Week 11: Combinatorics: additional exercises

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups A&B, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups C&E, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups D&F, benoit.tran@polytechnique.edu).

Exercise 1.

1. In how many ways can we write the 6 integers between 1 and 6 in the following squares

so that the first number is less than the second number?

2. In how many ways can we write the 6 integers between 1 and 6 in the following squares

so that the three numbers are in increasing order?

Solution of exercise 1.

1. We view the filling of squares as a permutation $\sigma \in S_6$ as follows:

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$		$\sigma(4)$	$\sigma(5)$	$\sigma(6)$

We want to find the number of permutations $\sigma \in S_6$ such that $\sigma(1) < \sigma(4)$:

- there are $\binom{6}{2}$ choices for the subset A of the images of 1 et 4 (there is only one ordering when these two integers are chosen, since we require $\sigma(1) < \sigma(4)$);
- once these two integers chosen, there are 4! bijections between $\{2,3,5,6\}$ and $\{1,\ldots,6\} \setminus A$ in order to fix the images of $2,3,4,5$ by σ.

The answer is therefore
$$\binom{6}{2}4! = 360.$$

2. Similarly, we get the number of permutations $\sigma \in S_6$ such that $\sigma(1) < \sigma(3) < \sigma(5)$ is
$$\binom{6}{3}3! = 120.$$

□

Exercise 2. Let $1 \leq n \leq p$ be integers. How many (strictly) increasing functions from $\{1,2,\ldots,n\} \to \{1,2,\ldots,p\}$ are there?

Solution of exercise 2. Such a function is entirely defined by its image, which is a subset of $\{1,2,\ldots,p\}$ having n elements. The answer is therefore $\binom{p}{n}$. □

Exercise 3. Let $n \geq 2$ be an integer and let us consider a deck of n cards numbered from 1 to n.

1. In how many ways is it possible to shuffle the deck so that the card with number 1 is further in the deck than the card 2?
2. In how many ways is it possible to shuffle the deck so that the cards with numbers 1 and 2 are neighbours ?

Solution of exercise 3. We may view a shuffling of the deck as a permutation $\sigma \in S_n$.

1. We want to count the number of elements of the set $A = \{\sigma \in S_n,\ \sigma(1) > \sigma(2)\}$. To this end, we partition A according to the value of $k = \sigma(1)$. Once this $k \geq 2$ has been chose, we have to chose:
   - the value of $\sigma(2)$, with $k-1$ choices (positive integers less that k) ,
   - then a bijection between $\{3,4,\ldots,n\}$ and $\{1,2,\ldots,n\} \setminus \{\sigma(1),\sigma(2)\}$, with $(n-2)!$ choices.

   Therefore
   $$\mathrm{Card}(A) = \sum_{k=2}^{n} (k-1)(n-2)! = (n-2)!\frac{(n-1)n}{2} = \frac{n!}{2}.$$
2. We want to count the number of elements of the set $B = \{\sigma \in S_n,\ |\sigma(1)-\sigma(2)| = 1\}$. To this end, we partition B according to the value of $k = \sigma(1)$ and then according to the value of $\sigma(2)$ (only one choice if $k = 1$ or $k = n$, two choices otherwise); it then remains to choose a bijection between $\{3,4,\ldots,n\}$ and $\{1,2,\ldots,n\} \setminus \{\sigma(1),\sigma(2)\}$. Therefore
   $$\mathrm{Card}(B) = (n-2)! + \sum_{k=2}^{n-1} 2(n-2)! + (n-2)! = (n-2)!(2n-2) = 2(n-1)!.$$

□

Exercise 4. Let $1 \leq p \leq n$ be integers. Let E be a set with n elements and A a subset of E with p elements.
1) How many subsets X of E such that $A \subset X$ are there?
2) If $p \leq m \leq m$, how many subsets X of E such that $A \subset X$ are there?
3) How many couples (X,Y) of subsets of E such that $X \cap Y = A$ are there?

Solution of exercise 4.

1) As many as the number of subsets of $E\backslash A$ (which correspond to the elements we add to A to obtain X), that is 2^{n-p}.

2) As many as the number of subsets of $E\backslash A$ having $m-p$ elements, that is $\binom{n-p}{m-p}$.

3) Once we have chosen a subset X of E such that $A \subset X$ and $\mathrm{Card}(X) = m$ ($\binom{n-p}{m-p}$ choices), we have to choose for Y a subset of $E\backslash A$ (2^{n-m} choices). The answer is therefore

$$\sum_{m=p}^{n} \binom{n-p}{m-p} 2^{n-m} = \sum_{k=0}^{n-p} \binom{n-p}{k} 2^{n-p-k} = (1+2)^{n-p} = 3^{n-p},$$

where we have used the Binomial theorem for the second equality. □

Exercise 5. Let $n \geq 2$ be an integer. Find the number of permutations $\sigma \in S_n$ such that 1 and n belong to the same orbit of σ (that is, such that there exists an integer $k \geq 1$ with $\sigma^k(1) = n$).

Solution of exercise 5. Let us partition the set depending on the size ℓ of the orbit $\mathcal{O}$ of 1 and n. Once $2 \leq \ell \leq n$ is fixed, we have to choose:

- the $\ell - 2$ other elements belonging to this orbit $\mathcal{O}$, that is $\binom{n-2}{\ell-2}$ choices,
- the circular permutation of these ℓ elements, that is $(\ell-1)!$ choices,
- then a permutation of $\{1, 2, \ldots, n\} \setminus \mathcal{O}$, that is $(n-\ell)!$ choices.

Therefore the answer is

$$\sum_{\ell=2}^{n} \binom{n-2}{\ell-2} (\ell-1)!(n-\ell)! = (n-2)! \sum_{\ell=2}^{n} (\ell-1) = \frac{n!}{2}.$$

□