Week 10: Permutations

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 8 & 7 & 6 & 2 & 1 \end{pmatrix}$

(1) Compute σ^{-1}. (2) Write σ as a product of cycles with disjoint supports.
(3) Write σ^2 in table notation and in cycle notation. (4) Compute σ^{2019}.

Definition. Let $\sigma \in S_n$ be a permutation. For $1 \leq i, j \leq n$, we say that (i, j) is an inversion if $i < j$ and $\sigma(i) > \sigma(j)$. We denote by $I(\sigma)$ the total number of inversions of σ, and we define $\epsilon(\sigma) = (-1)^{I(\sigma)}$ to be the *signature* of σ. If $\epsilon(\sigma) = 1$, we say that σ is an *even* permutation, and if $\epsilon(\sigma) = -1$, we say that σ is an *odd* permutation.

Theorem. If $\pi, \sigma \in S_n$ are two permutations, we have $\epsilon(\pi\sigma) = \epsilon(\pi)\epsilon(\sigma)$.
We say that ϵ is *multiplicative*. You should remember this result. See the course webpage for the (optional) proof, based on the identity $\epsilon(\pi) = \prod_{1 \leq i < j \leq n} \frac{\pi(j)-\pi(i)}{j-i}$.

Exercise 2. If σ is the permutation of exercise 1, find $I(\sigma)$ and $\epsilon(\sigma)$.

The following exercise gives a way to compute the signature of a permutation by using its cycle decomposition.

Exercise 3. Fix an integer $n \geq 1$.

1. Let $\tau_{i,j}$ be the transposition which exchanges i and j (with $1 \leq i < j \leq n$). Show that $\varepsilon(\tau_{i,j}) = -1$.
2. If $\sigma = \tau_1 \tau_2 \cdots \tau_N$ is a product of transpositions, show that $\varepsilon(\sigma) = (-1)^N$ (**you should remember this result**).
3. Let $\tau = (x_1, \ldots, x_p)$ be a p-cycle (with $p \geq 2$).
   a) Show that $\tau = (x_1, x_p) \circ (x_1, x_{p-1}) \circ \cdots (x_1, x_3) \circ (x_1, x_2)$.
   b) Show that the signature of a p-cycle is $(-1)^{p-1}$ (**you should remember this result**).
4. Use the previous questions to find the signature of the permutation σ defined in exercise 1.

Exercise 4. Fix an integer $n \geq 3$.

1. Find two permutations $\sigma, \tau \in \mathfrak{S}_n$ such that $\sigma \circ \tau = \tau \circ \sigma$.
2. Find two permutations $\sigma, \tau \in \mathfrak{S}_n$ such that $\sigma \circ \tau \neq \tau \circ \sigma$.

2 Homework exercise

You have to individually hand in the written solution of the next exercise to your TA on December, 9th.

Exercise 5. Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 7 & 4 & 6 & 2 \end{pmatrix}$.

(1) Write σ^{2019} as a product of cycles with disjoint support. (2) What is the signature of σ^{2019}?
Justify your answers (you can use without proof the results of the previous exercises)

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 10.

Exercise 6. 1) List all the elements of $\mathfrak{S}_4$ which are 3-cycles.

2) Fix two integers $2 \leq k \leq n$. How many different k-cycles of $\mathfrak{S}_n$ are there? Justify your answer.

Exercise 7. Fix an integer $n \geq 1$ and a permutation $\pi \in \mathfrak{S}_n$. Show that the function $f : \mathfrak{S}_n \to \mathfrak{S}_n$ defined by $f(\sigma) = \sigma \circ \pi$ for every $\sigma \in \mathfrak{S}_n$ is a bijection.

Exercise 8. Fix an integer $n \geq 2$, a permutation $\sigma \in \mathfrak{S}_n$ and consider a p-cycle $c = (a_1, a_2, \ldots, a_p)$. Show that $\sigma \circ c \circ \sigma^{-1} = \big(\sigma(a_1), \ldots, \sigma(a_p)\big)$.

Exercise 9. Let $n \geq 3$ be an integer. For $1 \leq i \neq j \leq n$, denote by $\tau_{i,j}$ the transposition exchanging i and j.

1. Show that for every $2 \leq i < j \leq n$, $\tau_{i,j} = \tau_{1,i} \circ \tau_{1,j} \circ \tau_{1,i}$. Deduce that every permutation of $\mathfrak{S}_n$ can be written as a product of transpositions of the form $\tau_{1,i}$ with $2 \leq i \leq n$.
2. Show that for every $2 \leq i < j \leq n$, $(1, i, j) = \tau_{1,j} \circ \tau_{1,i}$. Deduce that every even permutation can be written as a product of cycles of the form $\tau_{1,i,j}$ with $2 \leq i \neq j \leq n$.

Exercise 10. Fix an integer $n \geq 2$ and consider the circular permutation $c = (1, 2, \ldots, n-1, n)$. Find all the permutations $\sigma \in \mathfrak{S}_n$ that commute with c (that is $\sigma \circ c = c \circ \sigma$).

Exercise 11. Fix integers $1 \leq k \leq n$. If $\sigma \in \mathfrak{S}_n$ is a permutation and $1 \leq k \leq n$, we say that k is a record of σ if for every $1 \leq i \leq k-1$ we have $\sigma(i) < \sigma(k)$. Show that the number of permutations of $\mathfrak{S}_n$ having k cycles in their cycle decomposition is equal to the number of permutations of $\mathfrak{S}_n$ having k records.

Hint. Try to find a bijection!

Exercise 12. Let n be an odd integer and $\sigma \in \mathfrak{S}_n$. Show that 4 divises $\prod_{i=1}^{n} (\sigma(i)^2 - i^2)$.

Exercise 13. Fix an integer $n \geq 1$ and consider the sub-vector space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0\}$ of $\mathbb{R}^n$. Denote by $(e_i)_{1 \leq i \leq n}$ the canonical basis. For every $\sigma \in \mathfrak{S}_n$, let $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the endomorphism such that $f_\sigma(e_i) = e_{\sigma(i)}$ for every $1 \leq i \leq n$. Set $p = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f_\sigma$.

1. Show that p is a projector, that is $p \circ p = p$ (*hint:* You may use the result of exercise 7).
2. Find $\mathfrak{Im}\, p$ and $\ker p$.

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 10.

Exercise 14. The names of 100 mathematicians are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the mathematicians are led into the room; each may look in at most 50 boxes, but must leave the room exactly as she found it and is permitted no further communication with the others. The mathematicians have a chance to plot their strategy in advance, and they are going to need it, because unless every single mathematician finds her own name all will subsequently lose their funding. Find a strategy for them which has probability of success (mathematics survive) exceeding 30%.

Remark. If each mathematician examines a random set of 50 boxes, their probability of success is $\frac{1}{2^{100}}$ (each mathematician that opens 50 boxes at random among 100 has a probability $\frac{1}{2}$ to find her name), which is very very small.