Week 9 (Midterm exam): Tuesday, December 4th, 8am-10am

Very important:

- Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).
- Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1.

1) Show that if $f : X \to Y$ and $g : Y \to Z$ are two one-to-one functions, then $g \circ f$ is one-to-one.

2) Show that if $f : X \to Y$ and $g : Y \to Z$ are two onto functions, then $g \circ f$ is onto.

Solution of exercise 1.

1) Let $x, y \in X$ such that $g(f(x)) = g(f(y))$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$. Hence $g \circ f$ is one-to-one.

2) Let $z \in Z$. Since g is onto, there exists $y \in Y$ such that $g(y) = z$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Hence $g(f(x)) = z$, so that z has a pre-image by $g \circ f$. This shows that $g \circ f$ is onto. □

Part 2

Exercise 2. Let A, B, C be three sets. Show that $A \cap C = A \cup B \iff B \subset A$ and $A \subset C$.

Solution of exercise 2.

We show the double implication.

$\boxed{\Longrightarrow}$ Assume that $A \cap C = A \cup B$. Take $b \in B$. Then $b \in A \cup B$, so that $b \in A \cap C$, so $b \in A$. Hence $B \subset A$. Take $a \in A$. Then $a \in A \cup B$, so that $a \in A \cap C$, so $a \in C$. Hence $A \subset C$.

$\boxed{\Longleftarrow}$ Assume that $B \subset A$ and $A \subset C$. We show that $A \cap C = A \cup B$ by double inclusion. Since $A \cap C \subset A$, clearly $A \cap C \subset A \cup B$. Next, if $x \in A \cup B$:

First case. $x \in A$. Then $x \in C$ (since $A \subset C$), so that $x \in A \cap C$.

Second case. $x \in B$. Then $x \in A$ (since $B \subset A$) and $x \in C$ (since $A \subset C$). Hence $x \in A \cap C$.

Therefore $A \cup B \subset A \cap C$. □

Exercise 3. Let E be a set and $f : E \to E$ a function such that $f(f(x)) = f(x)$ for every $x \in E$.

1) Show that f is one-to-one if and only if f is onto.

2) Give an example of a set E and of a function $f : E \to E$ such that $f(f(x)) = f(x)$ for every $x \in E$ and such that f is a bijection (and explain why it is a bijection).

3) Give an example of a set E and of a function $f : E \to E$ such that $f(f(x)) = f(x)$ for every $x \in E$ and such that f is not a bijection (and explain why it is not a bijection).

Solution of exercise 3.

1) We argue by double implication.

$\boxed{\Longrightarrow}$ Assume that f is one-to-one. Fix $x \in E$. Then $f(x) = x$. Hence f is onto, since x has a pre-image by f.

$\boxed{\Longleftarrow}$ Assume that f is onto. We claim that $f(y) = y$ for every $y \in E$. To show this, fix $y \in E$. Since f is onto, there exists $x \in E$ such that $y = f(x)$. Since $f(f(x)) = f(x)$, we get that $f(y) = y$. Therefore f is the identity function, so that f is one-to-one.

2) Take $E = \{1\}$ and $f : E \to E$ such that $f(1) = 1$, which is clearly a bijection.

3) Take $E = \{1,2\}$ and $f : E \to E$ defined by $f(1) = 1$ and $f(2) = 1$ (indeed $f(f(x)) = f(x) = 1$ for every $x \in E$), which is not a bijection since it ie not one-to-one. □

Part 3

Exercise 4. Show that for every integer $n \geq 1$, $\displaystyle\sum_{k=1}^{n} k \times k! = (n+1)! - 1$.

Solution of exercise 4. We argue by induction on n. For $n \geq 1$, let $P(n)$ be the property "$\displaystyle\sum_{k=1}^{n} k \times k! = (n+1)! - 1$".

Basis step. For $n = 1$, we indeed have $1 = 2! - 1$.

Induction step. Let $n \geq 1$ be an integer such that $P(n)$ is true. Then, by the induction hypothesis:

$$\sum_{k=1}^{n+1} k \times k! = (n+1)! - 1 + (n+1) \times (n+1)! = (n+1)! \times (1 + n + 1) - 1 = (n+2)! - 1.$$

Hence $P(n+1)$ is true.

Therefore $P(n)$ is true for every $n \geq 1$. This completes the proof. □

Exercise 5. Professor B. goes to a shop to buy 3 different books about kittens and 2 different books about capybaras. In the shop there are 10 different books about kittens and 20 different books about capybaras.

Note: *in the following questions, please give numbers as answers (without binomial coefficients nor factorials) and justify your answers ("informal" proofs are allowed and recommended).*

1) In how many ways can Professor B. buy his books ?

2) Once home, Professor B. forms a (vertical) pile of his books. How many different piles can he make?

3) Same question as 2), if the books about kittens have to be at the top of the pile and the books about capybaras at the bottom of the pile.

4) Same question as 2), if two books about capybaras cannot be one on the other.

Solution of exercise 5.

1) There are $\binom{10}{3}$ choices for the kitten books and then $\binom{20}{2}$ choices for the capybara books. By the product rule, the total number of ways is therefore:

$$\binom{10}{3} \times \binom{20}{2} = \frac{8 \times 9 \times 10 \times 19 \times 20}{2 \times 3 \times 2} = 2 \times 3 \times 10 \times 20 \times 19 = 22800.$$

2) Since there are 5 books, there are $5! = 120$ ways of ordering them by the product rule.

3) There are $3!$ ways of ordering the books on kittens and $2!$ ways of ordering the books about capybaras, so the total number is $6 \times 2 = 12$ ways by the product rule.

4) We use the complement rule and count instead the number of ways to make a pile such that the two books about capybaras are one after the other. We treat the two books as one big book. Then there are $4!$ ways of arranging the 4 books, and then 2 ways of ordering the two books about capybaras within the "big book"n which gives $4! \times 2 = 48$ ways. By the complement rule, the number of piles such that two books about capybaras cannot be one on the other is

$$120 - 48 = 72.$$

□

Part 4

Exercise 6. This exercise studies a simplified model of a protein with 3 amino-acids. Let $n \geq 1$ be an integer. Let S_n be the set of all words with n letters formed by using the letters C, D, E and such that the following two conditions hold:

(∗) the words start with the letter C

(∗) there is never two times the same letter one after the other.

For example $S_3 = \{CDC, CDE, CED, CEC\}$.

Let C_n be the subset of S_n made of words finishing with the letter C, let D_n be the subset of S_n made of words finishing with the letter D and let E_n be the subset of S_n made of words finishing with the letter E. Finally set $c_n = \#C_n$, $d_n = \#D_n$, $e_n = \#E_n$.

1) Compute $c_1, d_1, e_1, c_2, d_2, e_2, c_3, d_3, e_3$.

2) What is the value of $\#S_n$? Justify your answer.

3) Show that $c_{n+1} = d_n + e_n$, $d_{n+1} = c_n + e_n$, $e_{n+1} = c_n + d_n$ for $n \geq 1$.

4) Show that $c_{n+2} = c_{n+1} + 2c_n$ for $n \geq 1$.

5) Show that $c_n = \frac{2^n}{6} + \frac{2}{3}(-1)^{n+1}$ for every $n \geq 1$ and find a similar simple formula for d_n and e_n.

Remark. If at some point you do not manage to solve a question, you can write on your sheet of paper that you assume that it is true and use it to solve a next question if needed.

Solution of exercise 6.

1) Since $S_1 = \{C\}$, $S_2 = \{CE, CD\}$ and $S_3 = \{CDC, CDE, CED, CEC\}$, we have $c_1 = 1$, $d_1 = e_1 = 0$; $c_2 = 0$, $d_2 = 1$, $e_2 = 1$, $c_3 = 2$, $d_3 = 1$ and $e_3 = 1$.

2) We have $\#S_n = 2^{n-1}$. Indeed, the first letter is fixed, and for the next letter we have each time two choices since we cannot repeat the same letter. By the multiplicative rule, we get the result.

3) These identities come from removing the last letter: a word with length $n+1$ finishing with C gives a word with length n finishing with D or E, which implies $c_{n+1} = d_n + e_n$ (formally, the function $C_{n+1} \to D_n \cup E_n$ defined by forgetting the last letter is a bijection). The other two equalities follow from the same reason.

4) By the previous question, $c_{n+2} = d_{n+1} + e_{n+1} = 2c_n + d_n + e_n = 2c_n + c_{n+1}$.

5) One checks this result by strong induction. Let $P(n)$ be the proposition "$c_n = \frac{2^n}{6} + \frac{2}{3}(-1)^{n+1}$".

Basis step. For $n = 1$ and $n = 2$ we indeed have $c_1 = 1$ and $c_2 = 0$.

Inductive step. Assume that the result holds up to an integer $n \geq 2$. Then write

$$c_{n+1} = c_n + 2c_{n-1} = \frac{2^n}{6} + \frac{2}{3}(-1)^{n+1} + \frac{2^n}{6} + \frac{4}{3}(-1)^n = \frac{2^{n+1}}{6} + (-1)^n\left(\frac{4}{3} - \frac{2}{3}\right) = \frac{2^{n+1}}{6} - \frac{2}{3}(-1)^{n+2}.$$

This shows that $P(n+1)$ is true. Hence $P(n)$ is true for every $n \geq 0$.

Since $d_n = e_n$ by symmetry (formally, the map which exchanges D's and E's is a bijection between D_n and E_n) and since $c_n + d_n + e_n = 2^{n-1}$, we get that

$$d_n = e_n = \frac{1}{2}\left(2^{n-1} - \frac{2^n}{6} + \frac{2}{3}(-1)^n\right) = \frac{2^n}{6} + \frac{1}{3}(-1)^n.$$

for $n \geq 1$. □

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Recall that a partition of an integer $n \geq 1$ into k parts is a sequence $(x_1, \ldots, x_k)$ of k positive integers such that $x_1 \geq x_2 \geq \cdots \geq x_k$ and $x_1 + x_2 + \cdots + x_k = n$. Show that the number of partitions of n into at most k parts is equal to the number of partitions of $n+k$ into exactly k parts.

Solution of exercise 7. If $(x_1, \ldots, x_m)$ is a partition of n in at most k parts (so that $m \leq k$), the function that sends $(x_1, \ldots, x_m)$ to $(x_1+1, \ldots, x_{m+1}, 1, 1, \ldots, 1)$ (so that there are k parts in the end) is a bijection from the set of partitions of n into at most k parts to the partitions of $n+k$ into exactly k parts, and the desired result follows. □

Exercise 8. Let $n \geq 1$ be an integer. How many words of length n can be formed from the letters A,B,C if the letter A has to occur an even number of times?

Solution of exercise 8. Let a_n be this number. One may form such a word by first specifying the positions of the A's ($\binom{n}{k}$ choices with k even) and then choosing B or C on the remaining positions (2^{n-k} choices). This gives the following formula

$$a_n = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^{n-k}.$$

To compute this sum, we use the trick consisting in expanding $(2-1)^n$ and $(2+1)^n$ using the Binomial theorem:

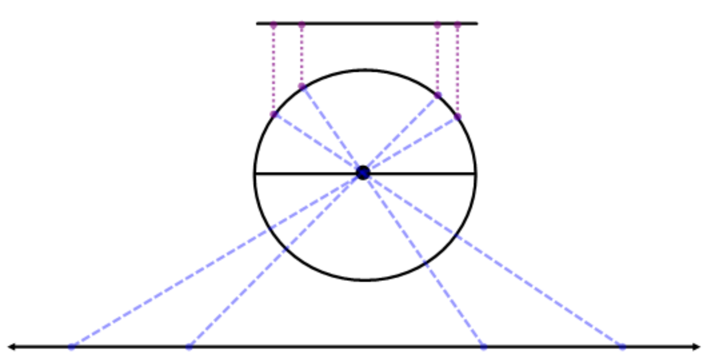
$$(2-1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(-1)^k, \qquad (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}.$$

Therefore

$$3^n + 1 = 2 \cdot \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^{n-k}.$$

We conclude that $a_n = \frac{3^n+1}{2}$. □

Exercise 9. What does the following image represent?



Solution of exercise 9. This image shows that any open interval is in bijection with $\mathbb{R}$. $\square$