1}^{i} p_{k_j} \leqslant \sup_{k_1\geqslant n/i} p_{k_1} \cdot \left(\sum_{k_2\geqslant A} p_{k_2}\right) \cdot \prod_{j=3}^{i} \left(\sum_{k_j=1}^{\infty} p_{k_j}\right) \leqslant C(n/i)^{-3/2} A^{-1/2},$$

for some constant $C > 0$ which is independent of $n$ and $i$. Hence, provided that $A < n^{1-\varepsilon}$,

$$\left| \left(\sum_{k_1+\cdots+k_i=n} \prod_{j=1}^{i} p_{k_j}\right) - i \left(\sum_{1\leqslant k_1,\ldots,k_{i-1}\leqslant A} p_{n-\sum_j k_j} \prod_{j=1}^{i-1} p_{k_j}\right) \right| \leqslant C i^{7/2} n^{-3/2} A^{-1/2}. \quad (7.15)$$

We apply this to $A = A(n) = n^{8\varepsilon}$ in such a way that the right-hand side of the above inequality is negligible in comparison with $ip_n$. Then note that for $1 \leqslant k_1, \ldots, k_{i-1} \leqslant A$, we have

$$n - n^{9\varepsilon} \leqslant n - \sum_{j=1}^{i-1} k_j \leqslant n.$$

Moreover, since $9\varepsilon < 1$, Lemma 7.1.4 gives $p_j/p_n \to 1$ uniformly in $n - n^{9\varepsilon} \leqslant j \leqslant n$ as $n \to \infty$. Thus, using Lemma 7.1.4 again

$$i \left(\sum_{1\leqslant k_1,\ldots,k_{i-1}\leqslant A} p_{n-\sum_j k_j} \prod_{j=1}^{i-1} p_{k_j}\right) \sim ip_n \left(\sum_{k=1}^{n^{8\varepsilon}} p_k\right)^{i-1} \sim ip_n \left(1 - \frac{n^{-4\varepsilon}}{\sqrt{\pi\sqrt{2}}}\right)^{i} \sim ip_n,$$

which completes the proof of the lemma. □

### 7.3.2 Applications

The "local convergence" given in Theorem 7.3.1 allows us to study "local" properties of large uniform dissections by reading them directly on the critical Galton-Watson tree conditioned to survive. We will focus our attention on the following two local properties of random uniform dissections: Vertex degrees and face degrees.

Let us introduce some notation. Recall that $\mathcal{D}_n$ stands for a uniformly distributed dissection of $P_{n+1}$. Denote by $\delta^{(n)}$ the degree of the face adjacent to the side $[1, e^{2i\pi/(n+1)}]$ in the random dissection $\mathcal{D}_n$ and by $D^{(n)}$ the maximal degree of a face of $\mathcal{D}_n$. Similarly, denote by $\partial^{(n)}$ the number of diagonals adjacent to the vertex corresponding to the complex number $1$ in $\mathcal{D}_n$ and by $\Delta^{(n)}$ the maximal number of diagonals adjacent to some vertex of $P_{n+1}$. Finally, for $b > 0$, we write $\log_b(\cdot)$ for $\ln(\cdot)/\ln(b)$.

We shall establish that $\delta^{(n)}$ and $\partial^{(n)}$ converge in distribution, and read their limiting distributions on the random infinite tree $\mathcal{T}_\infty$. We also provide sharp concentration estimates on $D^{(n)}$ and $\Delta^{(n)}$, confirming in particular a conjecture of [14] concerning $\Delta^{(n)}$.

#### Face degrees

**Proposition 7.3.4.** *As $n$ goes to infinity, $\delta^{(n)}$ converges in distribution to the random variable $X$ with distribution*

$$\mathbb{P}(X = k) = (k-1)\mu_{k-1} = (k-1)\left(\frac{2-\sqrt{2}}{2}\right)^{k-2}, \qquad k \geqslant 3.$$

*Proof.* This is an immediate consequence of Theorem 7.3.1 and the construction of the critical Galton-Watson tree conditioned to survive, after taking into account the fact that $\delta^{(n)} - 1$ is the number of children of $\varnothing$ in the dual tree of $\mathcal{D}_n$. □

**Proposition 7.3.5.** *Set $\beta = 1/\mu_2 = 2 + \sqrt{2}$. For every $c > 0$, we have*

$$\mathbb{P}\left(\log_\beta(n) - c\log_\beta\log_\beta(n) \leqslant D^{(n)} \leqslant \log_\beta(n) + c\log_\beta\log_\beta(n)\right) \xrightarrow[n\to\infty]{} 0.$$

*Proof.* By construction of the dual tree $\mathcal{T}_n$ of $\mathcal{D}_n$, we have $D^{(n)} - 1 = \max_{u\in t_n} k_u$. Thus, by Proposition 7.1.3, for every measurable function $F : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[F(D^{(n)} - 1)\right] = \mathbb{E}_\mu\left[F\left(\max_{u\in\tau} k_u\right) | \lambda(\tau) = n\right].$$

The result now follows from [70, Remark 7.3]. □

#### Vertex degrees

We are now interested in another graph-theoretical property of large uniform dissections, namely vertex degrees. Since these vertex degrees are read on the dual tree in a more complicated fashion than face degrees, the arguments are slightly more involved.

Recall that $\partial^{(n)}$ stands for the number of diagonals adjacent to the vertex corresponding to the complex number $1$ in the uniform dissection $\mathcal{D}_n$.

**Proposition 7.3.6.** *As* $n$ *goes to infinity,* $\partial^{(n)}$ *converges in distribution to the sum of two independent geometric random variables of parameter* $1-\mu_0 = \sqrt{2}-1$, i.e. *for any* $k \geqslant 0$ *we have*

$$\mathbb{P}(\partial^{(n)} = k) \quad \xrightarrow[n\to\infty]{} \quad (k+1)\mu_0^2(1-\mu_0)^k.$$

See also [14] for a closely related result.

*Proof.* If $\tau$ is a plane tree, we introduce the length $\ell(\tau)$ of the left-most path in $\tau$ starting at $\varnothing$ (that is following left-most children until we reach a leaf),

$$\ell(\tau) = \max\{i \geqslant 0 : 1_i \in \tau\}, \quad \text{where } 1_i = 1...1 \quad (i \text{ times}) \text{ with } 1_0 = \varnothing.$$

Using the bijection between a dissection and its dual tree, it is easy to see that the number of diagonals adjacent to the vertex $1$ of the random dissection $\mathcal{D}_n$ is $\ell(\mathcal{T}_n)-1$. By Theorem 7.3.1, for every $k \geqslant 1$, $[t_n]_k \to [\mathcal{T}_\infty]_k$ as $n \to \infty$. It follows that $\partial^{(n)} = \ell(\mathcal{T}_n)-1$ converges in distribution towards $\ell(\mathcal{T}_\infty)-1$. Let us identify the distribution of this variable using the first description of the law $\widehat{\mathbb{P}}_\mu$: The length $\ell(\mathcal{T}_\infty)-1$ can be decomposed into

$$\ell(\mathcal{T}_\infty) - 1 = (\ell_1 - 1) + \ell_2,$$

where $\ell_1$ is the smallest integer $i \geqslant 1$ such that the element $1_i = 1...1$ ($i$ times) is not on the *spine* of $\mathcal{T}_\infty$ and $\ell_2 = \ell(\mathcal{T}_\infty) - (\ell_1 - 1)$ is the length of the left-most path in the critical $\mu$-Galton-Watson tree grafted on the left of $1_i$. By the description in Section 7.3.1, the two variables $\ell_1 - 1$ and $\ell_2$ are independent. It is straightforward that $\ell_2$ is distributed according to a geometric variable of parameter $\sqrt{2}-1$. Let us now turn to $\ell_1 - 1$. Recall the notation introduced in Section 7.3.1. For $k \geqslant 0$, we have:

$$\begin{aligned}
\mathbb{P}(\ell_1 \geqslant k+1) &= \mathbb{P}(U_1 = 1, U_2 = 1, \dots, U_k = 1) \\
&= \prod_{i=0}^{k-1} \left( \sum_{j=1}^{\infty} \mathbb{P}(D_i = j)\mathbb{P}(U_{i+1} = 1 \mid D_i = j) \right) \\
&= \left( \sum_{j=2}^{\infty} (1 - 2^{-1/2})^{j-1} \right)^k = (1-\mu_0)^k.
\end{aligned}$$

We thus see that $\ell_1 - 1$ is also geometric with parameter $\sqrt{2}-1$ and the desired result follows. $\square$

Recall that $\Delta^{(n)}$ stands for the maximal number of diagonals adjacent to a vertex of $P_{n+1}$. The following theorem proves a conjecture of [14].

**Theorem 7.3.7.** *Set* $b = 1/(1-\mu_0) = \sqrt{2}+1$. *For every* $c > 0$, *we have*

$$\mathbb{P}\left(\Delta^{(n)} \geqslant \log_b(n) + (1+c)\log_b \log_b(n)\right) \quad \xrightarrow[n\to\infty]{} \quad 0.$$

*Proof.* Let $p, n \geqslant 1$ be integers. By rotational invariance, the degrees of the vertices of $\mathcal{D}_n$ are identically distributed random variables. It follows that

$$\mathbb{P}\left[\Delta^{(n)} \geqslant p\right] \quad \leqslant \quad (n+1)\mathbb{P}\left(\partial^{(n)} \geqslant p\right).$$

We have already noticed that the number of diagonals adjacent to the vertex 1 of the random dissection $\mathcal{D}_n$ corresponds to $\ell(\mathcal{T}_n) - 1$, where $\ell(t_n)$ denotes the length of the left-most path in $\mathcal{T}_n$ starting at $\varnothing$. Thus, by Proposition 7.1.3,

$$(n+1)\mathbb{P}\left(\partial_0^{(n)} \geqslant p\right) = (n+1)\mathbb{P}_\mu\left(\ell(\tau) \geqslant p+1 \mid \lambda(\tau) = n\right)$$

We now estimate the right-hand side and show that it tends to 0 when $n \to \infty$ and $p = p_n = \log_b(n) + (1+c)\log_b\log_b(n)$ for $c > 0$. If $\ell(\tau) = p$, define $\theta(\tau) = k_\varnothing + k_1 + k_{1_2} + \ldots + k_{1_{p-1}} - p$ ($\theta(\tau)$ can be interpreted as the total number of those children of vertices in the left-most path that are not in that path). Note that under $\mathbb{P}_\mu$, $\ell(\tau)$ is distributed according to a geometric random variable of parameter $\sqrt{2}-1$. In particular, for $\alpha = |4/\log(1-\mu_0)|$,

$$n^3\mathbb{P}_\mu\left[\ell(\tau) \geqslant \alpha\log(n)\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{7.16}$$

Note also that for positive integers $j, k$ we have $\mathbb{P}_\mu\left[\theta(\tau) = j \mid \ell(\tau) = k\right] = \mathbb{P}(Y^{*k} = j)$, where $Y^{*k}$ is distributed as the sum of $n$ independent random variables distributed according to $\gamma(j) = \mu_{j+1}/\mu([1,\infty])$.

Choose $\varepsilon > 0$ such that the conclusion of Lemma 7.3.3 holds. We first claim that if $A_n = \{\theta(\tau) \leqslant n^\varepsilon\}$, then $\mathbb{P}_\mu\left(A_n\right) \geqslant 1 - n^{-3}$ for $n$ large enough. To this end, write

$$\begin{aligned} n^3\mathbb{P}_\mu\left(\theta(\tau) > n^\varepsilon\right) &\leqslant n^3\mathbb{P}_\mu\left[\ell(\tau) > \alpha\log(n)\right] + n^3 \sum_{j=1}^{\lfloor\alpha\log(n)\rfloor} \mathbb{P}_\mu\left[\theta(\tau) > n^\varepsilon \mid \ell(\tau) = j\right]\mathbb{P}_\mu\left[\ell(\tau) = j\right] \\ &\leqslant n^3\mathbb{P}_\mu\left[\ell(\tau) > \alpha\log(n)\right] + n^3\alpha\log(n)\mathbb{P}_\mu\left[\theta(\tau) > n^\varepsilon \mid \ell(\tau) = \lfloor\alpha\log(n)\rfloor\right]. \end{aligned}$$

The first term tends to zero by (7.16), and the second one as well by the previous description of the law of $\theta(\tau)$ under the conditional probability distribution $\mathbb{P}_\mu\left[\,\cdot \mid \ell(\tau) = k\right]$ and a standard large deviation inequality. Thus our claim holds and it follows that, for $n$ large enough,

$$(n+1)\mathbb{P}_\mu(A_n^c \mid \lambda(\tau) = n) \leqslant (n+1)n^{-3}/\mathbb{P}_\mu(\lambda(\tau) = n) \underset{n\to\infty}{\longrightarrow} 0 \tag{7.17}$$

by Lemma 7.1.4. We now consider the event $A_n$. We have

$$\begin{aligned} &\mathbb{P}_\mu\left[\{\ell(\tau) = p\} \cap A_n \,|\, \lambda(\tau) = n\right] \\ &\qquad \leqslant \mu_0 \sum_{\substack{r_0,\ldots,r_{p-1}\geqslant 0 \\ \sum r_j \leqslant n^\varepsilon}} \mu_{r_0+1}\cdots\mu_{r_{p-1}+1} \frac{\mathbb{P}_{\mu,r_0+r_1+\cdots+r_{p-1}}\left(\lambda(\tau) = n-1\right)}{\mathbb{P}_\mu(\lambda(\tau) = n)}. \end{aligned}$$

We can then apply Lemma 7.3.3 to get that the quotient in the last display is bounded above by some constant $C_2$ times $\theta(\tau) = r_0 + \ldots + r_p$, so that

$$\begin{aligned} \mathbb{P}_\mu\left[\ell(\tau) = p, \theta(\tau) \leqslant n^\varepsilon \,|\, \lambda(\tau) = n\right] &\leqslant C_2\mu_0 \sum_{r_0,\ldots,r_{p-1}\geqslant 0} \mu_{r_0+1}\cdots\mu_{r_{p-1}+1}(r_0 + \ldots + r_{p-1}) \\ &= C_2 p\mu_0^2(1-\mu_0)^{p-1}, \end{aligned}$$

where we have successively calculated these sums by using

$$\sum_{k=1}^{\infty} \mu_{k+1} = 1 - \mu_0, \qquad \sum_{k=1}^{\infty} k\mu_{k+1} = \mu_0.$$

Consequently, setting $p_n = \log_b(n) + (1+c)\log_b\log_b(n)$ we deduce from the above estimates that

$$(n+1)\mathbb{P}_\mu\left(\ell(\tau) \geqslant p_n, \theta(\tau) \leqslant n^\varepsilon \mid \lambda(\tau) = n\right) \xrightarrow[n\to\infty]{} 0,$$

which together with (7.17) completes the proof of the theorem. □

**On $\log\log$ concentration**

In [14], it is shown that for every $c > 0$

$$\mathbb{P}\left(\Delta^{(n)} \leqslant \log_b(n) - (2+c)\log_b\log_b(n)\right) \xrightarrow[n\to\infty]{} 0.$$

Using the connection between uniform dissections and Galton-Watson trees conditioned on their number of leaves, it is possible to refine the above lower bound and to replace $(2+c)$ by $c$. However, we believe that the optimal concentration result is given by the following conjecture:

**Conjecture 7.3.8.** For every $c > 0$ we have

$$\mathbb{P}\left(\left|\Delta^{(n)} - (\log_b(n) + \log_b\log_b(n))\right| > c\log_b\log_b(n)\right) \xrightarrow[n\to\infty]{} 0.$$

If the degrees of vertices in $\mathcal{D}_n$ were independent, this concentration result would hold. However, the difficulty comes from the fact that this independence property does not exactly hold. Let us mention that this conjecture (for a different value of $b$) has been proved in the particular case of triangulations using generating function methods [47].

It is worth pointing out that although $\delta^n$ and $\partial^{(n)}$ have a similar limiting distribution (roughly speaking a size-biased geometric distribution), the maximal degree of a face and the maximal degree of a vertex in a random uniform dissection possess a different concentration behavior: $D^{(n)}$ is strongly concentrated around $\log_\beta(n) + o(\log\log(n))$, whereas $\Delta^{(n)}$ is (conjecturally) strongly concentrated around $\log_b(n)+\log_b\log_b(n)+o(\log\log(n))$. This comes from the heuristic observation that a "typical" vertex has a limiting distribution which is given by a sized-biased geometric distribution, whereas a "typical" face of $\mathcal{D}_n$ has a limiting distribution which is a geometric distribution (which is not size-biased). Let us give some details.

We start with the vertex degree. By Proposition 7.3.6 and by rotational invariance, a "typical" vertex has a limiting distribution which is given by a sized-biased geometric distribution. This is why $\Delta^{(n)}$ should have the same concentration behavior as $n$ independent random variables distributed as size-biased geometric laws, that is, $\Delta^{(n)}$ should be concentrated around $\log_b(n) + \log_b\log_b(n) + o(\log\log(n))$.

The situation is however different is the case of face degrees. Choosing the face adjacent to $[1, e^{2i\pi/(n+1)}]$ introduces a size-biasing in the distribution of a "typical" face (indeed, the face containing a given side of $P_n$ is not a typical face but a size-biased one; a typical face would be a face chosen "uniformly" among all faces of the dissection). In other words, a typical face of $\mathcal{D}_n$ follows a geometric distribution, but $\delta^{(n)}$ is a size-biased distribution of a typical face of $\mathcal{D}_n$. This is why $D^{(n)}$ follows the same concentration behavior as $n$ independent geometric variables (which are not size-biaised), and hence explains why $D^{(n)}$ is concentrated around $\log_\beta(n) + o(\log\log n)$.

# Bibliographie

# *Bibliographie*


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