INEQUALITIES THROUGH PROBLEMS

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1. Heuristics of problem solving

Strategy or tactics in problem solving is called *heuristics*. Here is a summary taken from *Problem-Solving Through Problems* by Loren C. Larson.

1. Search for a pattern.
2. Draw a figure.
3. Formulate an equivalent problem.
4. Modify the problem.
5. Choose effective notation.
6. Exploit symmetry.
7. Divide into cases.
8. Work backwards.
9. Argue by contradiction.
10. Check for parity.
11. Consider extreme cases.
12. Generalize.

2. Classical Theorems

Theorem 1. (**Schur**) *Let x, y, z be nonnegative real numbers. For any $r > 0$, we have*

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Theorem 2. (**Muirhead**) *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that*

$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1+a_2 \geq b_1+b_2, a_1+a_2+a_3 = b_1+b_2+b_3$.

Let x, y, z be positive real numbers. Then, we have $\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} \geq \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}$.

Theorem 3. (**The Cauchy-Schwarz inequality**) *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,*

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Theorem 4. (**AM-GM inequality**) *Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 5. (**Weighted AM-GM inequality**) *Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have*

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Theorem 6. (**Hölder's inequality**) *Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have*

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Theorem 7. (**Power Mean inequality**) *Let $x_1, \cdots, x_n > 0$. The power mean of order r is defined by*

$$M_{(x_1, \cdots, x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,, \quad M_{(x_1, \cdots, x_n)}(r) = \left(\frac{x_1^r + \cdots + {x_n}^r}{n} \right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(x_1, \cdots, x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Theorem 8. (**Majorization inequality**) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a, b]$. Then, we obtain*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Theorem 9. (**Bernoulli's inequality**) *For all $r \geq 1$ and $x \geq -1$, we have*

$$(1 + x)^r \geq 1 + rx.$$

Definition 1. (**Symmetric Means**) *For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t + x_1) \cdots (t + x_n)$ is called the i-th elementary symmetric function σ_i. This means that*

$$(t + x_1) \cdots (t + x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1} t + \sigma_n.$$

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 10. *Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have*

(1) (**Newton's inequality**) $\frac{S_i}{S_{i+1}} \geq \frac{S_{i-1}}{S_i}$,

(2) (**Maclaurin's inequality**) ${S_i}^{\frac{1}{i}} \geq {S_{i+1}}^{\frac{1}{i+1}}$.

Theorem 11. (**Rearrangement inequality**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have*

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

Theorem 12. (**Chebyshev's inequality**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have*

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 13. (**Hölder's inequality**) *Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

Theorem 14. (**Minkowski's inequality**) *If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ and $p > 1$, then*

$$\left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {y_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}}$$

3. Years 2001 ∼ 2005

Each problem that I solved became a rule, which served afterwards to solve other problems. Rene Descartes

1. (**BMO 2005, Proposed by Serbia and Montenegro**) $(a, b, c > 0)$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}$$

2. (**Romania 2005, Cezar Lupu**) $(a, b, c > 0)$

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

3. (**Romania 2005, Traian Tamaian**) $(a, b, c > 0)$

$$\frac{a}{b+2c+d}+\frac{b}{c+2d+a}+\frac{c}{d+2a+b}+\frac{d}{a+2b+c}\ge 1$$

4. (**Romania 2005, Cezar Lupu**) $\left(a+b+c\ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c},\ a, b, c > 0\right)$

$$a+b+c\ge \frac{3}{abc}$$

5. (**Romania 2005, Cezar Lupu**) $(1=(a+b)(b+c)(c+a),\ a, b, c > 0)$

$$ab+bc+ca\ge \frac{3}{4}$$

6. (**Romania 2005, Robert Szasz**) $(a+b+c=3,\ a, b, c > 0)$

$$a^2b^2c^2\ge (3-2a)(3-2b)(3-2c)$$

7. (**Romania 2005**) $(abc\ge 1,\ a, b, c > 0)$

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\le 1$$

8. (**Romania 2005, Unused**) $(abc=1,\ a, b, c > 0)$

$$\frac{a}{b^2(c+1)}+\frac{b}{c^2(a+1)}+\frac{c}{a^2(b+1)}\ge \frac{3}{2}$$

9. (**Romania 2005, Unused**) $(a+b+c\ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ a, b, c > 0)$

$$\frac{a^3c}{b(c+a)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)}\ge \frac{3}{2}$$

10. (**Romania 2005, Unused**) $(a+b+c=1,\ a, b, c > 0)$

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\ge \sqrt{\frac{3}{2}}$$

11. (**Romania 2005, Unused**) $(ab+bc+ca+2abc=1,\ a, b, c > 0)$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ge \frac{3}{2}$$

12. (**Chzech and Solvak 2005**) $(abc=1,\ a, b, c > 0)$

$$\frac{a}{(a+1)(b+1)}+\frac{b}{(b+1)(c+1)}+\frac{c}{(c+1)(a+1)}\ge \frac{3}{4}$$

13. (**Japan 2005**) $(a+b+c=1,\ a,b,c>0)$

$$a\,(1+b-c)^{\frac{1}{3}}+b\,(1+c-a)^{\frac{1}{3}}+c\,(1+a-b)^{\frac{1}{3}}\leq 1$$

14. (**Germany 2005**) $(a+b+c=1,\ a,b,c>0)$

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{b}\right)\geq\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}$$

15. (**Vietnam 2005**) $(a,b,c>0)$

$$\left(\frac{a}{a+b}\right)^3+\left(\frac{b}{b+c}\right)^3+\left(\frac{c}{c+a}\right)^3\geq\frac{3}{8}$$

16. (**China 2005**) $(a+b+c=1,\ a,b,c>0)$

$$10(a^3+b^3+c^3)-9(a^5+b^5+c^5)\geq 1$$

17. (**China 2005**) $(abcd=1,\ a,b,c,d>0)$

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq 1$$

18. (**China 2005**) $(ab+bc+ca=\frac{1}{3},\ a,b,c\geq 0)$

$$\frac{1}{a^2-bc+1}+\frac{1}{b^2-ca+1}+\frac{1}{c^2-ab+1}\leq 3$$

19. (**Poland 2005**) $(0\leq a,b,c\leq 1)$

$$\frac{a}{bc+1}+\frac{b}{ca+1}+\frac{c}{ab+1}\leq 2$$

20. (**Poland 2005**) $(ab+bc+ca=3,\ a,b,c>0)$

$$a^3+b^3+c^3+6abc\geq 9$$

21. (**Baltic Way 2005**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{a^2+2}+\frac{b}{b^2+2}+\frac{c}{c^2+2}\geq 1$$

22. (**Serbia and Montenegro 2005**) $(a,b,c>0)$

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}(a+b+c)}$$

23. (**Serbia and Montenegro 2005**) $(a+b+c=3,\ a,b,c>0)$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca$$

24. (**Bosnia and Hercegovina 2005**) $(a+b+c=1, a,b,c>0)$

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq \frac{1}{\sqrt{3}}$$

25. (**Iran 2005**) $(a,b,c>0)$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

26. (**Austria 2005**) $(a,b,c,d>0)$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}\geq \frac{a+b+c+d}{abcd}$$

27. (**Moldova 2005**) $(a^4+b^4+c^4=3,\ a,b,c>0)$

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca}\leq 1$$

28. (**APMO 2005**) $(abc=8,\ a,b,c>0)$

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}}+\frac{b^2}{\sqrt{(1+b^3)(1+c^3)}}+\frac{c^2}{\sqrt{(1+c^3)(1+a^3)}}\geq \frac{4}{3}$$

29. (**IMO 2005**) $(xyz\geq 1, x,y,z>0)$

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0$$

30. (**Poland 2004**) $(a+b+c=0, a,b,c\in\mathbb{R})$

$$b^2c^2+c^2a^2+a^2b^2+3\geq 6abc$$

31. (**Baltic Way 2004**) $(abc=1, a,b,c>0, n\in\mathbb{N})$

$$\frac{1}{a^n+b^n+1}+\frac{1}{b^n+c^n+1}+\frac{1}{c^n+a^n+1}\leq 1$$

32. (**Junior Balkan 2004**) $((x,y)\in\mathbb{R}^2-\{(0,0)\})$

$$\frac{2\sqrt{2}}{x^2+y^2}\geq \frac{x+y}{x^2-xy+y^2}$$

33. (**IMO Short List 2004**) $(ab + bc + ca = 1,\ a, b, c > 0)$

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

34. (**APMO 2004**) $(a, b, c > 0)$

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)$$

35. (**USA 2004**) $(a, b, c > 0)$

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$

36. (**Junior BMO 2003**) $(x, y, z > -1)$

$$\frac{1 + x^2}{1 + y + z^2} + \frac{1 + y^2}{1 + z + x^2} + \frac{1 + z^2}{1 + x + y^2} \geq 2$$

37. (**USA 2003**) $(a, b, c > 0)$

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8$$

38. (**Russia 2002**) $(x + y + z = 3,\ x, y, z > 0)$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx$$

39. (**Latvia 2002**) $\left(\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1,\ a, b, c, d > 0\right)$

$$abcd \geq 3$$

40. (**Albania 2002**) $(a, b, c > 0)$

$$\frac{1 + \sqrt{3}}{3\sqrt{3}}(a^2 + b^2 + c^2)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq a + b + c + \sqrt{a^2 + b^2 + c^2}$$

41. (**Belarus 2002**) $(a, b, c, d > 0)$

$$\sqrt{(a + c)^2 + (b + d)^2} + \frac{2|ad - bc|}{\sqrt{(a + c)^2 + (b + d)^2}} \geq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a + c)^2 + (b + d)^2}$$

42. (**Canada 2002**) $(a, b, c > 0)$

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

43. (**Vietnam 2002, Dung Tran Nam**) $(a^2+b^2+c^2=9,\ a,b,c\in\mathbb{R})$

$$2(a+b+c)-abc\le 10$$

44. (**Bosnia and Hercegovina 2002**) $(a^2+b^2+c^2=1, a,b,c\in\mathbb{R})$

$$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ca}+\frac{c^2}{1+2ab}\le\frac{3}{5}$$

45. (**Junior BMO 2002**) $(a,b,c>0)$

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\ge\frac{27}{2(a+b+c)^2}$$

46. (**Greece 2002**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\ge\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

47. (**Greece 2002**) $(bc\ne 0, \frac{1-c^2}{bc}\ge 0,\ a,b,c\in\mathbb{R})$

$$10(a^2+b^2+c^2-bc^3)\ge 2ab+5ac$$

48. (**Taiwan 2002**) $\left(a,b,c,d\in\left(0,\frac{1}{2}\right]\right)$

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)}\le\frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

49. (**APMO 2002**) $(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,\ x,y,z>0)$

$$\sqrt{x+yz}+\sqrt{y+zx}+\sqrt{z+xy}\ge\sqrt{xyz}+\sqrt{x}+\sqrt{y}+\sqrt{z}$$

50. (**Ireland 2001**) $(x+y=2,\ x,y\ge 0)$

$$x^2y^2(x^2+y^2)\le 2.$$

51. (**BMO 2001**) $(a+b+c\ge abc,\ a,b,c\ge 0)$

$$a^2+b^2+c^2\ge\sqrt{3}abc$$

52. (**USA 2001**) $(a^2+b^2+c^2+abc=4,\ a,b,c\ge 0)$

$$0\le ab+bc+ca-abc\le 2$$

53. (**Columbia 2001**) $(x,y\in\mathbb{R})$

$$3(x+y+1)^2+1\ge 3xy$$

54. (**KMO Winter Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{(a^2b + b^2c + c^2a)\,(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)\,(b^3 + abc)\,(c^3 + abc)}$$

55. (**IMO 2001**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

4. Years 1996 ∼ 2000

Life is good for only two things, discovering mathematics and teaching mathematics. Simeon Poisson

56. (**IMO 2000, Titu Andreescu**) $(abc = 1,\ a, b, c > 0)$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

57. (**Czech and Slovakia 2000**) $(a, b > 0)$

$$\sqrt[3]{2(a + b)\left(\frac{1}{a} + \frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$$

58. (**Hong Kong 2000**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1 + ab^2}{c^3} + \frac{1 + bc^2}{a^3} + \frac{1 + ca^2}{b^3} \geq \frac{18}{a^3 + b^3 + c^3}$$

59. (**Czech Republic 2000**) $(m, n \in N,\ x \in [0, 1])$

$$(1 - x^n)^m + (1 - (1 - x)^m)^n \geq 1$$

60. (**Macedonia 2000**) $(x, y, z > 0)$

$$x^2 + y^2 + z^2 \geq \sqrt{2}\,(xy + yz)$$

61. (**Russia 1999**) $(a, b, c > 0)$

$$\frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3$$

62. (**Belarus 1999**) $(a^2 + b^2 + c^2 = 3,\ a, b, c > 0)$

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}$$

63. (**Czech-Slovak Match 1999**) $(a, b, c > 0)$

$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\geq 1$$

64. (**Moldova 1999**) $(a, b, c > 0)$

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}\geq\frac{a}{c+a}+\frac{b}{b+a}+\frac{c}{c+b}$$

65. (**United Kingdom 1999**) $(p+q+r=1,\ p, q, r > 0)$

$$7(pq+qr+rp)\leq 2+9pqr$$

66. (**Canada 1999**) $(x+y+z=1,\ x, y, z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

67. (**Proposed for 1999 USAMO**, [AB, pp.25]) $(x, y, z > 1)$

$$x^{x^2+2yz}y^{y^2+2zx}z^{z^2+2xy}\geq(xyz)^{xy+yz+zx}$$

68. (**Turkey, 1999**) $(c\geq b\geq a\geq 0)$

$$(a+3b)(b+4c)(c+2a)\geq 60abc$$

69. (**Macedonia 1999**) $(a^2+b^2+c^2=1,\ a, b, c > 0)$

$$a+b+c+\frac{1}{abc}\geq 4\sqrt{3}$$

70. (**Poland 1999**) $(a+b+c=1,\ a, b, c > 0)$

$$a^2+b^2+c^2+2\sqrt{3abc}\leq 1$$

71. (**Canda 1999**) $(x+y+z=1, x, y, z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

72. (**Iran 1998**) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2,\ x, y, z > 1\right)$

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

73. (**Belarus 1998**, I. Gorodnin) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{b+c}+\frac{b+c}{a+b}+1$$

74. (**APMO 1998**) $(a, b, c > 0)$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

75. (**Poland 1998**) $\left(a+b+c+d+e+f=1,\ ace+bdf \geq \frac{1}{108}\ a, b, c, d, e, f > 0\right)$

$$abc+bcd+cde+def+efa+fab \leq \frac{1}{36}$$

76. (**Korea 1998**) $(x+y+z=xyz,\ x, y, z > 0)$

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}$$

77. (**Hong Kong 1998**) $(a, b, c \geq 1)$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{c(ab+1)}$$

78. (**IMO Short List 1998**) $(xyz=1,\ x, y, z > 0)$

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

79. (**Belarus 1997**) $(a, x, y, z > 0)$

$$\frac{a+y}{a+x}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z \geq x+y+z \geq \frac{a+z}{a+z}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z$$

80. (**Ireland 1997**) $(a+b+c \geq abc,\ a, b, c \geq 0)$

$$a^2+b^2+c^2 \geq abc$$

81. (**Iran 1997**) $(x_1x_2x_3x_4=1, x_1, x_2, x_3, x_4 > 0)$

$$x_1^3+x_2^3+x_3^3+x_4^3 \geq max\left(x_1+x_2+x_3+x_4, \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

82. (**Hong Kong 1997**) $(x, y, z > 0)$

$$\frac{3+\sqrt{3}}{9} \geq \frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)}$$

83. (**Belarus 1997**) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

84. (**Bulgaria 1997**) $(abc=1,\ a,b,c>0)$

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$$

85. (**Romania 1997**) $(xyz=1,\ x,y,z>0)$

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}\geq 2$$

86. (**Romania 1997**) $(a,b,c>0)$

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\geq 1\geq\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}$$

87. (**USA 1997**) $(a,b,c>0)$

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{1}{abc}.$$

88. (**Japan 1997**) $(a,b,c>0)$

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}$$

89. (**Estonia 1997**) $(x,y\in\mathbb{R})$

$$x^2+y^2+1>x\sqrt{y^2+1}+y\sqrt{x^2+1}$$

90. (**APMC 1996**) $(x+y+z+t=0, x^2+y^2+z^2+t^2=1, x,y,z,t\in\mathbb{R})$

$$-1\leq xy+yz+zt+tx\leq 0$$

91. (**Spain 1996**) $(a,b,c>0)$

$$a^2+b^2+c^2-ab-bc-ca\geq 3(a-b)(b-c)$$

92. (**IMO Short List 1996**) $(abc=1,\ a,b,c>0)$

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\leq 1$$

93. (**Poland 1996**) $\left(a+b+c=1,\ a,b,c\geq-\frac{3}{4}\right)$

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}$$

94. (**Hungary 1996**) $(a+b=1,\ a,b>0)$

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}$$

95. (**Vietnam 1996**) $(a,b,c\in\mathbb{R})$

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}\left(a^4+b^4+c^4\right)$$

96. (**Bearus 1996**) $(x+y+z=\sqrt{xyz},\ x,y,z>0)$

$$xy+yz+zx\geq 9(x+y+z)$$

97. (**Iran 1996**) $(a,b,c>0)$

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geq\frac{9}{4}$$

98. (**Vietnam 1996**) $(2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab=16, a,b,c,d\geq 0)$

$$a+b+c+d\geq\frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

5. Years 1990 ∼ 1995

Any good idea can be stated in fifty words or less. S. M. Ulam

99. (**Baltic Way 1995**) $(a,b,c,d>0)$

$$\frac{a+c}{a+b}+\frac{b+d}{b+c}+\frac{c+a}{c+d}+\frac{d+b}{d+a}\geq 4$$

100. (**Canda 1995**) $(a,b,c>0)$

$$a^ab^bc^c\geq abc^{\frac{a+b+c}{3}}$$

101. (**IMO 1995, Nazar Agakhanov**) $(abc=1,\ a,b,c>0)$

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

102. (**Russia 1995**) $(x,y>0)$

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}$$

103. (**Macedonia 1995**) $(a, b, c > 0)$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}} \geq 2$$

104. (**APMC 1995**) $(m, n \in \mathbb{N},\ x, y > 0)$

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n) \geq nm(x^{n+m-1}y+xy^{n+m-1})$$

105. (**Hong Kong 1994**) $(xy+yz+zx=1,\ x, y, z > 0)$

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}$$

106. (**IMO Short List 1993**) $(a, b, c, d > 0)$

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}$$

107. (**APMC 1993**) $(a, b \geq 0)$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \leq \frac{a+\sqrt{ab}+b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

108. (**Poland 1993**) $(x, y, u, v > 0)$

$$\frac{xy+xv+uy+uv}{x+y+u+v} \geq \frac{xy}{x+y}+\frac{uv}{u+v}$$

109. (**IMO Short List 1993**) $(a+b+c+d=1,\ a, b, c, d > 0)$

$$abc+bcd+cda+dab \leq \frac{1}{27}+\frac{176}{27}abcd$$

110. (**Italy 1993**) $(0 \leq a, b, c \leq 1)$

$$a^2+b^2+c^2 \leq a^2b+b^2c+c^2a+1$$

111. (**Poland 1992**) $(a, b, c \in \mathbb{R})$

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

112. (**Vietnam 1991**) $(x \geq y \geq z > 0)$

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y} \geq x^2+y^2+z^2$$

113. (**Poland 1991**) $(x^2 + y^2 + z^2 = 2,\ x, y, z \in \mathbb{R})$

$$x + y + z \leq 2 + xyz$$

114. (**Mongolia 1991**) $(a^2 + b^2 + c^2 = 2,\ a, b, c \in \mathbb{R})$

$$|a^3 + b^3 + c^3 - abc| \leq 2\sqrt{2}$$

115. (**IMO Short List 1990**) $(ab + bc + cd + da = 1,\ a, b, c, d > 0)$

$$\frac{a^3}{b + c + d} + \frac{b^3}{c + d + a} + \frac{c^3}{d + a + b} + \frac{d^3}{a + b + c} \geq \frac{1}{3}$$

6. Supplementary Problems

Every Mathematician Has Only a Few Tricks. *A long time ago an older and well-known number theorist made some disparaging remarks about Paul Erdös′s work. You admire Erdos′s contributions to mathematics as much as I do, and I felt annoyed when the older mathematician flatly and definitively stated that all of Erdos′s work could be reduced to a few tricks which Erdös repeatedly relied on in his proofs. What the number theorist did not realize is that other mathematicians, even the very best, also rely on a few tricks which they use over and over. Take Hilbert. The second volume of Hilbert′s collected papers contains Hilbert′s papers in invariant theory. I have made a point of reading some of these papers with care. It is sad to note that some of Hilbert′s beautiful results have been completely forgotten. But on reading the proofs of Hilbert′s striking and deep theorems in invariant theory, it was surprising to verify that Hilbert′s proofs relied on the same few tricks.* **Even Hilbert had only a few tricks!** Gian-Carlo Rota, *Ten Lessons I Wish I Had Been Taught*, Notices of the AMS, January 1997

116. (**Lithuania 1987**) $(x, y, z > 0)$

$$\frac{x^3}{x^2 + xy + y^2} + \frac{y^3}{y^2 + yz + z^2} + \frac{z^3}{z^2 + zx + x^2} \geq \frac{x + y + z}{3}$$

117. (**Yugoslavia 1987**) $(a, b > 0)$

$$\frac{1}{2}(a + b)^2 + \frac{1}{4}(a + b) \geq a\sqrt{b} + b\sqrt{a}$$

118. (**Yugoslavia 1984**) $(a, b, c, d > 0)$

$$\frac{a}{b + c} + \frac{b}{c + d} + \frac{c}{d + a} + \frac{d}{a + b} \geq 2$$

119. (**IMO 1984**) $(x + y + z = 1,\ x, y, z \geq 0)$

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$$

120. (**USA 1980**) $(a, b, c \in [0,1])$

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

121. (**USA 1979**) $(x + y + z = 1,\ x, y, z > 0)$

$$x^3 + y^3 + z^3 + 6xyz \geq \frac{1}{4}.$$

122. (**IMO 1974**) $(a, b, c, d > 0)$

$$1 < \frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < 2$$

123. (**IMO 1968**) $(x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in \mathbb{R}, x_1y_1 > {z_1}^2, x_2y_2 > {z_2}^2)$

$$\frac{1}{x_1y_1 - {z_1}^2} + \frac{1}{x_2y_2 - {z_2}^2} \geq \frac{8}{(x_1+x_2)(y_1+y_2) - (z_1+z_2)^2}$$

124. (**Nesbitt's inequality**) $(a, b, c > 0)$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

125. (**Polya's inequality**) $(a \neq b, a, b > 0)$

$$\frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right) \geq \frac{\ln b - \ln a}{b-a}$$

126. (**Klamkin's inequality**) $(-1 < x, y, z < 1)$

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2$$

127. (**Carlson's inequality**) $(a, b, c > 0)$

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}$$

128. (**[ONI], Vasile Cirtoaje**) $(a, b, c > 0)$

$$\left(a + \frac{1}{b} - 1\right)\left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right)\left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right)\left(a + \frac{1}{b} - 1\right) \geq 3$$

129. (**[ONI], Vasile Cirtoaje**) $(a, b, c, d > 0)$

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0$$

130. (**Elemente der Mathematik, Problem 1207, S̃efket Arslanagić**) $(x, y, z > 0)$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

131. ($\sqrt{WURZEL}$, **Walther Janous**) $(x + y + z = 1,\ x, y, z > 0)$

$$(1 + x)(1 + y)(1 + z) \geq (1 - x^2)^2 + (1 - y^2)^2 + (1 - z^2)^2$$

132. ($\sqrt{WURZEL}$, **Heinz-Jürgen Seiffert**) $(xy > 0, x, y \in \mathbb{R})$

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2}$$

133. ($\sqrt{WURZEL}$, **Šefket Arslanagić**) $(a, b, c > 0)$

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}$$

134. ($\sqrt{WURZEL}$, **Šefket Arslanagić**) $(abc = 1, a, b, c > 0)$

$$\frac{1}{a^2(b+c)} + \frac{1}{b^2(c+a)} + \frac{1}{c^2(a+b)} \geq \frac{3}{2}.$$

135. ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) $(abc = 1, a, b, c > 0)$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{2}(a+b)(c+a)(b+c) - 1.$$

136. ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) $(x + y + z = 3, x^2 + y^2 + z^2 = 7, x, y, z > 0)$

$$1 + \frac{6}{xyz} \geq \frac{1}{3}\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)$$

137. ($\sqrt{WURZEL}$, **Šefket Arslanagić**) $(a, b, c > 0)$

$$\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{3(a+b+c)}{a+b+c+3}.$$

138. (**[ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva**) $(a, b, c > 0)$

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1 + a)(1 + b)(1 + c)$$

139. (**Gazeta Matematicã**) $(a, b, c > 0)$

$$\sqrt{a^4 + a^2b^2 + b^4} + \sqrt{b^4 + b^2c^2 + c^4} + \sqrt{c^4 + c^2a^2 + a^4} \geq a\sqrt{2a^2 + bc} + b\sqrt{2b^2 + ca} + c\sqrt{2c^2 + ab}$$

140. (**C[1]2362, Mohammed Aassila**) $(a, b, c > 0)$

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{1+abc}$$

141. (**C2580**) $(a, b, c > 0)$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab}$$

142. (**C2581**) $(a, b, c > 0)$

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c$$

143. (**C2532**) $(a^2+b^2+c^2 = 1,\ a, b, c > 0)$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3+b^3+c^3)}{abc}$$

144. (**C3032, Vasile Cirtoaje**) $(a^2+b^2+c^2 = 1,\ a, b, c > 0)$

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2}$$

145. (**C2645**) $(a, b, c > 0)$

$$\frac{2(a^3+b^3+c^3)}{abc} + \frac{9(a+b+c)^2}{(a^2+b^2+c^2)} \geq 33$$

146. $(x, y \in \mathbb{R})$

$$-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$$

147. $(0 < x, y < 1)$

$$x^y + y^x > 1$$

148. $(x, y, z > 0)$

$$\sqrt[3]{xyz} + \frac{|x-y| + |y-z| + |z-x|}{3} \geq \frac{x+y+z}{3}$$

149. $(a, b, c, x, y, z > 0)$

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}$$

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[1]CRUX with MAYHEM

150. $(x, y, z > 0)$

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\le 1$$

151. $(x+y+z=1,\ x, y, z > 0)$

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}}\ge\sqrt{\frac{3}{2}}$$

152. $(a, b, c \in \mathbb{R})$

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\ge\frac{3\sqrt{2}}{2}$$

153. $(a, b, c > 0)$

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}\ge\sqrt{a^2+ac+c^2}$$

154. $(xy+yz+zx=1,\ x, y, z > 0)$

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\ge\frac{2x(1-x^2)}{(1+x^2)^2}+\frac{2y(1-y^2)}{(1+y^2)^2}+\frac{2z(1-z^2)}{(1+z^2)^2}$$

155. $(x, y, z \ge 0)$

$$xyz\ge(y+z-x)(z+x-y)(x+y-z)$$

156. $(a, b, c > 0)$

$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)}\ge\sqrt{4abc+(a+b)(b+c)(c+a)}$$

157. (**Darij Grinberg**) $(x, y, z \ge 0)$

$$\left(\sqrt{x\,(y+z)}+\sqrt{y\,(z+x)}+\sqrt{z\,(x+y)}\right)\cdot\sqrt{x+y+z}\ge 2\sqrt{(y+z)\,(z+x)\,(x+y)}.$$

158. (**Darij Grinberg**) $(x, y, z > 0)$

$$\frac{\sqrt{y+z}}{x}+\frac{\sqrt{z+x}}{y}+\frac{\sqrt{x+y}}{z}\ge\frac{4\,(x+y+z)}{\sqrt{(y+z)\,(z+x)\,(x+y)}}.$$

159. (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2\,(b+c)}{(b^2+c^2)\,(2a+b+c)}+\frac{b^2\,(c+a)}{(c^2+a^2)\,(2b+c+a)}+\frac{c^2\,(a+b)}{(a^2+b^2)\,(2c+a+b)}>\frac{2}{3}.$$

160. (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2}{2a^2+(b+c)^2}+\frac{b^2}{2b^2+(c+a)^2}+\frac{c^2}{2c^2+(a+b)^2}<\frac{2}{3}.$$

161. (**Vasile Cirtoaje**) $(a, b, c \in \mathbb{R})$

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$