Week 9 (Midterm exam): Tuesday, November 28th, 8am-10am

Very important:

- Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).
- Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1.

1) Give an example of a function which is not onto and an example of a function which is not one-to-one.

2) Show that if $f : E \to F$ is a one-to-one function and A is a subset of E, then $A = f^{-1}(f(A))$.

Solution of exercise 1.

1. For example $f : \{0,1\} \to \{0,1\}$ defined by $f(0) = f(1) = 0$ is neither onto (because 1 does not have a pre-image) neither one-to-one (because $0 \neq 1$ and $f(0) = f(1)$).

2. We show the double inclusion.

   First take $x \in A$. To show that $x \in f^{-1}(f(A))$, we need to show that $f(x) \in f(A)$. Since $x \in A$, we indeed have $f(x) \in f(A)$.

   Now take $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Therefore there exists $a \in A$ such that $f(x) = f(a)$. Since f is one-to-one, we have $x = a$. Therefore $x \in A$.

□

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$ for every $x \in \mathbb{R}$.

1) Give a simple expression of the set $f([-2,2])$. Justify your answer.

2) Give a simple expression of the set $f^{-1}([-1,2])$. Justify your answer.

Solution of exercise 2.

1. We have $f([-2,2]) = [0,2]$. To justify this equality of sets, we argue by double inclusion.

   First, if $y \in f([-2,2])$, there exists $x \in [-2,2]$ such that $y = |x|$, so that $0 \leq y \leq 2$. Hence $f([-2,2]) \subset [0,2]$.

Second, if $y \in [0,2]$, we have $f(y) = y$, so that $y \in f([-2,2])$.

2. We have $f^{-1}([-1,2]) = [-2,2]$. To justify this equality of sets, we argue by double inclusion.

   First, if $x \in f^{-1}([-1,2])$, we have $-1 \leq |x| \leq 2$, so that $-2 \leq x \leq 2$. Hence $x \in [-2,2]$.

   Second, if $x \in [-2,2]$, $x \in f^{-1}(|x|) \in f^{-1}([-1,2])$ because $0 \leq |x| \leq 2$.

Remark. It is also possible to justify these answers graphically from the graphical representation of f. □

Part 2

Exercise 3. Let P be the following mathematical assertion

P : "There exists a real number x such that for every real number y we have $x + y = y$".

1) Write P using quantifiers.
2) Write the negation of P using quantifiers.
3) Is P true? Justify your answer.

Solution of exercise 3. 1)

$$P : \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = y.$$

2)

$$\neg P : \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq y.$$

3) P is true : Indeed, $x = 0$ satisfies $\forall y \in \mathbb{R}, x + y = y$.

□

Exercise 4. Fix two integers $a, b \in \mathbb{Z}$. Show that

if $a^2(b^2 - b)$ is odd, then a, b are both odd.

Hint. You may use the method of proof by contrapositive.

Solution of exercise 4. In order to prove that $A \implies B$, where

- $A : a^2(b^2 - b)$ is odd.
- $B : a$ is odd and b is odd,

we argue by contrapositive and show that $\neg B \implies \neg A$, where

$$\neg B : a \text{ is even or } b \text{ is even}, \qquad \neg A : a^2(b^2 - b) \text{ is even}.$$

To this end, assume that $\neg B$ is true. We have three cases:

- **Case 1:** a **even,** b **even.** Then we can write $a = 2k$, $b = 2k'$ for some integers k, k'. Therefore

$$a^2(b^2 - b) = (2k)^2(b^2 - b) = 2 \times 2k^2(b^2 - b)$$

  which is even: $\neg A$ is true.

- **Case 2:** a **even,** b **odd.** This case is identical: we can write $a = 2k$ for some integer k, and

$$a^2(b^2 - b) = (2k)^2(b^2 - b) = 2 \times 2k^2(b^2 - b)$$

  which is even: $\neg A$ is true.

- **Case 3:** a **odd,** b **even.** Then we can write $a = 2k + 1, b = 2k'$ for some integers k, k'. We have

$$a^2(b^2 - b) = (2k+1)^2((2k)^2 - 2k) = (2k+1)^2(2k^2 - k) \times 2,$$

  which is even: $\neg A$ is true.

To conclude, we have proved $\neg B \implies \neg A$, *i.e.* $A \implies B$. □

Part 3

Exercise 5.

1) Show that for every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

2) How many integers $1 \leq a, b, c, d \leq 100$ such that $a < b$ and $a < c$ and $a < d$ are there?

Hint. If you do not manage to solve the first question, you can assume that the result of the first question is true in order to solve this second question.

Solution of exercise 5.

1) We argue by induction. For $n \geq 1$, let $P(n)$ by the property

$$P(n) : \text{“} \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \text{”}.$$

Basis step. For $n = 1$, we have $1^3 = 1 = \frac{1^2 2^2}{4}$.

Inductive step. Fix an integer $n \geq 1$ and assume that $P(n)$ is true. We show that $P(n+1)$ is true by writing:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4},$$

which finally yields

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}.$$

This shows that $P(n+1)$ is true and this completes the proof.

2) For a given choice of a, there are $(100-a)^3$ choices of (b,c,d). The total number of choices is therefore

$$\sum_{a=1}^{100}(100-a)^3 = \sum_{a=0}^{99} a^3 = \sum_{a=1}^{99} a^3 = \frac{99^2 \cdot 100^2}{4} = \frac{9801 \cdot 10000}{4} = 9801 \cdot 2500 = 24502500.$$

More formal solution. Set $E = \{(a,b,c,d) : 1 \leq a,b,c, \leq 100, a < b \text{ and } a < c \text{ and } a < d\}$, and for $1 \leq i \leq 1000$, write $E_i = \{(i,b,c,d) : 1 \leq b,c, \leq 100, i < b \text{ and } i < c \text{ and } i < d\}$. Then $E = \cup_{i=1}^{100} E_i$ and the union is disjoint. Therefore

$$\#E = \sum_{i=1}^{100} \#E_i$$

and $\#E_i = (100-i)^3$. □

Part 4

Exercise 6. Let g be the function defined by

$$\begin{array}{rccc} g & : & \mathbb{R}\times\mathbb{R} & \to & \mathbb{R}\times\mathbb{R} \\ & & (x,y) & \mapsto & (x+y, xy) \end{array}$$

1) Is g one-to-one? Justify your answer.

2) Is g onto? Justify your answer.

Solution of exercise 6.

1) The function g is not one-to-one, since $f(0,1) = f(1,0)$ but $(0,1) \neq (1,0)$.

2) The function g is not onto since $(0,1)$ does not have a pre-image. Indeed, argue by contradiction and assume that there exists $(x,y) \in \mathbb{R}^2$ such that $x+y=0$ and $xy=1$. Then $y=-x$, so that $x^2=-1$, which is a contradiction. □

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Fix an integer $n \geq 2$. In how many ways can we choose two subsets A and B of $\{1, 2, \ldots, n\}$ such that $A \subset B$?

Solution of exercise 7. Choosing such subsets amounts to choosing for each integer $1 \leq i \leq n$ whether it belongs to neither A, nor B, or to A, or to B but not to A. This gives 3 choices for every integer $1 \leq i \leq n$, and the multiplicative rule implies that the total number of ways is 3^n.

Alternative proof (using the Binomial theorem). We distinguish cases according to $\mathrm{Card}(A)$. For $0 \leq k \leq n$, there are $\binom{n}{k}$ ways of choosing $A \subset \{1, 2, \ldots, n\}$ with k elements, and then there are 2^{n-k} ways to "complete" A into B (for every one of the $n-k$ elements in $\{1, 2, \ldots, n\} \backslash A$, we can either add it or not). As a consequence, the number of ways we can choose two subsets A and B of $\{1, 2, \ldots, n\}$ such that $A \subset B$ is

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 1^k = (2+1)^n = 3^n.$$

□

Exercise 8. If A is a set, recall that $\mathcal{P}(A)$ denotes the set of all subsets of A. Let $f : E \to F$ be a function. Set $\mathcal{S} = \{X \subset E : f^{-1}(f(X)) = X\}$. Define the function g as follows:

$$\begin{array}{rcccl} g & : & \mathcal{S} & \to & \mathcal{P}(f(E)) \\ & & A & \mapsto & f(A) \end{array}$$

Show that g is a bijection.

Solution of exercise 8. We first show that g is one-to-one. Let $A, B \in \mathcal{S}$ such that $g(A) = g(B)$. Then $f(A) = f(B)$. Hence, since $A, B \in \mathcal{A}$,

$$A = f^{-1}(f(A)) = f^{-1}(f(B)) = B.$$

Hence $A = B$, so that g is one-to-one.

We next show that f is onto. Take $B \in \mathcal{P}(f(E))$, so that $B \subset f(E)$. Set $A = f^{-1}(B)$.

★ We first check that $f(A) = B$, or, equivalently, that $f(f^{-1}(B)) = B$. We argue by double inclusion.

▹ Take $y \in B$. Since $B \subset f(E)$, there exists $x \in E$ such that $y = f(x)$. In addition, since $y \in B$, we have $x \in f^{-1}(B)$. Therefore $y \in f(f^{-1}(B))$. Thus $B \subset f(f^{-1}(B)) = f(A)$.

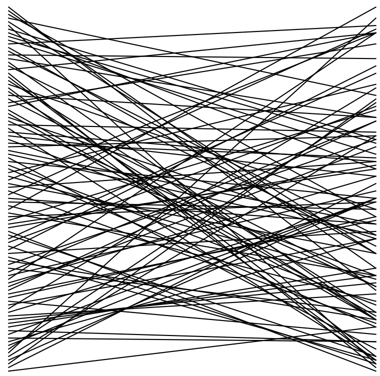
▹ Now take $y \in f(f^{-1}(B))$. There exists $x \in f^{-1}(B)$ such that $y = f(x)$. But $x \in f^{-1}(B)$, so that $f(x) \in B$. Therefore $y \in B$. Hence $f(f^{-1}(B)) \subset B$.

⋆ We finally check that $A \in \mathcal{S}$ and show that $f^{-1}(f(A)) = A$. To this end, using the fact that $f(f^{-1}(B)) = B$, write

$$f^{-1}(f(A)) = f^{-1}(f(f^{-1}(B))) = f^{-1}(B) = A.$$

This completes the proof. □

Exercise 9. What does the following image represent?



Solution of exercise 9. The image represents a random function (every point with coordinates $(0, i/n)$ is mapped to a point with coordinates $(0, X_i/n)$ with X_i being chosen uniformly at random among $\{1, 2, 3, \ldots, n\}$), all the choices being independent; here $n = 100$, see the code below that generated this image (using the software Mathematica). □

```
n = 100;
dessin2 = {};
For[i = 1, i ≤ n, i++,
  r = RandomInteger[{1, n}];
  dessin2 = Append[dessin2, Line[{{0, i}, {n, r}}]];
 ];

Graphics[dessin2]
```

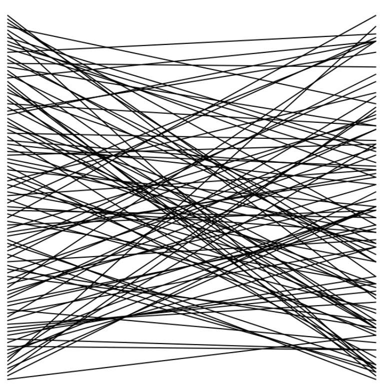


Figure 1: The Mathematica code used to generate the image of exercise 9.