Week 17 (Final exam): Thursday, January 23, 14:00-16:00

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Advice. Use draft paper before writing your answers in the final form. Reread your work. Do not forget that what is graded is what is written, not what is in your head.

In this exam, "informal" counting arguments are OK (for non-counting arguments, proofs should be precise)

Part 1

Exercise 1.

1) Give the definition of a probability on a finite state space Ω.

2) Show that for every $n \geq 6$ it is possible to cut a square into n smaller squares.

3) Let $(\Omega, \mathbb{P})$ be a finite probability space. Show that for every $A, B \subseteq \Omega$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Solution of exercise 1.

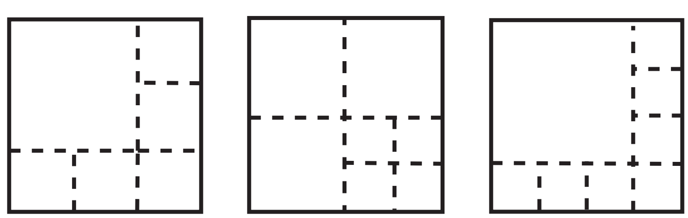
1) A probability $\mathbb{P}$ on a finite state space Ω is a function $\mathbb{P} : \mathcal{P}(\Omega) \to [0,1]$ such that:

– $\mathbb{P}(\Omega) = 1$

– for every $A, B \subseteq E$, if $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

2) We argue by induction. For an integer $n \geq 6$, let $P(n)$ be the assertion "it is possible to cut a square into n, $n+1$ and $n+2$ smaller squares".

Basis step. As the following figure shows, $P(6)$ is true.



Inductive step. Let $n \geq 6$ be an integer such that $P(n)$ is true. We show that $P(n+1)$ is true. By the inductive hypothesis, we already know that it is possible to cut a square into $n+1$ and $n+2$ squares. To show that is possible to cut a square into $n+3$ squares, consider a cutting of a square into n squares (possible thanks to the inductive hypothesis), and then simply cut one of the squares into 4, thus giving a cutting into $n+3$ squares.

Remark. It is also possible to prove that $P(n)$ is true for every $n \geq 6$ by using a strong induction.

3) We write $A \cup B = (A \backslash A \cap B) \cup (A \cap B) \cup (B \backslash A \cap B)$ and observe that this is a union of pairwise disjoint events. Hence, by the result stated in the lecture (namely: if $n \geq 2$ and $(A_k)_{1 \leq k \leq n}$ are pairwise disjoint events; then $\mathbb{P}(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mathbb{P}(A_k)$),

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \backslash A \cap B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \backslash A \cap B).$$

We have $\mathbb{P}(A) = \mathbb{P}(A \backslash A \cap B) + \mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = \mathbb{P}(B \backslash A \cap B) + \mathbb{P}(A \cap B)$ (indeed, as was seen in the lecture, if E, F are two events such that $E \subseteq F$, then $\mathbb{P}(F) = \mathbb{P}(F \backslash E) + \mathbb{P}(E)$). Therefore

$$\mathbb{P}(A \cup B) = (\mathbb{P}(A) - \mathbb{P}(A \cap B)) + \mathbb{P}(A \cap B) + (\mathbb{P}(B) - \mathbb{P}(A \cap B)),$$

and the desired result follows.

□

Part 2

Exercise 2. Let A, B and C be three sets. Show that $(A \cup B = A \cup C \text{ and } A \cap B = A \cap C) \iff B = C$.

Solution of exercise 2. We argue by double implication.

For the converse implication, it is clear that if $B = C$ then $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

For the direct implication, assume that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. We show that $B = C$ by double inclusion.

– Take $x \in B$. We argue by contradiction and assume that $x \notin C$. We have $x \in A \cup B$, so $x \in A \cup C$. Hence $x \in A$. Therefore $x \in A \cap B$, so $x \in A \cap C$ and thus $x \in C$, which is a contradiction. We conclude that $B \subseteq C$.

Since B and C play symmetric roles, the same proof shows that $C \subseteq B$. To be complete, here is the proof:

– Take $x \in C$. We argue by contradiction and assume that $x \notin B$. We have $x \in A \cup C$, so $x \in A \cup B$. Hence $x \in A$. Therefore $x \in A \cap C$, so $x \in A \cap B$ and thus $x \in B$, which is a contradiction. We conclude that $C \subseteq B$.

□

Exercise 3. Let E, F, G, H be four sets and $f : E \to F$, $g : F \to G$ and $h : G \to H$ be three functions. Assume that $g \circ f$ is bijective and that $h \circ g$ is bijective.

1) Show that g is bijective.

Remark. If you have not managed to solve this question, if needed, you can assume that g is bijective for the next question.

2) Show that f is bijective.

Solution of exercise 3.

1) We show that g is one-to-one and that g is onto.

– Take $x, y \in F$ and assume that $g(x) = g(y)$. Then $h(g(x)) = h(g(y))$. Since $h \circ g$ is bijective, it is one-to-one, so $x = y$. Hence g is one-to-one.

– Take $y \in G$. We show that y has a preimage by g. Since $g \circ f$ is bijective, it is onto, so there exists $x \in E$ such that $g(f(x)) = y$. In particular, $f(x)$ is a preimage of y by g. Hence g is onto.

2) Since g is bijective and since a composition of bijective functions is bijective, $f = g^{-1} \circ (g \circ f)$ is bijective. □

Part 3

Exercise 4. Let $n \geq 2$ be an integer. In a group of n people, every person throws a fair dice which has n faces labelled $\{1, 2, \ldots, n\}$.

1) Give a probability space to model this experiment.

2) What is the probability that all the results on the dices are different? Justify your answer.

3) What is the probability that no one gets the same result as Prof. B. (who is one of the people of the group)? Justify your answer.

Solution of exercise 4.

1) We take $\Omega = \{1, 2, \ldots, n\}^n$ equipped with the uniform probability $\mathbb{P}$ (the i-th coordinate of an element of Ω represents the result of the i-th person). Note that $\mathrm{Card}(\Omega) = n^n$.

2) Let A be the event "all the results all different". Since $\mathbb{P}(A) = \frac{\mathrm{Card}(A)}{n^n}$, we count the number of elements of A. To obtain n different results, we have n choices for the result of the first person, $n-1$ for the second one, etc. up to the last one (1 choice), so we get $n!$ choices in total (product rule). Hence

$$\mathbb{P}(A) = \frac{n!}{n^n}.$$

3) Let B be the event "no one gets the same result as Prof. B.". Since $\mathbb{P}(B) = \frac{\mathrm{Card}(B)}{n!}$, we count the number of elements of B. We have n choices for the result of Prof. B. and $n-1$ choices for the results of all the other people, so we get $n(n-1)^{n-1}$ choices in total. Hence

$$\mathbb{P}(B) = \frac{n(n-1)^{n-1}}{n^n} = \left(1 - \frac{1}{n}\right)^{n-1}.$$

□

Exercise 5. Let $n \geq 1$ be an integer.

1) For an integer $1 \leq k \leq n$, show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

2) Show that $\sum_{k=1}^{n} k\binom{n}{k}2^{n-k} = n3^{n-1}$.

3) How many couples (X,Y) of subsets of $\{1,2,\ldots,n\}$ such that $\mathrm{Card}(X \cap Y) = 1$ are there? Give the simplest possible expression and justify your answer.

Remark. If you have not managed to solve a question, if needed, you can assume that it is true.

Solution of exercise 5.

1) We have

$$k\binom{n}{k} = k\frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-1-(k-1))(k-1)!} = n\binom{n-1}{k-1}.$$

2) We use the first question and the Binomial theorem:

$$\begin{aligned}\sum_{k=1}^{n} k\binom{n}{k}2^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1}2^{n-k} \\ &= n2^{n-1}\sum_{k=0}^{n-1}\binom{n-1}{k}2^{-k} \\ &= n2^{n-1}\left(1+\frac{1}{2}\right)^{n-1} \\ &= n3^{n-1}.\end{aligned}$$

3) To construct a couple (X,Y) of subsets of $\{1,2,\ldots,n\}$ such that $\mathrm{Card}(X \cap Y) = 1$, we first choose a subset X of $\{1,2,\ldots,n\}$ with k elements (with $1 \leq k \leq n$), then choose the element which will be in common with Y (k choices) and that build Y by adding a subset of $\{1,2,\ldots,n\}\backslash X$ (2^{n-k} choices). The answer is therefore

$$\sum_{k=1}^{n} k\binom{n}{k}2^{n-k} = n3^{n-1}.$$

□

Part 4

Exercise 6. Let $n \geq 2$ be an integer. We denote by $\mathcal{S}_n$ the set of all permutations of $\{1,2,\ldots,n\}$ and by $\varepsilon(\sigma)$ the signature of a permutation $\sigma \in \mathcal{S}_n$. We say that σ is *even* if $\varepsilon(\sigma) = 1$ and odd if $\varepsilon(\sigma) = -1$. We denote by A_n the set of all even permutations of $\mathcal{S}_n$ and by O_n the set of all odd permutations of $\mathcal{S}_n$.

1) Is the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$ even or odd? Justify your answer.

2) Show that the function

$$\begin{array}{rccc} F : & A_n & \longrightarrow & O_n \\ & \sigma & \longmapsto & \sigma \circ (1,2) \end{array}$$

is a bijection (here $(1,2)$ is a transposition). Deduce that $\mathrm{Card}(A_n) = \frac{n!}{2}$.

3) Let $k \geq 1$ be an integer. What is the probability that the product of k permutations chosen uniformly at random is an even permutation? Justify your answer (do not forget to mention the probability space).

Remark. If you have not managed to solve question 2), if needed, you can assume that $\mathrm{Card}(A_n) = \frac{n!}{2}$.

Solution of exercise 6.

1) To find the signature of π we write it as a product of cycles and use the multiplicativity of the signature: $\pi = (1,3,4)(2,5)$, so $\varepsilon(\pi) = (-1)^{3-1}(-1)^{2-1} = -1$.

2) First, since the signature of a transposition is -1 and since the signature is multiplicative, if $\sigma \in A_n$, then $\sigma \circ (1,2) \in O_n$.

To show that F is a bijection we show that it is one-to-one and onto.

– Take $\sigma, \tau \in A_n$ such that $F(\sigma) = F(\tau)$. Then $\sigma \circ (1,2) = \tau \circ (1,2)$, hence $\sigma \circ (1,2) \circ (1,2) = \tau \circ (1,2) \circ (1,2)$, so $\sigma = \tau$ (because $(1,2) \circ (1,2)$ is the identity). Hence F is one-to-one.

– Take $\tau \in O_n$. Then $\tau \circ (1,2) \in A_n$ is a preimage of τ by F, so F is onto.

As a consequence, since S_n has cardinality $n!$ and is the disjoint union of A_n and O_n, we have $\mathrm{Card}(A_n) = \mathrm{Card}(O_n) = n!/2$.

3) We take $\Omega = \mathcal{S}_n^k$ equipped with the uniform probability. We are interested in the probability of the event $E = \{(\sigma_1, \ldots, \sigma_k) \in \mathcal{S}_n^k : \epsilon(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k) = 1\}$.

To simplify notation, set $\ell = \lfloor k/2 \rfloor$ (so that $k = 2\ell$ if k is even and $k = 2\ell + 1$ if k is odd). By the multiplicativity property of the signature, $\epsilon(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k) = 1$ if and only if an even number of the σ_i's are odd. As a consequence, we can write E as the pairwise disjoint union

$$E = E_0 \cup E_1 \cup \cdots \cup E_\ell,$$

where

$$E_j = \{(\sigma_1, \ldots, \sigma_k) \in \mathcal{S}_n^k : \mathrm{Card}(\{1 \leq i \leq k : \varepsilon(\sigma_i) = -1\} = 2j\})$$

for $0 \leq j \leq \ell$.

To compute the cardinality of E_j we first choose the $2j$ indices among k which will carry an odd permutation ($\binom{k}{2j}$ choices), and then we choose the corresponding permutations ($n!/2$ choices for every index between 1 and k, since $\mathrm{Card}(A_n) = \mathrm{Card}(O_n) = n!/2$). Hence

$$\mathrm{Card}(E_j) = \binom{k}{2j} \left(\frac{n!}{2}\right)^k.$$

But

$$\sum_{j=0}^{\ell} \binom{k}{2j} = 2^{k-1}. \tag{1}$$

Indeed, if we set

$$A = \sum_{j=0, j \text{ even}}^{k} \binom{k}{j}, \qquad B = \sum_{j=0, j \text{ odd}}^{k} \binom{k}{j},$$

by the Binomial theorem we have $2^k = (1+1)^k = A + B$ and $0 = (1-1)^k = A - B$, so we get (1).

Therefore

$$\mathrm{Card}(E) = 2^{k-1} \left(\frac{n!}{2}\right)^k.$$

We conclude that

$$\mathbb{P}(E) = \frac{\mathrm{Card}(E)}{(n!)^k} = \frac{1}{2}.$$

□

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,2,3,4,5,6,7,8,9,10\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish (a subset A of a set E is called proper if $A \neq E$).

Solution of exercise 7. The answer is 55.

Indeed, let $[n]$ denote the set $\{1,2,\dots,n\}$, and let f_n denote the number of minimal selfish subsets of $[n]$. Then the number of minimal selfish subsets of $[n]$ not containing n is equal to f_{n-1}.

On the other hand, since 1 and n cannot both occur in a minimal selfish set of $[n]$, for any minimal selfish subset of $[n]$ containing n, by subtracting 1 from each element, and then taking away the element $n-1$ from the set, we obtain a minimal selfish subset of $[n-2]$. Conversely, any minimal selfish subset of $[n-2]$ gives rise to a minimal selfish subset of $[n]$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $[n]$ containing n is f_{n-2}. Thus we obtain $f_n = f_{n-1} + f_{n-2}$. Since $f_1 = f_2 = 1$, we have $f_n = F_n$, where F_n denotes the n-th term of the Fibonacci sequence.

Since $F_{10} = 55$, the desired result follows.

Remarks. – Using the fact that in a minimal selfish set, its smallest element is its cardinality, one can explicitly count the number of minimal selfish sets of $\{1,2,\dots,10\}$ by distinguishing cases according to their smallest element.

– This is essentially Exercice B-1 from 1996 Putnam competition. □

Exercise 8. Let $n \geq 1$ be an integer. Denote by M_n the number of words one can create by using an alphabet of n letters such that all the letters in the word are different. Show that $M_n = \lfloor en! \rfloor - 1$.

Remark. You may use the fact that $e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$

Solution of exercise 8. Let $M_{n,k}$ be the number of words of k letters such that all the letters are different. Clearly, $M_{n,k} = 0$ if $k > n$. If $k \leq n$, obtaining a word of k letters such that all the letters are different amounts to choosing the first letter (n choices), then the second letter ($n-1$ choices), etc. up to the k-th letter ($n-k+1$ choices). Hence

$$M_{n,k} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

We conclude that

$$M_n = \sum_{k=1}^{n} M_{n,k} = n! \sum_{k=1}^{n} \frac{1}{(n-k)!} = n! \sum_{k=0}^{n} \frac{1}{k!} - 1.$$

Now, define

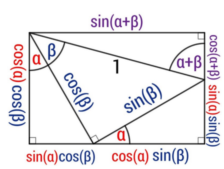
$$u_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad v_n = u_n + \frac{1}{n!}.$$

One readily checks that (u_n) and (v_n) are adjacent, meaning that (u_n) is increasing, (v_n) is decreasing and $v_n - u_n \to 0$. In particular, since $u_n \to e$, it follows that for every $n \geq 1$, $u_n < e < v_n$. Thus

$$M_n < n!e - 1 < M_n + 1,$$

which implies that $M_n = \lfloor en! \rfloor - 1$. □

Exercise 9. What does the following image prove?



Solution of exercise 9. The image proves that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ and $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. □