Week 4: Functions: injectivity, surjectivity, bijectivity

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Give an example of a function:

a) which is one-to-one but not onto; b) which is onto but not one-to-one;

c) which is bijective; d) which is neither one-to-one, nor onto.

Exercise 2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ (recall that $\mathbb{R}^2$ denotes the set of all ordered couples (x, y) with $x \in \mathbb{R}$ and $y \in \mathbb{R}$) be the function defined by $F(x, y) = (x + y, x - y)$ for every $(x, y) \in \mathbb{R}^2$. Is F a bijection?

Exercise 3. Let I be an interval and $f : I \to \mathbb{R}$ be an increasing function. Show that f is one-to-one.

Exercise 4. Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ be two functions.

1) Show that if $g \circ f$ is one-to-one, then f is one-to-one.

2) Show that if $g \circ f$ is onto, then g is onto.

Definition.

Let $f : X \to Y$ be a bijection. Recall that we define the function $f^{-1} : Y \to X$, called the inverse (bijection) of f, as follows. Fix $y \in Y$. Let x be the unique pre-image of y by f, and set $f^{-1}(y) = x$.

Exercise 5. Let $f : X \to Y$ be a bijection. Show that:

a) $\forall x \in X$, $f^{-1} \circ f(x) = x$

b) $\forall y \in Y$, $f \circ f^{-1}(y) = y$.

Exercise 6. Let $f : [0, +\infty) \to \mathbb{R}_+$ be the function defined by $f(x) = (\sqrt{x^2 + 1} + 2)^2$ for every $x \geq 0$. Show that f is a bijection between $[0, +\infty)$ and $[9, +\infty)$, and give a simple expression of its inverse bijection.

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on October, 21th.

Exercise 7. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $f(x) = x^2 + 4x + 4$ for every $x \geq 0$.

a) Prove that f is bijection between $[0, +\infty)$ and $[4, +\infty)$.

b) Give a simple expression of its inverse.

Exercise 8. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Show that if $g \circ f$ is one-to-one and f is onto, then g is one-to-one. Is the converse always true? Justify your answer.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 4.

Exercise 9. Let $n \geq 2$ and $k \geq 2$ be integers.

a) How many functions $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$ can one define?

b) How many functions $f : \{1,2,\ldots,n\} \to \{1,2\}$ are onto?

c) How many functions $f : \{1,2,3\} \to \{1,2,\ldots,k\}$ are one-to-one?

Exercise 10. Set $\mathbb{N} = \{1,2,\ldots\}$ and recall that $\mathbb{N}^2$ denotes the set of all ordered couples (x,y) with $x \in \mathbb{N}$ and $y \in \mathbb{N}$. Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be defined by $\phi(u,v) = 2^{u-1} \times (2v-1)$ for $u,v \in \mathbb{N}$. Prove that ϕ is a bijection.

(Hint: You can use the fact that any integer can be represented in exactly one way, up to the order of the factors, as a product of prime powers.)

Exercise 11. Recall that for a given point $M = (a,b)$ in the plane, the coordinates of the symmetric point to M with respect to the straight line with equation $y = x$ are (b,a).

Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that the two graphical representations of f and f^{-1} in the plane are symmetric with respect to the straight line $\{y = x\}$.

Exercise 12. If A and B are two sets, we denote by A^B the set of all functions from B to A.

a) Let E,F,G be sets with $E \neq \emptyset$, and $f : F \to G$ a function. Show that f is one-to-one if and only if

$$\forall g,h \in F^E, \quad f \circ g = f \circ h \implies g = h.$$

b) Let F,G,H be sets such that H has at least two different elements, and $f : F \to G$ a function. Show that f is onto if and only if

$$\forall g,h \in H^G, \quad g \circ f = h \circ f \implies g = h.$$

4 Fun exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 4.

Exercise 13. Does there exist a bijection between $(0,1)$ and $[0,1]$?

Exercise 14. It is clearly possible to cover an 8×8 chessboard with 32 dominos of size 2×1 (see the left picture below). Is it possible cover the chessboard on the right (in which two diagonally opposite corners have been removed) with 31 dominos?