## 2) Subcritical case

Recall that if $c>0$ and $\sum_{i\geq 0} c^i \mu(i) < \infty$, $\mu^{(c)}$ defined by

$$\mu^{(c)}(i) = \frac{c^i \mu(i)}{\sum_{i\geq 0} c^i \mu(i)}$$ satisfies $\mathbb{P}_\mu(\cdot \mid |T|=n) = \mathbb{P}_{\mu^{(c)}}(\cdot \mid |T|=n)$

- If we can find $c>0$ s.t. $\mu^{(c)}$ is critical, then $T_n$ converges to $T_\infty$ for $\mu^{(c)}$.
- If not, a condensation phenomenon occurs: $T_\infty$ has now a <u>finite</u> spine.

This is typically the case when

$$\sum_{i\geq 0} i\mu(i) < \infty \quad \text{and} \quad \mu(n) \underset{n\to\infty}{\sim} \frac{c}{n^\alpha}$$

with $\alpha > 1$.

# VI Scaling limits of GW trees

Here $\mu$ is a critical, <u>aperiodic</u> offspring distribution with variance $0<\sigma^2<\infty$

$W_n = X_1 + \dots + X_n$ is a random walk with $\mathbb{P}(X_1 = i) = \mu(i+1)$, $i \geq -1$.

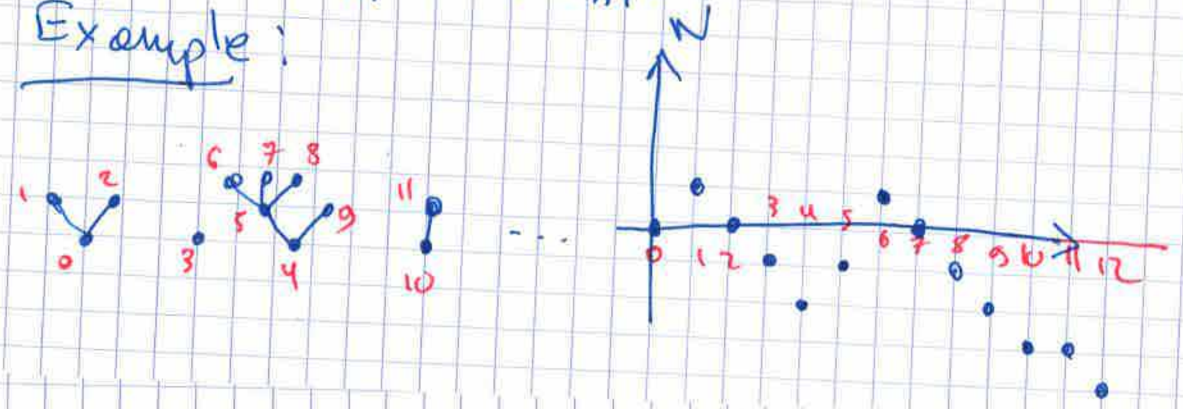
The goal is to show that a $GW_\mu(\cdot \mid |T|=n)$ "grows like $\sqrt{n}$", and rescaled converges to a random limiting tree with Hausdorff dimension 2.

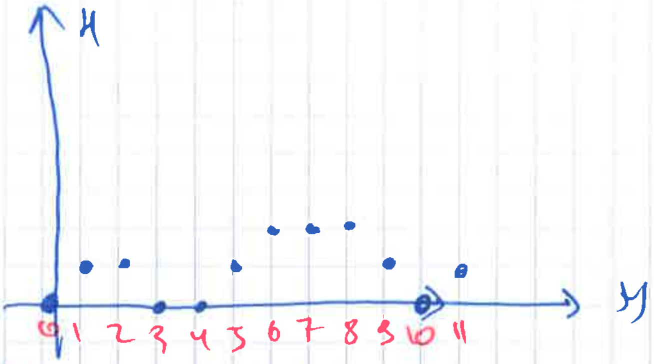
# 1) Asymptotics of the height function.

From now on, we assume that $(W_n)_{n\geq 0}$ is the Lukasiewicz path of a sequence of iid $GW_\mu$ trees. Denote by $u_0, u_1, u_2, \dots,$ its vertices ordered in lexicographical order. Let $(H_n)_{n\geq 0}$ be the height function, defined by

$$H_n = |u_n|.$$

Example:






**Key proposition:** For every $n \geq 0$,

$$H_n = \left| \left\{ 0 \leq k \leq n-1 \, ; \, W_k = \inf_{k \leq j \leq n} W_j \right\} \right|$$

Idea of proof: this comes from the fact that vertex $u_i$ is an ancestor of $u_j$ (for $i<j$) if and only if $W_i \leq \min_{i \leq k \leq j} W_k$

How to study $H_n$? Idea: use time-reversal.

Write, for $n \geq 0$, $\widehat{W}^{(n)} = (W_n - W_{n-i} \, ; \, 0 \leq i \leq n)$.

Then $\widehat{W}^{(n)} \overset{(d)}{=} (W_0, W_1, \ldots, W_n)$, and

$$\begin{aligned} H_n &= \left| \left\{ 0 \leq k \leq n-1 \, ; \, W_n - W_k = \max_{k \leq j \leq n} (W_n - W_j) \right\} \right| \\ &= \left| \left\{ 1 \leq k \leq n \, ; \, \widehat{W}^{(n)}_k = \max_{0 \leq j \leq k} \widehat{W}^{(n)}_j \right\} \right| \\ &\overset{(d)}{=} \underbrace{\left| \left\{ 1 \leq k \leq n \, ; \, W_k = \max_{0 \leq j \leq k} W_j \right\} \right|}_{=: R_n} \end{aligned}$$

$R_n$ is the number of (weak) records between times 1 and $n$.

Goal: study $R_n$.

To this end, set $T_0 = 0$, and $T_i = \inf\{ n > T_{i-1} \, ; \, W_n \geq W_{T_{i-1}} \}$ for $i \geq 1$, which are stopping times.

In particular, $\{ R_n = i \} = \{ T_i \leq n < T_{i+1} \}$.

Finally, set $\bar{\mu}(k) = \mu([k+1, +\infty))$ for $k \geq 0$, which is a probability measure since $\mu$ is critical.

**Proposition** The random variables $(W_{T_i} - W_{T_{i-1}} ; i \geq 1)$ are iid, and

$$\mathbb{P}(W_{T_1} = k) = \overline{\mu}(k) \quad \text{for } k \geq 0.$$

Proof: We take for granted that $(W_n)_{n\geq 0}$ is recurrent (we will prove it later). In particular $T_i < \infty$ a.s $\forall i \geq 1$. Also, the counting measure on $\mathbb{Z}$ is an invariant measure:

$$\forall x \in \mathbb{Z}, \quad x = \sum_{y \in \mathbb{Z}} x \cdot \mathbb{P}(\text{jump from } x \text{ to } y).$$

If $\zeta_0 = \inf\{n \geq 1 ; W_n = 0\}$, it is well known that $i \mapsto \mathbb{E}\left[\sum_{n=0}^{\zeta_0 - 1} \mathbb{1}_{\{W_n = i\}}\right]$ defines an invariant measure.

Since $(W_n)$ is recurrent, these measures are unique up to a constant.

$$\Rightarrow \exists c \geq 0 \text{ s.t } \mathbb{E}\left[\sum_{n=0}^{\zeta_0 - 1} \mathbb{1}_{\{W_n = i\}}\right] = c. \quad \forall i \in \mathbb{Z}$$

By taking $i = 0$, we get $\underline{c = 0}$.

Key observation: $T_1 \leq \zeta_0$ and $W$ is positive on $]T_1, \zeta_0[$

Hence $\mathbb{E}\left[\sum_{n=0}^{T_1 - 1} \mathbb{1}_{\{S_n = i\}}\right] = 1$ if $\underline{i \leq 0}$

In particular, since $W$ is $\leq 0$ on $[0, T_1[$, we have for every function $g : \mathbb{Z} \to \mathbb{Z}_+$,

$$\mathbb{E}\left[\sum_{n=0}^{T_1 - 1} g(W_n)\right] = \sum_{i=0}^{-\infty} g(i).$$

Now, for any function $f : \mathbb{Z} \to \mathbb{Z}_+$, write

$$\mathbb{E}[f(W_{T_1})] = \mathbb{E}\left[\sum_{k \geq 0} \mathbb{1}_{\{k < T_1\}} f(W_{k+1}) \mathbb{1}_{W_{k+1} \geq 0}\right)$$

$$\left(\mathbb{1}_{\{k < T_1, W_{k+1} \geq 0\}} = \mathbb{1}_{\{T_1 = k+1\}}\right)$$

$$= \sum_{k=0}^{\infty} \mathbb{E}\left[ \mathbb{1}_{\{k<T_1\}} \, g(W_{k+1}) \, \mathbb{1}_{\{W_{k+1} \geq 0\}} \right]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}\left[ \mathbb{1}_{\{k<T_1\}} \sum_{j=0}^{\infty} v(j) \, g(W_k + j) \, \mathbb{1}_{\{W_k + j \geq 0\}} \right]$$

(Markov prop at time $k$)

$$= \sum_{j=0}^{\infty} v(j) \, \mathbb{E}\left[ \sum_{k=0}^{T_1 - 1} g(W_k + j) \, \mathbb{1}_{\{W_k + j \geq 0\}} \right]$$

$$= \sum_{j=0}^{\infty} v(j) \sum_{i=0}^{-\infty} g(i+j) \, \mathbb{1}_{\{i+j \geq 0\}}$$

$$= \sum_{m=0}^{\infty} g(m) \, \bar{\mu}(m). \qquad \blacksquare$$