

Week 11: Combinatorics: additional exercises

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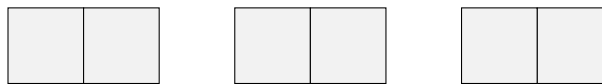
Exercise 1.

- In how many ways can we write the 6 integers between 1 and 6 in the following squares



so that the first number is less than the second number?

- In how many ways can we write the 6 integers between 1 and 6 in the following squares



so that the three numbers are in increasing order?

Solution of exercise 1.

- We view the filling of squares as a permutation $\sigma \in S_6$ as follows:



We want to find the number of permutations $\sigma \in S_6$ such that $\sigma(1) < \sigma(4)$:

- there are $\binom{6}{2}$ choices for the subset A of the images of 1 et 4 (there is only one ordering when these two integers are chosen, since we require $\sigma(1) < \sigma(4)$);
- once these two integers chosen, there are $4!$ bijections between $\{2, 3, 5, 6\}$ and $\{1, \dots, 6\} \setminus A$ in order to fix the images of 2, 3, 4, 5 by σ .

The answer is therefore

$$\binom{6}{2} 4! = 360.$$

- Similarly, we get the number of permutations $\sigma \in S_6$ such that $\sigma(1) < \sigma(3) < \sigma(5)$ is

$$\binom{6}{3} 3! = 120.$$

□

Exercise 2. Let $1 \leq n \leq p$ be integers. How many (strictly) increasing functions from $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, p\}$ are there?

Solution of exercise 2. Such a function is entirely defined by its image, which is a subset of $\{1, 2, \dots, p\}$ having n elements. The answer is therefore $\binom{p}{n}$. □

Exercise 3. Let $n \geq 2$ be an integer and let us consider a deck of n cards numbered from 1 to n .

1. In how many ways is it possible to shuffle the deck so that the card with number 1 is further in the deck than the card 2?
2. In how many ways is it possible to shuffle the deck so that the cards with numbers 1 and 2 are neighbours?

Solution of exercise 3. We may view a shuffling of the deck as a permutation $\sigma \in S_n$.

1. We want to count the number of elements of the set $A = \{\sigma \in S_n, \sigma(1) > \sigma(2)\}$. To this end, we partition A according to the value of $k = \sigma(1)$. Once this $k \geq 2$ has been chose, we have to chose:
 - the value of $\sigma(2)$, with $k - 1$ choices (positive integers less that k),
 - then a bijection between $\{3, 4, \dots, n\}$ and $\{1, 2, \dots, n\} \setminus \{\sigma(1), \sigma(2)\}$, with $(n - 2)!$ choices.

Therefore

$$\text{Card}(A) = \sum_{k=2}^n (k-1)(n-2)! = (n-2)! \frac{(n-1)n}{2} = \frac{n!}{2}.$$

2. We want to count the number of elements of the set $B = \{\sigma \in S_n, |\sigma(1) - \sigma(2)| = 1\}$. To this end, we partition B according to the value of $k = \sigma(1)$ and then according to the value of $\sigma(2)$ (only one choice if $k = 1$ or $k = n$, two choices otherwise); it then remains to choose a bijection between $\{3, 4, \dots, n\}$ and $\{1, 2, \dots, n\} \setminus \{\sigma(1), \sigma(2)\}$. Therefore

$$\text{Card}(B) = (n-2)! + \sum_{k=2}^{n-1} 2(n-2)! + (n-2)! = (n-2)!(2n-2) = 2(n-1)!.$$

□

Exercise 4. Let $1 \leq p \leq n$ be integers. Let E be a set with n elements and A a subset of E with p elements.

- 1) How many subsets X of E such that $A \subset X$ are there?
- 2) If $p \leq m \leq n$, how many subsets X of E such that $A \subset X$ are there?
- 3) How many couples (X, Y) of subsets of E such that $X \cap Y = A$ are there?

Solution of exercise 4.

1) As many as the number of subsets of $E \setminus A$ (which correspond to the elements we add to A to obtain X), that is 2^{n-p} .

2) As many as the number of subsets of $E \setminus A$ having $m - p$ elements, that is $\binom{n-p}{m-p}$.

3) Once we have chosen a subset X of E such that $A \subset X$ and $\text{Card}(X) = m$ ($\binom{n-p}{m-p}$ choices), we have to choose for Y a subset of $E \setminus A$ (2^{n-m} choices). The answer is therefore

$$\sum_{m=p}^n \binom{n-p}{m-p} 2^{n-m} = \sum_{k=0}^{n-p} \binom{n-p}{k} 2^{n-p-k} = (1+2)^{n-p} = 3^{n-p},$$

where we have used the Binomial theorem for the second equality. □

Exercise 5. Let $n \geq 2$ be an integer. Find the number of permutations $\sigma \in S_n$ such that 1 and n belong to the same orbit of σ (that is, such that there exists an integer $k \geq 1$ with $\sigma^k(1) = n$).

Solution of exercise 5. Let us partition the set depending on the size ℓ of the orbit \mathcal{O} of 1 and n . Once $2 \leq \ell \leq n$ is fixed, we have to choose:

- the $\ell - 2$ other elements belonging to this orbit \mathcal{O} , that is $\binom{n-2}{\ell-2}$ choices,
- the circular permutation of these ℓ elements, that is $(\ell - 1)!$ choices,
- then a permutation of $\{1, 2, \dots, n\} \setminus \mathcal{O}$, that is $(n - \ell)!$ choices.

Therefore the answer is

$$\sum_{\ell=2}^n \binom{n-2}{\ell-2} (\ell - 1)! (n - \ell)! = (n - 2)! \sum_{\ell=2}^n (\ell - 1) = \frac{n!}{2}.$$

□