Week 5: Functions (images and preimages of sets)

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x-2|$ for $x \in \mathbb{R}$.

1) Find $f\Big((1,6)\Big), f\Big((0,4]\Big), f\Big(\mathbb{Q}\Big)$.
2) Find $f^{-1}\Big(\{5\}\Big), f^{-1}\Big([1,+\infty)\Big)$.

Exercise 2. If U is a set, recall that $\mathrm{card}(U)$ denotes the number of its elements and that $\mathcal{P}(U)$ denotes the set of all subsets of U. Let f be the function defined by

$$\begin{array}{rccc} f & : & \mathcal{P}(\{1,2,3\}) & \longrightarrow & \{0,1,2,3\} \\ & & X & \longmapsto & \mathrm{card}(X) \end{array}$$

1) Is f onto? Is f one-to-one?
2) Find $f^{-1}(1)$, $f^{-1}(\{1\})$, $f^{-1}(\{0\})$ and $f^{-1}(\emptyset)$.

Exercise 3. Let $f : E \to F$ be a function and $B \subseteq F$.

1) Show that $f\left(f^{-1}(B)\right) \subseteq B$.
2) Do we always have $f\left(f^{-1}(B)\right) = B$?
3) Show that if f is onto, then $f\left(f^{-1}(B)\right) = B$.

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on November, 4th.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x}{2+x^2}$. Find $f^{-1}(\mathbb{R})$ and $f(\mathbb{R})$.

Exercise 5. Let $f : E \to F$ be a function. Let $A \subseteq E$.

1) Show that $A \subseteq f^{-1}(f(A))$.
2) Do we always have $f^{-1}(f(A)) \subseteq A$?
3) Show that if f is one-to-one, then $A = f^{-1}(f(A))$.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 5.

Exercise 6. Let $f : E \to F$ be a function and $B \subseteq F$. Show that $f^{-1}(F\backslash B) = E\backslash f^{-1}(B)$.

Exercise 7. Let X be a set and $f : X \to \mathcal{P}(X)$ a function, where we recall that $\mathcal{P}(X)$ denotes the set of all subsets of X. Show that f is not onto.

Hint. You may consider the set $A = \{x \in X : x \notin f(x)\}$.

Exercise 8. Let $K \geq 1$ be a fixed integer and let f be a one-to-one correspondence from $\{1,2,\dots,K\}$ to itself. We set $f^{(0)} = \mathrm{Id}$, where Id is the identity function defined by $\mathrm{Id}(x) = x$ for every $x \in \{1,2,\dots,K\}$, and, for every $n \geq 0$, $f^{(n+1)} = f \circ f^{(n)}$.

1) Explain briefly why $f^{(n)}$ is also a one-to-one correspondence.

2) How many one-to-one correspondences from $\{1,2,\dots,K\}$ to itself are there?

3) Prove that there exist two integers $i \neq j$ such that $f^{(i)} = f^{(j)}$.

4) Deduce from the above that there exists $n \geq 1$ such that $f^{(n)} = \mathrm{Id}$.

Exercise 9. Let $f : X \to Y$, $g : Y \to Z$ be two functions.

1) Show that for every $A \subseteq X$, $g \circ f(A) = g(f(A))$.

2) Show that for every $B \subseteq Z$, $(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$.

Exercise 10. Let E, F be sets and $f : E \to F$ a function. Show that for every $u, v \in F$, if $u \neq v$, then $f^{-1}(\{u\}) \cap f^{-1}(\{v\}) = \emptyset$.

Exercise 11. Let E, F be two sets and $f : E \to F$, $g : F \to E$ be two functions such that $f \circ g(x) = x$ for every $x \in F$. Show that $(g \circ f)(E) = g(F)$.

Exercise 12. If A, B are two subsets of a set E, recall that $A \Delta B$ is the subset of E defined by $A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$, where, to simplify notation, we denote by $\overline{C}$ the complement of C in E. When $A', B' \subseteq F$, we define $A' \Delta B'$ in a similar way (and also denote by $\overline{C'}$ the complement of C' in F when $C' \subseteq F$). Let E, F be two sets and let $f : E \to F$ be a function.

1) Show that for every $A', B' \subseteq F$, we have $f^{-1}(A' \Delta B') = f^{-1}(A') \Delta f^{-1}(B')$.

2) Show that f is one-to-one if and only if for every $A, B \subseteq E$ we have $f(A \Delta B) = f(A) \Delta f(B)$.

Exercise 13. Let E be a set. Let $A, B \subseteq E$ be two subsets of E. Let f be the function defined by

$$\begin{array}{rccc} f & : & \mathcal{P}(E) & \longrightarrow & \mathcal{P}(A) \times \mathcal{P}(B) \\ & & X & \longmapsto & (X \cap A, X \cap B). \end{array}$$

1) Find a necessary and sufficient condition on A and B for f to be one-to-one (recall that an assertion Q is a necessary and sufficient condition for P when $P \Leftrightarrow Q$ is true).

2) Find a necessary and sufficient condition on A and B for f to be onto.

3) Find a necessary and sufficient condition on A and B for f to be a bijection.

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 5.

Exercise 14. Chicken McNuggets are sold by boxes of 4, 6, 9 or 20 pieces. We say that $n \geq 1$ is a *McNugget* number if one can make an order of exactly n McNuggets.

Find all the positive integers which are not McNugget numbers.