Week 17 (Final exam): Thursday, January 23, 14:00-16:00

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Advice. Use draft paper before writing your answers in the final form. Reread your work. Do not forget that what is graded is what is written, not what is in your head.

In this exam, "informal" counting arguments are OK (for non-counting arguments, proofs should be precise)

Part 1

Exercise 1.
1) Give the definition of a probability on a finite state space Ω.
2) Show that for every $n \geq 6$ it is possible to cut a square into n smaller squares.
3) Let $(\Omega, \mathbb{P})$ be a finite probability space. Show that for every $A, B \subseteq \Omega$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Part 2

Exercise 2. Let A, B and C be three sets. Show that $(A \cup B = A \cup C$ and $A \cap B = A \cap C) \iff B = C$.

Exercise 3. Let E, F, G, H be four sets and $f : E \to F$, $g : F \to G$ and $h : G \to H$ be three functions. Assume that $g \circ f$ is bijective and that $h \circ g$ is bijective.

1) Show that g is bijective.

Remark. If you have not managed to solve this question, if needed, you can assume that g is bijective for the next question.

2) Show that f is bijective.

Part 3

Exercise 4. Let $n \geq 2$ be an integer. In a group of n people, every person throws a fair dice which has n faces labelled $\{1, 2, \ldots, n\}$.

1) Give a probability space to model this experiment.

2) What is the probability that all the results on the dices are different? Justify your answer.

3) What is the probability that no one gets the same result as Prof. B. (who is one of the people of the group)? Justify your answer.

Exercise 5. Let $n \geq 1$ be an integer.

1) For an integer $1 \leq k \leq n$, show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

2) Show that $\sum_{k=1}^{n} k\binom{n}{k}2^{n-k} = n3^{n-1}$.

3) How many couples (X, Y) of subsets of $\{1, 2, \ldots, n\}$ such that $\mathrm{Card}(X \cap Y) = 1$ are there? Give the simplest possible expression and justify your answer.

Remark. If you have not managed to solve a question, if needed, you can assume that it is true.

Part 4

Exercise 6. Let $n \geq 2$ be an integer. We denote by $\mathcal{S}_n$ the set of all permutations of $\{1, 2, \ldots, n\}$ and by $\varepsilon(\sigma)$ the signature of a permutation $\sigma \in \mathcal{S}_n$. We say that σ is *even* if $\varepsilon(\sigma) = 1$ and odd if $\varepsilon(\sigma) = -1$. We denote by A_n the set of all even permutations of $\mathcal{S}_n$ and by O_n the set of all odd permutations of $\mathcal{S}_n$.

1) Is the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$ even or odd? Justify your answer.

2) Show that the function

$$\begin{array}{cccc} F : & A_n & \longrightarrow & O_n \\ & \sigma & \longmapsto & \sigma \circ (1, 2) \end{array}$$

is a bijection (here $(1, 2)$ is a transposition). Deduce that $\mathrm{Card}(A_n) = \frac{n!}{2}$.

3) Let $k \geq 1$ be an integer. What is the probability that the product of k permutations chosen uniformly at random is an even permutation? Justify your answer (do not forget to mention the probability space).

Remark. If you have not managed to solve question 2), if needed, you can assume that $\mathrm{Card}(A_n) = \frac{n!}{2}$.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish (a subset A of a set E is called proper if $A \neq E$).

Exercise 8. Let $n \geq 1$ be an integer. Denote by M_n the number of words one can create by using an alphabet of n letters such that all the letters in the word are different. Show that $M_n = \lfloor en! \rfloor - 1$.

Remark. You may use the fact that $e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$

Exercise 9. What does the following image prove?