FINITELY PRESENTED WREATH PRODUCTS AND DOUBLE COSET DECOMPOSITIONS

YVES DE CORNULIER

Abstract. We characterize which permutational wreath products $G \ltimes W^{(X)}$ are finitely presented. This is the case if and only if G and W are finitely presented, G acts on X with finitely generated stabilizers, and with finitely many orbits on the cartesian square X^2.

On the one hand, this extends a result of G. Baumslag about standard wreath products; on the other hand, this provides nontrivial examples of finitely presented groups. For instance, we obtain two quasi-isometric finitely presented groups, one of which is torsion-free and the other has an infinite torsion subgroup.

Motivated by the characterization above, we discuss the following question: what finitely generated groups can have a finitely generated subgroup with finitely many double cosets? The discussion involves properties related to the structure of maximal subgroups, and to the profinite topology.

1. Introduction

Let G be a group, and X a G-set. Let W be another group. Then G acts on the direct sum $W^{(X)}$ by permutations of factors. The *(permutational) wreath product* $W \wr_X G$ is defined to be the semidirect product $W^{(X)} \rtimes G$. When the action of G on X is simply transitive, it is called the *standard* wreath product (this special case is sometimes called the wreath product) and denoted by $W \wr G$.

By a result of G. Baumslag [Ba61], a standard wreath product $W \wr G$ with $W \neq 1$ and G infinite is never finitely presented. In contrast, permutational wreath products provide nontrivial examples:

Theorem 1.1. *If $W \neq 1$, the wreath product $W \wr_X G$ is finitely presented if and only if*

(i) W and G are finitely presented,

(ii) G acts on X with finitely generated stabilizers, and

(iii) the product action of G on the cartesian square X^2 has finitely many orbits.

Note that this result extends Baumslag's result: indeed, if G acts simply transitively on X, then (iii) implies that X is finite.

We provide (see Examples 3.4, 3.5, 3.6) groups G with an infinite G-set X satisfying the hypotheses of Theorem 1.1.

Theorem 1.1 provides new examples of finitely presented groups. For instance, it allows to prove the existence of two quasi-isometric finitely presented groups, one of which is torsion-free and the other has an infinite torsion subgroup (see Proposition 2.12).

A general motivating question is: what are pairs (G, X) satisfying the hypotheses of Theorem 1.1? Trivial examples are pairs (G, X) where G is finitely presented and

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X a finite G-set, thus we focus on nontrivial cases, namely those for which X is infinite.

Section 3 is devoted to discuss obstructions, for a given group G, to the existence of an infinite G-set X satisfying (ii) and (iii) of Theorem 1.1 (resp. satisfying (iii)). It is, in the major part, written as a survey, including many examples.

For instance, if G is a finitely generated linear solvable group, there exists no infinite G-set satisfying (iii) of Theorem 1.1. If G is a free group there exists an infinite G-set satisfying (iii) of Theorem 1.1, but none can satisfy both (ii) and (iii).

2. Finitely presented wreath products

2.1. **Proof of Theorem 1.1.** For completeness, we first recall the following easy result.

Proposition 2.1. *If $X \neq \emptyset$, the wreath product $W \wr_X G$ is finitely generated if and only if G and W are finitely generated, and G has a finite number of orbits on X.*

Proof: If the conditions are satisfied, and if n denotes the number of G-orbits in X, then $W \wr_X G$ can be written as a quotient of the free product $W^{*n} * G$, where W^{*n} denotes the free product of n copies of W.

Conversely, suppose that $W \wr_X G$ is finitely generated. Being a quotient of $W \wr_X G$, G is also finitely generated. Since $X \neq \emptyset$, W embeds in $W \wr_X G$, hence is countable. If it is not finitely generated, it can be written as the union of a strictly increasing sequence of subgroups W_n. Therefore $W \wr_X G$ is the union of the strictly increasing sequence of subgroups $W_n \wr_X G$, and hence is not finitely generated. ■

Let us now look at a presentation for the wreath product $W \wr_X G$. For the sake of simplicity, we first suppose that G acts transitively on X, so that we can write $X = G/H$. It is easy to check that a presentation for $W \wr_{G/H} G$ is given by

$$\langle G, W, \mid [H, W], [W, gWg^{-1}] \, \forall g \in G - H \rangle \quad (2.1)$$ [1].

Using the relation $[H, W] = 1$; it is immediate that, in the family of relations $[W, gWg^{-1}]$ with $g \in G - H$, it suffices to take g in $G/H - \{H\}$. In fact, we can do better: we can take g in $H\backslash G/H$: this is obtained by conjugating the relation $[W, gWg^{-1}]$ by an element of H. With these remarks, we can prove:

Theorem 2.2. *Let G, W be finitely presented groups. Let G act on a set X, with finitely generated stabilizers. Suppose that the product action of G on X^2 has a finite number of orbits. Then $W \wr_X G$ is finitely presented.*

Proof: We begin by the case when G is transitive on X, so that we can write $X = G/H$. Since W and H are finitely generated, $[H, W] = 1$ in the presentation (2.1) reduces to a finite number of relators. The hypothesis that the product action of G on X^2 has a finite number of orbits reads as: $H\backslash G/H$ is finite. Then the result follows from the remarks above: the family of relations $[W, gWg^{-1}]$ of the presentation (2.1) reduces to the finite family $[W, g_iWg_i^{-1}]$, where (g_i) is a finite system of representants of the double classes modulo H in G, except the class H.

We now indicate how to deal with the case when G is not necessarily transitive on X, which makes no essential difference. Choosing a point in each orbit, we can

[1] This concise notation must be understood as: $W \wr_{G/H} G$ is the quotient of the free product $G * W$ by the given relations.

write $X = \coprod_{i\in I} G/H_i$, where $I = G\backslash X$. For all $i \in I$, consider a copy W_i of W. Then it is easy to check that a presentation for $W \wr_\alpha G$ is given by the quotient of the free product of G and all W_i by the relations:

$$[H_i, W_i] \;\; (i \in I), \quad [W_i, gW_ig^{-1}] \;\; (i \in I, \; g \in G - H_i), \quad [W_i, gW_jg^{-1}] \;\; (i, j \in I, \; i \neq j).$$

If we forget for a few seconds the two latter families of relations, we get the generalized free product with amalgamation $G *_{(H_i)} (H_i \times W_i)$. Given that I is finite, that G and W are finitely presented, and that all H_i are finitely generated, this free product with amalgamation is clearly finitely presented.

Choose $R \subset G$ such that, for every $i, j \in I$, every double coset H_igH_j is equal to $H_ig'H_j$ for some $g' \in R$.

Then the last two families of relations follow from their subfamilies when g ranges over R. On the other hand, the G-action on X^2 having a finite number of orbits is equivalent to saying that all double quotients $H_i\backslash G/H_j$ are finite, so that R can be chosen finite. Thus, since W is finitely generated, these reduce to finitely many relations. ■

We are now going to show that the converse of Theorem 2.2 is true. We need some elementary preliminaries on graph products.

Let Γ be a graph, that is, a set $\Gamma^0 = I$, whose elements are called *vertices*, along with a subset Γ^1 of subsets of cardinality two of Γ^0, called *edges*. For each $i \in \Gamma^0$, let W_i be a group. Following [Gre91], the graph product $P = (W_i)_{i\in I}^{\langle\Gamma\rangle}$ of all W_i is by definition the quotient of the free product of all W_i by the relations $[W_i, W_j] = 1$ if $\{i, j\} \in \Gamma^1$. Denote by σ_i the obvious morphism $W_i \to P$. Observe that if Γ is the totally disconnected graph, then P is the free product of all W_i, and if Γ is the complete graph, then Γ is the direct sum (sometimes called the restricted direct product) of all W_i. When all W_i are equal to a single group W; we denote the graph product by $W^{\langle\Gamma\rangle}$.

Lemma 2.3.

(1) *For all i, $\sigma_i : W_i \to P$ is injective.*
(2) *If $\{i, j\} \notin \Gamma^1$, the natural morphism $\sigma_i * \sigma_j \to P$ is injective.*
(3) *If $\{i, j\} \in \Gamma^1$, the natural morphism $\sigma_i \times \sigma_j \to P$ is injective.*
(4) *If $\{i, k\} \notin \Gamma^1$ and $\{j, k\} \notin \Gamma^1$, the natural morphism $W_i * W_j * W_k \to P$, or $(W_i \times W_j) * W_k \to P$ (according as whether $\{i, j\}$ belongs to Γ^1) is injective.*

Proof: It suffices to observe that all these morphisms are split, as we see by taking the quotient of P by the normal subgroup generated by all W_l for $l \neq i$ (resp. for $l \neq i, j$) (resp. for $l \neq i, j, k$). ■

Lemma 2.3 shows that we can recover the graph Γ from the structure of P, provided that all groups W_i are nontrivial. More precisely, let Γ' be another graph structure on the same set of vertices: $\Gamma'^0 = \Gamma^0 = I$. Suppose in addition that $\Gamma'^1 \supseteq \Gamma^1$. There is a natural morphism p from $P = (W_i)_{i\in I}^{\langle\Gamma\rangle}$ to $P' = (W_i)_{i\in I}^{\langle\Gamma'\rangle}$, which is obviously surjective. Lemma 2.3(2) yields:

Lemma 2.4. *Suppose that $W_i \neq 1$ for all $i \in I$. Then the morphism p is bijective if and only if $\Gamma'^1 = \Gamma^1$.*

Proof: Let $\{i,j\}$ be an edge in Γ^1. Then $[W_i,W_j]=\{1\}$ in P. By injectivity, we get that $[W_i,W_j]=\{1\}$ in P'. Since $W_i\neq\{1\}$ and $W_j\neq\{1\}$, we obtain that W_i and W_j cannot generate their free product in P', so that, by Lemma 2.3(2), $\{i,j\}\in\Gamma'^1$. ■

Now denote by Q the kernel of the natural morphism $P=(W_i)_{i\in I}^{\langle\Gamma\rangle}\to\bigoplus_{i\in I}W_i$. We want to show that Q often contains a free non-abelian group. Assume, from now on, that $W_i\neq\{1\}$ for all i. It already follows from Lemma 2.4 that if Γ is not the complete graph, then $Q\neq\{1\}$. Now denote by Γ_{op} the complement graph; namely, $\Gamma_{\mathrm{op}}^0=\Gamma^0=I$, and, for all $i\neq j\in I$, $\{i,j\}\in\Gamma_{\mathrm{op}}^1$ if and only if $\{i,j\}\notin\Gamma^1$. Note that a decomposition of Γ (resp. Γ_{op}) into connected components corresponds to a decomposition of P into a free product (resp. a direct sum).

Lemma 2.5. *Suppose that $W_i\neq\{1\}$ for all i. The following are equivalent.*

(i) *Q does not contain a non-abelian free subgroup.*

(ii) *All connected components of Γ_{op} have at most 2 elements, and whenever $\{i,j\}$ is a 2-element connected component of Γ_{op}, then W_i and W_j are isomorphic to C_2, the cyclic group on two elements.*

Proof: Suppose that (i) holds. Let J be the union of 1-element connected components of Γ_{op}, and $K\subset I-J$ a subset intersecting each 2-element connected connected component of Γ_{op} in exactly one element. Then Q can be identified to the kernel of the natural morphism $(D_\infty)^{(K)}\to(C_2\times C_2)^{(K)}$, where $D_\infty\simeq C_2*C_2$ denotes the infinite dihedral group, and thus is abelian (isomorphic to $\mathbf{Z}^{(K)}$) and cannot contains free subgroups.

Conversely, suppose that (ii) is satisfied.

a) Suppose that there exists a connected component of Γ_{op} with at least 2 elements, and with at least one element i such that W_i is not cyclic on two elements. Pick j such that $\{i,j\}\in\Gamma_{\mathrm{op}}^1$. The following fact is immediate.

Fact 2.6. Let G be a group with at least three elements. Then it has a subgroup isomorphic to either $\mathbf{Z}$, C_p (the cyclic group of prime odd order p), C_4, or $C_2\times C_2$.

Pick any nontrivial cyclic subgroup Z_j in W_j, and any subgroup Z_i of W_i as in Fact 2.6. By Lemma 2.3, there is a natural embedding of Z_i*Z_j into P, which is mapped to the abelian group $Z_i\times Z_j$ in $\bigoplus_{i\in I}W_i$. Since Z_i*Z_j contains a non-abelian free subgroup, so does its derived subgroup which is mapped to Q, so that Q contains a non-abelian free subgroup.

b) Otherwise, suppose that there exists a connected component of Γ_{op} with at least 3 elements. Take $i,j,k\in I$, distinct, such that $\{i,k\}$ and $\{j,k\}$ belong to Γ_{op}^1. We can suppose that W_i,W_j,W_k are cyclic on two elements, otherwise we can argue as in a). By Lemma 2.3, we get an embedding of $(C_2\times C_2)*C_2$ or $C_2*C_2*C_2$ into P, mapping to the abelian subgroup $C_2\times C_2\times C_2$ in $\bigoplus_{i\in I}W_i$. As in a), since both $(C_2\times C_2)*C_2$ and $C_2*C_2*C_2$ contain non-abelian free subgroups, we obtain that Q contains a non-abelian free subgroup. ■

When Γ is the totally disconnected graph, Lemma 2.5 reduces as:

Lemma 2.7. *Let $(W_i)_{i\in I}$ be a family of nontrivial groups, and let Q be the kernel of the natural morphism from the free product of all W_i to the direct sum of all W_i. Suppose that I has at least 2 elements, and, if all W_i are cyclic on 2 elements, that I has at least 3 elements. Then Q contains a non-abelian free group.* ■

Lemma 2.8. *Let X be a set, and Γ_n a increasing family of graph structures on X: that is, $\Gamma_n^0 = X$, and $\Gamma_n^1 \subseteq \Gamma_{n+1}^1$ for all n. Suppose that X can be written as a finite disjoint union $X = \coprod_{i=1}^k X_i$ such that, for all n, the complement graph $(\Gamma_n)_{\mathrm{op}}$ can be written as a disjoint union of subgraphs $\Lambda_{n,i}$, with $\Lambda_{n,i}^0 = X_i$ and $\Lambda_{n,i}$ has constant finite degree. Then the sequence (Γ_n) is eventually constant.*

Proof: Let $d_{n,i}$ denote the degree of $\Lambda_{n,i}$. The sequence $(\sum_{i=1}^k d_{n,i})_{n\in\mathbf{N}}$ decreases, hence is eventually constant. Thus eventually, all sequences $(d_{n,i})_{n\in\mathbf{N}}$ are constant. Observe that if $d_{n,i} = d_{n+1,i}$, then $\Lambda_{n,i} = \Lambda_{n+1,i}$. Accordingly, the sequence (Γ_n) is eventually constant. ■

Now suppose that all W_i are equal to a single group $W \neq \{1\}$, and suppose that a group G acts on Γ, i.e. acts on $\Gamma^0 = I$ preserving Γ^1. Then the semidirect product $W^{\langle\Gamma\rangle} \rtimes G$ is well-defined.

We have to describe, given a G-set, what are the graph structures preserved by G. Let X be a set. Define an *edge set* on X to be a subset of $X \times X$ which is symmetric and does not intersect the diagonal; an edge set obviously defines a structure of graph on X. Suppose now that X is a G-set. Decompose X into its G-orbits: $X = \coprod X_i$ $(i \in I)$, and choose some base-point x_i in each X_i so that we can write $X_i = G/H_i$.

Lemma 2.9. *If E is a G-invariant edge set on X, and if $(i,j) \in I^2$, define $B_{ij} = B_{ij}(E) = \{g \in G,\ (x_i, gx_j) \in E\}$. Then the subsets $B_{ij} \subset G$ satisfy: for all $i,j \in I$, $B_{ij}^{-1} = B_{ji}$, $H_iB_{ij} = B_{ij}$, $H_i \cap B_{ii} = \emptyset$.*

Conversely, for every family $(V_{ij})_{i,j\in I}$ of subsets of G satisfying these three conditions, there exists a unique G-invariant edge set E such that $V_{ij} = B_{ij}(E)$ for all $i,j \in I$, given by $(gx_i, g'x_j) \in E$ if and only if $g^{-1}g' \in V_{ij}$.

Proof: All verifications are straightforward. ■

We can now prove the converse of Theorem 2.2. It is essentially contained in the slightly stronger result:

Proposition 2.10. *Let G, W be groups, and X a G-set with finitely many orbits. Suppose that $W \neq \{1\}$, $X \neq \emptyset$, and that one of the following conditions is satisfied.*

(1) *The group G has infinitely many orbits on X^2.*
(2) *For some $x \in X$, the stabilizer G_x is not finitely generated.*

Then, for every finitely presented group mapping onto $W \wr_X G$, the kernel contains a non-abelian free group. In particular, $W \wr_X G$ is not finitely presented.

Proof: We keep the notations defined above: $X = \coprod G/H_i$.

Suppose that (1) is satisfied. Then, for some k, l, $H_k\backslash G/H_l$ is infinite. Define, for $i, j \in I$, $n \in \mathbf{N}$, subsets V_{ij}^n of G as follows.

If $k \neq l$, take a strictly increasing sequence (U_n) of finite subsets of $H_k\backslash G/H_l$ whose union is all of $H_k\backslash G/H_l$. Define $V_{kl}^n = U_n$, $V_{lk}^n = U_n^{-1}$.

If $k = l$, take a strictly increasing sequence (U_n) of finite subsets of $H_k\backslash G/H_k - \{H_k\}$ which are symmetric under inversion, so that the union of all U_n is all of $H_k\backslash G/H_k - \{H_k\}$. Define $V_{kk}^n = U_n$.

In both cases, for all i, j such that $\{i,j\} \neq \{k,l\}$, define V_{ij}^n to be all of $H_i\backslash G/H_j$ if $i \neq j$, and $H_i\backslash G/H_i - \{H_i\}$ if $i = j$.

Let E_n be the G-invariant edge set on X corresponding, by Lemma 2.9, to the family $(V_{ij}^n)_{i,j\in I}$, and denote by X_n the corresponding graph. Observe that (E_n) is a strictly increasing sequence of G-invariant edge sets whose union is the full edge set $E_\infty = X^2 - \mathrm{diag}(X)$. Hence, the sequence of surjective morphism between finitely generated groups $W^{\langle X_n\rangle} \rtimes G \to W^{\langle X_{n+1}\rangle} \rtimes G$ converges to $W^{\langle X_\infty\rangle} \rtimes G = W^{(X)} \rtimes G$. This already proves that $W^{(X)} \rtimes G$ is not finitely presented: more precisely, if a finitely presented group maps onto $W^{(X)} \rtimes G$, then the map factorizes through $W^{\langle X_n\rangle} \rtimes G$ for some n.

Now, if the kernel of $W^{\langle X_n\rangle} \to W^{(X)}$ does not contain a non-abelian free subgroup, then, by Lemma 2.5, the complement graph of X_n has all its vertices of degree at most 1. Since this degree is constant on every G-orbit of X, the hypotheses of Lemma 2.8 are satisfied, and thus the sequence of graphs (X_n) stabilizes, a contradiction. Therefore, for all n, the kernel of $W^{\langle X_n\rangle} \rtimes G \to W^{(X)} \rtimes G$ does not contain a non-abelian free subgroup. Since, for every finitely presented group mapping onto $W \wr_X G$, the map must factorize through $W^{\langle X_n\rangle} \rtimes G$ for some n, we obtain the desired conclusion.

Suppose that (2) is satisfied: fix i such that H_i is not finitely generated. Write H_i as a strictly increasing union of subgroups $H_{i,n}$. Define X_n as the disjoint union $\coprod_{j\neq i} G/H_j \sqcup G/H_{i,n}$, and endow it with the edge set defined as: $x \sim y$ unless $x = y$ or $x, y \in G/H_{i,n}$ and $x \in yH_i$. Let Q be the kernel of the natural map $W^{\langle X_n\rangle} \to W^{(X)}$. It coincides with the kernel of the natural map from the graph product $W^{\langle G/H_{i,n}\rangle}$ to $W^{(G/H_i)}$, and hence contains the kernel of the natural map from the free product $W^{*H_i/H_{i,n}}$ to W. Noting that $H_{i,n}$ has infinite index in H_i, by Lemma 2.7, Q contains a non-abelian free group.

Accordingly the kernel of $W^{\langle X_n\rangle} \rtimes G \to W^{(X)} \rtimes G$ also contains a non-abelian free group for all n, and since $W^{\langle X_n\rangle} \rtimes G$ is a sequence of finitely generated groups converging to $W \wr_X G$, we can conclude as we did for (1): if a finitely presented group maps onto $W^{(X)} \rtimes G$, then the map factorizes through $W^{\langle X_n\rangle} \rtimes G$ for some n. ∎

Theorem 2.11. *Let G, W be groups. Let G act on a nonempty set X. Suppose that $W \wr_X G$ is finitely presented. Then G and W are finitely presented, and, if $W \neq \{1\}$, then the action of G on X has finitely generated stabilizers, and the product action of G on X^2 has a finite number of orbits.*

Proof: By Proposition 2.1, G and W are finitely generated, and G has finitely many orbits on X.

Now observe that G is finitely presented, since it is obtained from $W \wr_X G = W^{(X)} \rtimes G$ by killing a finite generating subset of $W^I \subset W^{(X)}$, where $I \subset X$ is a finite set which contains one point in each orbit.

Suppose now that W is not finitely presented. Then there is a sequence of non-injective surjective morphisms $W_n \to W_{n+1}$ between finitely generated groups, whose limit is W. Then, the sequence of non-injective surjective morphisms between finitely generated groups: $W_n \wr_X G \to W_{n+1} \wr_X G$ converges to $W \wr_X G$, contradicting that $W \wr_X G$ is finitely presented.

Now Proposition 2.10 allows to conclude. ∎

2.2. **Applications.** Our main application consists in proving that the property of being torsion-free is not weakly geometric among finitely presented groups. The examples of [Dy00] are standard wreath products, so are infinitely presented.

Let F be the Thompson group of the dyadic interval (see Example 3.4), and $F_{1/2}$ the stabilizer of $1/2$. The homogeneous space $F/F_{1/2}$ can be identified with the set I of all dyadic numbers contained in the interval $(0,1)$, and the action of F is transitive on ordered pairs (a,b), $a<b$, that is, $F_{1/2}$ has exactly three cosets in T.

Proposition 2.12. *The finitely presented groups* $\mathbf{Z}\wr_I F$ *and* $D_\infty \wr_I F$ *are bi-Lipschitz-equivalent. The first is torsion-free, while the second contains an infinite subgroup of exponent 2.*

Proof: The finite presentation follows from Theorem 2.2. The second assertion reduces, by Proposition A.2, to the fact that $\mathbf{Z}$ and D_∞ are bi-Lipschitz-equivalent. The last assertion is clear. ■

Let S be any non-abelian simple, finitely presented group (possibly finite). Let G be a finitely presented group, with an infinite index, finitely generated subgroup H, such that $H\backslash G/H$ is finite, and such that the action of G on G/H is faithful.

Set $\Gamma = S \wr_{G/H} G$. This group has the following properties:

Proposition 2.13. *1)* Γ *is finitely presented.*

2) Any nontrivial normal subgroup of Γ *contains* $N = S^{(G/H)}$.

Proof: 1) follows from Theorem 2.2.

2) Since the action of G on N is purely outer (that is, the morphism $G \to \mathrm{Out}(N)$ is injective), every nontrivial normal subgroup of G intersects nontrivially N.

On the other hand, any normal subgroup N' intersecting non-trivially N contains it: let us recall the standard argument. For $x \in G/H$ and $s \in S$, denote by $\delta_x(s)$ the function $X \to S$ sending x to s and every $y \neq x$ to 1. If $(s_x)_{x\in G/H}$ be a nontrivial element in $N' \cap N$, then, taking the commutator with a suitable $\delta_x(s)$, we obtain that N' contains $\delta_x(t)$ for some $x \in G/H$ and some $1 \neq t \in S$. Such an element clearly generates N as a normal subgroup. ■

Note that the lattice structure of Γ is obtained from that of G by adding a point "at the bottom".

An example of direct application of Proposition 2.10 is the following well-known result, initially proved in [Shm].

Corollary 2.14. *The free d-solvable group on r generators* $(d, n \geq 2)$ *is not finitely presented.*

Proof: It suffices to observe that if A is a finitely presented group which maps onto the free d-solvable group on r generators $(d, n \geq 2)$, then A contains a free group on two generators. Indeed, such a group maps onto $\mathbf{Z}\wr\mathbf{Z}$, while every finitely presented group mapping onto $\mathbf{Z}\wr\mathbf{Z}$ must contain a free subgroup by Proposition 2.10. ■

3. Subgroups of finite biindex and related properties

3.1. **Definitions and examples.** Theorems 2.2 and 2.11 raise the following question: what finitely presented groups G have an infinite index finitely generated subgroup H such that G acts on $(G/H)^2$ with a finite number of orbits? It is also natural to ask the same question without assuming H finitely generated. These questions seem to have never been systematically investigated, but related properties give useful information for our purposes; for instance subgroup separability, which has been extensively studied for other motivations, such as the generalized word problem. Hence, the purpose of the following definitions is to present various obstructions for a group G to have an almost 2-transitive action on an infinite set.

Definition 3.1. Define a *pair of groups* as a pair (G, H), where G is a group and H a subgroup.

We say that a pair is *finitely presented* if G is finitely presented *and* H is finitely generated.

We say H has *finite biindex* in G if $H\backslash G/H$ is finite. We also say that the pair (G, H) is *almost 2-transitive*; this is equivalent to say that G has finitely many orbits on $(G/H)^2$.

We say H is *almost maximal* in G if there are only finitely many subgroups of G containing H. We also say that the pair (G, H) is *almost primitive.*

We say that a subgroup H of G has *finite proindex* if the profinite closure of H in G (that is, the intersection of all finite index subgroups of G containing H) has finite index in G.

Lemma 3.2. *For pairs (G, H), we have the implications: (H has finite index) $\Rightarrow$ (H has finite biindex) $\Rightarrow$ (H is almost maximal) $\Rightarrow$ (H has finite proindex).*

Proof: The first one is trivial. For the second one, suppose that H has finite biindex m. Every subgroup containing H is an union of double cosets of H; accordingly the number of possible subgroups is bounded by 2^m. For the third implication, observe that if a group has profinite closure of infinite index, this profinite closure must be the intersection of infinitely many finite index subgroups. ■

Remark 3.3. None of these implications is an equivalence, even when G is finitely presented.

- For examples of infinite index subgroups of finite biindex, see Examples 3.4, 3.5, and 3.6 below.
- If $G \neq \{1\}$ has no proper subgroup of finite index (for instance, G is infinite and simple), then $\{1\}$ has finite proindex in G, but is not almost maximal.
- Recall that, for a group G and a subgroup H, the pair (G, H) is called a Hecke pair if, for all $g \in G$, gHg^{-1} and H are commensurable, i.e. they have a common finite index subgroup; equivalently this means that the orbits of H in G/H are finite. On the other hand, H having finite biindex means that there are finitely many such orbits. Thus if (G, H) is a Hecke pair and H has infinite index, then H has infinite biindex. Now it is known that, for any prime p, $(\mathrm{SL}_2(\mathbf{Z}[1/p]), \mathrm{SL}_2(\mathbf{Z}))$ is a Hecke pair, and that $\mathrm{SL}_2(\mathbf{Z})$ is a maximal subgroup of infinite index in $\mathrm{SL}_2(\mathbf{Z}[1/p])$, hence also has infinite biindex.

We only know a restricted sample of faithful almost 2-transitive finitely presented pairs.

Example 3.4. Let G be the Thompson group F. This is the group of piecewise linear increasing homeomorphisms of $[0, 1]$ with singularities in $\mathbf{Z}[1/2] \cap [0, 1]$ and slopes powers of 2. This group is finitely presented and torsion-free, does not contain any non-abelian free subgroup, and has simple derived subgroup (see [CFP96]). The group F acts on $[0, 1] \cap \mathbf{Z}[1/2]$, fixing 0 and 1, and acting transitively on pairs $(a, b) \in \mathbf{Z}[1/2]$ satisfying $0 < a < b < 1$. The stabilizer $F_{1/2}$ of $1/2$ is easily seen to be isomorphic to $F \times F$. So the pair $(F, F_{1/2})$ is almost 2-transitive and finitely presented.

Example 3.5. Let G be the Thompson group T (see [CFP96]) of the circle, which is finitely presented and simple. This is the group of piecewise linear oriented homeomorphisms of the circle $\mathbf{R}/\mathbf{Z}$ with singularities in $\mathbf{Z}[1/2]/\mathbf{Z}$ and slopes powers of 2. The stabilizer H of $0 = 1 \in \mathbf{R}/\mathbf{Z}$ is isomorphic to the Thomposon group F of Example 3.4. Then T acts two-transitively on $T/F = \mathbf{Z}[1/2]/\mathbf{Z}$.

Example 3.6 (Houghton groups)**.** Fix an integer $n \geq 1$. Let $\mathbf{N}$ denote the non-negative integers, and set $\Omega_n = \mathbf{N} \times \{1, \dots, n\}$. We think at Ω_n as the disjoint union of n copies $\mathbf{N}_1, \dots, \mathbf{N}_n$ of $\mathbf{N}$. Let G_n be the group of all permutations σ of Ω_n such that, for all i, $\sigma(\mathbf{N}_i)\Delta\mathbf{N}_i$ is finite, and σ is eventually a translation on $\mathbf{N}_i$.

When $n = 1$, G_1 is the group of permutations with finite support of $\mathbf{N}_1$, while G_n is finitely generated if $n \geq 2$ and finitely presented if $n \geq 3$ (see [Bro87]; Brown attributes the finite presentation when $n = 3$ to R. Burns and D. Solitar; the finite generation is due to Houghton). For an explicit presentation when $n = 3$, see [Jo97].

Note that, for $n \geq 2$, the derived (resp. second derived) subgroup of G coincides with the group of permutations (resp. even permutations) with finite support of Ω_n. In particular, the action of G_n on Ω_n is k-transitive for all k.

On the other hand, as an extension of $\mathbf{Z}^{n-1}$ by a locally finite group, G_n is elementary amenable (but not virtually solvable). The stabilizer H_n of a point is isomorphic to G_n; in particular, it is finitely generated for $n \geq 2$.

Example 3.7. In [RW98], a 3-manifold group Γ together with an infinite index surface subgroup Λ are exhibited; it is proved in [NW98] that $\Lambda\backslash\Gamma/\Lambda$ is finite. (We do not know if the Γ-action on Γ/Λ is faithful.)

Example 3.8. A refinement by D. Wise [Wi03] of a construction of Rips shows that, for every finitely presented group Q, there exists a finitely presented, residually finite, torsion-free, $C'(1/6)$ small cancellation group G and a surjective map $p : G \to Q$, such that $\mathrm{Ker}(p)$ is a finitely generated subgroup of G.

Accordingly, if K is a finitely generated subgroup of finite biindex and infinite index in Q, then $p^{-1}(K)$ is a finitely generated subgroup of finite biindex and infinite index in Q.

Thus, starting from any of the above examples, we obtains examples of almost 2-transitive finitely presented pairs (G, H) with G/H infinite and G torsion-free, word hyperbolic, satisfying the $C'(1/6)$ small cancellation property.

3.2. **Related definitions.** We first introduce some obstructions to the existence of a infinite index subgroup of finite biindex.

Definition 3.9. We say that a group G has Property (PF) (resp. (MF)) (resp. (BF)) if every finite proindex (resp. almost maximal) (resp. finite biindex) subgroup H has finite index.

We also recall that a group is (ERF) if every subgroup is closed for the profinite topology (ERF stands for "Extended Residually Finite").

As a consequence of Lemma 3.2, we have the following implications.

$$ERF \Longrightarrow PF \Longrightarrow MF \Longrightarrow BF$$

Note that these properties are inherited by quotients. Note also that Properties ERF and PF are invariant by commensurability, and that Property ERF is also inherited by subgroups. We show below (Proposition 3.14) that, for finitely generated groups, Properties (PF) and (MF) are equivalent.

Example 3.10. 1) A free non-abelian group is not (BF) [Di90].

2) In [MS81], it is proved that a finitely generated group which is linear over a commutative ring, and not virtually solvable, has a maximal subgroup of infinite index, thus does not have Property (MF).

3) By a result of Ol'shanskii [Ol00], any non-elementary word hyperbolic group has an infinite quotient with no proper subgroup of finite index. In particular, it has a maximal subgroup of infinite index, hence does not satisfy Property (MF).

4) Hall [Ha59] has exhibited finitely generated 3-solvable groups with infinite index maximal subgroups, hence without Property (MF).

5) If G is a virtually solvable group which is not virtually polycyclic, then it is proved in [Al99] that G has a subgroup H conjugate to a proper subgroup of itself. In particular, G is not ERF.

6) A virtually polycyclic group is ERF (Malcev [Mal]). We do not know if there are other examples of finitely generated ERF groups.

7) We prove (Proposition 3.20) that if a finitely generated group Γ is an extension with virtually polycyclic quotient and nilpotent kernel, then Γ has Property (MF). In particular, this holds when Γ is a linear virtually solvable group.

8) The first Grigorchuk group Γ has Property (PF) (Pervova [Per00], Grigorchuk and Wilson [GW03]). It is not ERF: indeed, it has a subgroup isomorphic to a direct sum $\bigoplus C_{2^{n_i}}$ with $n_i \to \infty$, thus mapping onto the quasi-cyclic group C_{2^∞} which is not residually finite. Accordingly, Γ has a subgroup which is not ERF, hence Γ is neither ERF. On the other hand, it is an open question to find a group of subexponential growth which does not have Property (PF); equivalently to find group of subexponential growth with a maximal subgroup of infinite index.

We now introduce similar obstructions to the existence of a *finitely generated* infinite index subgroup of finite biindex.

Definition 3.11. We say that a group G has Property (LPF) (resp. (LMF)) (resp. (LBF)) if every finite proindex (resp. almost maximal) (resp. finite biindex) *finitely generated* subgroup H has finite index.

We also recall that a group is LERF if every finitely generated subgroup is closed for the profinite topology. (LERF is also called "subgroup separable"). In these four abbreviations, the letter L stands for "locally".

Again as a consequence of Lemma 3.2, we have the following implications.

$$\begin{array}{ccccccc}
ERF & \Longrightarrow & PF & \Longrightarrow & MF & \Longrightarrow & BF \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
LERF & \Longrightarrow & LPF & \Longrightarrow & LMF & \Longrightarrow & LBF
\end{array}$$

Note that the properties in the second row are no longer inherited by quotients: indeed, free groups are LERF (see the example below) but do not have Property BF (see Example 3.10). Note also that Properties LERF and LPF are invariant by commensurability, and that Property LERF is also inherited by subgroups.

In the literature, a group is defined to have the *engulfing* Property if every proper finitely generated subgroup is contained in a proper finite index subgroup. Clearly, a group has Property (LPF) if and only if all its finite index subgroups have the engulfing Property.

Example 3.12. 1) A free non-abelian group is LERF [HaJr49].

2) The first Grigorchuk group Γ is LERF (Pervova [Per00], Grigorchuk and Wilson [GW03]). On the other hand, non-residually finite groups of subexponential growth appear in [Ers04].

3) If G is the Baumslag-Solitar group $BS(1,p)$ ($|p| \geq 2$), then G is not LERF, since its subgroup $\mathbf{Z}[1/p]$ is not LERF (its quotient by a cyclic subgroup is divisible). If G is a standard wreath product $A \wr \mathbf{Z}$, with A finitely generated abelian, then G is LERF but not ERF (Proposition 3.19).

4) In [NW98], an example of free-by-cyclic 3-manifold group which fails to satisfy Property (LPF) is given.

5) If $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ is a lattice, then Γ has Property (LMF). More precisely, for every finitely generated subgroup of infinite index $\Lambda \subset \Gamma$, there exists a strictly decreasing sequence of subgroups $\Lambda \subset \Lambda_n \subset \Gamma$. All this follows from the proof of [GSS05, Theorem 1.1]. On the other hand, it is not known if Γ is always LERF, or even has (LPF).

6) Example 3.8, and the bare existence of finitely generated groups without Property (LBF) (Examples 3.4, 3.5, 3.6) imply the existence of torsion-free word hyperbolic groups without Property (LBF).

3.3. **Nearly maximal subgroups.** Recall [Ril69] that a subgroup H of a group G is *nearly maximal* if H is maximal among infinite index subgroups. A standard verification shows that every infinite index subgroup of a finitely generated group G is contained in a nearly maximal subgroup.

Observation 3.13. If H is a nearly maximal subgroup of a group G, then either H is closed or has finite proindex.

This obvious result has the following consequence. Suppose that a finitely generated group does not have Property (PF). Let H be an infinite index subgroup with finite proindex. Then H is contained in a nearly maximal subgroup M. Clearly, M has also finite proindex. So the only subgroups containing M are those which contain $\overline{M}$, and there are finitely many, so that M is almost maximal. This proves:

Proposition 3.14. *Let G be a finitely generated group. The following are equivalent.*
(i) G has Property (PF).
(ii) Every nearly maximal subgroup of G is profinitely closed.
(iii) G has Property (MF).

Remark 3.15. 1) On the other hand, it is not clear whether Property (LMF) implies Property (LPF). We actually conjecture that it is not true. A possible counterexample could be a free product $G * G$, where G is any nontrivial finitely generated group without any nontrivial finite quotient, but we do not know how to prove Property (LMF) for such a group. Note also that although $SL_n(\mathbf{Z})$ ($n \geq 3$) is known not to have Property (LPF) [SV00], whether it has Property (LMF) is open (by [MS81] it does not have Property (MF)).

2) Note that the (infinitely generated) quasi-cyclic group $C_{p^\infty} = \mathbf{Z}[1/p]/\mathbf{Z}$ is (PF) but not (MF).

A consequence of Proposition 3.14 is that, for finitely generated groups, Property (MF) is a commensurability invariant. We do not know if this it is true for Property (LMF); however, we have:

Proposition 3.16. *Property (LMF) is inherited by subgroups of finite index.*

Lemma 3.17. *Let G be a group without Property (LMF). Then G has a finitely generated nearly maximal subgroup which is almost maximal.*

Proof: Let H_0 be a finitely generated, almost maximal subgroup of infinite index. If H_0 is not nearly maximal, it is properly contained in a subgroup K_1 of infinite index; define H_1 as the subgroup generated by H_0 and one element in $K_0 - H_0$. Go on defining an increasing sequence of finitely generated subgroups of infinite index. This processus stops, since H is almost maximal. So, for some n, H_n is nearly maximal, and, since it contains H_0, it is almost maximal. ■

Proof of Proposition 3.16 Note that a non-finitely generated group has necessarily Property (LMF). Let G be a finitely generated group, and H a subgroup of finite index. Suppose that H does not have Property (LMF). By Lemma 3.17, let M be a finitely generated nearly maximal, almost maximal subgroup of H. Then M is contained in a nearly maximal subgroup M' of G. Since M' has infinite index in G and H has finite index in G, $M' \cap H$ has infinite index in H, so that $M' \cap H = M$. In particular, M has finite index in M', so that M' is also finitely generated.

It is clear that M' is not profinitely closed in G: otherwise, so would be $M = M' \cap H$, and M would also be closed in H. ■

3.4. **Finitely generated solvable groups.**

Lemma 3.18. *Let G be a finitely generated group which has a surjective morphism p onto an abelian group A, with abelian kernel K. Let H be a subgroup of G such that $p(H) = A$. Then H is closed for the profinite topology.*

Proof: The assumption implies that $H \cap K$ is normal in G. Upon replacing G by $G/(H \cap K)$, we can suppose that $H \cap K = \{1\}$, so that $G = H \ltimes K$. To see that H is closed for the profinite topology, it clearly suffices to show that its profinite closure has trivial intersection with K. Thus, let $k \in K - \{1\}$ belong to the profinite closure of H.

Since G is finitely generated and metabelian, it is residually finite (see [Rob82]). So, there exists a finite index subgroup J of H and a finite index subgroup L of K, normalized by J, such that $k \notin L$. If $h \in H$, observe that hL only depends on the class of h modulo J. Set $M = \bigcap_{h \in H/J} hL$. Then $H \ltimes M$ contains H, has finite index in G, and does not contain k. This is a contradiction. ■

Proposition 3.19. *Let G be a standard wreath product $A \wr \mathbf{Z}$, with $A \neq \{1\}$ finitely generated abelian. Then G is LERF, but not ERF.*

Proof: 1) It is not ERF because the subgroup $A^{(\mathbf{N})}$ of $A^{(\mathbf{Z})} \rtimes \mathbf{Z}$ is not closed for the profinite topology, since it is conjugate to a proper subgroup of itself.

2) Let H be a finitely generated subgroup of G. Let us show that H is closed for the profinite topology.

First case: H is not contained in $A^{(\mathbf{Z})}$. Then the projection of H in $\mathbf{Z}$ is a subgroup $n\mathbf{Z}$ of $\mathbf{Z}$ ($n \geq 1$). It clearly suffices to show that H is closed in $A^{(\mathbf{Z})} \rtimes n\mathbf{Z}$, and this is a consequence of Lemma 3.18.

Second case: H is contained in A_0, where $A^{(\mathbf{Z})} = \bigoplus_{i \in \mathbf{Z}} A_i$. There exists a finite abelian group K and a morphism $p_0 : A_0 \to K$, whose kernel contains H but not h. By the identification $A_i = A = A_0$, this gives rise to a morphism $p_i : A_i \to K$. Let

$p = \bigoplus_i p_i$. Extend p to G by putting $\mathbf{Z}$ in the kernel. Then p is a morphism of G to a finite group whose kernel contains H but not h. Observe that in this special case, h is not contained in the normal subgroup generated by H.

Third case: H is contained in $A^{(\mathbf{Z})}$. Take a finite subset F of $\mathbf{Z}$ containing all supports of generators of H and h. Let n be greater than the diameter of F. Then H is contained in the finite index subgroup $A^{(\mathbf{Z})} \rtimes n\mathbf{Z}$, which is isomorphic to $A^n \wr \mathbf{Z}$. This permits to reduce to the second case. ■

Example 3.12(4) indicates that it is not obvious how to generalize Proposition 3.19. It would be interesting to characterize LERF groups among finitely generated solvable groups; even in the case of metabelian groups this is open.

Here is now a result about Property (MF) for a class of finitely generated solvable groups.

Proposition 3.20. *Let G be group, which is nilpotent-by-(virtually polycyclic) (i.e. lies in a extension with nilpotent kernel and virtually polycyclic quotient). Then G has Property (MF).*

Note that every finitely generated, virtually solvable group which is linear over a field is nilpotent-by-(virtually abelian), hence belongs to this class. In particular, this encompasses a result of Margulis and Soifer (the easier implication in main Theorem of [MS81]). The main ingredient to prove Proposition 3.20 is the following deep result:

Theorem 3.21 (Rosenblade, [Ros73])**.** *Let H be a virtually polycyclic group, and let M be a simple $\mathbf{Z}H$-module. Then M is finite.*

Proof of Proposition 3.20: let G be a finitely generated group, N a nilpotent, normal subgroup, such that G/N is virtually polycyclic.

Suppose by contradiction that G does not have Property (MF). Passing to a subgroup of finite index if necessary, we can suppose that G has a maximal subgroup M of infinite index. We can suppose that M contains no nontrivial normal subgroup of G. The centre $Z(N)$ of N is normal in G. Since M does not contain any nontrivial normal subgroup, M does not contain $Z(N)$. By maximality, $MZ(N) = G$. Thus, since $M \cap Z(N)$ is normalized by both M and $Z(N)$, it is a normal subgroup of G contained in M, hence is trivial. Accordingly, G is the semidirect product of M by $Z(N)$. Since M is a maximal subgroup, $Z(N)$ is a simple M-module, and actually a simple $M/(M \cap N)$-module since N acts trivially on its centre. Since $M/(M \cap N)$ is virtually polycyclic (as a subgroup of G/N), by Theorem 3.21, $Z(N)$ is finite. Hence M has finite index, contradiction. ■

Remark 3.22. P. Hall has constructed [Ha59] a 3-solvable group G with a maximal subgroup of infinite index which is finitely generated and metabelian. In particular, G does not have Property (LMF), so that "nilpotent-by-polycyclic" cannot be replaced by "3-solvable" in Proposition 3.20.

We do not know if there exists a finitely generated solvable group with Property (BF). However, using standard arguments, we have the following result.

Proposition 3.23. *The following are equivalent.*

(1) There exists a finitely generated n-solvable group without Property (BF).

(2) There exists a finitely generated n-solvable group without Property (LBF).

(3) There exists a finitely generated $(n-1)$-solvable group Γ, and an infinite Γ-module V, such that the action of Γ on V has finitely many orbits.

Proof: (2)$\Rightarrow$(1) is trivial.

(3)$\Rightarrow$(2). Observe that Γ is a finitely generated subgroup of finite biindex in $\Gamma \ltimes V$.

Suppose (1). Let G be a finitely generated solvable group, and M a subgroup of finite biindex and infinite index. Replacing M by a larger subgroup if necessary, we can suppose it nearly maximal, and upon replacing G by the profinite closure of M, we can suppose M maximal. Moreover, taking the quotient by a normal subgroup if necessary, we can suppose the only normal subgroup of G contained in M is $\{1\}$, i.e. G acts faithfully on G/M.

Let A be the last nontrivial term of the derived series of G. Then A is a normal subgroup and $A \neq \{1\}$, so that A is not contained in M. Accordingly, $MA = G$. Observe that $M \cap A$ is normalized both by M (since A is normal) and by A (since A is abelian). It follows that $M \cap A$ is normal in G; therefore $M \cap A = \{1\}$, and $G \simeq M \ltimes A$. Since M has finite biindex in G, M acts with finitely many orbits on A. ■

For $n \geq 3$, we leave as open whether the equivalent statements of Proposition 3.23 are true. For $n \leq 2$, they are false as a consequence of Proposition 3.20. We record this in the following:

Question 3.24. 1) Does there exist a finitely generated, solvable group without Property (LBF)?

2) Does there exist a finitely presented solvable group without Property (LBF)?

3) Does there exist a finitely presented solvable group without Property (MF)?

The existence of a finitely generated solvable group without Property (LBF) would permit to construct solvable finitely presented wreath products, and would imply, arguing as in Proposition 2.12, that the class of virtually solvable groups is not invariant under quasi-isometries within the class of finitely presented groups.

3.5. **Amalgams and obstructions to Property ((L)BF).** The following theorem is due to M. Hall in the case of free groups, P. Scott in the case of surface groups, and to Brunner, Burns, and Solitar [BBS84] for the general case.

Theorem 3.25. *Let G be the amalgam of two free groups over a cyclic subgroup. Then G is LERF.*

In contrast, Burger and Mozes [BM00] have constructed amalgams of two free groups over a finite index subgroup which are finitely presented simple groups. We do not know if these groups have Property (LMF).

These examples indicate that amalgams may have very different behaviours, so that it seems that no general statement can be made.

The following result is a particular case of Theorem 2 in [KS73].

Theorem 3.26 (Karrass, Solitar (1973))**.** *Let G be a finitely generated group which splits as a nontrivial amalgam over a finite subgroup. Then G has Property (LBF).*

In contrast, if F is a non-abelian free group, there exist many 2-transitive pairs (G, H) with H infinitely generated: it suffices to map F to any two-generated group which has a 2-transitive action on a infinite set, as in Example 3.6. It is shown in [Di90] that, in a suitable sense, a generic subgroup on $n \geq 2$ generators of the symmetric group $\mathrm{Sym}(\mathbf{N})$ is free and 2-transitive, showing that free groups also have *faithful* 2-transitive actions on infinite sets.

Example 3.27. Let G be a infinite group, all of whose subgroups are either finite or of finite index. Then G has clearly Property (BF). If, moreover, G has no subgroup of finite index, G has a maximal subgroup which is finite; in particular G does not have Property (LMF). There exist nontrivial examples of such groups: infinite two-generator groups all of whose nontrivial proper subgroups are isomorphic to $\mathbf{Z}/p\mathbf{Z}$, p a big prime, have been constructed by Ol'shanskii (see [Ol91]). All known examples of such groups are infinitely presented.

3.6. **Fibre products.** Let G_1, G_2, Q be groups and $p_i : G_i \to Q$ a surjection for $i = 1, 2$. We are interested in the pair (G, H), where $G = G_1 \times G_2$ and H is the fibre product $G_1 \times_Q G_2 = \{(x, y) \in G_1 \times G_2,\ p_1(g_1) = p_2(g_2)\}$.

Proposition 3.28. *1) There is a natural order-preserving bijection between the set of subgroups of $G_1 \times G_2$ containing $H = G_1 \times_Q G_2$ and the set of normal subgroups of Q. It induces a bijection between finite index normal subgroups of Q and finite index subgroups of $G_1 \times G_2$ containing $G_1 \times_Q G_2$.*

2) Suppose that G_1 and G_2 are finitely generated. If Q is finitely presented, then $G_1 \times_Q G_2$ is finitely generated. Conversely, if G_1 and G_2 are in addition finitely presented and if $G_1 \times_Q G_2$ is finitely generated, then Q is finitely presented.

Proof: 1) If K is a subgroup of $G_1 \times G_2$ containing $H = G_1 \times_Q G_2$, then set $u(K) = p_1(K \cap (G_1 \times \{1\}))$. This is a normal subgroup of Q, because $K \cap (G_1 \times \{1\})$ is normal in G_1 (identified with $G_1 \times \{1\}$). Indeed, let x belong to $K \cap (G_1 \times \{1\})$. This means that $(x, 1) \in K$. Fix $y \in G_1$ and let us check that $yxy^{-1} \in K \cap (G_1 \times \{1\})$, i.e. $(yxy^{-1}, 1) \in K$. Choose $a \in G_2$ such that $p_2(a) = p_1(y)$. Then $(y, a) \in H \subseteq K$, so that $(yxy^{-1}, 1) = (y, a)(x, 1)(y, a)^{-1}$ also belongs to K.

If N is a normal subgroup of Q, set $v(Q) = G_1 \times_{Q/N} G_2$. This is a subgroup of $G_1 \times G_2$ containing H. We claim that u and v are inverse bijections (clearly, they preserve the order).

- $K \subseteq v(u(K))$: Let (x, y) belong to K. Write $p_2(y) = p_1(a)$ for some $a \in G_1$, so that $(a, y) \in H \subseteq K$. Then $(xa^{-1}, 1) = (x, y)(a, y)^{-1} \in K$. Thus $p_1(xa^{-1}) = p_1(x)p_2(y)^{-1} \in u(K)$. This means that $(x, y) \in G_1 \times_{Q/u(K)} G_2 = v(u(K))$.
- $v(u(K)) \subseteq K$: Let (x, y) belong to $v(u(K)$. This means that $p_1(x)p_2(y)^{-1} \in u(K)$, i.e. $p_1(x)p_2(y)^{-1} = p_1(a)$ for some $a \in G_1$ such that $(a, 1) \in K$. Therefore $(xa^{-1}, y) \in H \subseteq K$, so that $(x, y) = (xa^{-1}, y)(a, 1) \in K$.
- $N \subseteq u(v(N))$: Let α belong to N. Write $\alpha = p_1(x)$ for some $x \in X$. Then $(x, 1) \in G_1 \times_{Q/N} G_1 = v(N)$, so that $\alpha \in u(v(N))$.
- $u(v(N)) \subseteq N$: Let α belong to $u(v(N))$. This means that $\alpha = p_1(x)$, for some $x \in G_1$ such that $(x, 1) \in G_1 \times_{Q/N} G_2$, so that $p_1(x) = \alpha \in N$.

2) Suppose that G_1 and G_2 are finitely generated, and Q is finitely presented. For $i = 1, 2$, write $N_i = \mathrm{Ker}(p_i)$. Since Q is finitely presented and G_i finitely generated, N_i generated as a normal subgroup in G_i by a finite subset R_i. Besides, take a finite subset S of $H = G_1 \times_Q G_2$ such that $p_i(S)$ generates Q for $i = 1, 2$. Then $(R_1 \times \{1\}) \cup (\{1\} \times R_2) \cup S$ is a finite generating subset for a subgroup M of H. We claim that $M = H$. Let (x, y) belong to H. The hypothesis on S implies that

there exists $z \in G_2$ and $a \in N_1$ such that $(ax, z) \in M$. Since $(ax, z) \in M \subseteq H$, $p_1(x) = p_2(z)$, so that $z^{-1}y \in N_2$. Hence $(x, y) = (a, 1)^{-1}(ax, z)(1, z^{-1}y)$. We claim that $(a, 1) \in M$. Indeed, $a \in N_1$, and, using that $p_1(S)$ generates G_1 and R_1 generates N_1 as a normal subgroup, we obtain that $(a, 1) \in M$. Similarly, $(1, z^{-1}y) \in M$, and therefore $(x, y) \in M$.

Conversely, suppose that G_1 and G_2 are finitely presented and suppose that H is finitely generated. There exists a finitely presented group Q', a surjective map $q : Q' \to Q$, surjective maps $q_i : G_i \to Q'$, $i = 1, 2$, such that $p_i = q \circ q_i$ for $i = 1, 2$. If Q is not finitely presented, then the kernel of q can be written as a union of an increasing sequence of subgroups M_n, normal in Q'. By (1), the normal subgroups intermediate between $\mathrm{Ker}(q_1)$ and $\mathrm{Ker}(p_1)$ correspond bijectively with the subgroups intermediate between $G_1 \times_{Q'} G_2$ and $G_1 \times_Q G_2$. Accordingly, the later is not finitely generated. ■

Corollary 3.29. *Let G_1, G_2 be finitely generated groups. The subgroup $G_1 \times_Q G_2$ has finite proindex (resp. is almost maximal) in $G_1 \times G_2$ if and only if Q has a minimal finite index subgroup (resp. Q has a finite number of normal subgroups).*

Remark 3.30. The question of finite presentability of a fibre product $G_1 \times_Q G_2$ is not trivial at all. It is easy to show that G_1 and G_2, and Q must necessarily be finitely presented, but the converse is not true. For instance, take the Baumslag-Solitar group $BS(1, p) = \mathbf{Z} \ltimes_p \mathbf{Z}[1/p]$ with $p \geq 2$, which has presentation $\langle x, y \mid {}^x y = y^p \rangle$. There are two morphisms p_+, p_- of this group onto $\mathbf{Z}$. This gives, up to isomorphism, two possible fiber products over $\mathbf{Z}$, which we denote by $BS(1, p) \times_{\mathbf{Z}++} BS(1, p)$ and $BS(1, p) \times_{\mathbf{Z}+-} BS(1, p)$. Then the former is finitely presented, while the second is not. The first has presentation $\langle x, y, z \mid [y, z] = 1,\ {}^x y = y^p,\ {}^x z = z^p \rangle$, while the second has a finitely generated central extension by $\mathbf{Z}[1/p]$, given by the semidirect product of the diagonal subgroup of $SL_2(\mathbf{Z}[1/p])$ by the Heisenberg group $H_3(\mathbf{Z}[1/p])$, hence is not finitely presented. For more about the finite presentation of fibre products, see [Gru78, BBHM03, BrG04].

Proposition 3.31. *The subgroup $H = G_1 \times_Q G_2$ is has finite biindex in $G_1 \times G_2$ if and only if Q has a finite number of conjugacy classes.*

Proof: It suffices to check that every double coset of H contains an element of $G_1 \times \{1\}$, and that two elements $(x, 1)$ and $(y, 1)$ of $G_1 \times \{1\}$ are in the same double coset if and only if the images of x and y in Q are conjugate. ■

Remark 3.32. Examples of infinite, finitely generated groups with finitely many conjugacy classes have been constructed by S. Ivanov (see [Ol91, Theorem 41.2]), and examples with exactly one nontrivial conjugacy class have recently been announced by D. Osin [Os04]. But it is an open question to find infinite finitely presented groups with finitely many conjugacy classes.

3.7. **Hereditary properties.**

Lemma 3.33. *If N is normal in G, then the following statements are equivalent: (i) N is almost maximal, (ii) N has finite index.*

Proof: It suffices to show (i)⇒(ii), which is equivalement to the statement: every infinite group has infinitely many subgroups. If G/N is torsion, then it is the union of its finite subgroups, so they are infinitely many. Otherwise, G/N contains an infinite cyclic subgroup, which contains infinitely many subgroups. ■

Lemma 3.34. *Suppose H_1, H_2 are subgroups of G, $H_1 \subset H_2$. Suppose that H_1 has finite biindex (resp. is almost maximal) in G. Then H_2 has finite biindex (resp. is almost maximal) in G and H_1 has finite biindex (resp. is almost maximal) in H_2.*

Proof: The statement for with "almost maximal" is trivial.

Suppose that H_1 has finite biindex in G. Trivially, so has H_2. Write $G = \bigcup_{i \in I} H_1 g_i H_1$, with I finite, and set $J = \{i \in I, g_i \in H_2\}$. Then $H_2 = \bigcup_{i \in J} H_1 g_i H_1$, so that H_1 has finite biindex in G.

Lemma 3.35. *Suppose H_1, H_2 are subgroups of G, H_1 is contained as a subgroup of finite index in H_2, and H_2 has finite biindex in G. Then H_1 has finite biindex in G.*

Proof: Write $G = \bigcup_i H_2 g_i H_2$, $H_2 = \bigcup_j h_j H_1 = \bigcup_k H_1 h'_k$, all these unions being finite. Then $G = \bigcup_{i,j,k} H_1 h'_k g_i h_j H_1$. ■

Remark 3.36. The converse of Lemma 3.34 is false in both cases.

In the case of finite biindex, let $G = SL(2, K)$, K an algebraically closed field, T the subgroup of upper triangular matrices in G, D the diagonal matrices in G. Then G is two-transitive on $G/T \simeq P^1(K)$, and T is two-transitive on $T/D \simeq K$, the affine line. But, by a dimension argument, the action of D on G/D cannot have a finite number of orbits. On the other hand, we do not know any counterexample with G finitely generated.

For a counterexample with almost maximal subgroups, which also shows that the analogue of Lemma 3.35 is false with almost maximal subgroups, take an infinite group G with a finite maximal subgroup H. Such groups are constructed in [Ol91]. So H is almost maximal in G and $\{1\}$ has finite index in H. But, by Lemma 3.33 $\{1\}$ is not almost maximal in G.

Remark 3.37. Here is a trivial consequence of Lemma 3.34. Let G_1 be a group, G_2 is a finite index subgroup of G_1, and H a subgroup of G_2. Then, if H has finite biindex (resp. is almost maximal) in G_2, it has also finite biindex (resp. is almost maximal) in G_1.

The point is that we do not know, in both cases, if the converse is true.

Lemma 3.38. *Suppose that, for $i = 1, 2$, H_i has finite biindex (resp. is almost maximal) in G_i. Then $H_1 \times H_2$ has finite biindex (resp. is almost maximal) in $G_1 \times G_2$.*

Proof: This is obvious with finite biindex. Suppose that, for $i = 1, 2$, H_i is almost maximal in G_i. If there are infinitely many subgroups containing $H_1 \times H_2$, infinitely many have the same intersection K_i and projection P_i on G_i for $i = 1, 2$. Note that K_i is normal in P_i. Since, as a consequence of Lemma 3.34, K_i is almost maximal in P_i, this implies, by Lemma 3.33, that P_i/K_i is finite for $i = 1, 2$. Since only finitely many subgroups can exist between $K_1 \times K_2$ and $P_1 \times P_2$, we have a contradiction. ■

Lemma 3.39. *If H has finite biindex (resp. is almost maximal) in G and N is a normal subgroup of G, then $H/(H \cap N)$ has finite biindex (resp. is almost maximal) in G/N.*

Proof: For the case of finite biindex, pass the expression $G = \bigcup_{i=1}^n H g_i H$ to the quotient. The statement for almost maximal subgroups is trivial. ■

Proposition 3.40. *Properties (BF) and (LBF) are inherited from finite index subgroups.*

Proof: Let G be a group and N a finite index subgroup. Suppose that N is (L)BF. Let H be a (finitely generated) almost maximal subgroup in G. By Lemma 3.35, $H \cap N$ has finite biindex in G (and is also finitely generated), so has finite biindex in N by Lemma 3.34. Since N has Property (L)BF, $H \cap N$ has finite index in N, so that H has finite index in G. ■

Remark 3.41. We do not know if Properties (BF) and (LBF) are inherited by finite index subgroups. This motivates the following question.

Question 3.42. Let G be a group, N a subgroup of finite index, and H a subgroup of N.

If H has finite biindex in N, must it have finite biindex in G?

Remark 3.43. We are especially interested to an answer to these questions under the assumption that G is finitely generated.

If Question 3.42 has a positive answer, then Properties (BF) and (LBF) are inherited by finite index subgroups.

3.8. **Faithful almost 2-transitive pairs.** We could define weaker analogs of Properties (PF) through (LBF), say (fPF), etc., by only considering subgroups H such that the action of G on G/H is faithful.

We do not know much about these properties. It follows from the result of Dixon [Di90] noticed in §3.5 that non-abelian free groups do not have Property (fBF). The non-existence of a faithful primitive action, which is, for infinite groups, a priori slightly stronger to Property (fMF), is widely investigated in [GG05].

Examples 3.4, 3.5, and 3.6 provide essentially the only examples of finitely presented groups which we know not to have Property (fLBF). In the finitely generated case, we also have the groups $G \times G$ when the infinite group G has finitely many conjugacy classes, and has trivial center (this latter assumption is always satisfied modulo a finite normal subgroup). Note that, in all these examples, G has very few normal subgroups: for F and the Houghton groups, G has simple derived subgroup; T is itself simple, and the groups with finitely many conjugacy classes have finitely many normal subgroups. We therefore ask:

Question 3.44. Does there exist a residually finite group which acts almost 2-transitively and faithfully on an infinite set, with finitely generated stabilizers?

The answer is yes when "almost 2-transitively" is replaced by "primitively", as shows the example, pointed out in [GG05], of the action of $PSL_2(\mathbf{Z}[1/p])$ on $PSL_2(\mathbf{Z}[1/p])/PSL_2(\mathbf{Z})$.

Appendix A. Length of words in wreath products

We consider the wreath product $A = W \wr_{G/H} G$, where W and G are finitely generated. We write W additively although it is not necessarily abelian. We write the elements of A: (f, c), where $f \in W^{(G/H)}$ and $c \in G$; we denote by x_0 the base-point of G/H. If $w \in W$ and $c \in G$, by the abusive notation (w, c), we mean $(\delta_{x_0}(w), c)$, where δ_{x_0} is the natural inclusion of W into the x_0-component of $W^{(G/H)}$. The product in A is given by $(f_1, c_1)(f_2, c_2) = (f_1 + c_1 f_2, c_1 c_2)$. Fix a symmetric finite generating set S of G. We call a path of length n in G a sequence $(g_0,\dots,g_n)$ such that $g_0=1$ and $g_{i+1}g_i^{-1}\in S$ for all $i=0,\dots,n-1$. For any finite subset F of X and $c\in G$, let $K(F,c)$ be the minimal length of a path $(g_0,\dots,g_n)$ in G such that $g_n=c$ and $F\subset\{g_0x_0,\dots,g_nx_0\}$. On the other hand, fix a finite symmetric generating subset T of W, and denote by $|\cdot|$ the corresponding word length. For $f\in W^{(X)}$, set $|f|=\sum_{x\in X}|f(x)|$. Fix, as generating set of A, the union of $(t,1)$ for $t\in T$ and $(0,s)$ for $s\in S$. Denote again by $|\cdot|$ the word length in A.

Lemma A.1. *For $f\in W^{(G/H)}$ and $c\in G$, we have $|(f,c)|=K(\mathrm{supp}(f),c)+|f|$.*

Proof: Set $n=K(\mathrm{supp}(f),c)$.

Let $1=g_0,g_1\dots,g_n$ be a path of length n such that $g_n=c$ and, whenever $x\in\mathrm{Supp}(f)$, $x=g_ix_0$ for some i. For all i, set $g_{i+1}=s_ig_i$ with $s_i\in S$, and $x_i=g_ix_0$.

Then
$$\begin{aligned}(f,c)&=(f(x_0),s_1)(f(x_1),s_2)\dots(f(x_{n-1}),s_n)(f(x_n),1)\\&=(f(x_0),1)(0,s_1)(f(x_1),1)(0,s_2)\dots(f(x_{n-1}),1)(0,s_n).\end{aligned}$$

In both cases, (f,c) can be expressed as a product of $K(\mathrm{supp}(f),c)+|f|$ generators. Accordingly, for all (f,c), $|(f,c)|\le K(\mathrm{supp}(f),c)+|f|$. Conversely suppose that (f,c) can be expressed as a product of n generators. Putting generators of W_{x_0} together, we get $(f,c)=(w_1,1)(0,s_1)(w_2,1)(0,s_2)\dots(w_m,1)(0,s_m)(w_{m+1},1)=(w_1,s_1)\dots(w_m,s_m)(w_{m+1},1)$ with each $w_i\in W$, with each $s_i\in S$. Set $g_i=\prod_{j=1}^i s_i$.

Then $(g_0,g_1,\dots,g_m)$ is a path joining 1 to $g_m=c$. Besides, $f=w_1+g_1w_2+\dots+g_mw_{m+1}$, so that $\mathrm{Supp}(f)$ is contained in $\{g_0x_0,g_1x_0,\dots,g_{m-1}x_0,g_mx_0\}$. Accordingly, $K(\mathrm{Supp}(f),c)\le m$, and $n=\sum_i|w_i|+m\ge|f|+K(\mathrm{Supp}(f),c)$. ■

This immediately implies the following result, first observed by A. Erschler-Dyubina [Dy00] in the case of standard wreath products.

Proposition A.2. *Let G be a group, H a subgroup, and W_1,W_2 two bi-Lipschitz-equivalent groups. Then $W_1\wr_{G/H}G$ and $W_2\wr_{G/H}G$ are bi-Lipschitz-equivalent.*

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Yves de Cornulier
EPF Lausanne, IGAT
E-mail: `decornul@clipper.ens.fr`