Week 14: Independence and conditional probabilities

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. A fair dice is thrown. For $1 \le i \le 6$, let A_i be the event "the result is at least equal to i". Compute the following probabilities:

a) $\mathbb{P}(A_4|A_2)$ b) $\mathbb{P}(A_2|A_4)$ c) $\mathbb{P}(A_4|\text{the result is odd})$.

Solution of exercise 1.

First step: modelisation. We take $\Omega = \{1,2,3,4,5,6\}$ and $\mathbb{P}$ the uniform probability on Ω.

Second step: computations. a)

$$\mathbb{P}(A_4|A_2) = \frac{\mathbb{P}(A_4 \cap A_2)}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\{4,5,6\})}{\mathbb{P}(\{2,3,4,5,6\})} = \frac{3/6}{5/6} = \frac{3}{5}.$$

b)

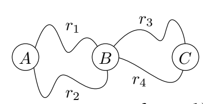
$$\mathbb{P}(A_2|A_4) = \frac{\mathbb{P}(A_2 \cap A_4)}{\mathbb{P}(A_4)} = \frac{\mathbb{P}(\{4,5,6\})}{\mathbb{P}(\{4,5,6\})} = \frac{3/6}{3/6} = 1.$$

c)

$$\mathbb{P}(A_4|\text{the result is odd}) = \frac{\mathbb{P}(A_4 \cap \text{the result is odd})}{\mathbb{P}(\text{the result is odd})} = \frac{\mathbb{P}(\{5\})}{\mathbb{P}(\{1,3,5\})} = \frac{1/6}{3/6} = \frac{1}{3}.$$

□

Exercise 2. Here is a network of four roads joining cities A, B, C:



Each of the four roads is blocked by snow with probability p (where $0 \le p \le 1$ is fixed), all independently. Let $X \leftrightarrow Y$ denote the event "there is an open route from X to Y", and denote by $X \not\leftrightarrow Y$ the complementary event. Compute a) $\mathbb{P}(A \leftrightarrow B)$ b) $\mathbb{P}(A \leftrightarrow C)$ c) $\mathbb{P}(A \not\leftrightarrow B \mid A \not\leftrightarrow C)$.

Remark. In such exercises, one starts by assuming the existence of a probability space in which the required conditions are satisfied (without giving the probability space explicitly). For example, here we assume the existence of a probability space $(\Omega, \mathbb{P})$ and for $1 \le i \le 4$, events C_i which are "the road r_i is closed", such that the events C_1, C_2, C_3, C_4 are independent, and $\mathbb{P}(C_i) = p$ for every $1 \le i \le 4$.

Solution of exercise 2.

a) We have

$$\begin{aligned}\mathbb{P}(A \leftrightarrow B) &= 1 - \mathbb{P}(A \not\leftrightarrow B)\\ &= 1 - \mathbb{P}(C_1 \cap C_2)\\ &= 1 - \mathbb{P}(C_1) \times \mathbb{P}(C_2) \qquad \text{(independence of } C_1, C_2\text{)}\\ &= 1 - p^2.\end{aligned}$$

b) First note that the events $A \leftrightarrow C$ and $A \leftrightarrow B \cap B \leftrightarrow C$ are equal and that the events $A \leftrightarrow B$, $B \leftrightarrow C$ are independent since they involve independent roads. (Ici on utilise quelque chose qu'on n'a pas vu en cours, qui est une sorte de principe des "coalitions"/principe des "paquets' 'pour les événements de type: si A, B, C, D sont indépendants, alors des événements de "type" $f(A, B)$ et $g(C, D)$ sont indépendants. Si besoin n'hésitez pas à préciser cela et expliquer pourquoi intuitivement c'est vrai, et dire qu'ils verront plus tard un résultat plus général) They also have the same probability. Thus we find

$$\begin{aligned} \mathbb{P}(A \leftrightarrow C) &= \mathbb{P}(A \leftrightarrow B \cap B \leftrightarrow C) \\ &= \mathbb{P}(A \leftrightarrow B) \times \mathbb{P}(B \leftrightarrow C) \\ &= (\mathbb{P}(A \leftrightarrow B))^2 \\ &= (1-p^2)^2. \end{aligned}$$

c) In the previous questions we have found

$$\mathbb{P}(A \not\leftrightarrow B) = p^2, \qquad \mathbb{P}(A \not\leftrightarrow C) = 1-(1-p^2)^2.$$

Now, using the fact that the events $A \not\leftrightarrow B \ \cap \ A \not\leftrightarrow C$ and $A \not\leftrightarrow B$ are the same,

$$\mathbb{P}(A \not\leftrightarrow B \mid A \not\leftrightarrow C) = \frac{\mathbb{P}(A \not\leftrightarrow B \ \cap \ A \not\leftrightarrow C)}{\mathbb{P}(A \not\leftrightarrow C)} = \frac{\mathbb{P}(A \not\leftrightarrow B)}{\mathbb{P}(A \not\leftrightarrow C)} = \frac{p^2}{1-(1-p^2)^2}.$$

□

Exercise 3. **(Law of total probability)** Let $B_1, \ldots, B_n$ be events of a finite probability space $(\Omega, \mathbb{P})$ such that $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ and $B_i \cap B_j = \varnothing$ for $i \neq j$ (we say that $B_1, \ldots, B_n$ is a partition of Ω). Assume that $\mathbb{P}(B_i) > 0$ for every $1 \leq i \leq n$ and let A be an event.

1) Show that $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i)$.

2) Show that $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$.

Remark. The law of total probability is the probabilistic version of the "sum rule", and therefore plays an equally important role.

3) (*Application*) Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

NB. As in Exercise 2, you may assume the existence of a probability space satisfying the required assumptions.

Solution of exercise 3.

1) The events $A \cap B_1, \ldots, A \cap B_n$ are pairwise disjoint (indeed, if $i \neq j$, $(A \cap B_i) \cap (A \cap B_j) = A \cap B_i \cap B_j = A \cap \varnothing = \varnothing$. Therefore, as seen in the course and in the previous exercise sheet,

$$\sum_{i=1}^{n} \mathbb{P}(A \cap B_i) = \mathbb{P}\left(\bigcup_{i=1}^{n} (A \cap B_i) \right).$$

By Morgan's law:

$$\bigcup_{i=1}^{n}(A\cap B_i) = A\cap\left(\bigcup_{i=1}^{n} B_i\right) = A\cap\Omega = A.$$

The desired result follows.

2) This readily follows from 1) by using the fact that $\mathbb{P}(A\cap B_i) = \mathbb{P}(A|B_i)\mathbb{P}(B_i)$ for every $1\le i\le n$.

3) *Modelisation.* We work on a probability space with events A, B_X, B_Y defined as follows: A is the event "the purchased bulb works for longer than 5000 hours", B_X is the event that the purchased bulb was manufactured by factory X and B_Y is the event that the purchased bulb was manufactured by factory Y. By assumption,

$$\mathbb{P}(B_X) = \frac{6}{100}, \quad \mathbb{P}(B_Y) = \frac{4}{10}, \quad \mathbb{P}(A|B_X) = \frac{99}{100}, \quad \mathbb{P}(A|B_Y) = \frac{95}{100}.$$

Computation. By the law of total probability (with $n=2$):

$$\mathbb{P}(A) = \mathbb{P}(A \mid B_X)\cdot\mathbb{P}(B_X) + \mathbb{P}(A \mid B_Y)\cdot\mathbb{P}(B_Y) \tag{1}$$

$$= \frac{99}{100}\cdot\frac{6}{10} + \frac{95}{100}\cdot\frac{4}{10} = \frac{594+380}{1000} = \frac{974}{1000} \tag{2}$$

□

Exercise 4. **(Particular case of Bayes' formula)** Let A and B be events of a finite probability space such that $0 < \mathbb{P}(A) < 1$ and $\mathbb{P}(B) > 0$.

1) Show that $\mathbb{P}(A|B) = \mathbb{P}(B|A)\frac{\mathbb{P}(A)}{\mathbb{P}(B)}$.

2) Show that $\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}\left(B|\overline{A}\right)\mathbb{P}\left(\overline{A}\right)$.

3) Show the following formula (which is a particular case of Bayes' formula):

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}\left(B|\overline{A}\right)\mathbb{P}\left(\overline{A}\right)}.$$

4) (*Application*) An urn has two dices: one is fair, while the other always gives 6 (but you cannot tell the difference just by looking at them). Take one dice at random and throw it. If you get 6, what is the probability that the dice is fair?

Remark. Roughly speaking, Bayes' formula is useful for "retroactive" reasoning: if we can measure the consequence of A on B, then we can measure the consequence of B on A.

Solution of exercise 4.

1) Write

$$\mathbb{P}(B|A)\frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B\cap A)}{\mathbb{P}(A)}\frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B\cap A)}{\mathbb{P}(B)} = \mathbb{P}(A|B).$$

2) This is just question 2) of Exercise 3.

3) This readily follows by combining 1) and 2).

4) Let A be the event "the chosen dice is fair" and B the event "we get 6". Then $\mathbb{P}(B|A) = \frac{1}{6}$,

$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}\left(B|\overline{A}\right) = 1$, $\mathbb{P}\left(\overline{A}\right) = \frac{1}{2}$. Using the formula of question 3), we get

$$\mathbb{P}(A|B) = \frac{\frac{1}{6}\cdot\frac{1}{2}}{\frac{1}{6}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{1}{1+6} = \frac{1}{7}.$$

□

2 Homework exercises

There are no homework exercises this time ☹.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 14.

Exercise 5. (**Compound probability theorem**) Let $A_1, \ldots, A_n$ be events of a finite probability space such that $\mathbb{P}(A_1 \cap \cdots \cap A_n) > 0$.

1) Show that for every $1 \le i \le n$, $\mathbb{P}(A_1 \cap \cdots \cap A_i) > 0$.

2) Show that $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)\cdots\mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$.

Remark. This identity is called the compound probability theorem and is for instance useful in situations where the past has an influence on the future (and is in some sense the probabilistic version of the "multiplicative rule").

3) (*Application*) Consider an urn with 6 identical blue balls and 4 identical red balls. Take one after the other 3 balls at random. What is the probability that the 3 balls are red ?

Hint. You may introduce the event A_i: "the i-the ball is red".

Solution of exercise 5.

1) Fix $1 \le i \le n$. Since $A_1 \cap \cdots \cap A_n \subset A_1 \cap \cdots \cap A_i$, we have $0 < \mathbb{P}(A_1 \cap \cdots \cap A_n) < \mathbb{P}(A_1 \cap \cdots \cap A_i)$.

2) Simply write, by definition of the conditional expectation:

$$\begin{aligned}
&\mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)\cdots\mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})\\
&\quad = \mathbb{P}(A_1)\frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_{1)}}\frac{\mathbb{P}(A_1 \cap A_2 \cap A_3)}{\mathbb{P}(A_1 \cap A_2)}\cdots\frac{\mathbb{P}(A_1 \cap \cdots \cap A_n)}{\mathbb{P}(A_1 \cap \cdots \cap A_{n-1})}\\
&\quad = \mathbb{P}(A_1 \cap \cdots \cap A_n)
\end{aligned}$$

by noticing a telescoping product.

3) We are interested in the probability of the event $A_1 \cap A_2 \cap A_3$. We have $\mathbb{P}(A_1) = \frac{4}{10}$, $\mathbb{P}(A_2|A_1) = \frac{3}{9} = \frac{1}{3}$ (since once a red ball has been taken, there remains 6 blue balls and 3 red balls) and $\mathbb{P}(A_3|A_1 \cap A_2) = \frac{2}{8} = \frac{1}{4}$ (since once two red balls have been taken, there remains 6 blue balls and 2 red balls). Therefore, by the compound probability theorem,

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\cdot\mathbb{P}(A_2|A_1)\cdot\mathbb{P}(A_3|A_1 \cap A_2) = \frac{2}{5}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{30}.$$

□

Exercise 6. Fix $\alpha \in (0,1)$. A particle switches randomly between two locations a, b in the following way: at each time step, the particle changes its location with probability α and stays still with probability $1-\alpha$ (independently from the past). We assume that at time $n = 0$, the particle is at a.

For $n \geq 0$ we define the event $A_n = \{\text{at time } n, \text{the particle is at position } a\}$, and we put $p_n = \mathbb{P}(A_n)$. Observe that by assumption we have $p_0 = \mathbb{P}(A_0) = 1$.

1) Prove that for $n \geq 0$ we have $p_{n+1} = (1-\alpha)p_n + \alpha(1-p_n)$. *(Hint: you may use exercise 3 and notice that there are two cases according to the position of the particle at time n.)*

2) Prove by induction that for every $n \geq 0$, we have $p_n = \frac{1}{2}(1-2\alpha)^n + \frac{1}{2}$. What happens when $n \to +\infty$? What does this mean?

Solution of exercise 6.

1. Write
$$A_{n+1} = (A_n \cap \{\text{ the particle stays still}\}) \bigcup \left(\overline{A_n} \cap \{\text{ the particle switches}\}\right),$$
and observe that these two events are disjoint. Therefore
$$\begin{aligned}\mathbb{P}(A_{n+1}) &= \mathbb{P}(A_n \cap \{\text{ the particle stays still}\}) + \mathbb{P}\left(\overline{A_n} \cap \{\text{ the particle switches}\}\right)\\ &= \mathbb{P}(\text{the particle stays still}|A_n) \cdot \mathbb{P}(A_n) + \mathbb{P}\left(\text{the particle switches}|\overline{A_n}\right) \cdot \mathbb{P}\left(\overline{A_n}\right).\end{aligned}$$
(note that this formula is a particular case of Exercise 4) 2)) Thus
$$p_{n+1} = (1-\alpha) \times p_n + \alpha(1-p_n).$$

2. For $n = 0$ the formula is correct:
$$\tfrac{1}{2}(1-2\alpha)^0 + \tfrac{1}{2} = \tfrac{1}{2} \times 1 + \tfrac{1}{2} = 1 = p_0.$$
Assume that for a fixed $n \geq 0$, $p_n = \frac{1}{2}(1-2\alpha)^n + \frac{1}{2}$. Then
$$\begin{aligned}p_{n+1} &= (1-\alpha) \times p_n + \alpha(1-p_n)\\ &= p_n(1-2\alpha) + \alpha\\ &= \left(\tfrac{1}{2}(1-2\alpha)^n + \tfrac{1}{2}\right)(1-2\alpha) + \alpha\\ &= \tfrac{1}{2}(1-2\alpha)^{n+1} + \tfrac{1}{2}(1-2\alpha) + \alpha\\ &= \tfrac{1}{2}(1-2\alpha)^{n+1} + \tfrac{1}{2} - \alpha + \alpha\\ &= \tfrac{1}{2}(1-2\alpha)^{n+1} + \tfrac{1}{2},\end{aligned}$$
and the formula is correct at step $n+1$.

3. We have $|1-2\alpha| < 1$, so $\lim_{n\to+\infty}(1-2\alpha)^n = 0$. This yields $\lim_{n\to+\infty} p_n = \frac{1}{2}$. If we wait long enough the particle is roughly equally likely to be at a or b, no matter the value of α!

□

Exercise 7. Let A, B, C be independent events. Show that the two events $A \cup B$ and C are independent.

Solution of exercise 7. We have to check that
$$\mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A \cup B) \times \mathbb{P}(C). \qquad (\star)$$

Step 1. We compute the left-hand side of (⋆). The goal is to transform unions into intersections so that we can use the hypothesis of independence. Using that (Morgan law)

$$(A \cup B) \cap C \quad = \quad (A \cap C) \cup (B \cap C)$$

we obtain that

$$\begin{aligned} \mathbb{P}((A \cup B) \cap C) &= \mathbb{P}((A \cap C) \cup (B \cap C)) \\ &= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}((A \cap C) \cap (B \cap C)) \\ &= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C), \end{aligned}$$

where we used at list line that A, B, C ar independent.
Step 2. We compute the right-hand side of (⋆).

$$\begin{aligned} \mathbb{P}(A \cup B) \times \mathbb{P}(C) &= (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) \times \mathbb{P}(C) \\ &= \mathbb{P}(A)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C), \end{aligned}$$

which is equal to the left-hand side of (⋆). □

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 14.

Exercise 8. **(The duelling idiots)**
(Taken from: *Duelling idiots and other probability puzzlers* P.J.Nahin, Princeton Univ.Press (2000))
A and B decide to duel but they have just one gun (a six shot revolver) and only one bullet. Being dumb, this does not deter them and they agree to "duel" as follows: They will insert the lone bullet into the gun's cylinder, A will then spin the cylinder and shoot at B (who, standing inches away, is impossible to miss). If the gun doesn't fire then A will give the gun to B, who will spin the cylinder and then shoot at A. This back and forth duel will continue until one fool shoots the other.
What is the probability that A will win?

Solution of exercise 8. We condition on the first try:

$$\begin{aligned} \mathbb{P}(A \text{ wins}) &= \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ succeeds}) + \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ fails}) \\ &= \mathbb{P}(\text{ 1st try succeeds}) + \mathbb{P}(A \text{ wins} \cap \text{ 1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails}) \\ &= 1/6 + \mathbb{P}(A \text{ wins} \mid \text{1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails})) \times \mathbb{P}(\text{ 1st try of } A \text{ fails} \cap \text{ 1st try of } B \text{ fails}) \\ &= 1/6 + \mathbb{P}(A \text{ wins}) \times 5/6 \times 5/6, \end{aligned}$$

since after two misses the whole process starts again. Finally we solve this last equation and find

$$\mathbb{P}(A \text{ wins}) = 6/11 = 0.545454\ldots$$

□