A Tour de Force in Geometry

Thomas Mildorf

January 4, 2006

We are alone in a desert, pursued by a hungry lion. The only tool we have is a spherical, adamantine cage. We enter the cage and lock it, securely. Next we perform an inversion with respect to the cage. The lion is now in the cage and we are outside.

In this lecture we try to capture some geometric intuition. To this end, the first section is a brief outline (by no means complete) of famous results in geometry. In the second, we motivate some of their applications.

1 Factoids

First, we review the canonical notation. Let ABC be a triangle and let $a = BC, b = CA, c = AB$. K denotes the area of ABC, while r and R are the inradius and circumradius of ABC respectively. G, H, I, and O are the centroid, orthocenter, incenter, and circumcenter of ABC respectively. Write r_A, r_B, r_C for the respective radii of the excircles opposite A, B, and C, and let $s = (a + b + c)/2$ be the semiperimeter of ABC.

1. Law of Sines:
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R$$

2. Law of Cosines:
$$c^2 = a^2 + b^2 - 2ab\cos(C) \qquad \text{or} \qquad \cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

3. Area (h_a height from A):
$$\begin{aligned} K &= \frac{1}{2}ah_a = \frac{1}{2}ab\sin(C) = \frac{1}{2}ca\sin(B) = \frac{1}{2}ab\sin(C) \\ &= 2R^2\sin(A)\sin(B)\sin(C) \\ &= \frac{abc}{4R} \\ &= rs = (s-a)r_A = (s-b)r_B = (s-c)r_C \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)} \\
&= \sqrt{r r_A r_B r_C}
\end{aligned}
$$

4. R, r, r_A, r_B, r_C:

$$
\begin{aligned}
&4R + r = r_A + r_B + r_C \\
&\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} \\
&1 + \frac{r}{R} = \cos(A) + \cos(B) + \cos(C)
\end{aligned}
$$

5. G, H, I, O:

$$
\begin{aligned}
&Euler\, Line\, OGH: \vec{OH} = 3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC} \\
&3(GA^2 + GB^2 + GC^2) = a^2 + b^2 + c^2 \\
&HG^2 = 4R^2 - \frac{4}{9}(a^2 + b^2 + c^2) \\
&OH^2 = 9R^2 - (a^2 + b^2 + c^2) \\
&Euler: OI^2 = R^2 - 2rR \\
&IG^2 = r^2 + \frac{1}{36}\left(5(a^2 + b^2 + c^2) - 6(ab + bc + ca)\right) \\
&AH = 2R\cos(A) \\
&Feuerbach: \vec{IH} \cdot \vec{IG} = -\frac{2}{3}r(R - 2r)
\end{aligned}
$$

6. Trigonometric Identities (x, y, z arbitrary):

$$
\begin{aligned}
\sin(x) + \sin(y) + \sin(z) - \sin(x+y+z) &= 4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right) \\
\cos(x) + \cos(y) + \cos(z) + \cos(x+y+z) &= 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{y+z}{2}\right)\cos\left(\frac{z+x}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cos^2(A) + \cos^2(B) + \cos^2(C) + 2\cos(A)\cos(B)\cos(C) = 1 \\
&\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C)
\end{aligned}
$$

7. Mass-point Geometry (D, E, and F are points on lines BC, CA, and AB respectively):

$$
\begin{aligned}
&Ceva: \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \iff AD, BE, CF \text{ concurrent.} \\
&Menelaus: \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1 \iff D, E, F \text{ collinear}
\end{aligned}
$$

8. Stewart's Theorem (D on $\overline{BC}$, $AD = d, BD = m, DC = n$):

$$ad^2 + amn = b^2m + c^2n \qquad \text{or} \qquad man + dad = bmb + cnc$$

9. Power of a Point (Lines l_1, l_2 through P; l_1 intersects a circle ω at A_1, B_1; and l_2 intersects ω at A_2, B_2):

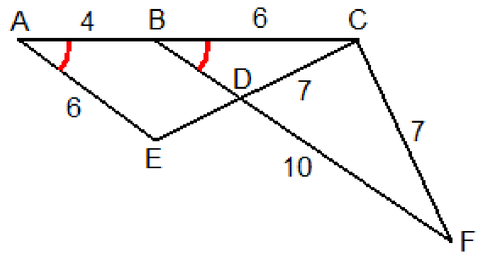
$$PA_1 \cdot PB_1 = PA_2 \cdot PB_2 \qquad \text{or} \qquad \triangle PA_1A_2 \sim \triangle PB_2B_1$$

10. Cyclic Quadrilaterals (A, B, C, D on a circle in that order; $s = (a + b + c + d)/2$):

$$Ptolemy: \ AB \cdot CD + AD \cdot BC = AC \cdot BD$$
$$Brahmagupta: \ K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

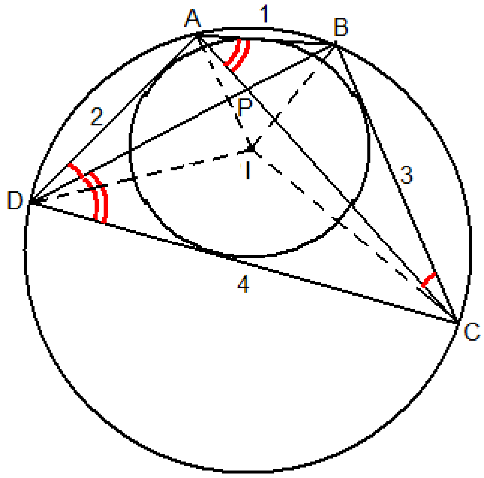
2 Selected Problems

The preceeding facts summarize a great many implications; understanding these implications is fundamental to "intuition" in geometry. I will provide solutions to problems 1.3, 2.3, 2.4, 2.6, 3.2, 3.7, 3.8, and 4.8 from "A Geometry Problem Set," breaking them down to my own thought process.



Problem 1.3. Note that $AC = AB + BC = 10$, so that triangle AEC is congruent to triangle BCF. Now $[ABDE] = [AEC] - [BDC] = [BCF] - [BDC] = [CDF]$, so the desired ratio is $\frac{1}{1}$ and the answer is 2. □

Basically, we have to notice that the two triangles are congruent - we need to locate D somehow. The solution above proceeds about as cleanly as possible after making the required observation. A less magical approach would be to use similar triangles ACE and BCD to place D, then compute the ratios of the areas of the desired figures relative to $[BCD]$. Another bit of experience to gather from this problem is not to let goofy answers psyche you out. This problem was inspired by an old ARML question, which asked for the same conversion of 1/1.



Problem 2.3. Because $ABCD$ has an incircle, $AD+BC = AB + CD = 5$. Suppose that $AD : BC = 1 : \gamma$. Then $3 : 8 = BP : DP = (AB \cdot BC) : (CD \cdot DA) = \gamma : 4$. We obtain $\gamma = \frac{3}{2}$, which substituted into $AD + BC = 5$ gives $AD = 2$, $BC = 3$. Now, the area of $ABCD$ can be obtained via Brahmagupta's formula: $s = \frac{1+2+3+4}{2} = 5$, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{24}$ and $K = rs = 5r$, where r is the inradius of $ABCD$. Thus, $r = \frac{\sqrt{24}}{5}$ from which its area $\frac{24\pi}{25}$ yields the answer $24 + 25 = 49$. □

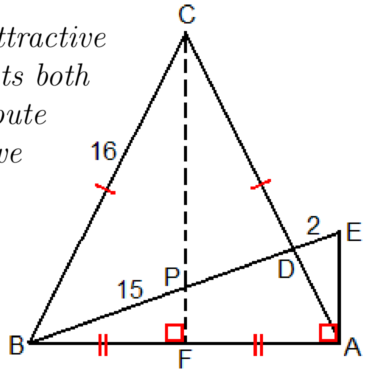
One of the main ideas in this problem is that power of a point does not capture the full strength of equal angles and similar triangles in circles. We have $\angle ACB = \angle ADB$, so triangles PDA and PCB are similar. Thus, $AP/PB = AD/BC$. And since $\angle BAC = \angle BDC$, triangles PAB and PDC are similar. Thus, $PB/PC = AB/DC = 1/4$. This allows us to compute the sides of $ABCD$. Now to compute the inradius of $ABCD$, we use area as an intermediary. We know little about the incircle except that it is tangent to every side of $ABCD$, which means triangles AIB, BIC, CID, and DIA have equal heights from I, all being a radius. Knowing the sides of $ABCD$ and how to compute its area, we arrive at a natural solution.

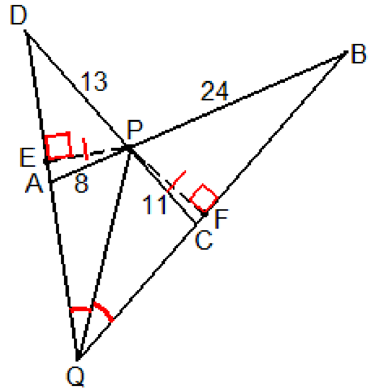
Problem 2.4. Draw in altitude $\overline{CF}$ and denote its intersection with $\overline{BD}$ by P. Since ABC is isosceles, $AF = FB$. Now, since BAE and BFP are similar with a scale factor of 2, we have $BP = \frac{1}{2}BE = \frac{17}{2}$, which also yields $PD = BD - BP = 15 - \frac{17}{2} = \frac{13}{2}$. Now, applying Menelaus to triangle ADB and collinear points C, P, and F, we obtain

$$\begin{aligned} \frac{AC}{CD}\frac{DP}{PB}\frac{BF}{FA} &= \frac{AC}{CD}\frac{DP}{PB} = -1 \\ \Longrightarrow |CD| = AC \cdot \frac{DP}{PB} &= 16 \cdot \frac{\left(\frac{13}{2}\right)}{\left(\frac{17}{2}\right)} = \frac{208}{17} \end{aligned}$$

where the minus sign was a consequence of directed distances.[1] The answer is therefore $208 + 17 = 225$. □

[1] A system of linear measure in which for any points A and B, $AB = -BA$.





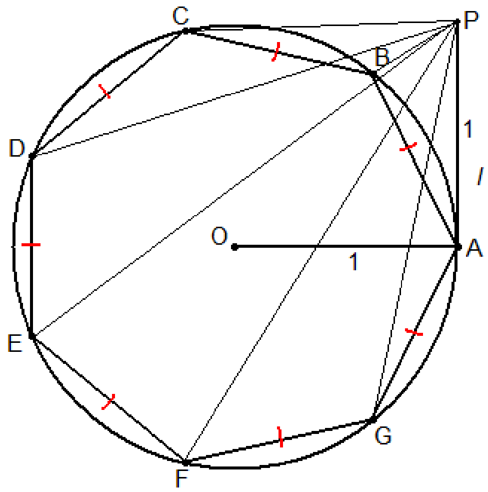
The altitude $\overline{CF}$ is particularly attractive because it is parallel to $\overline{AE}$ and bisects both $\overline{AB}$ and $\overline{BE}$. This allows us to compute BP and PD easily, and from there we recognize that any ratio along $\overline{CDA}$ would allow us to compute the desired length. We recognize Menelaus as a convenient choice, although mass points would also suffice. Alternatively, we could approach this problem with coordinates. Let B the the origin and write $A = (x, 0), E = (x, y)$. D is easily computed, and since $C = (\frac{x}{2}, z)$, we can obtain the desired ratio easily.

Problem 2.6. Let E and F be the projections of P onto AD and BC respectively. Note that the angle bisector condition is equivalent to $PE = PF$. It follows that $\frac{AD}{BC} = \frac{[APD]}{[BPC]} = \frac{AP \cdot DP}{BP \cdot CP} = \frac{13}{33}$ so the answer is 46. $\square$

One of the first observations we make is that due to congruent angles at P, we can easily find the ratio of the areas of triangles APD and BPC. Now a bit of wishful thinking: AD/BC would be precisely this ratio if the heights from P were equal. This we immediately recognize as a consequence of the angle bisector condition. The problem could also be done by extending $\overline{QP}$ to $\overline{BD}$ and using the angle bisector theorem followed by numerous applications of Ceva and Menelaus or mass point geometry.

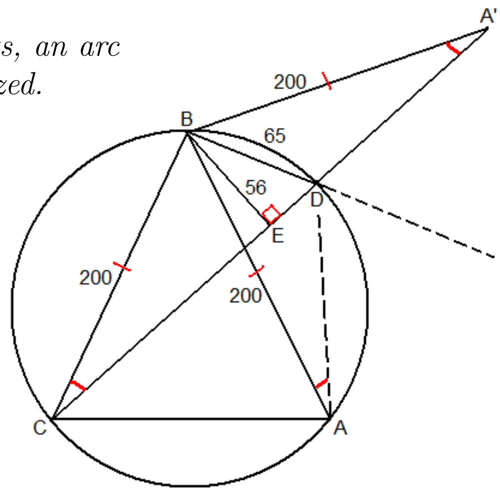
Problem 3.2. Overlay the complex number system with $O = 0 + 0i, A = 1 + 0i$, and $P = 1 + i$. The solutions to the equation $z^7 = 1$ are precisely the seven vertices of the heptagon. Letting a, b, c, d, e, f, and g denote the complex numbers for A, B, C, D, E, F, and G respectively, this equation rewrites as $(z - a)(z - b)(z - c)(z - d)(z - e)(z - f)(z - g) = z^7 - 1 = 0$. The magnitude of the factored product represents the product of the distances from the arbitrary point represented by z. Thus, plugging in $1 + i$, we have $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP = |(1+i)^7 - 1| = |8 - 8i - 1| = |7 - 8i| = \sqrt{7^2 + 8^2} = \sqrt{113}$. It follows that the answer is 113. $\square$

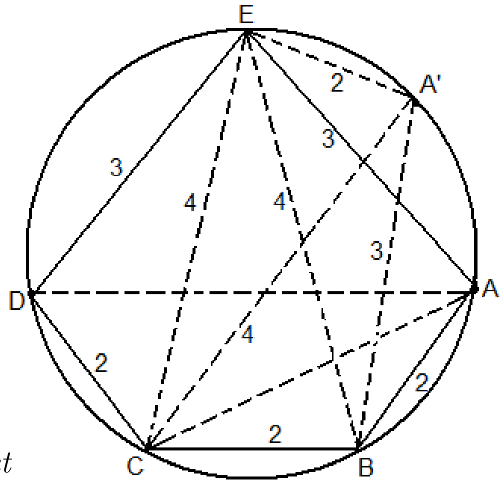
The trigonometric functions evaluated at $\pi/7$ do not have simple radical expressions, although the symmetric polynomial identities can be derived by expanding $(\cos(\theta) + i\sin(\theta))^7 = 0$. Thus, a highly computational trigonometric solution is possible. However, the presence of regular heptagon, its [unit!] circumcircle, and a product involving lengths of segments ending at those vertices constitutes a particularly strong reason to inspect for a complex numbers solution. Indeed, the problem is solved almost immediately after imposing the system.



Problem 3.7. Reflect A over $\overline{BD}$ to A'. Then $m\angle BDA' = m\angle ADB = \pi - m\angle BCA = \pi - m\angle BDC$. Therefore, A' lies on line CD. It follows that $CE = EA' = ED + DA' = ED + DA$. Thus, $AD = CE - ED$. Now Pythagoras gives $AD = \sqrt{200^2 - 56^2} - \sqrt{65^2 - 56^2} = 192 - 33 = 159$. □

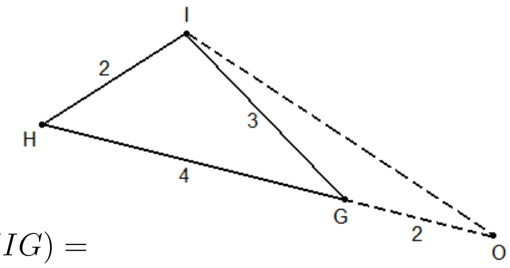
We have lots of information, perpendicular segments, an arc midpoint, and a cyclic quadrilateral, but it is disorganized. We seek a point which ties this information together in a useful way. Opposite angles adding to 180° suggests a line, and isosceles triangles together with a perpendicular often lead to the foot of that perpendicular serving as a midpoint. By a gracious act of serendipity, reflecting A over $\overline{BD}$ produces just the A' we are looking for. Another suitable point is D' obtained by reflecting D over $\overline{BE}$. (Why?) The alternative to this approach is computing just about everything: use right triangles BED and BEC to compute $\sin(\angle BDE)$ and $\sin(\angle BCE)$, from which one computes $\sin(\angle DCA)$, and combine this with the circumradius of $ACBD$ (found via the area of ABC after computing AC) to obtain AD.





Problem 3.8. Construct A' on minor arc AE such that $A'E = 2$ and $A'B = 3$. Now $BE = EC = CA' = x$ because each intercepts the same pair of arcs. Ptolemy on $BCEA'$ gives $x = 4$. Now Ptolemy on $BCEA$ gives $AC = 7/2$. And, since $AC = BD$, a third Ptolemy on $ABCD$ gives $AD = \frac{33}{8}$ for an answer of 41. $\square$

Our key observation is that many of the given pentagon's sides are equal. This suggests that the idea of equal segments could be useful. By inspection $BE = CE$ and $AC = BD$, so there are just 3 distinct diagonal lengths. Next we ask how those diagonals of $ABCDE$ could be computed. Ptolemy comes to mind, but that applies only to quadrilaterals, and in our original picture Ptolemy leads only to a system of quadratic equations in several variables each. Again, the information is not in a useful format, but if we could isolate one variable we would be set. The construction of A' above was cooked up to introduce $A'C = CE = BE$, allowing us to compute BE and CE in the original picture, from which our solution follows easily.



Problem 4.8. Examining the Euler line, O lies on line HG such that $GO = 2$. Now Stewart's theorem on G, H, I, O yields $4OI^2 + 8 = 54 + 48$ from which $OI^2 = \frac{47}{2}$. Another famous result of Euler is that $OI^2 = R(R - 2r)$. Finally, a corollary of Feuerbach's theorem gives $-\frac{2}{3}r(R-2r) = \vec{IG} \cdot \vec{IH} = IG \cdot IH \cos(HIG) = \frac{HG^2 - IG^2 - IH^2}{-2} = -\frac{3}{2}$. It follows that $\cos(A) + \cos(B) + \cos(C) = 1 + \frac{r}{R} = 1 + \frac{9}{94} = \frac{103}{94}$, so the answer is 197. $\square$

The main difficulty in this problem is overcoming the sheer befuddlement that one feels being asked a question about triangle ABC about which one is told only the relative position of three centers. Indeed, even drawing the complete diagram is extremely difficult. Thus, we restrict our attention to reasonable points and completely disregard ABC in our diagram. Now, how could we possibly answer a question about ABC? The identity $\cos(A)+\cos(B)+\cos(C) = 1+\frac{r}{R}$ seems to be the only hope: we can look for the ratio r/R. What equations for the inradius and circumradius in terms of $G, H,$ and I do we know? The esoteric corollary of Feuerbach's theorem $\vec{IG} \cdot \vec{IH} = -\frac{2}{3}r(R-2r)$. But we need another equation. Maybe $OI^2 = R(R-2r)$? We don't know anything about O... until we construct the Euler line! The equations for r and R even share a factor that cancels, and the solution is now evident. In hindsight, the only point we could even reasonably have considered constructing is O: it too is a center of ABC, and we happen to know where it is.

[†]Generally, such magical exercises are never given on contests. Especially ones requiring obscure theorems, as well as the insight to forgo the urge to even draw the triangle ABC, the only triangle mentioned in the problem statement!

Over time, with good practice, one's intuition for solving geometry problems increases. Mustering insight from the formulae given in this lecture, however, is nontrivial. For this, I have provided 32 homemade problems ranging from quite easy to monstrously difficult. Their solutions, as well as everything else from this lecture, can be found at my website:

http://web.mit.edu/~tmildorf/www/