ON THE VANISHING OF REDUCED 1-COHOMOLOGY FOR BANACHIC REPRESENTATIONS

YVES CORNULIER AND ROMAIN TESSERA

Abstract. A theorem of Delorme states that every unitary representation of a connected Lie group with nontrivial reduced first cohomology has a finite-dimensional subrepresentation. More recently Shalom showed that such a property is inherited by cocompact lattices and stable under coarse equivalence among amenable coutable discrete groups. In this note we give a new geometric proof of Delorme's theorem which extends to a larger class of groups, including solvable p-adic algebraic groups, and finitely generated solvable groups with finite Prüfer rank. Moreover all our results apply to isometric representations in a large class of Banach spaces, including reflexive Banach spaces. As applications, we obtain an ergodic theorem in for integrable cocycles, as well as a new proof of Bourgain's Theorem that the 3-regular tree does not embed quasi-isometrically into a superreflexive Banach space.

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1. Introduction

1.1. **Background.** Let G be a locally compact group. We consider representations of G into Banach spaces. It is thus convenient to call G-module a Banach space V endowed with a representation ρ of G, by bounded automorphisms, in

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a way that the mapping $g \mapsto gv = \rho(g)v$ is continuous for all $v \in V$. We will denote by V^G the subspace of G-fixed points.

The space $Z^1(G,V)$ (also denoted $Z^1(G,\rho)$) of 1-cocycles is the set of continuous maps $b : G \to V$ satisfying the 1-cocycle condition $\rho(gh) = \rho(g)b(h) + b(g)$. It is endowed with the topology of uniform convergence on compact subsets. The subspace of coboundaries $B^1(G,V)$ consists of those b of the form $b(g) = v - \rho(g)v$ for some $v \in V$. It is not always closed, and the quotient of $Z^1(G,V)$ by its closure is called the first reduced cohomology space $\overline{H^1}(G,V)$ (or $\overline{H^1}(G,\rho)$). See notably the reference book [G80].

Vanishing properties of the first reduced cohomology has especially been studied in the context of unitary representations on Hilbert spaces. If G satisfies Kazhdan's Property T, it is a classical result of Delorme that $\overline{H^1}(G,V) = 0$ (and actually H^1 itself vanishes) for every unitary Hilbert G-module V. See also [BHV08, Chapter 2]. For G a discrete finitely generated group, the converse was established by Mok and Korevaar-Schoen [M95, KS97]: if G fails to satisfy Kazhdan's Property T then $\overline{H^1}(G,V) \neq 0$ for some unitary Hilbert G-module V. A more metrical proof was provided by Gromov in [G03], and Shalom [S00] extended the result to the setting of non-discrete groups, see also [BHV08, Chap. 3].

The groups in which we will be interested will usually be amenable (and non-compact), so that this non-vanishing result holds. However, it often happens that unitary representations with non-vanishing 1-cohomology for a given group are rare. For instance, it is an easy observation of Guichardet [G72] that if G is abelian, or more generally nilpotent, and V is a Hilbert G-module with $V^G = 0$, then $\overline{H^1}(G,\pi) = 0$. In particular, the only irreducible unitary representation with non-vanishing $\overline{H^1}$ is the trivial 1-dimensional representation. Shalom [Sh04] thus introduced the following terminology: if G satisfies the latter property, it is said to satisfy Property $\mathcal{H}_{\mathrm{t}}$. He also introduced a natural slightly weaker invariant: G has Property $\mathcal{H}_{\mathrm{fd}}$ if for every unitary Hilbert G-module with no G-submodule (= G-invariant closed subspace) of positive finite dimension, we have $\overline{H^1}(G,V) = 0$. This can also be characterized as follows: G has Property $\mathcal{H}_{\mathrm{fd}}$ if and only if every irreducible unitary representation with $\overline{H^1} \neq 0$ has finite dimension, and there are only countably many up to equivalence. Nontrivial examples were provided by the following theorem of P. Delorme.

Theorem ([D77], Th. V6, Cor. V2)**.** *Let G be a connected solvable Lie group and let V be an irreducible unitary G-module. Assume that V is not a tensor power of a character occurring as quotient of the adjoint representation. Then V has zero first reduced cohomology.*

Using Shalom's subsequent terminology, it follows that connected solvable Lie group have Property $\mathcal{H}_{\mathrm{fd}}$. Florian Martin [Ma06] extended this (and the above theorem) to arbitrary amenable connected Lie groups.

Delorme's proof takes more than 10 pages, involving a lot of ad-hoc analytical arguments and strongly relies on representation theory of the Lie algebra. It implies in particular that such G has Property $\mathcal{H}_{\mathrm{fd}}$. Shalom proved in [Sh04] that Property $\mathcal{H}_{\mathrm{fd}}$ is invariant under passing to cocompact lattices. As a consequence, it is satisfied by virtually polycyclic groups: indeed such a group Γ has a finite index subgroup Γ' embedding as a cocompact lattice in a group G as in the theorem, and then Property $\mathcal{H}_{\mathrm{fd}}$ thus successively passes from G to Γ' and then to Γ.

A motivation for Property $\mathcal{H}_{\mathrm{fd}}$ is to find interesting finite-dimensional representations. More precisely, if G is an infinite discrete amenable group with Property $\mathcal{H}_{\mathrm{fd}}$, it is easy to deduce that G admits an infinite virtually abelian quotient.

In this work, we provide a new, simpler proof of Delorme's theorem based on geometric/dynamical considerations. This allows to extend the previous results in two directions: first our approach allows to encompass a much larger class of groups, and second it allows to generalize it to uniformly bounded representations in more general Banach spaces.

A crucial feature which is used all the time in the context of unitary representations is the notion of "orthogonal complement". Although this notion does not survive in the more general framework of Banach spaces, a weaker and yet very powerful property holds for a large class of Banach G-modules: the subspace of invariant vectors is a factor, in fact it admits a canonical complement.

Recall that a G-module (V,π) is called weakly almost periodic (WAP for short) if for every $v \in V$, the orbit $\pi(G)v$ is relatively compact in V in the weak topology. Note that this does not depend on a choice of topology on G. As observed in [BRS13], it implies that ρ is a uniformly bounded representation: $\sup_{g\in G}\|\rho(g)\| < \infty$, and in case V is reflexive, this is equivalent to being a uniformly bounded representation. WAP representations turn out to be a convenient wide generalization of unitary representations.

Definition 1. We say that a locally compact group G has Property $\mathcal{WAP}_{\mathrm{t}}$ if every WAP G-module V with $V^G = \{0\}$ has $\overline{H^1}(G,V) = 0$.

Property $\mathcal{WAP}_{\mathrm{t}}$ is a strengthening of Property $\mathcal{H}_{\mathrm{t}}$. Using that there a G-invariant complement ([BRS13, Theorem 14], see §2.4), this means that for every WAP G-module V, the reduced 1-cohomology is "concentrated" in V^G. Observe that a 1-cocycle valued in V^G is just a continuous group homomorphism. It follows, for instance that for a compactly generated, locally compact group G without Kazhdan's Property T (e.g., amenable and non-compact), the condition $\mathrm{Hom}(G,\mathbf{R}) = 0$ is an obstruction to Property $\mathcal{H}_{\mathrm{t}}$, and hence to Property $\mathcal{WAP}_{\mathrm{t}}$.

This explains why it is, in the context of unitary representations, natural to deal with the more flexible Property $\mathcal{H}_{\mathrm{fd}}$. Defining an analogue in this broader context leads to some technical difficulties, which leads us to introduce two distinct notions. First, recall that a Banach G-module is almost periodic if all orbits closures are compact (in the norm topology). This is obviously a strengthening of being WAP. This is satisfied by finite-dimensional uniformly bounded modules, and it can be checked (Corollary 2.8) that a uniformly bounded Banach G-module is almost periodic if and only if the union of its finite-dimensional submodules is dense. In general, the set of vectors whose G-orbit has compact closure is a closed submodule, denoted $V^{G,\mathrm{ap}}$, and, by the above, equals the closure of the union of finite-dimensional submodules.

Definition 2. Let G be a locally compact group.

- G has Property $\mathcal{WAP}_{\mathrm{ap}}$ if for every WAP Banach G-module V with $V^{G,\mathrm{ap}} = 0$ we have $\overline{H^1}(G, V) = 0$;
- G has Property $\mathcal{WAP}_{\mathrm{fd}}$ if for every WAP Banach G-module V and every 1-cocycle b that is nonzero in $\overline{H^1}(G, V)$, there exists a closed submodule W of nonzero finite codimension such that the projection of b on V/W is unbounded.

Since finite-codimensional submodules are complemented (Proposition 2.12), Property $\mathcal{WAP}_{\mathrm{fd}}$ implies Property $\mathcal{WAP}_{\mathrm{ap}}$. We do not know if the converse holds (see Question 4.5 and the discussion around it).

1.2. **Main results.** In the sequel, we abbreviate "compactly generated locally compact" as "CGLC".

Our main result is the fact that a relatively large class of CGLC groups, including connected solvable Lie groups, algebraic solvable p-adic groups, and finitely generated solvable groups with finite Prüfer rank, satisfy property $\mathcal{WAP}_{\mathrm{ap}}$. Even in the case of connected solvable Lie groups, the proof is not merely an adaptation of Delorme's proof, which is specific to the Hilbert setting. Instead it uses a dynamical phenomenon which is very specific to these groups.

In order to illustrate this dynamical phenomenon, let us examine the simplest example where it arises: the affine group $\mathrm{Aff}(\mathbf{R}) := U \rtimes A$, where $U \simeq A \simeq \mathbf{R}$, and where the group law is given by $(x, t)(y, s) = (x + e^t y, t + s)$. The important feature of this group is the fact that the normal subgroup U is "contracted" by the action of A: i.e. given $a_t = (0, t) \in A$ and $u = (x, 0) \in U$, one has

$$a_t^{-1} u a_t = (e^{-t} x, 0),$$

from which we deduce that $a_t^{-1} u a_t \to (0, 0)$ as $t \to \infty$. The group $\mathrm{Aff}(\mathbf{R})$ turns out to have Property $\mathcal{WAP}_{\mathrm{t}}$, and, roughly speaking, the proof consists in proving that U behaves as if it did not exist at all, so that everything boils down to the fact that A itself satisfies $\mathcal{WAP}_{\mathrm{t}}$.

An elaboration of this argument applies to the following ad-hoc class of groups:

Definition 3. Denote by $\mathfrak{C}$ the class of (solvable) locally compact groups G having two closed subgroups U and N such that

(1) U is normal and $G = UN$;
(2) N is a CGLC compact-by-nilpotent group, i.e., is compactly generated and has a compact normal subgroup such that the quotient is nilpotent;

(3) U decomposes as a finite direct product $\prod U_i$, where each U_i is normalized by the action of N and is an open subgroup of a unipotent group $\mathbb{U}_i(\mathbf{K}_i)$ over some non-discrete locally compact field of characteristic zero $\mathbf{K}_i$.
(4) U admits a cocompact subgroup V with, for some k, a decomposition $V = V_1 V_2 \dots V_k$ where each V_i is a subset such that there is an element $t = t_i \in N$ such that $t^{-n} v t^n \to 1$ as $n \to \infty$ for all $v \in V_i$.

This notably includes (see Proposition 6.1)

- real triangulable connected Lie groups;
- groups of the form $G = \mathbb{G}(\mathbf{Q}_p)$, where $\mathbb{G}$ is a solvable connected linear algebraic group defined over the p-adic field $\mathbf{Q}_p$ such that G is compactly generated;
- mixtures of the latter, such as the semidirect product $(\mathbf{K}_1 \times \mathbf{K}_2) \rtimes_{(t_1,t_2)} \mathbf{Z}$, where $\mathbf{K}_i$ is a nondiscrete locally compact field and $|t_i| \neq 1$.

Let us also pinpoint that in many cases, the method applies without the characteristic zero assumption in (3). Namely, assuming that $U = U_0 \times \bigoplus_p U_p$ where U_0 is the characteristic zero part and U_p is the p-torsion (p ranges over primes, with only finitely many p for which $U_p \neq 1$), this applies if for every $p > 0$, U_p is $(p-1)$-step-nilpotent, so that it naturally has a Lie algebra, and this Lie algebra has a G-invariant structure of Lie algebra over $\mathbf{F}_p(\!(t)\!)$.

Here is our first main result

Theorem 4. Locally compact groups in the class $\mathfrak{C}$ have Property $\mathcal{WAP}_{\mathrm{t}}$. In particular, they have Property $\mathcal{H}_{\mathrm{t}}$.

The proof of Theorem 4 involves several steps of independent interest, including the existence of "strong controlled Følner subsets" for groups in the class $\mathfrak{C}$ (Theorem 6.10).

In order to extend Theorem 4, we use induction methods as in [Sh04, BFGM07] to obtain

Theorem 5. (see Section 4)

(1) Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{ap}}$ are inherited by closed cocompact subgroups such that the quotient has an invariant probability measure. In particular these properties are inherited by closed cocompact subgroups among amenable groups.
(2) Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{ap}}$ are invariant (i.e., both the property and its negation is stable) under taking quotients by compact normal subgroups.
(3) Let G be a locally compact group with a closed normal cocompact subgroup N. If N has Property $\mathcal{WAP}_{\mathrm{t}}$ then G has Property $\mathcal{WAP}_{\mathrm{ap}}$.
(4) Let Λ be a countable discrete group with Property $\mathcal{WAP}_{\mathrm{ap}}$ and Γ a countable discrete group admitting an RCE (random cocompact embedding, see

§3.2.1) into Λ. Then Γ also has Property $\mathcal{WAP}_{\mathrm{ap}}$. In particular, Property $\mathcal{WAP}_{\mathrm{ap}}$ is stable under coarse equivalence among countable discrete amenable groups.

We introduce the terminology RCE to name a slight reinforcement of uniform measure equivalence used in [Sh04].

Definition 6. Let $\mathfrak{C}'$ be the larger class consisting of those locally compact groups G such that there exists a sequence of copci (= continuous, proper with cocompact image) homomorphisms $G \to G_1 \leftarrow G_2 \to G_3$ such that the image of $G_2 \to G_1$ is normal in G_1 and G_3 belongs to the class $\mathfrak{C}$.

This may sound a bit artificial, but the point is that this definition ensures that all amenable, virtually connected Lie groups belong to the class $\mathfrak{C}'$ (Proposition 6.2), as well as all groups with a open finite index subgroup in the class $\mathfrak{C}$.

Corollary 7. Locally compact groups in the class $\mathfrak{C}'$ have Property $\mathcal{WAP}_{\mathrm{fd}}$. In particular, they have Property $\mathcal{H}_{\mathrm{fd}}$.

Corollary 8. Every virtually connected amenable Lie group G has Property $\mathcal{WAP}_{\mathrm{fd}}$.

In view of Proposition 6.1, we deduce

Corollary 9. Real-triangulable Lie groups and compactly generated amenable Zariski-(closed connected) subgroups of $\mathrm{GL}_n(\mathbf{Q}_p)$ have $\mathcal{WAP}_{\mathrm{t}}$.

We combine these results to obtain the following Banach space version of Delorme's theorem.

Corollary 10. (see Corollary 7.5) Let G be a connected solvable Lie group. Then every WAP G-module with nonzero first cohomology has a 1-dimensional factor (with nonzero first cohomology).

We can also apply Theorem 4 to deduce many new examples of discrete groups with Property $\mathcal{WAP}_{\mathrm{ap}}$.

Corollary 11. The class of groups satisfying Property $\mathcal{WAP}_{\mathrm{ap}}$ includes all discrete groups that are virtually cocompact lattices in a finite direct product of connected Lie groups and algebraic groups over $\mathbf{Q}_p$ (for various primes p). This includes polycyclic groups and more generally all amenable groups already known to satisfy Property $\mathcal{H}_{\mathrm{fd}}$ (see [Sh04]). Some of these, being cocompact lattices in groups in the class $\mathfrak{C}$, actually have Property $\mathcal{WAP}_{\mathrm{t}}$: this includes for instance of SOL, solvable Baumslag-Solitar groups and lamplighter groups $(\mathbf{Z}/n\mathbf{Z}) \wr \mathbf{Z}$.

We can also, along with additional structural work, deduce the following result, which answers a question of Shalom [Sh04] (who asked whether these groups have $\mathcal{H}_{\mathrm{fd}}$).

Recall that a group has finite Prüfer rank if for some k, all its finitely generated subgroups admit a generating k-tuple. Let us abbreviate "virtually solvable of finite Prüfer rank" to "VSP".

Theorem 12. (Corollary 8.11) Every finitely generated, VSP group has Property $\mathcal{WAP}_{\mathrm{fd}}$, and in particular has Property $\mathcal{H}_{\mathrm{fd}}$.

Finitely generated amenable (or equivalently, virtually solvable) subgroups of $\mathrm{GL}(d,\mathbf{Q})$ are notably covered by the theorem: more precisely, these are precisely (when d is allowed to vary) the virtually torsion-free groups in the class of finitely generated VSP groups. Actually, the theorem precisely consists of first proving it in this case, and deduce the general case using a recent result of Kropholler and Lorensen [KL17]: every finitely generated VSP group is quotient of a virtually torsion-free finitely generated VSP group.

We deduce the following strengthening of [Sh04, Theorem 1.3], which is the particular case of polycyclic groups.

Corollary 13. Let Λ be a finitely generated, (virtually) solvable group of finite Prüfer rank. Let Γ be a finitely generated group quasi-isometric to Λ. Then Γ has a finite index subgroup with infinite abelianization.

Proof. This consists in combining Theorem 12 with two results of Shalom:

- [Sh04, Theorem 4.3.1], which says that every infinite finitely generated amenable group with Property $\mathcal{H}_{\mathrm{fd}}$ has a finite index subgroup with infinite abelianization;
- [Sh04, Theorem 4.3.3]: among finitely generated amenable groups, Property $\mathcal{H}_{\mathrm{fd}}$ is a quasi-isometry invariant. $\square$

1.3. **Cocompact hull of amenable subgroups of** $\mathrm{GL}(d,\mathbf{Q})$**.** Our proof of Theorem 12 relies on a construction of independent interest. We start introducing a second variant of the class $\mathfrak{C}$.

Definition 14. Let $\mathfrak{C}''$ be the class of compactly generated locally compact groups defined as the class $\mathfrak{C}$ (Definition 3) but replacing (3) with: N has polynomial growth.

Theorem 15. Every finitely generated amenable (= virtually solvable) subgroup of $\mathrm{GL}_m(\mathbf{Q})$ embeds as a compact lattice into a locally compact group G with an open subgroup of finite index G' in the class $\mathfrak{C}''$.

This is a key step in the proof of Theorem 12. Let us pinpoint other consequences of the existence of this cocompact hull.

Let G be a compactly generated locally compact group, and let λ be the left representation of G on real-valued functions on G, namely $\lambda(g)f(x) = f(g^{-1}x)$. We let S be a compact symmetric generating subset of G. For any $1 \leq p \leq \infty$, and any subset A of G, define

$$J_p(A) = \sup_f \frac{\|f\|_p}{\sup_{s\in S} \|f - \lambda(s)f\|_p},$$

where f runs over functions in $L^p(G)$, supported in A. Recall [T08] that the L^p-isoperimetric profile inside balls is given by

$$J^b_{G,p}(n) = J_p(B(1,n)).$$

Corollary 16. (see Corollary 8.12) For every finitely generated VSP group G equipped with a finite generating subset S, we have

$$J^b_{G,p}(n) \succeq n,$$

i.e., there exists $c > 0$ such that $J^b_{G,p}(n) \geq cn$ for all n.

Corollary 16 has a consequence in terms of equivariant L^p-compression rate. Recall that the equivariant L^p-compression rate $B_p(G)$ of a locally compact compactly generated group is the supremum of those $0 \leq \alpha \leq 1$ such that there exists a proper isometric affine action σ on some L^p-space satisfying, for all $g \in G$, $\|\sigma(g).0\|_p \geq |g|_S^\alpha - C$ for some constant $C < \infty$. It follows from [T11, Corollary 13] that for a group G with $J^b_{G,p}(n) \succeq n$, we have $B_p(G) = 1$; hence

Corollary 17. Let $1 \leq p < \infty$, and G be a finitely generated VSP group. Then $B_p(G) = 1$.

See also [T11, Theorem 10] for a finer consequence.

1.4. **An mean ergodic theorem in L^1.** Let X be a probability space and let $T : X \to X$ be a measure-preserving ergodic self-map of X. Recall that Birkhoff's theorem states that for all $f \in L^1(X)$, the sequence $\frac{1}{n}\sum_{i=0}^{n-1} T^i(f)$ converges a.e. and in L^1 to the integral of f. Observe that the map $n \mapsto c(n) = \sum_{i=0}^{n-1} T^i(f) \in L^1(X)$ (a priori well defined on positive integers, and more generally on $\mathbf{Z}$ if T is invertible) satisfies the cocycle relation: $c(n+1) = T(c(n)) + c(1)$. Hence, assuming that T is invertible, Birkhoff's ergodic theorem can be restated in a more group-theoretic fashion: given an ergodic measure measure-preserving action of $\mathbf{Z}$ on a probability space X, every continuous cocycle $c \in Z^1(\mathbf{Z}, L^1(X))$ is such that

$$\frac{1}{|n|}\left(c(n)(x) - \int c(n)(x')d\mu(x')\right) \to 0,$$

both a.e. and in L^1. By measure-preserving action of G on a probability space X, we mean a measurable map $G \times X \to X$, denoted $(g,x) \mapsto g.x$ (G being endowed with the Lebesgue σ-algebra), such that for every $g, h \in G$, the functions $g.(hx)$ and $(gh)x$ coincide outside a subset of measure zero. This makes $L^p(G)$ a Banach G-module for all $1 \leq p < \infty$.

A generalization of this result is due to Boivin and Derriennic [BD91] for $\mathbf{Z}^d$ (and similarly for $\mathbf{R}^d$). To obtain almost sure convergence, stronger integrability conditions are required when $d > 1$ (see [BD91, Theorems 1 and 2]). Here however, we focus on convergence in L^1:

Definition 18. A CGLC group G satisfies the *mean ergodic theorem for cocycles in* L^1 if for every ergodic measure-preserving action of G on a probability space X, and every continuous cocycle $c \in Z^1(G, L^1(X))$, we have

$$\lim_{|g|\to\infty} \frac{1}{|g|}\left(c(g)(x) - \int_X c(g)(x')d\mu(x')\right) = 0,$$

where the convergence is in $L^1(X)$, and $|g|$ denotes a word length on G associated to some compact generating subset.

In [BD91, Theorem 4], Boivin and Derriennic prove that $\mathbf{Z}^d$ and $\mathbf{R}^d$ satisfy the mean ergodic theorem in L^1. We start by the following observation.

Proposition 19. A group G with Property $\mathcal{WAP}_{\mathrm{ap}}$ satisfies the mean ergodic theorem for 1-cocycles in L^1 if and only if G satisfies $\mathcal{WAP}_{\mathrm{t}}$.

The *if* part immediately follows from the well-known fact [EFH15, Corollary 6.5] that the representation of G on $L^1(X)$ is WAP. The "only if" part is more anecdotical, see §9 for the proof.

Corollary 20. Groups in the class $\mathfrak{C}$ and their closed cocompact subgroups satisfy the ergodic theorem for cocycles in L^1.

To our knowledge, this is new even for the group SOL. For nilpotent groups, it can be easily deduced from Proposition 19 together with the fact that these groups have $\mathcal{WAP}_{\mathrm{t}}$, an observation due to [BRS13].

1.5. **Bourgain's theorem on tree embeddings.** We obtain a new proof of the following result of Bourgain.

Corollary 21 (Bourgain, [Bo86])**.** The 3-regular tree does not quasi-isometrically embed into any superreflexive Banach space.

The idea is to use a CGLC group in the class $\mathfrak{C}$ that is quasi-isometric to the 3-regular tree, and make use of amenability and Property $\mathcal{WAP}_{\mathrm{t}}$. In [CTV07], the authors and Valette used a similar argument based on property $\mathcal{H}_{\mathrm{t}}$ to show Bourgain's result in the case of a Hilbert space. See Section 9 for the proof.

2. Preliminaries on Banach modules

2.1. **First reduced cohomology versus affine actions.** Let G be a locally compact group, and (V, π) be a Banach G-module.

Observe that, given a continuous function $b : G \to V$, we can define for every $g \in G$ an affine transformation $\alpha_b(g)v = \pi(g)v + b(g)$. Then the condition $b \in Z^1(G, \pi)$ is a restatement of the condition $\alpha_b(g)\alpha_b(h) = \alpha_b(gh)$ for all $g, h \in G$, meaning that α is an action by affine transformations. Then the subspace $B^1(G, \pi)$ is the set of b such that α_b has a G-fixed point, and its closure $\overline{B^1}(G, \pi)$ is the set of 1-cocycles b such that the action α_b almost has fixed points, that is, for every $\varepsilon > 0$ and every compact subset K of G, there exists a vector $v \in V$ such that for every $g \in K$,

$$\|\alpha_b(g)v - v\| \leq \varepsilon.$$

If G is compactly generated and if S is a compact generating subset, then this is equivalent to the existence of a sequence of almost fixed points, i.e. a sequence v_n of vectors satisfying

$$\lim_{n\to\infty} \sup_{s\in S} \|\alpha_b(s)v_n - v_n\| = 0.$$

2.2. **Almost periodic actions.**

Definition 2.1 (Almost periodic actions)**.** Let G be a group acting on a metric space X. Denote by $X^{G,\mathrm{ap}}$ the set of $x \in X$ whose G-orbit has a compact closure in X. Say that X is almost (G-)periodic if $X^{G,\mathrm{ap}} = X$.

Note that this definition does not refer to any topology on G. Although we are mainly motivated by Banach G-modules, some elementary lemmas can be established with no such restriction.

The following lemma is well-known when $X = V$ is a WAP Banach G-module.

Lemma 2.2. Let G be a group and X a complete metric space with a uniformly Lipschitz G-action (in the sense that $C < \infty$, where C is the supremum over g of the Lipschitz constant C_g of the map $x \mapsto gx$). Then $X^{G,\mathrm{ap}}$ is closed in X.

Proof. Let v be a point in the closure of $X^{G,\mathrm{ap}}$. Choose $v_j \in X^{G,\mathrm{ap}}$ with $v = \lim_j v_j$.

Let (g_n) be a sequence in G. We have to prove that $(g_n v)$ has a convergent subsequence. First, up to extract, we can suppose that $(g_n v_j)$ is convergent for all j. Then for all m, n, j we have

$$\begin{aligned} d(g_n v, g_m v) \leq & d(g_n v, g_n v_j) + d(g_n v_j, g_m v_j) + d(g_m v_j, g_m v) \\ \leq & 2C d(v, v_j) + d(g_n v_j, g_m v_j). \end{aligned}$$

Now fix $\varepsilon > 0$ and choose j such that $d(v, v_j) \leq \varepsilon/3$. Then there exists n_0 such that for all $n, m \geq n_0$, we have $d(g_n v_j, g_m v_j) \leq \varepsilon/3$. It then follows from the above inequality that for all $n, m \geq n_0$, we have $d(g_n v, g_m v) \leq \varepsilon$. □

Lemma 2.3. Let G be a locally compact group and H a closed cocompact subgroup. Let X be a metric space with a uniformly Lipschitz, separately continuous G-action. Then $X^{H,\mathrm{ap}} = X^{G,\mathrm{ap}}$.

Proof. Choose $x \in X^{H,\mathrm{ap}}$. Choose a compact subset $K \subset G$ such that $G = KH$. Let $(g_n x)$ be any sequence in the G-orbit of x. Write $g_n = k_n h_n$ with $k_n \in K$, $h_n \in H$. We can find an infinite subset I of integers such that $k_n \xrightarrow[n\to\infty]{n\in I} k$ and

$h_n v \underset{n\to\infty}{\overset{n\in I}{\longrightarrow}} w$. Then, denoting C the supremum of Lipschitz constants, we have

$$\begin{aligned} d(g_n x, kw) = d(k_n h_n x, kw) \leq & d(k_n h_n x, k_n w) + d(k_n w, kw) \\ & \leq C d(h_n x, w) + C d(w, k_n^{-1} kw) \underset{n\to\infty}{\overset{n\in I}{\longrightarrow}} 0. \end{aligned}$$

Thus the G-orbit of x has compact closure, that is, $x \in V^{G,\mathrm{ap}}$, showing the nontrivial inclusion. □

Proposition 2.4. Let G be a topological group and (X, π) a metric space with a separately continuous uniformly Lipschitz G-action (denoted $gx = \pi(g)x$). Then X is almost G-periodic if and only if the action of G on X factors through a compact group K with a separately continuous action of K on X and a continuous homomorphism with dense image $G \to K$.

Proof. The "if" part is immediate. Conversely, suppose that X is almost periodic. Let $\mathcal{O}$ be the set of G-orbit closures in X. Each $Y \in \mathcal{O}$ is a compact metric space; let I_Y be its isometry group; we thus have a continuous homomorphism $G \to I_Y$. So we have a canonical continuous homomorphism $q : G \to \prod_{Y\in\mathcal{O}} Y$; let K be the closure of its image; this is a compact group. We claim that the representation has a unique separately continuous extension to K. The uniqueness is clear. To obtain the existence, consider any net (g_i) in g such that $(q(g_i))$ converges in K, $\pi(g_i)v$ converges in X for every $v \in X$. Indeed, we have $d(\pi(g_i^{-1} g_j)v, v) \to 0$ when $i, j \to \infty$, and hence, since the representation is uniformly bounded, $d(\pi(g_j)v, \pi(g_i)v)$ tends to 0. So $(\pi(g_i)v)$ is Cauchy and thus converges; the limit only depends on v and on the limit k of $(q(g_i))$; we define it as $\tilde{\pi}(k)v$.

Also, if c is the supremum of all Lipschitz constants, then, as a pointwise limit of c-Lipschitz maps, $\tilde{\pi}(k)$ is c-Lipschitz.

Now let us show that it defines an action of K, namely $\tilde{\pi}(k)\tilde{\pi}(\ell) = \tilde{\pi}(k\ell)$ for all k, ℓ in K. We first claim that $k \mapsto \tilde{\pi}(k)v$ is continuous for every fixed $v \in X$. Indeed, if Y is the closure of the orbit of v, then this map can be identified to the orbital map of the action of the image of k in I_Y, which is continuous. The same argument shows that $(k, \ell) \mapsto \tilde{\pi}(k)\tilde{\pi}(\ell)v$ is continuous, and also this implies, by composition, that $(k, \ell) \mapsto \tilde{\pi}(k\ell)v$ is continuous. Since these two maps coincide on the dense subset $q(G) \times q(G)$, they agree. Thus $\tilde{\pi}$ defines an action, and we have also checked along the way that it is separately continuous. □

2.3. **Almost periodic Banach modules.** We recall the following well-known fact.

Lemma 2.5. Let G be a group and (V, π) a Banach G-module. If every G-orbit is bounded, then the representation is uniformly bounded. In particular,

- every WAP representation is uniformly bounded;
- for G compact, every Banach G-module is uniformly bounded.

Proof. Define $V_n = \{v \in V : \forall g \in G, \|gv\| \leq n\}$. Since the action is by bounded (=continuous) operators, V_n is closed for all n. Since all G-orbits are bounded, $\bigcup_n V_n = V$. By Baire's theorem, there exists V_n with non-empty interior. Since $V_n = -V_n$ and $V_k + V_\ell \subset V_{k+\ell}$ for all k, ℓ, the set V_{2n} contains the centered ball of radius ε for some $\varepsilon > 0$. This implies that $\sup_{g \in G} \|\pi(g)\| \leq 2n/\varepsilon$.

(Note that the last consequence was not a trivial consequence of the definition, since the map $g \mapsto \pi(g)$ often fails to be continuous for the norm topology on operators.) $\square$

Lemma 2.6. Let V be a Banach G-module. Then $V^{G,\mathrm{ap}}$ is a subspace of V.

Proof. This is clear since $G(\lambda v + w) \subset \lambda\overline{Gv} + \overline{Gw}$ for all v, w. $\square$

Theorem 2.7. [Shi55, Theorem 2] Let G be a compact group and (V, π) be a Banach G-module. Then the sum of finite-dimensional irreducible G-submodules of V is dense in V.

Let now G be an arbitrary topological group. Recall that a Banach G-module is almost periodic if every G-orbit is relatively compact in the norm topology.

Corollary 2.8. Let V be a uniformly bounded Banach G-module. Then $V^{G,\mathrm{ap}}$ is the closure of the sum of all finite-dimensional submodules of V, and is also the closure of the sum all irreducible finite-dimensional submodules of V.

Proof. Let $V^{G,\mathrm{ap}}, V_2, V_3$ be the three subspaces in the corollary. That $V_3 \subset V_2$ is clear.

That every finite-dimensional submodule is contained in $V^{G,\mathrm{ap}}$ is clear (even without assuming V uniformly bounded). So the sum of all finite-dimensional submodules is contained in $V^{G,\mathrm{ap}}$, and hence its closure by Lemma 2.2, since V is uniformly bounded. So $V_2 \subset V^{G,\mathrm{ap}}$.

For the inclusion $V^{G,\mathrm{ap}} \subset V_3$, we use that the G-action on $V^{G,\mathrm{ap}}$ factors through a compact group (Proposition 2.4), and then invoke Theorem 2.7. $\square$

Definition 2.9. Let G be a locally compact group and let (V, π) be a Banach G-module. Define $V^*_{[G]}$ as the set of $f \in V^*$ such that the orbital function $\nu_f : g \mapsto g \cdot f$ is continuous on G.

Lemma 2.10. Let G be a locally compact group and (V, π) be a Banach G-module. Then $V^*_{[G]}$ is a closed subspace of V^* (and thus is a Banach G-module). Moreover, $V^*_{[G]}$ separates the points of G.

Proof. That $V^*_{[G]}$ is a subspace is clear.

Write $c = \sup_{g \in G} \|\pi(g)\|$ (it is finite by Lemma 2.5). For $f, f' \in V^*$ and $v \in V$, we have

$$\nu_f(g)(v) - \nu_{f'}(g)(v) = (g \cdot f)(v) - (g \cdot f')(v) = (f - f')(g^{-1}v);$$

hence

$$\|\nu_f(g)(v) - \nu_{f'}(g)(v)\| \leq \|f - f'\|\|g^{-1}v\| \leq c\|f - f'\|\|v\|$$

and thus

$$\|\nu_f(g) - \nu_{f'}(g)\| \le c\|f - f'\|$$

Suppose that (f_n) converges to f, with $f_n \in V^*_{[G]}$ and $f \in V^*$. The above inequality shows that ν_{f_n} converges uniformly, as a function on G, to ν_f. By assumption, ν_{f_n} is continuous, and thus ν_f is continuous, meaning that $f \in V^*_{[G]}$.

For the separation property, we consider $v \in V \smallsetminus \{0\}$ and have to find an element in $V^*_{[G]}$ not vanishing on v. First, we choose $f \in V^*$ such that $f(v) = 1$.

For every $g \in G$, $(g \cdot f)(v) = f(g^{-1}v)$, and hence $q : g \mapsto (g \cdot f)(v)$ is continuous on G; we have $q(1) = 1$. Let U be a compact neighborhood of 1 in G on which q takes values $\ge 1/2$. Let φ be a non-negative continuous function on G, with support in U, and integral 1 (G being endowed with a left Haar measure).

For $\xi \in V$, define $u(\xi) = \int_G \varphi(g) f(g^{-1}\xi) dg$. Then u is clearly linear, and $\|u\| \le c\|f\|$, so u is continuous. We have $u(v) = \int_G h_n q \ge 1/2$.

It remains to show that $u \in V^*_{[G]}$. It is enough to check that $h \mapsto h \cdot u$ is continuous at 1. We have, for $h \in G$ and $\xi \in V$

$$\begin{aligned} u(\xi) - (h \cdot u)(\xi) &= \int_G \varphi(g) f(g^{-1}\xi) dg - \int_G \varphi(g) f(g^{-1}h^{-1}\xi) dg \\ &= \int_G \varphi(g) f(g^{-1}\xi) dg - \int_G \varphi(h^{-1}g) f(g^{-1}\xi) dg \\ &= \int_G (\varphi(g) - \varphi(h^{-1}g)) f(g^{-1}\xi) dg. \end{aligned}$$

Define $\varepsilon_h = \sup_{g \in G} |\varphi(g) - \varphi(h^{-1}g)|$. Since φ has compact support, it is uniformly continuous. Hence ε_h tends to 0 when $h \to 1$. We conclude

$$\|u(\xi) - (h \cdot u)(\xi)\| \le \varepsilon_h \int_G |f(g^{-1}\xi)| dg \le \varepsilon_h c \|f\| \|\xi\|,$$

so $\|u - h \cdot u\| \le \varepsilon_h c\|f\|$, which tends to 0 when $h \to 1$. □

Proposition 2.11. Let V be an almost periodic Banach G-module. Then every finite-dimensional submodule is complemented in V as G-module.

Proof. By Proposition 2.4, we can suppose that G is compact.

Let C be a finite-dimensional submodule; let us show, by induction on $d = \dim(C)$, that C is complemented. This is clear if $d = 0$; assume now that C is irreducible. Beware that V^* need not be a Banach G-module (G does not always act continuously). We consider the subspace $V^*_{[G]} \subset V$ of Definition 2.9, which is a Banach G-module by Lemma 2.10. Let $F \subset V^*_{[G]}$ the sum of all irreducible finite-dimensional submodules. By Theorem 2.7, F is dense in $V^*_{[G]}$, and by Lemma 2.10, the latter separates the points of V. So there exists an element in F that does not vanish on C. In turn, this means that there is an irreducible finite-dimensional submodule M of F that does not vanish on C, or equivalently whose orthogonal W does not contain C. Note that $W \subset V$ is closed and that M is isomorphic, as

G-module, to the dual of V/W; in particular, V/W is an irreducible G-module; in other words, W is maximal among proper G-submodules of V. It follows that $V = C \oplus W$.

If C is not irreducible, let C' be a nonzero proper submodule. Then by induction, C/C' is complemented in V/C', which means that $V = C + W$ with W a G-submodule and $W \cap C = C'$. By induction, we can write $W = C' \oplus W'$ with W' a submodule. Hence $V = C \oplus W'$. □

Proposition 2.12. Let V be an almost periodic Banach G-module. Then every finite-codimensional submodule is complemented in V as G-module.

Proof. Let W be a submodule of finite codimension. Lift a basis of the quotient to a family in the complement $(e_1, \dots, e_n)$. Then there exists an open ball B_i around e_i such that for every $(e'_1, \dots, e'_n) \in \prod B_i$, the family $(e'_1, \dots, e'_n)$ projects to a basis of V/W. Since by Corollary 2.8, the union of finite-dimensional submodules of V is dense, we can choose e'_i to belong to a finite-dimensional submodule F_i, and define $F = \sum F_i$.

Then F is a finite-dimensional G-submodule and $F + W = V$. Then G preserves a scalar product on F, so preserves the orthogonal F' of $F \cap W$ for this scalar product. Thus $V = F' \oplus W$. □

2.4. **Canonical decompositions.**

We use the following known results.

Theorem 2.13. Let G be a locally compact group and V a WAP G-module. Then

1. V^G has a canonical complement, consisting of those v such that 0 belongs to the closure of the convex hull of the orbit Gv;
2. $V^{G,\mathrm{ap}}$ has a canonical complement, consisting of those v such that 0 belongs to the weak closure of the orbit Gv.

Here by complement of a subspace W_1 in a Banach space V we mean a closed submodule W_2 such that the canonical map $W_1 \oplus W_2 \to V$ is an isomorphism of Banach spaces. The complement being here defined in a "canonical" way, it follows that if G preserves these complements.

These statements are Theorems 14 and 12 in [BRS13]. The part (1) is due to Alaoglu-Birkhoff [AB40] in the special case superreflexive G-modules (that is, whose underlying Banach space is superreflexive), and [BFGM07, Proposition 2.6] in general. Part (2) is a generalized version of a theorem of Jacobs and de Leeuw-Glicksberg, stated in this generality in [BJM].

3. Induction

3.1. **A preliminary lemma.**

Lemma 3.1. Let G be a locally compact group, let (E, π) be a continuous Banach G-module. Let $b : G \to E$ satisfy the cocycle relation $b(gh) = \pi(g)b(h) + b(g)$.

If b is measurable and locally integrable, then it is continuous. Moreover if a sequence (b_n) in $Z^1(G,\pi)$ is locally uniformly bounded, i.e. for every compact subset $K \subset G$, we have $\sup_{g\in K, n\in \mathbf{N}} \|b_n(g)\| < \infty$, and if (b_n) converges pointwise to b, then the convergence is uniform on compact subsets.

Proof. Pick a probability measure ν on G with compactly supported continuous density ϕ and define $\tilde{b}(g) = \int b(h)(\phi(g^{-1}h) - \phi(h))dh$. One checks (see the proof of Lemma 5.2 in [T09]) that $\tilde{b}$ is continuous and that $\tilde{b}(g) - b(g) = \pi(g)v - v$, where $v = \int b(h)d\nu(h)$. Hence b is continous.

For the second statement, we first note that $v_n = \int b_n(h)d\nu(h)$ converges to $v = \int b(h)d\nu(h)$, by Lebesgue's dominated convergence theorem. So we are left to consider $(\tilde{b}_n)$, which converges pointwise for the same reason. We conclude observing that the $\tilde{b}_n$ are equicontinuous. Indeed, let $g_1, g_2 \in G$ and define

$$C = \sup\{\|b_n(g)\|;\ g \in g_1\mathrm{supp}(\phi) \cup g_2\mathrm{supp}(\phi), n \in \mathbf{N}\}.$$

We have $\|\tilde{b}_n(g_1) - \tilde{b}_n(g_2)\|$

$$= \left\| \int b_n(h)(\phi(g_1^{-1}h) - \phi(g_2^{-1}h))dh \right\| \leq C \sup_{h\in G} |\phi(g_1^{-1}h) - \phi(g_2^{-1}h)|,$$

and we conclude thanks to the fact that $g \mapsto \phi(g^{-1})$ is uniformly continuous. □

3.2. **Measure equivalence coupling.** For the notions introduced in this subsection, we refer to [Sh04].

3.2.1. *ME coupling and ME cocycles.* Given countable discrete groups Γ and Λ, a measure equivalence (ME) coupling between them is a nonzero σ-finite measure space (X,μ), which admits commuting free μ-preserving actions of Γ and Λ which both have finite-measure fundamental domains, respectively X_Γ and X_Λ. Let $\alpha : \Gamma \times X_\Lambda \to \Lambda$ (resp. $\beta : \Lambda \times X_\Gamma \to \Gamma$) be the corresponding cocycle defined by the rule: for all $x \in X_\Lambda$, and all $\gamma \in \Gamma$, $\alpha(\gamma, x)\gamma x \in X_\Lambda$ (and symmetrically for β). If, for any $\lambda \in \Lambda$ and $\gamma \in \Gamma$, there exists finite subsets $A_\lambda \subset \Gamma$ and $B_\gamma \subset \Lambda$ such that $\lambda X_\Gamma \subset A_\lambda X_\Gamma$ and $\gamma X_\Lambda \subset B_\gamma X_\Lambda$, then we say the coupling is uniform, and call it a UME coupling, in which case the groups Γ and Λ are called UME. We now introduce the following reinforcement of UME.

Definition 3.2. A random cocompact embedding of Λ inside Γ is a UME coupling satisfying in addition $X_\Gamma \subset X_\Lambda$.

3.2.2. *Induction of WAP modules.* We assume that Λ and Γ are ME and we let $\alpha : \Gamma \times X_\Lambda \to \Lambda$ be the corresponding cocycle. Now let (V,π) be an Λ-module. The induced module is the Γ-module $(W, \mathrm{Ind}_\Lambda^\Gamma \pi)$ defined as follows: W is the space $L^1(X_\Lambda, \mu, V)$ of measurable maps $f : X_\Lambda \to V$ such that

$$\int_{X_\Lambda} |f(x)| d\mu(x) < \infty,$$

and we let Γ act on W by

$$\mathrm{Ind}_\Lambda^\Gamma \pi(g) f(x) = \pi(\alpha(g,x))(f(g^{-1}\cdot x).$$

Proposition 3.3. If (V,π) is WAP, then so is $(W, \mathrm{Ind}_\Lambda^\Gamma \pi)$.

Proof. This follows from the main result of [Tal84]. □

3.2.3. *Induction of cohomology.* For simplicity, we shall restrict our discussion to cohomology in degree one. We now assume that Λ and Γ are UME, and we let α and β be the cocycles associated to some UME-coupling. Assume (V,π) is a Banach Λ-module. Following [Sh04], we define a topological isomophism $I : H^1(\Gamma,\pi) \to H^1(\Gamma, \mathrm{Ind}_\Lambda^\Gamma\pi)$ as follows: for every $b \in Z^1(\Lambda,\pi)$ define

$$Ib(g)(x) = b(\alpha(g,x)),$$

for a.e. $x \in X$ and all $g \in \Gamma$. Note that Ib is continuous by Lemma 3.1.

Observe that the UME assumption ensures that $Ib(g)$ has finite norm for all $g \in G$ and therefore is a well-defined 1-cocycle. Now assume in addition that the coupling satisfies $X_\Gamma \subset X_\Lambda$ (i.e. that Λ randomly cocompact embeds inside Γ).

Then one can define an inverse T of I, defined for all $c \in Z^1(\Gamma, \mathrm{Ind}_\Lambda^\Gamma\pi)$ and $h \in \Gamma$ by

$$Tc(h) = \int_{X_\Gamma} c(\beta(h,y))(y) d\mu(y).$$

The fact that I and T induce inverse maps in cohomology follows from the proof of [Sh04, Theorem 3.2.1].

3.3. **Induction from a closed cocompact subgroup.** Let G be a LCSC group and H a closed cocompact subgroup of finite covolume.

Let (V,π) be a WAP H-module. Let μ be a G-invariant probability measure on the quotient G/H. Let $E = L^2(G/H, V, \mu)$ be the space of Bochner-measurable functions $f : G/H \to V$ such that $\|f\| \in L^2(G/H,\mu)$. Let $D \subset G$ be a bounded fundamental domain for the right action of H on G; let $s : G/H \to D$ be the measurable section. Define the cocycle $\alpha : G \times G/H \to H$ by the condition that

$$\alpha(g,x) = \gamma \Longleftrightarrow g^{-1}s(x)\gamma \in D,$$

We can now define a G-module $(E, \mathrm{Ind}_H^G\pi)$ *induced* from (V,π) by letting an element $g \in G$ act on $f \in E$ by

$$(gf)(kH) = \pi(\alpha(g,kH))f(g^{-1}kH).$$

The fact that the induced representation is WAP follows from [Tal84].

Note that one can similarly induce affine actions (the same formula holds, replacing π by an affine action σ). The corresponding formula for 1-cocycles (corresponding to the orbit of 0) is as follows: given $b \in Z^1(H,\pi)$, one defines the induced cocycle $\tilde{b} \in Z^1(G, \mathrm{Ind}_H^G\pi)$ by

$$\tilde{b}(h)(gH) = b(\alpha(h,gH)).$$

This defines a continuous cocycle by Lemma 3.1.

The map $b \to \tilde{b}$ induces a topological isomorphism in 1-cohomology (by Lemma 3.1). The inverse is defined as follows: given a cocycle $c \in Z^1(G, \mathrm{Ind}_H^G \pi)$, one gets a cocycle $\bar{c} \in Z^1(H, \pi)$ by averaging (see for instance [Sh04, Theorem 3.2.2]):

$$\bar{c}(\gamma) = \int_D c(x\gamma x^{-1})(x) d\mu(x). \tag{3.4}$$

Observe that that $\bar{\tilde{b}} = b$.

4. Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{ap}}$

4.1. **The definitions.**

Definition 4.1. Let G be a locally compact group. We say that G has

- Property $\mathcal{WAP}_{\mathrm{t}}$ if $\overline{H^1}(G, V) = 0$ for every WAP Banach G-module V such that $V^G = 0$;
- Property $\mathcal{WAP}_{\mathrm{ap}}$ if $\overline{H^1}(G, V) = 0$ for every WAP Banach G-module V such that $V^{G,\mathrm{ap}} = 0$;
- Property $\mathcal{WAP}_{\mathrm{fd}}$ if for every G-module V and $b \in Z^1(G, V)$ that is not an almost coboundary, there exists a closed G-submodule of positive finite codimension modulo which b is unbounded.
- Property $\mathcal{AP}_{\mathrm{fd}}$: same as $\mathcal{WAP}_{\mathrm{fd}}$, but assuming that V is almost periodic.

There is a convenient restatement of the definition of $\mathcal{WAP}_{\mathrm{fd}}$, in view of the following lemma:

Lemma 4.2. Let V be an almost periodic Banach G-module and b a 1-cocycle. The following are equivalent:

(1) there exists a G-module decomposition $V = V_1 \oplus V_2$ such that $\dim(V_1) < \infty$ and, under the corresponding decomposition $b = b_1 + b_2$, we have b_1 unbounded;
(2) there is a closed G-submodule $W \subset V$ of positive finite codimension such that the projection of b in V/W is unbounded.

Proof. Clearly (1) implies (2), and the converse follows from the fact that W is complemented (Proposition 2.12). □

Proposition 4.3. Property $\mathcal{WAP}_{\mathrm{fd}}$ is equivalent to the conjunction of Properties $\mathcal{WAP}_{\mathrm{ap}}$ and $\mathcal{AP}_{\mathrm{fd}}$.

Proof. It is obvious that $\mathcal{WAP}_{\mathrm{fd}}$ implies both other properties. Conversely, assume that G has both latter properties. Let V be a G-module and let b be a 1-cocycle that is not an almost coboundary. By Theorem 2.13(2), we have $V = V^{G,\mathrm{ap}} \oplus W$ for some G-submodule W. Let $b = b_1 + b_2$ be the corresponding decomposition of b. Since $W^{G,\mathrm{ap}} = 0$, Property $\mathcal{WAP}_{\mathrm{ap}}$ implies that b_2 is an almost coboundary. Hence b_1 is not an almost coboundary for the almost periodic G-module $V^{G,\mathrm{ap}}$. Then Property $\mathcal{AP}_{\mathrm{fd}}$ yields the conclusion. □

Proposition 4.4. *G has Property $\mathcal{WAP}_{\mathrm{t}}$ (resp. $\mathcal{WAP}_{\mathrm{ap}}$) if and only if, for every G-module V and $b \in Z^1(G,V)$ that is not an almost coboundary, $V^G \neq 0$ (resp. V admits a nonzero finite-dimensional submodule).*

Proof. The case $\mathcal{WAP}_{\mathrm{t}}$ is trivial and only stated to emphasize the analogy.

Suppose that G satisfies the given property (in the second case). Let V be a WAP G-module with $V^{G,\mathrm{ap}} = 0$. Let b be a 1-cocycle. Since the condition $V^{G,\mathrm{ap}} = 0$ implies that V has no nonzero finite-dimensional subrepresentation, the assumption implies that b is an almost coboundary.

Conversely, suppose that G has Property $\mathcal{WAP}_{\mathrm{ap}}$. Let V and b be as in the assumptions. By Theorem 2.13(2), write $V = V^{G,\mathrm{ap}} \oplus W$ with W its canonical complement. Decompose $b = b_1 + b_2$ accordingly. Then by Property $\mathcal{WAP}_{\mathrm{ap}}$, b_2 is an almost coboundary. So b_1 is not a coboundary. Hence $V^{G,\mathrm{ap}} \neq 0$. Hence, it admits a nonzero finite-dimensional subrepresentation, by Corollary 2.8. $\square$

As a consequence, we have the implications

$$\mathcal{WAP}_{\mathrm{t}} \Rightarrow \mathcal{WAP}_{\mathrm{fd}} \Rightarrow \mathcal{WAP}_{\mathrm{ap}}.$$

The left-hand implication is not an equivalence, for instance the infinite dihedral group is a counterexample.

Question 4.5. Are Properties $\mathcal{WAP}_{\mathrm{fd}}$ and $\mathcal{WAP}_{\mathrm{ap}}$ equivalent?

This is the case for the unitary Hilbert analogue, because any almost periodic unitary Hilbert G-module can be written as an ℓ^2-direct sum of finite-dimensional ones. A positive answer to the question, even with some restrictions on the class of G-modules considered, would be interesting (at least if the given class has good stability properties under induction of actions).

In view of Proposition 4.3, a positive answer would follow from a positive answer to:

Question 4.6. Does Property $\mathcal{AP}_{\mathrm{fd}}$ hold for an arbitrary locally compact group G?

In turn, a positive answer would result from the following less restrictive question:

Question 4.7. Let G be a locally compact group and V an almost periodic Banach G-module. Consider $b \in Z^1(G,V) \smallsetminus \overline{B^1(G,V)}$. Does there exist a G-submodule of finite codimension $W \subset V$ such that the image of b in $Z^1(G,V/W)$ is unbounded?

For instance, the answer is positive in the case of unitary Hilbert G-modules. See §4.6 for more on Property $\mathcal{AP}_{\mathrm{fd}}$.

4.2. Extension by a compact normal subgroup.

Proposition 4.8. Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of locally compact groups, and assume that K is compact. Then G has property $\mathcal{WAP}_{\mathrm{t}}$ (resp. $\mathcal{WAP}_{\mathrm{ap}}$, resp. $\mathcal{WAP}_{\mathrm{fd}}$) if and only if Q does.

Proof. These properties are obviously stable under taking quotients.

For the converse, consider a WAP G-module (V,π). Let b be a 1-cocycle. Since K is compact, we can find a cohomologous 1-cocycle b' that vanishes on K. Then b' takes values in V^K: indeed, for $g \in G$ and $k \in K$, $b(kg) = \pi(k)b(g) + b(k) = \pi(k)b(g)$, so

$$\pi(k)b(g) = b(kg) = b(gg^{-1}kg) = \pi(g)b(g^{-1}kg) + b(g) = b(g).$$

If we assume that Q has Property $\mathcal{WAP}_{\mathrm{t}}$, and we assume $V^G = 0$, then $(V^K)^G = 0$ and we deduce that b is an almost coboundary, showing that G has Property $\mathcal{WAP}_{\mathrm{t}}$. If we assume that Q has Property $\mathcal{WAP}_{\mathrm{ap}}$, and assume $V^{G,\mathrm{ap}} = 0$, we deduce that $(V^K)^{Q,\mathrm{ap}} = 0$. It follows that b' is an almost coboundary, and hence b as well.

If we assume that Q has Property $\mathcal{WAP}_{\mathrm{fd}}$, we first invoke Theorem 2.13(1): we have $V = V^K \oplus W$, where W is a canonically defined complement. Then by Property $\mathcal{WAP}_{\mathrm{fd}}$ of Q, we have $V^K = W' \oplus W''$, where W' is a finite-dimensional G-submodule and b has an unbounded projection on W' modulo W''. This shows that G has Property $\mathcal{WAP}_{\mathrm{ap}}$. ☐

4.3. Invariance of $\mathcal{WAP}_{\mathrm{t}}$ under central extension.

The material of this section uses some trick which was exploited in [ANT13] in the case of Heisenberg's group. See [Sh04, Theorem 4.1.3] in the Hilbert setting and [BRS13, Theorem 2] for a more general statement (involving reduced cohomology in any degree).

Proposition 4.9. Let G be a locally compact group with a compactly generated, closed central subgroup Z such that G/Z has Propery $\mathcal{WAP}_{\mathrm{t}}$. Then G has Property $\mathcal{WAP}_{\mathrm{t}}$.

Proof. We start with the case when Z is discrete cyclic. Let $1 \to \mathbf{Z} \to G \to Q \to 1$ be a central extension where Q has property $\mathcal{WAP}_{\mathrm{t}}$. Let V be a weakly almost periodic Banach space and let (V,π) be a G-module with $V^G = 0$, and let b be a cocycle. Let $V^{\mathbf{Z}}$ be the subspace of V consisting of fixed $\mathbf{Z}$-vectors. Because $\mathbf{Z}$ is central, V decomposes as a G-invariant direct sum $V = V_1 \oplus V_2$, where $V_1 = V^{\mathbf{Z}}$ and V_2 is its canonical complement (Proposition 2.13(1)).

Let us decompose $\pi = \pi_1 \oplus \pi_2$ and $b = b_1 \oplus b_2$ accordingly, with $b_i \in Z^1(G,\pi_i)$. Since $\overline{H}^1(G,\pi) = \oplus_i \overline{H}^1(G,\pi_i)$, it is enough to show that both terms in the direct sum vanish. Let z be a generator of $\mathbf{Z}$. Let us first show that $\overline{H}^1(G,\pi_2) = 0$, showing that under this assumption the sequence $x_n = \frac{1}{n}\sum_{i=0}^{n-1} b(z^i)$ is almost fixed by the affine action σ of G associated to b. The cocycle relation together with the fact that $\mathbf{Z}$ is central imply that

$$\sigma(g)x_n = \frac{1}{n}\sum_{i=0}^{n-1} b(gz^i) = \frac{1}{n}\sum_{i=0}^{n-1} b(z^i g) = \frac{1}{n}\sum_{i=0}^{n-1} \pi(z)^i b(g)$$

which goes to zero by the ergodic theorem (which states in general that this converges to a G-invariant vector and is an immediate verification). Let $\phi : \overline{H}^1(G,\pi_1) \to \overline{H}^1(\mathbf{Z},\pi_1)$ be the map in cohomology obtained by restricting cocycles to the central subgroup $\mathbf{Z}$. Using that $\mathbf{Z}$ is central and that the restriction of π_1 to $\mathbf{Z}$ is trivial, we deduce that $b_1(z)$ is a $\pi(G)$-invariant vector, which therefore equals 0. It follows that b_1 induces a cocycle $\tilde{b}_1$ for the representation $\tilde{\pi}_1$ of Q. It is easy to see that b_1 is an almost coboundary if and only if $\tilde{b}_1$ is an almost coboundary. So we conclude thanks to the fact that Q has $\mathcal{WAP}_{\mathrm{t}}$.

Let us now prove the general case. As a CGLC abelian group, Z has a cocompact discrete subgroup Λ isomorphic to $\mathbf{Z}^d$ for some d. Then G/Z is quotient of G/Λ with compact kernel, and hence by Proposition 4.8, G/Λ has Property $\mathcal{WAP}_{\mathrm{t}}$. Then by an iterated application of the case with discrete cyclic kernel, G has Property $\mathcal{WAP}_{\mathrm{t}}$ as well. $\square$

Corollary 4.10. Among compactly generated locally compact groups, the class of compactly presented groups with Property $\mathcal{WAP}_{\mathrm{t}}$ is closed under taking central extensions.

Proof. Let G be compactly generated, with a central subgroup Z such that G/Z is also compactly presented and has Property $\mathcal{WAP}_{\mathrm{t}}$. Since G/Z is compactly presented and G is compactly generated, Z is compactly generated. Hence Proposition 4.9 applies (and G is compactly presented). $\square$

Since CGLC nilpotent groups are compactly presented, we deduce

Corollary 4.11. [[BRS13], Theorem 8] CGLC nilpotent groups have Property $\mathcal{WAP}_{\mathrm{t}}$.

4.4. **Cocompact subgroups.**

Lemma 4.12. Let G be a locally compact group, H a closed normal cocompact subgroup. Let (V,π) be a Banach G-module. If $b \in Z^1(G,V)$ is an almost coboundary in restriction to H, then it is an almost coboundary.

Proof. Use a bounded measurable section $P \subset G$, so that $P \times H \to G$ is a measurable bijective map (with measurable inverse). Denote by $y \mapsto \hat{y}$ the section $G/H \to P$. Let (v_n) be a sequence of H-almost fixed vectors. Let S be a compact generating subset of H; enlarging S if necessary, we can suppose that PS contains a compact generating subset of G.

Define $\xi_n = \int_{x\in P} \alpha(\hat{x})v_n dx$. Then for $y \in G/H$ and $s \in S$, we have

$$\begin{aligned}
\xi_n - \alpha(\hat{y}s)\xi_n &= \int_{x\in P} \alpha(x)v_n dx - \alpha(\hat{y}s)\int_{x\in P} \alpha(x)v_n dx \\
&= \int_{x\in G/H} \alpha(\hat{x})v_n - \int_{x\in G/H} \alpha(\hat{y}s\hat{x})v_n dx \\
&= \int_{x\in G/H} \alpha(\hat{x})v_n - \int_{x\in G/H} \alpha(\hat{y}s\widehat{y^{-1}x})v_n dx \\
&= \int_{x\in G/H} (\alpha(\hat{x})v_n - \alpha(\hat{x}\nu(x,y,s))v_n) dx \\
&= \int_{x\in G/H} \pi(\hat{x})(v_n - \alpha(\nu(x,y,s))v_n) dx,
\end{aligned}$$

with $\nu(x,y,s) = \hat{x}^{-1}\hat{y}s\widehat{y^{-1}x}$. Since $\nu(x,y,s)$ belongs to some fixed ball of H (independently of x,y,s), we have $\|v_n - \alpha(\nu(x,y,s))v_n\| \le \varepsilon_n$ for some sequence (ε_n) depending only of this ball, tending to zero. Thus $\|\xi_n - \alpha(\hat{y}s)\xi_n\| \le c\varepsilon_n$, where $c = \sup_{g\in G}\|\pi(g),\|$ and hence (ξ_n) is a sequence of almost fixed points for G. □

Proposition 4.13. Let G be a locally compact group and H a closed cocompact subgroup.

(1) if H has Property $\mathcal{WAP}_{\mathrm{ap}}$, so does G;
(2) if H has Property $\mathcal{WAP}_{\mathrm{t}}$ and is normal in G, then G has Property $\mathcal{WAP}_{\mathrm{fd}}$.

Proof. Suppose that H has Property $\mathcal{WAP}_{\mathrm{ap}}$. Let V be a G-module with $V^{G,\mathrm{ap}} = 0$ and $b \in Z^1(G,\pi)$. Since H is cocompact in G, we have $V^{H,\mathrm{ap}} \subset V^{G,\mathrm{ap}}$ and hence $V^{H,\mathrm{ap}} = 0$. By Property $\mathcal{WAP}_{\mathrm{ap}}$ of H, b is an almost coboundary in restriction to H, and hence on G by Lemma 4.12. Hence G has Property $\mathcal{WAP}_{\mathrm{ap}}$.

Now assume that H is normal and has Property $\mathcal{WAP}_{\mathrm{t}}$. Let V be a WAP G-module and b a 1-cocycle that is not an almost coboundary. Since H is normal, Theorem 2.13(1) implies that V decomposes as a G-invariant direct sum $V = V^H \oplus V_2$. Decompose $b = b_1 + b_2$ accordingly. Since H has Property $\mathcal{WAP}_{\mathrm{t}}$, b_2 is an almost coboundary in restriction to H, and hence, by Lemma 4.12, b_2 is an almost coboundary on G. Hence b_1 is not an almost coboundary (on G). But b_1 is a group homomorphism in restriction to H, and since H is CGLC, $b_1(H)$ generates a finite-dimensional subspace F of V. Since H is normal, this subspace is $\pi(G)$-invariant. By Proposition 2.11, F as a complement W in V^H as a G-module, and under the decomposition $V = F \oplus (W \oplus V_2)$ we have $b = b_1 + (0 + b_2)$, where b_1 is unbounded. This shows Property $\mathcal{WAP}_{\mathrm{fd}}$. □

Theorem 4.14. Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{fd}}$ are inherited by closed cocompact subgroups $H \subset G$ of finite covolume.

Proof. We start with Property $\mathcal{WAP}_{\mathrm{t}}$. Let (V,π) be a WAP H-module and c a 1-cocycle. We use the notation of §3.3; in particular $(E, \mathrm{Ind}_H^G \pi)$ is the induced G-module. Assuming that c is nonzero in the reduced cohomology, we get that $\tilde{c}$ is also nonzero in reduced cohomology. Decompose the cocycle $\tilde{c} = \tilde{c}_1 + \tilde{c}_2$ according to the decomposition $E = E_1 \oplus E_2$ (see Theorem 2.13(1)). Since G has $\mathcal{WAP}_{\mathrm{t}}$, and $\tilde{c}$ is nonzero in reduced cohomology, we obtain that $\tilde{c}_1$ is a nonzero group homomorphism. Therefore, integrating $\tilde{c}_1$ over μ as in (3.4) gives back a non-zero group homomorphism $H \to V$ which, being a 1-cocycle, is valued in V^H. Hence $V^H \neq 0$, proving Property $\mathcal{WAP}_{\mathrm{t}}$.

Now suppose G has $\mathcal{WAP}_{\mathrm{fd}}$. We argue in the same way, but instead with $E_1 = E^{G,\mathrm{ap}}$ (using Theorem 2.13(2) instead). We obtain a decomposition $E = F \oplus W$ of G-module, with F finite-dimensional, such that the corresponding decomposition $\tilde{c} = \tilde{c}_1 + \tilde{c}_2$ has $\tilde{c}_1$ unbounded (hence not an almost coboundary). Write $c_i = \overline{\tilde{c}_i} \in Z^1(H,\pi)$. Then c_1 is also not an almost coboundary, and in addition, has its range contained in a finite-dimensional subspace. Clearly the subspace V_1 spanned by the range of c_1, being the affine hull of the orbit of 0 in the affine action defined by c_1, is π-invariant. By Proposition 2.11, we can find an H-module complement $V = V_1 \oplus V_2$, and the projection of c_1 is just c_1. On the other hand, $\tilde{c}_2$ being an almost boundary, so is c_2, as well as its projections. Since $c = \bar{\tilde{c}}$, we have $c = c_1 + c_2$, and the projection of c to V_1 differs from c_1 by a bounded function, and hence is unbounded. This proves that H has Property $\mathcal{WAP}_{\mathrm{fd}}$. ☐

Remark 4.15. We could not adapt this proof to Property $\mathcal{WAP}_{\mathrm{ap}}$.

Theorem 4.16. Every compactly generated, locally compact group G with polynomial growth has Property $\mathcal{WAP}_{\mathrm{fd}}$.

Proof. By Losert's theorem [Lo87] (due to Gromov in the discrete case), for such G, there exists a copci (proper continuous with cocompact image) homomorphism to a locally compact group of the form $N \rtimes K$ with N a simply connected nilpotent Lie group and K a compact Lie group. Let W be the kernel of such a homomorphism.

Then N has Property $\mathcal{WAP}_{\mathrm{t}}$ by Corollary 4.11, and hence $N \rtimes K$ has Property $\mathcal{WAP}_{\mathrm{fd}}$ by Proposition 4.13(2), and hence G/W has Property $\mathcal{WAP}_{\mathrm{fd}}$ by Theorem 4.14, and in turn G has Property $\mathcal{WAP}_{\mathrm{fd}}$ by Proposition 4.8. ☐

4.5. **Stability under RCE.**

Theorem 4.17. If a countable group Λ has Property $\mathcal{WAP}_{\mathrm{fd}}$ and Λ randomly cocompactly embeds inside another countable group Γ, then Λ has $\mathcal{WAP}_{\mathrm{fd}}$ as well.

Proof. The proof is almost identical to that of Theorem 4.14 but we reproduce it for the sake of completeness. Assume that Λ randomly cocompactly embeds inside Γ and that Γ has Property $\mathcal{WAP}_{\mathrm{fd}}$. Consider a WAP Λ-module (V,π) and a 1-cocycle $c \in \mathbf{Z}^1(\Lambda, \pi)$. Assuming that c is nonzero in the reduced cohomology, we get that $\tilde{c} := Ic$ is also nonzero in reduced cohomology. By Proposition 3.3, the induced Γ-module $(E, Ind_\Lambda^\Gamma \pi)$ is WAP. Hence there is a decomposition $E = F \oplus W$ of Γ-module (see Theorem 2.13(2)), with F finite-dimensional, such that the corresponding decomposition $\tilde{c} = \tilde{c}_1 + \tilde{c}_2$ has $\tilde{c}_1$ unbounded (hence not an almost coboundary). Write $c_i = T\tilde{c}_i \in Z^1(H, \pi)$. Then c_1 is also not an almost coboundary, and in addition, has its range contained in a finite-dimensional subspace. Clearly the subspace V_1 spanned by the range of c_1, being the affine hull of the orbit of 0 in the affine action defined by c_1, is π-invariant. By Theorem 2.11, we can find an Λ-module complement $V = V_1 \oplus V_2$, and the projection of c_1 is just c_1. On the other hand, $\tilde{c_2}$ being an almost boundary, so is c_2, as well as its projections. Since $c = TIc$, we have $c = c_1 + c_2$, and the projection of c to V_1 differs from c_1 by a bounded function, and hence is unbounded. This proves that H has Property $\mathcal{WAP}_{\mathrm{fd}}$. $\square$

4.6. **Property $\mathcal{AP}_{\mathrm{fd}}$.** The following part is notably motivated by Question 4.6.

Let G be a locally compact group. Let $\mathcal{K}(G)$ be the intersection of all kernels of continuous homomorphisms into compact groups. Then $\mathcal{K}(G)$ is itself such a kernel (using a product). Note that by the Peter-Weyl theorem, it is also the intersection of all kernels of continuous homomorphisms into the finite-dimensional orthogonal groups $\mathrm{O}(n)$.

Next, define $\mathcal{K}^\dagger(G)$ to be the intersection of all kernels of continuous homomorphisms into the finite-dimensional isometry groups $\mathbf{R}^n \rtimes \mathrm{O}(n)$. Clearly, $H = G/\mathcal{K}^\dagger(G)$ is the largest quotient of G such that $\mathcal{K}^\dagger(H) = 1$.

Recall that $g \in G$ is *distorted* if there exists a compact subset S of G containing g such that $\lim |g^n|_S/n = 0$, where $|\cdot|_S$ is the word length with respect to S (in particular, this includes elements of finite order and more generally elliptic elements, for which $(|g^n|_S)$ is bounded for suitable S).

Proposition 4.18. $\mathcal{K}(G)/\mathcal{K}^\dagger(G)$ is abelian, and contains no nontrivial element that is distorted in $G/\mathcal{K}^\dagger(G)$.

Proof. We can suppose that $\mathcal{K}^\dagger(G)$ is trivial. So we have to prove that $\mathcal{K}(G)$ is abelian and has no nontrivial distorted element.

If $u, v \in \mathcal{K}(G)$ do not commute, we can find n and a continuous homomorphism $G \to \mathbf{R}^n \rtimes \mathrm{O}(n)$ such that $[u, v]$ is not in the kernel. Since $u, v \in \mathcal{K}(G)$, both are mapped to translations, and we have a contradiction. Also, if $u \in \mathcal{K}(G) \smallsetminus \{1\}$, we can find a continuous homomorphism as above such that u is not in the kernel, and hence u maps to a translation. Then we have a contradiction since the translation is undistorted in the group of Euclidean isometries. $\square$

This yields a method to "approach" $G/\mathcal{K}^\dagger(G)$ from G: first mod out by the closure of the derived subgroup $\overline{[\mathcal{K}(G), \mathcal{K}(G)]}$. Then mod out by the closure of the subgroup of the abelian subgroup $\mathcal{K}\Big(G/\overline{[\mathcal{K}(G),\mathcal{K}(G)]}\Big)$ consisting of those elements that are distorted in $G/\overline{[\mathcal{K}(G),\mathcal{K}(G)]}$.

Example 4.19. Let G be a real triangulable Lie group. Then $\mathcal{K}^\dagger(G)$ is equal to the derived subgroup (because the derived subgroup is equal to $\mathcal{K}(G)$ and its elements are at least quadratically distorted).

If $G = \mathbb{G}(\mathbf{Q}_p)$ for some linear algebraic $\mathbf{Q}_p$-group $\mathbb{G}$, let $\mathbb{H} = \mathbb{G}/\mathbb{N}$ be the largest quotient with no simple factor of positive $\mathbf{Q}_p$-rank, with abelian unipotent radical, and whose maximal $\mathbf{Q}_p$-split torus centralizes the unipotent radical. (By Borel-Tits, G is compactly generated if and only if : $\mathbb{H}$ is reductive.) Then $\mathbb{N}(G)$ is contained in $\mathcal{K}^\dagger(G)$. (If G is compactly generated, then $\mathbb{H}(\mathbf{Q}_p)$ is compact-by-abelian.)

Proposition 4.20. G has Property $\mathcal{AP}_{\mathrm{fd}}$ if and only if $G/\mathcal{K}^\dagger(G)$ has Property $\mathcal{AP}_{\mathrm{fd}}$. If N is any closed normal subgroup contained in $\mathcal{K}^\dagger(G)$, this is also equivalent to: G/N has Property $\mathcal{AP}_{\mathrm{fd}}$.

Proof. It is trivial that Property $\mathcal{AP}_{\mathrm{fd}}$ passes to quotients, hence it passes from G to G/N and from G/N to $G/\mathcal{K}^\dagger(G)$. Now suppose that $G/\mathcal{K}^\dagger(G)$ has Property $\mathcal{AP}_{\mathrm{fd}}$. Let V be an almost periodic G-module and b a 1-cocycle that is not an almost coboundary. By Proposition 2.4, the G-representation factors through a compact group; in particular, it is trivial on $\mathcal{K}(G)$. So on $\mathcal{K}(G)$, b is given by a continuous group homomorphism $\mathcal{K}(G) \to V$. We claim that b vanishes on $\mathcal{K}^\dagger(G)$. Assume the contrary by contradiction: pick $g \in \mathcal{K}^\dagger(G)$ with $b(g) \neq 0$.

Then by Lemma 2.10, there exists $f \in V^*_{[G]}$ such that $f(b(g)) \geq 2$. Since $V^*_{[G]}$ is almost periodic, the union of finite-dimensional submodules is dense (Corollary 2.8), and hence there is $f' \in V^*_{[G]}$ inside a finite-dimensional submodule $M \subset V^*_{[G]}$ such that $f'(b(g)) \geq 1$. Let W be the orthogonal (for duality) of M, it has finite codimension and $b(g) \notin W$. This means that the projection of b in V/W is nonzero. Hence g is not in the kernel of the affine action on V/W. Hence $g \notin \mathcal{K}^\dagger(G)$, a contradiction. $\square$

Say that G has Property $\mathcal{AP}_{\mathrm{fd}}$ if every almost periodic Banach G-module V with $V^G = 1$, we have $\overline{H^1}(G,V) = 0$.

The same proof also shows:

Proposition 4.21. Let G be a locally compact group. Then G has Property $\mathcal{AP}_{\mathrm{fd}}$ if and only if $G/\mathcal{K}^\dagger(G)$ has Property $\mathcal{AP}_{\mathrm{fd}}$. In particular, if G has Property $\mathcal{WAP}_{\mathrm{ap}}$, it has Property $\mathcal{WAP}_{\mathrm{fd}}$ if and only if $G/\mathcal{K}^\dagger(G)$ has Property $\mathcal{AP}_{\mathrm{fd}}$.

Let us now provide information about $\mathcal{K}^\dagger(G)$ in more specific examples.

Lemma 4.22. Let G be a compactly generated locally compact group in the class $\mathfrak{C}''$. Then $G/\mathcal{K}^\dagger(G)$ has polynomial growth.

Proof. We first see that $\mathrm{Contr}(G) \subset \mathcal{K}^\dagger(G)$. This amounts to showing that in $H = \mathbf{R}^n \rtimes \mathrm{O}(n)$ we have $\mathcal{K}^\dagger(H) = \{1\}$: indeed first in $L = \mathrm{O}(n)$ we have $\mathcal{K}^\dagger(L) = \{1\}$ because L is compact and hence has closed conjugacy classes; it follows that $\mathcal{K}^\dagger(H) \subset \mathbf{R}^n$, and clearly the conjugacy classes of H contained in $\mathbf{R}^n$ are compact.

By Lemma 7.3, we deduce that $U_{\mathrm{div}} \subset \mathcal{K}^\dagger(G)$. Since the class $\mathfrak{C}''$ is stable under taking quotients, we are reduced to proving that if G belongs to the class $\mathfrak{C}''$ satisfies $U_{\mathrm{div}} = \{1\}$, then G has polynomial growth. Indeed in this case, U is compact and since $G = UN$ and N has polynomial growth, the conclusion follows. $\square$

The previous lemma is a way to show that $G/\mathcal{K}^\dagger(G)$ is small in many relevant examples. In contrast, the following proposition shows that it is often large in the setting of discrete groups.

Proposition 4.23. Let Γ be a discrete and finitely generated group. Then $\mathcal{K}(\Gamma) = \mathcal{K}^\dagger(\Gamma)$ is the intersection of all finite index subgroups of G.

Proof. Denote by $R(\Gamma)$ the intersection of all finite index subgroups of Γ. Clearly, $\mathcal{K}^\dagger(\Gamma) \subset \mathcal{K}(\Gamma) \subset R(\Gamma)$. The remaining inclusion $R(\Gamma) \subset \mathcal{K}^\dagger(\Gamma)$ follows from Malcev's theorem that finitely generated linear groups are residually finite. $\square$

5. A dynamical criterion for property $\mathcal{WAP}_{\mathrm{ap}}$

This section contains the central ideas of this paper. It culminates with Theorem 5.12, which provides dynamical criteria for Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{ap}}$. The latter is designed to apply to groups from the class $\mathfrak{C}$. The following lemmas are the key ingredients. It starts with Lemma 5.2, an analogue of Mautner's phenomenon.

5.1. Mautner's phenomenon.

Definition 5.1. Let G be a locally compact group, and let N be a subgroup of G. Denote by $\mathrm{Contr}(N)$ the set of elements $g \in G$ such that there exists a sequence $a_n \in N$ such that $a_n^{-1} g a_n \to 1$. Such an element g is called N-contracted.

Note that $\mathrm{Contr}(N)$ is stable under inversion; it is not always a subgroup.

Lemma 5.2. (Mautner's phenomenon) Let G be a locally compact group, N a subgroup, and L the subgroup generated by $\mathrm{Contr}(N)$. Consider a separately continuous, uniformly Lipschitz action of G on a metric space X. Then the subspace $X^{N,\mathrm{ap}}$ of almost periodic points (Definition 2.1) is contained in the subspace X^L of L-fixed points.

Proof. Let C be the supremum of all Lipschitz constants of the G-action. Fix $u \in \mathrm{Contr}(N)$, a sequence (a_n) with $a_n \in N$ and $\varepsilon_n = a_n^{-1} u a_n \to 1$.

Consider $x \in X^{N,\mathrm{ap}}$. Write $ux = a_n\varepsilon_n a_n^{-1}x$. Let J be an infinite subset of integers such that $(a_n^{-1}x)_{j\in J}$ converges, say to x'. Then

$$\begin{aligned} d(a_n\varepsilon_n a_n^{-1}x, x) \leq & d(a_n\varepsilon_n a_n^{-1}x, a_n\varepsilon_n x') + d(a_n\varepsilon_n x', a_n x') + d(a_n x', x) \\ \leq & Cd(a_n^{-1}x, x') + Cd(\varepsilon_n x', x') + Cd(x', a_n^{-1}x) \xrightarrow[n\to\infty]{n\in J} 0. \end{aligned}$$

This shows that $ux = a_n^{-1}\varepsilon_n a_n x = x$. □

Lemma 5.2 generalizes Mautner's phenomenon, as well as [Sh04, Lemma 5.2.6]; both were written in a more specific context for G; in addition in Mautner's result the metric space is a Hilbert space with a unitary representation; in Shalom's result X is an arbitrary metric space with an isometric action; in both cases the result takes the form $X^N \subset X^L$. We will use the following consequence of Lemma 5.2.

Lemma 5.3. Let G a locally compact group, and N a subgroup of G. Define M as the normal subgroup generated by $\mathrm{Contr}(N)$, and $H = \overline{MN}$. Consider a separately continuous, uniformly Lipschitz action of G on a metric space X. Then

(1) if $H = G$, then $X^N = X^G$;
(2) if H is cocompact in G, then $X^{N,\mathrm{ap}} = X^{G,\mathrm{ap}}$.

Proof. By definition N acts trivially on X^N, and by Lemma 5.2, M acts trivially on X^N. Hence, in the context of (1), MN is dense and hence G acts trivially on X^N.

Assume now as in (2), and write $H = \overline{MN}$, which is cocompact. Then by Lemma 5.2, M acts trivially on $X^{N,\mathrm{ap}}$. In particular H preserves $X^{N,\mathrm{ap}}$, and the H-action on $X^{N,\mathrm{ap}}$ factors through H/M. Since N has a dense image in H/M, it follows that the closure of H-orbits in $X^{N,\mathrm{ap}}$ coincide with closure of N-orbits, which are compact by assumption. This shows that $X^{N,\mathrm{ap}} = X^{H,\mathrm{ap}}$.

Finally, we have $X^{H,\mathrm{ap}} = X^{G,\mathrm{ap}}$ by Lemma 2.3. □

5.2. **Controlled Følner sequences and sublinearity of cocycles.** We now recall some material from [CTV07], also used in [T09].

Definition 5.4. [CTV07] Let G be a locally compact group generated by some compact subset S and equipped with some left Haar measure μ. A sequence of compact subsets $F_n \subset G$ of positive measure is called a controlled Følner sequence if either G is compact, or $\mathrm{Diam}(F_n) \to \infty$, and there exists a constant $C \geq 1$ such that for all $n \in \mathbf{N}$ and all $s \in S$,

$$\mu(sF_n \vartriangle F_n) \leq C\frac{\mu(F_n)}{\mathrm{Diam}(F_n)}.$$

Remark 5.5. Let G be a compactly generated group with a compact generating subset S. For n, let $f(n)$ be the smallest m such that the m-ball contains a compact subset F of positive measure such that $\mu(sF_n \vartriangle F_n) \leq \mu(F_n)/n$. Then

G is amenable if and only if $f(n) < \infty$ for all n, and admits a controlled Følner sequence if and only if $\liminf f(n)/n < \infty$. Actually in all examples we are aware of groups with controlled Følner sequences, we indeed have $f(n) = O(n)$; notably strong controlled Følner sequences of Definition 6.3 satisfy this property.

The following is [CTV07, Corollary 3.7] in the case of unitary Hilbert modules.

Proposition 5.6. (Sublinearity versus triviality of cocycles) Let G be a CGLC group and let (V, π) be a uniformly bounded Banach G-module. Let $b \in Z^1(G, \pi)$ be a 1-cocycle.

If b is an almost coboundary, then it is sublinear, i.e. $\|b(g)\| = o(|g|)$ when $|g|$ goes to infinity. The converse holds if G admits a controlled Følner sequence: if b is sublinear, then it is an almost coboundary.

Proof. Both implications are adaptations of the original proof for unitary Hilbert G-modules, up to some technical points, which we emphasize below.

Denote $C = \sup_{g \in G} \|\pi\|$, and by S a compact generating subset of G; let $|\cdot|$ be the word length in G with respect to S. If $b \in Z^1(G, \pi)$ and $T \subset G$, denote $\|b\|_T = \sup_{s \in S} \|b(s)\|_T$. Then we have, for all $g \in G$, the inequality $\|b(g)\| \leq C|g|\|b\|_S$.

Suppose that b is an almost coboundary. For $\varepsilon > 0$, there exists a bounded cocycle b' such that $\|b - b'\| \leq \varepsilon/C$ on S. Say, $\|b'\|_G \leq c_\varepsilon$. Then, using the previous inequality for the cocycle $b - b'$, we have for all $g \in G$, $\|b(g)\| \leq \|b'(g)\| + \|(b - b')(g)\| \leq c + |g|\varepsilon$. Thus, for $|g| \geq c_\varepsilon \varepsilon^{-1}$, we have $\frac{\|b(g)\|}{|g|} \leq 2\varepsilon$.

Now assume that G admits a controlled Følner sequence and let us prove the converse. Suppose that b is sublinear. Let (F_n) be a controlled Følner sequence in G. We need to define a sequence $(v_n) \in V^{\mathbf{N}}$ by

$$v_n = \frac{1}{\mu(F_n)} \int_{F_n} b(g) dg.$$

Here is the technical issue: since V is not assumed to be reflexive, we have to be more careful to claim that this integral is well-defined. Since F_n is compact, it follows that on F_n, we can write the function b (or any continuous function to a normed space) as a uniform limit of simple functions. This implies that $g \mapsto b(g)$ is Bochner-integrable.

We claim that (v_n) defines a sequence of almost fixed points for the affine action σ defined by $\sigma(g)v = \pi(g)v + b(g)$ (which is equivalent to saying that b is an almost coboundary). Indeed, we have (noting that $\sigma(s)b(g) = b(sg)$)

$$\begin{aligned} \|\sigma(s)v_n - v_n\| &= \left\| \frac{1}{\mu(F_n)} \int_{F_n} \sigma(s)b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &= \left\| \frac{1}{\mu(F_n)} \int_{F_n} b(sg)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &= \left\| \frac{1}{\mu(F_n)} \int_{s^{-1}F_n} b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &\leq \frac{1}{\mu(F_n)} \int_{s^{-1}F_n \triangle F_n} \|b(g)\| dg. \end{aligned}$$

Since $F_n \subset S^n$, we obtain that

$$\|\sigma(s)v_n - v_n\| \leq \frac{C}{n} \sup_{|g|_S \leq n+1} \|b(g)\|,$$

which converges to 0. □

5.3. **Combing.** An important feature of the groups studied in this article is the following "combing" property.

Definition 5.7. Let G be a locally compact group generated by some compact subset S, and let $H \leq G$ be a closed subgroup. We say that G is H-combable if there exists an integer $k \in \mathbf{N}$ and a constant $C \geq 1$ such that every element $g \in G$ can be written as a word $w = w_1 \dots w_k$ in the alphabet $S \cup H$ with

$$\sum_{i=1}^{k} |w_i|_S \leq C|g|_S. \tag{5.8}$$

Remark 5.9. It is easy to check that being H-combable does not depend on the choice of S. Moreover, assuming for convenience that S is symmetric with 1, it is equivalent to the existence of constants k, ℓ such that $S^n \subset ((S^{\ell n} \cap H)S)^k$ for all n. In most examples, we actually have a stronger property: there exist constants k, ℓ such that $S^n \subset ((S^\ell \cap H)^n S)^k$ for all n.

If G is H-combable with k as above, then $G \subset (HS)^k$. In particular, when H is compact, then G is H-combable if and only if G is compact. However, there are many interesting cases where G is H-combable with H being nilpotent and G having exponential growth.

The following lemma is immediate, but we emphasize it to show how this property can be used.

Lemma 5.10. Let G be a CGLC group and H a closed subgroup such that G is H-combable. Let ℓ be a length function on G that is sublinear on H (with respect to the restriction to H of the word length of G). Then it is sublinear on G.

In particular, if H is compactly generated and ℓ is sublinear on H (with respect to its intrinsic word length), then ℓ is sublinear on G. □

5.4. **Injectivity of the restriction map in 1-cohomology.**

Lemma 5.11. (Restriction in 1-cohomology) Let G be a locally compact group and N a closed subgroup. Suppose that G has a controlled Følner sequence and is N-combable. Let V be a WAP G-module. Then the restriction map $\overline{H^1}(G,V) \to \overline{H^1}(N,V)$ is injective.

Proof. We can change the norm to an equivalent G-invariant norm. Consider a cocycle $b \in Z^1(G,\pi)$. Suppose that in restriction to N, b is an almost coboundary. Then by Proposition 5.6, b is sublinear in restriction to N, i.e. $\|b(a)\| = o(|a|_S)$. Note that $\|b\|$ is sub-additive, because the norm is G-invariant. Since G is N-combable, one thus deduces from Lemma 5.10 that b is sublinear on all of G, and therefore is an almost coboundary by Proposition 5.6, using that G has a controlled Følner sequence. □

5.5. **Dynamical criteria for Properties $\mathcal{WAP}_{\mathrm{t}}$ and $\mathcal{WAP}_{\mathrm{ap}}$.** We are now ready to state and prove the main result of this section.

Theorem 5.12. Let G be locally compact group, and let N be a closed subgroup. Let H be the closure $\overline{MN}$ of the subgroup generated by $N \cup M$, where M is the normal subgroup generated by $\mathrm{Contr}(N)$. Assume:

(1) G has a controlled Følner sequence;
(2) G is N-combable;
(3) N has Property $\mathcal{WAP}_{\mathrm{t}}$;
(4) H is dense in G.

Then G has Property $\mathcal{WAP}_{\mathrm{t}}$.

Still assume (1) and (2) along with:

(3') N has Property $\mathcal{WAP}_{\mathrm{ap}}$;
(4') H is cocompact in G.

Then G has Property $\mathcal{WAP}_{\mathrm{ap}}$.

Proof. Suppose that (1), (2), (3), (4) hold. Let V be a Banach G-module with $V^G = 0$. By (4) and Lemma 5.3(1), $V^N = 0$. By (3), $\overline{H^1}(N,V) = 0$. By Lemma 5.11 and (1), (2), it follows that $\overline{H^1}(G,V) = 0$. So G has Property $\mathcal{WAP}_{\mathrm{t}}$.

Similarly, suppose that (1), (2), (3'), (4') hold. Let V be a Banach G-module with $V^{G,\mathrm{ap}} = 0$. By (4') and Lemma 5.3(2), $V^{N,\mathrm{ap}} = 0$. By (3'), $\overline{H^1}(N,V) = 0$. By Lemma 5.11 and (1), (2), it follows that $\overline{H^1}(G,V) = 0$. So G has Property $\mathcal{WAP}_{\mathrm{ap}}$. □

6. Groups in the class $\mathfrak{C}$

6.1. **Families of examples in the class $\mathfrak{C}$.** We start proving that two important classes of groups belong to the class $\mathfrak{C}$ introduced in Definition 3.

Proposition 6.1. The following groups belong to the class $\mathfrak{C}$:

(1) real triangulable connected Lie groups;
(2) groups of the form $G = \mathbb{G}(\mathbf{Q}_p)$, where $\mathbb{G}$ is a connected linear algebraic group defined over $\mathbf{Q}_p$ such that G is compactly generated and amenable.

Proof. (1) Let G be a real triangulable connected Lie group. Let N be a Cartan subgroup and $U = V$ its exponential radical. So U is the intersection of the lower central series, N is nilpotent, and $G = UN$. The N-action on the Lie algebra $\mathfrak{g}$ induces a grading on $\mathfrak{g}$, valued in the dual $\mathrm{Hom}(\mathfrak{n}, \mathbf{R})$ of the abelianization of $\mathfrak{n}$, for which $\mathfrak{n} = \mathfrak{g}_0$ and $\mathfrak{u}$ is the subalgebra generated by the $\mathfrak{g}_\alpha$ for $\alpha \neq 0$. By definition, a weight is an element α in $\mathrm{Hom}(\mathfrak{n}, \mathbf{R})$ such that $\mathfrak{g}_\alpha \neq \{0\}$. Then there exists k and a (possibly non-injective) family of nonzero weights $\alpha_1, \dots, \alpha_k$ such that, denoting $V_i = V_{\alpha_i}$, we have $U = V_1 \dots V_k$; we omit the details since this is already covered by [CT17, §6.B]. The existence of elements in N acting as contraction on V_i is straightforward, and thus all the data fulfill the requirements to belong to the class $\mathfrak{C}$.

(2) Write $\mathbb{G} = \mathbb{U} \rtimes (\mathbb{D}\mathbb{K})$, where $\mathbb{U}$ is the unipotent radical, and some reductive Levi factor is split into its $\mathbf{Q}_p$-split part $\mathbb{D}$ (a torus, by the amenability assumption) and its $\mathbf{Q}_p$-anisotropic part $\mathbb{K}$ (so $\mathbb{K}(\mathbf{Q}_p)$ is compact). Define $N = \mathbb{D}(\mathbf{Q}_p)$ and $V = \mathbb{U}(\mathbf{Q}_p)$. Thus $U = V \rtimes \mathbb{K}(\mathbf{Q}_p)$, so V is closed cocompact in U. Then in the grading on $\mathfrak{g}$ induced by the D-action (which takes values in a free abelian group of rank $\dim(D)$, namely the group of multiplicative characters of D), $\mathfrak{v}$ is generated by the nonzero weights: as a consequence of the assumption that G is compactly generated. Then the proof can be continued as in the real case (being also covered by the work in [CT17, §6.B]). $\square$

Proposition 6.2. Every virtually connected amenable Lie group G belongs to the class $\mathfrak{C}'$ (Definition 6).

Proof. By [CT17, Lemma 3.A.1] (based on [C08, Lemmas 2.4 and 6.7]), there exist copci homomorphisms $G \leftarrow G_1 \rightarrow G_2 \leftarrow G_3$ with G_3 triangulable. In addition, $G_1 \rightarrow G_2$ has normal image (as it is mentioned in the proof of that lemma that G_2 is generated by its center and the image of G_1). Thus G belongs to the class $\mathfrak{C}'$. $\square$

6.2. Controlled Følner sequences for groups in the class $\mathfrak{C}$".

Definition 6.3. In a CGLC group G with compact generating symmetric subset S with 1 and left Haar measure μ, we call strong controlled Følner sequence, a sequence of positive measure compact subsets (F_n) such that F_n belongs to a ball of radius $O(n)$ and

$$\frac{\mu(F_n')}{\mu(F_n)} = O(1),$$

where $F_n' = F_n S^n$ is the n-tubular neighborhood of F_n with respect to word metric associated to S.

In [T11], the pair (F_n, F'_n) is called a controlled Følner pair. An easy argument [T11, Proposition 4.9] shows that if (F_n) is a strong controlled Følner sequence, then there exists $k_n \in \{1, \dots, n\}$ such that $(F_n S^{k_n})$ is a controlled Følner sequence (Definition 5.4).

We shall need the following result from [CT17] (which is proved there for a smaller class of groups but the proof readily extends to our setting).

Lemma 6.4. [CT17, Theorem 6.B.2] Fix a CGLC group $G = UN$ in the class $\mathfrak{C}$". Let S and T be compact generating subsets of respectively G and N. There exist constants μ_1 and $\mu_2 > 1$ such that the following holds. For every small enough norm $\|\cdot\|$ on $\mathfrak{u}$, denoting by $U[r]$ the exponential of the r-ball in $(\mathfrak{u}, \|\cdot\|)$, we have, for all n, the inclusions

$$S^n \subset U[\mu_2^n]T^n, \quad U[\mu_1^n]T^n \subset S^{2n}.$$

By a straightforward application of the Baker-Campbell-Hausdorff formula, we obtain the following lemma.

Lemma 6.5. Let $\mathfrak{u}$ be a finite-dimensional nilpotent Lie algebra over a finite product of non-archimedean local fields of characteristic zero. Consider a norm $\|\cdot\|$ on $\mathfrak{u}$. There exists $d \in \mathbf{N}$ such that for all $r \geq 2$

$$\langle U[r] \rangle \subset U[r^d],$$

where $\langle U[r] \rangle$ is the (compact) subgroup generated by $U[r]$.

In the proof of [Gu73, Theorem II.1], Guivarc'h provides an asymptotic description of K^n, where K is a compact symmetric generating subset of a nilpotent connected Lie group G, which in particular implies the following lemma.

Lemma 6.6. Let $\mathfrak{u}$ be a finite-dimensional nilpotent Lie algebra over $\mathbf{R}$. Consider a norm $\|\cdot\|$ on $\mathfrak{u}$. For every compact symmetric generating subset K of U, there exists $C > 1$ and $\varepsilon > 0$ such that for all integers $r \geq 2$,

$$U[\varepsilon r^{1/C}] \subset K^r \subset U[r^C].$$

We deduce the following corollaries:

Corollary 6.7. Under the assumptions of Lemma 6.4, and assuming in addition that U is totally disconnected, there exist constants λ_1 and $\lambda_2 > 1$ such that the following holds. For every norm $\|\cdot\|$ on $\mathfrak{u}$, denoting by $U[r]$ the exponential of the r-ball in $(\mathfrak{u}, \|\cdot\|)$, we have, for all large enough n, the inclusions

$$S^n \subset \langle U[\lambda_2^n] \rangle T^n, \quad \langle U[\lambda_1^n] \rangle T^n \subset S^{2n}.$$

Corollary 6.8. We keep the assumptions of Lemma 6.4, assuming in addition that U is connected, and that K is a compact generating set of U. Then there exist constants β_1 and $\beta_2 > 1$ such that the following holds. For all large enough n, we have the inclusions

$$S^n \subset K^{\beta_1^n} T^n, \quad K^{\beta_2^n} T^n \subset S^{2n}.$$

Finally, we shall use

Lemma 6.9. Under the assumptions of Lemma 6.4, there exists $\lambda > 1$ such that for all $r \geq 1$, $n \in \mathbf{N}$, $g \in U[r]$, and $h \in T^n$,

$$h^{-1}gh \in U[\lambda^n r].$$

In particular, if U is connected, and K is a compact generating subset of U, there exist $\alpha, b \geq 0$ such that for all integers $r \geq 2$, $n \geq 1$, for all $g \in K^r$ and $h \in T^n$,

$$h^{-1}gh \in K^{\lceil \alpha^n r^b \rceil}.$$

Proof. Let λ be the supremum over all $t \in T$ of the operator norm of t acting on the normed vector space $(\mathfrak{u}, \|\cdot\|)$. The first statement is now a direct consequence of the fact that the operator norm is submultiplicative. The second statement follows from Lemma 6.6. $\square$

Theorem 6.10. Every CGLC group $G = NU$ in the class $\mathfrak{C}$" (in particular in the class $\mathfrak{C}$) admits a strong controlled Følner sequence.

Proof. We write $U = U_{\mathbf{R}} \times U_{\mathrm{na}}$, where $U_{\mathbf{R}}$ is connected, and U_{na} is totally disconnected. Fix some large enough integer μ (to be specified later) and define

$$F_n = (K^{\mu^n} \times \langle U_{\mathrm{na}}[\mu^n] \rangle) T^n,$$

where K is a compact symmetric generating subset of $U_{\mathbf{R}}$.

By Corollaries 6.7 and 6.8, there exists $C > 0$ such that $F_n \subset S^{Cn}$. Now observe that if μ is large enough, Lemma 6.9 implies that

$$\langle U_{\mathrm{na}}[\mu^n] \rangle T^n \langle U_{\mathrm{na}}[\lambda_1^n] \rangle = \langle U_{\mathrm{na}}[\mu^n] \rangle T^n.$$

On the other hand, assuming $\mu \geq \beta_1^b \lambda$, we have

$$K^{\mu^n} T^n K^{\beta_1^n} \subset K^{\mu^n + \lambda^n \beta_1^{bn}} T^n \subset K^{2\mu^n} T^n.$$

We deduce that

$$F_n' \subset (\langle U_{\mathrm{na}}[\mu^n] \rangle \times K^{2\mu^n}) T^{2n}.$$

Finally, in order to conclude, we observe that by the doubling property for both N and $U_{\mathbf{R}}$, there exists an integer k such that for all all n there exist finite subsets $X_n \subset N$ and $Y_n \subset U_{\mathbf{R}}$ of cardinality at most k such that $T^{2n} \subset T^n X_n$ and $K^{2\mu^n} \subset Y_n K^{\mu^n}$. Hence we have

$$F_n' \subset Y_n F_n X_n,$$

from which we deduce that

$$|F_n'| \leq k^2 |F_n|. \qquad \square$$

6.3. Combability of groups in the class $\mathfrak{C}$".

Theorem 6.11. CGLC groups in the class $\mathfrak{C}$" are N-combable in the sense of Definition 5.7.

Proof. Let $G \in \mathfrak{C}$ and let S be a compact generating subset of G. For convenience, we shall assume that the generating set $S = S_U \cup S_N$, where $S_U \subset U$ (resp. $S_N \subset N$). We assume that $i = 1$, namely that $G = UN$, with U being a nilpotent algebraic group over some local field $\mathbf{K}$; the general case being similar. Let $q : G \to M := G/U$. For every $g \in G$ of size n, its projection has length $\leq n$ with respect to $q(S)$, hence g can be written as a product $g = um$, such that m has length equal to $|q(g)| \leq n$, and u has length $\leq n + |q(g)| \leq 2n$. Therefore, it is enough to prove (5.8) for $g = u$.

Consider a finite-dimensional faithful representation of U as unipotent matrices in $M_d(\mathbf{K})$ and equip the latter with a submultiplicative norm $\|\cdot\|$. We shall use the notation $\preceq$ and $\simeq$ to mean "up to multiplicative and additive constants".

Moreover, an easy calculation (using that U is unipotent) shows that for all $u_1, \ldots, u_n \in U$,

$$\|u_1 \ldots u_n\| \preceq n^d \max_i \|u_i\|. \tag{6.12}$$

We shall also use the fact that given a norm $\|\cdot\|_{\mathrm{Lie}}$ on the Lie algebra $\mathfrak{u}$ of U, one has

$$\log \|u\| \simeq \log \|\log(u)\|_{\mathrm{Lie}},$$

where $\log : U \to \mathfrak{u}$ is the inverse of the exponential map. This estimate follows from the Baker-Campbell-Hausdorff formula, using that log and exp are polynomial maps. The action by conjugation of N on U induces a group homomorphism $N \to \mathrm{Aut}(\mathfrak{u})$. Let $\|\cdot\|_{\mathrm{op}}$ be the operator norm on $\mathrm{End}(\mathfrak{u})$, that by abuse of notation we consider as a norm on elements of N. Let $C = \max_{m \in S_N} \|m\|_{\mathrm{op}}$ and $K = \max_{u \in S_U} \|u\|_{\mathrm{Lie}}$.

Lemma 6.13. For all $u \in U$, $|u|_S \succeq \log \|u\|$.

Proof. Assume that $|u|_S = n$, and so that $u = m_1 u_1 \ldots m_n u_n$, where the $u_i \in S_U$ and $m_i \in S_N$. Denote $h^g = g^{-1}hg$. One has the following formula

$$u = u_1^{m_1} u_2^{m_1 m_2} \ldots u_n^{m_1 \ldots m_n}.$$

by (6.12), by submultiplicativity, one has

$$\|u\| \preceq n^d \max_i \|u_i^{m_1 \ldots m_i}\|.$$

On the other hand, for every i, $\|u^{m_1 \ldots m_i}\|_{\mathrm{Lie}} \leq \|m_1 \ldots m_i\|_{\mathrm{op}} \|u_i\|_{\mathrm{Lie}} \leq KC^n$. The lemma follows. □

In addition, we have

Lemma 6.14. [CT17, Lemma 6.B.3.] There exists an integer ℓ such that every element $x \in U$ can be written as $v_1 \ldots v_\ell$ with $v_i \in \bigcup_j V_j$, and $\max \|v_i\| \preceq \|u\|$.

Thanks to the previous lemma, it is enough to treat the case where $U = V_j$. Therefore we can assume that there exists $t \in N$ contracting all of U. Up to replacing it by some power, we can assume that $\|t\|_{\mathrm{op}} \leq 1/2$. For convenience, let us assume that S_U contains all elements $u \in U$ such that $\|u\|_{\mathrm{Lie}} \leq 1$. Given an element $u \in U$ such that $\|u\|_{\mathrm{Lie}} \leq 2^n$, it follows that the element u^{t^n} belongs to S_U. It follows from Lemma 6.13, that every element $u \in U$ such that $|u|_S \leq n$ can be written as $t^{n'}u_0t^{-n'}$, where $n' \preceq n$ and $u_0 \in S_U$. This finishes the proof that G is N-combable. □

7. Proof of Theorem 4 and other results

7.1. **Proof of Theorem 4.** We need to check all four requirements of Theorem 5.12:

- G has a controlled Følner sequence: this is done in §6.2.
- G is N-combable: this is done in §6.3.
- N has Property $\mathcal{WAP}_{\mathrm{t}}$: this is easy (Corollary 4.11 with Proposition 4.8)

The last requirement, (4), can actually fail, but we can arrange it to hold enlarging N, replacing it with $N' = NW$ for some suitable compact subgroup W normalized by N. Namely, we use:

Lemma 7.1. Let U be a open subgroup in a finite product L of unipotent real and p-adic fields. Then the divisible subgroup U_{div} of U is closed, cocompact in U and contains the real component.

Proof. Since U decomposes as a product over the components, we can suppose L is real or p-adic for a single p. In the real case, necessarily $U = L$. Suppose that L is p-adic. Then $U_{\mathrm{div}} = \bigcap_n \phi^n(U)$, where $\phi(u) = u^p$. Since u is a self-homeomorphism of L, $\phi^n(U)$ is closed and hence U_{div} is closed; this implies that it is a p-adic subgroup, and it easily follows that it is Zariski-closed. An extension of two divisible nilpotent groups is divisible. In particular, in the quotient L/U_{div}, the open subgroup group U/U_{div} has no p-divisible element. Since ϕ contracts to 0, we can deduce that U/U_{div} is compact. □

Lemma 7.2. Let G be a group in the class $\mathfrak{C}$, with U, N as in the definition. Then G has a compact subgroup $W \subset U$, normalized by N, such that $U = U_{\mathrm{div}}W$.

Proof. The quotient U/U_{div} is compact, and is a product of various unipotent p-adic groups. So the N-action is necessarily distal (only eigenvalues of modulus 1).

Let U_{na} be the elliptic radical of U, so $U_{\mathrm{na}} \times U^\circ$. Let U_1 be the distal part of the N-action on U_{na}. Then the restriction to U_{na} of the quotient map $U \to U/U_{\mathrm{div}}$ is surjective. Moreover, U_1 is an increasing union of its N-invariant compact open subgroups. Hence there exists an N-invariant compact open subgroup W of U_1 whose image in U/U_{div} is surjective. □

To conclude, we define $N' = NW$. This does not affect the first condition, nor the second since we pass to a larger subgroup N. Since Property $\mathcal{WAP}_{\mathrm{t}}$ is also invariant under extensions by compact kernels (Proposition 4.8), N' has Property $\mathcal{WAP}_{\mathrm{t}}$. Finally, the last verification (Lemma 7.3) is that the subgroup generated by $\mathrm{Contr}(N')$ is equal to U_{div} and we deduce that $H = G$.

Lemma 7.3. The subgroup V generated by $\mathrm{Contr}(N)$ is equal to U_{div}.

Proof. Clearly, the image of $\mathrm{Contr}(N)$ in the compact group U/U_{div} is trivial. Hence $\mathrm{Contr}(N) \subset U_{\mathrm{div}}$. Thus $\bar{V}$ is cocompact in U_{div}. It is easy to see that U_{div} has no proper cocompact subgroup: indeed, if U_{div} is abelian, its quotient by $\bar{V}$ is a compact, divisible totally disconnected abelian group and apart from the trivial group, this does not exist (since nontrivial profinite groups have nontrivial finite quotients). So, when U_{div} is abelian, we deduce that V is dense. But in this case, it is clear that V is closed, since it is generated by some eigenspaces. So $V = U_{\mathrm{div}}$ when U is abelian.

In general, we deduce that $V[U_{\mathrm{div}}, U_{\mathrm{div}}] = U_{\mathrm{div}}$, that is, V generates the nilpotent group U_{div} modulo commutators, and deduce that $V = U_{\mathrm{div}}$. ☐

7.2. **Other results.**

Theorem 7.4. Every compactly generated locally compact group G having an open subgroup G' of finite index in the class $\mathfrak{C}''$ (Definition 14) has Property $\mathcal{WAP}_{\mathrm{fd}}$.

Proof. Along with Theorem 4.16 and using the second part of Theorem 5.12 the above proof shows that G' has Property $\mathcal{WAP}_{\mathrm{ap}}$. That G' has Property $\mathcal{WAP}_{\mathrm{ap}}$ follows from Proposition 4.13(1).

Then we observe that $G/\mathcal{K}^{\dagger}(G)$ has polynomial growth (see §4.6 for the definition of $\mathcal{K}^{\dagger}(G)$), by Lemma 4.22, and hence has Property $\mathcal{WAP}_{\mathrm{fd}}$ by Theorem 4.16. Hence $G/\mathcal{K}^{\dagger}(G)$ has Property $\mathcal{AP}_{\mathrm{fd}}$, and in turn, by Proposition 4.20, G has Property $\mathcal{AP}_{\mathrm{fd}}$. Since G has Property $\mathcal{WAP}_{\mathrm{ap}}$, this shows that G has Property $\mathcal{WAP}_{\mathrm{fd}}$. ☐

Note that by Proposition 4.21, we have a criterion whether G has Property $\mathcal{WAP}_{\mathrm{t}}$, namely if and only if the group of polynomial growth $G/\mathcal{K}^{\dagger}(G)$ has Property $\mathcal{WAP}_{\mathrm{t}}$.

Proof of Corollary 7. The statement is that locally compact groups in the class $\mathfrak{C}'$ (Definition 6) have Property $\mathcal{WAP}_{\mathrm{fd}}$. Indeed, consider $G \to G_1 \leftarrow G_2 \to G_3$ as in Definition 6. Since G_3 belongs to the class $\mathfrak{C}$, it has Property $\mathcal{WAP}_{\mathrm{t}}$ by Theorem 4 Since $G_2 \to G_3$ is copci and G_3 has Property $\mathcal{WAP}_{\mathrm{t}}$, G_2 has Property $\mathcal{WAP}_{\mathrm{t}}$ by Theorem 4.14 and Proposition 4.8. Since $G_2 \to G_1$ is copci with normal image, it follows that G_1 has Property $\mathcal{WAP}_{\mathrm{fd}}$ by Proposition 4.13 and Proposition 4.8. By Theorem 4.14 again (for Property $\mathcal{WAP}_{\mathrm{fd}}$ this time) and Proposition 4.8, it follows that G has Property $\mathcal{WAP}_{\mathrm{fd}}$. ☐

Proof of Corollary 8. This follows from Corollary 7 in combination with Proposition 6.2. □

Theorem 7.5. Let G be a connected solvable Lie group. Then every WAP Banach G-module with nonzero reduced first cohomology has a 1-dimensional factor (with nonzero first cohomology).

Proof. This follows from Corollary 8, using that finite-dimensional unitary irreducible representations of connected solvable Lie groups have complex dimension one. □

8. Subgroups of $\mathrm{GL}(n, \mathbf{Q})$

8.1. **Unipotent closure.** Let (G_i) be a family of locally compact groups, with given compact open subgroups K_i. The corresponding semirestricted product is the subgroup of $\prod_i G_i$ consisting of families of whose coordinates are in K_i with finitely many exceptions (in other words, it is the subgroup generated by $\prod K_i$ and $\bigoplus G_i$); it has a natural group topology for which $\prod K_i$ is a compact open subgroup. We denote it by $\prod_i^{(K_i)} G_i$.

If $H_i \subset G_i$ is a family of closed subgroups, $\prod_i^{(K_i \cap H_i)} H_i$ naturally occurs as a closed subgroup of $\prod_i^{(K_i)} G_i$. We call it a standard subgroup (according to this given decomposition).

Now assume that, p ranging over the prime numbers, G_p is a p-elliptic locally compact group (in the sense that every compact subset of G is contained in a pro-p-subgroup). Then we have

Lemma 8.1. Every closed subgroup H of $\prod^{(K_p)} G_p$ is standard.

Proof. Let us show that H is closed under taking under all projections. Fix a prime q. That $\mathbf{Z}$ is dense in $\prod_p \mathbf{Z}_p$ implies that there exists a sequence (n_i) in $\mathbf{Z}$ such that $n_i \to 1$ in $\mathbf{Z}_q$ and $n_i \to 0$ in $\mathbf{Z}_p$ for all $p \neq q$. Then for every $x \in \prod^{(K_p)} G_p$, the sequence x^{n_i} tends to the projection of x on G_p. In particular, if $x \in H$, then this projection also belongs to H.

Now let H_p be the projection of H on G_p. Then the closed subgroup generated by the H_p contains $\prod_p (H_p \cap K_p)$ and contains $\bigoplus_p H_p$. Thus $\prod^{(K_p \cap H_p)} H_p$ is contained in H. Conversely, H is contained in both $\prod_p H_p$ and $\prod^{(K_p)} G_p$, and the intersection of these two is by definition $\prod^{(K_p \cap H_p)} H_p$, so H is contained in $\prod^{(K_p \cap H_p)} H_p$; we conclude that these subgroups are equal. □

Recall that the ring of adeles is the semirestricted product $\mathbf{A} = \prod_p^{(\mathbf{Z}_p)} \mathbf{Q}_p$; the diagonal inclusion embeds $\mathbf{Q}$ as a dense subring into $\mathbf{A}$ and as a discrete cocompact subring in $\mathbf{A} \times \mathbf{R}$.

Definition 8.2. Let H be a subgroup of $\mathrm{GL}_m(\mathbf{Q})$. We define its fine closure as the subgroup $\mathcal{C}(H)$ of $\mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$ generated by the closures of the various p-adic projections $\pi_p(H)$, and the Zariski closure $\mathcal{C}_0(H)$ of the real projection.

Lemma 8.3. Let H be a subgroup of $\mathrm{GL}_m(\mathbf{Q})$ conjugate to a subgroup of upper unipotent matrices. Then H is cocompact in its fine closure $\mathcal{C}(H)$.

(It is well-known that the condition is equivalent to assuming that each element of H is unipotent.)

Proof. Denote by $\mathcal{C}_p(H)$ the closure of the projection of H in $\mathrm{GL}_m(\mathbf{Q}_p)$ and $\mathcal{C}_+(H)$ the closure of subgroup they generate in $\mathrm{GL}_m(\mathbf{A})$, so that $\mathcal{C}(H) = \mathcal{C}_+(H) \times \mathcal{C}_0(H)$. Denote by π_+ and π the natural embeddings $\mathrm{GL}_m(\mathbf{Q}) \to \mathrm{GL}_m(\mathbf{A})$ and $\mathrm{GL}_m(\mathbf{Q}) \to \mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$.

By the assumption, $\mathcal{C}_p(H)$ is p-elliptic for every prime p. Then by Lemma 8.1, $\pi_+(H)$ is dense in $\mathcal{C}_+(H) = \prod_p^{(\mathcal{C}_p(H)\cap\mathbf{Z}_p)} \mathcal{C}_p(H)$. Let K be the compact open subgroup $\prod_p(\mathcal{C}_p(H) \cap \mathbf{Z}_p)$. Then this implies that $\mathcal{C}_+(H) = \pi_+(H)K$. Therefore, $\mathcal{C}(H) = \pi(H)(K \times \mathcal{C}_0(H))$.

We claim that the projection of $\pi(H) \cap (K \times \mathcal{C}_0(H))$ is cocompact in $\mathcal{C}_0(H)$. Indeed, in a connected unipotent real group U, a subgroup is cocompact if and only if it is Zariski-dense, if and only it is not contained in the kernel any nonzero homomorphism $U \to \mathbf{R}$. Assume by contradiction we have such a homomorphism f on $\mathcal{C}_0(H)$. Pick $(u, b) \in \pi(H)$ with $u \in \mathcal{C}_+(H)$ and $b \in \mathcal{C}_0(H)$ with $f(b) \neq 0$. Then there exists $n \geq 1$ such that $u^n \in K$. Hence $(u^n, b^n) \in \pi(H) \cap (K \times \mathcal{C}_0(H))$ but $f(b^n) = nf(b) \neq 0$, a contradiction.

So the projection of $\pi(H) \cap (K \times \mathcal{C}_0(H))$ is cocompact in $\mathcal{C}_0(H)$. This implies (pulling back by a quotient homomorphism with compact kernel) that $\pi(H) \cap (K \times \mathcal{C}_0(H))$ is cocompact in $K \times \mathcal{C}_0(H)$. Since $K \times \mathcal{C}_0(H)$ is an open subgroup and $\mathcal{C}(H) = \pi(H)(K \times \mathcal{C}_0(H))$, this implies that $\pi(H)$ is cocompact in $\mathcal{C}(H)$. □

Now, let Γ be a finitely generated, virtually solvable subgroup of $\mathrm{GL}_m(\mathbf{Q})$. Let U be its unipotent radical (the intersection with the unipotent radical of its Zariski closure, which is also the largest normal subgroup of Γ consisting of unipotent elements). So Γ/U is finitely generated and virtually abelian. Identify Γ with its image in $\mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$.

Proposition 8.4. $\mathcal{C}(U)$ is open in $\Gamma\mathcal{C}(U)$, which is closed in $\mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$, and Γ is cocompact in $\Gamma\mathcal{C}(U)$.

Proof. Choose a partial flag in $\mathbf{Q}^m$ that is Γ-invariant with irreducible successive quotients. This yields an upper block-triangular decomposition. Then U acts trivially on the irreducible subquotients, which means it acts by matrices with identity diagonal blocks. Let ϕ be the homomorphism mapping a matrix that is upper triangular in this decomposition to its "diagonal trace", that is, replacing all upper unipotent blocks with 0. Then the kernel of $\phi : \Gamma \to \phi(\Gamma)$ is exactly U. Moreover, ϕ extends to $\Gamma\mathcal{C}(U)$ and $\phi(\Gamma\mathcal{C}(U)) = \phi(\Gamma)$, which is discrete. Hence the kernel $\mathcal{C}(U)$ of $\phi : \Gamma\mathcal{C}(U) \to \phi(\Gamma)$ is open in $\Gamma\mathcal{C}(U)$. This implies in particular that $\Gamma\mathcal{C}(U)$ is closed in $\mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$.

Lemma 8.3 ensures that U is cocompact in $\mathcal{C}(U)$, and the cocompactness statement follows. ☐

8.2. **Partial splittings.**

Lemma 8.5. Let M be a locally compact group and s a contracting automorphism of M. Define $f(g) = gs(g)^{-1}$. Then f is a self-homeomorphism of M.

Proof. We wish to define its inverse as $F(g) = gs(g)s^2(g)\dots$; we need to check that this product is "summable" (uniformly on compact subsets, namely that $\prod_{k=n}^{n+\ell} s^k(g)$ tends to 1 when n tends to $+\infty$, uniformly in ℓ and for g in any given compact subset.

If M is totally disconnected, then it has a compact open subgroup K such that $s(K) \subset K$, and then $\bigcap_{n\ge 0} s^n(K) = \{1\}$. Then the summability condition immediately follows.

If M is connected, then M is a finite-dimensional real vector space with a linear contraction and the summability is a standard verification.

The general case follows from the fact that M decomposes canonically as topological direct product of M° and its elliptic radical, which is totally disconnected [Sie86, Prop. 4.2].

Once the summability is established, it is immediate that $F \circ f$ and $f \circ F$ are both the identity of G. ☐

Lemma 8.6. Let G be a locally compact group in an extension $1 \to M \to G \to A \to 1$, with M and A abelian. Assume that some element g of G right-acts on M as a contraction. Then the centralizer of g is a section of the extension.

Proof. Let H be the centralizer of g. Clearly $H \cap M = \{1\}$. So it is enough to show that $HM = G$. Equivalently, letting h be any element of G, we have to show that the equation $[hm, g] = 1$ has a solution $m \in M$. Here the commutator is defined as $[X, Y] = X^{-1}Y^{-1}XY$ and satisfies the identity $[XY, Z] = [X, Z]^Y.[Y, Z]$. Then $[hm, g] = [h, g][m, g]$. By Lemma 8.5, $m \mapsto [m, g]$ is a self-homeomorphism of M. So indeed we obtain a unique solution. ☐

In the following lemma, we refer to Definition 5.1 for the definition of $\mathrm{Contr}(F)$.

Lemma 8.7. Let G be a locally compact group in an extension $1 \to U \to G \to A \to 1$, with A compactly generated abelian and U sub-unipotent (over a finite product of adic and real fields), i.e., a closed subgroup of a unipotent group containing the real component. Then G has a compactly generated, closed subgroup F such that $FU = G$ and F has polynomial growth, and $F \cap U^\circ$ is the distal part of U°. Moreover, if G is compactly generated, the subgroup generated by F and $\mathrm{Contr}(F)$ is cocompact.

Proof. We first prove the result with F not assumed compactly generated (so polynomial growth means that all its open, compactly generated subgroups have polynomial growth). The result immediately follows, since we can replace then

F with a large enough compactly generated open subgroup F' still satisfying $F'U = G$ (since G/U is compactly generated).

We argue by induction on the dimension of U (the sum of its real and p-adic dimensions for various p). If $\dim(U) = 0$, then $U = 1$ and the result is trivial. Let Z be the last term of the lower central series of U. Then Z has positive dimension. If the action of A on Z is distal (i.e., all eigenvalues have modulus 1), we define $M = Z$; otherwise, in the Lie algebra we find an irreducible non-distal submodule, which corresponds to an irreducible submodule M of Z; in both cases M has positive dimension. We can argue by induction for $1 \to U/M \to G/M \to A \to 1$ to get a closed subgroup L/M of polynomial growth with $LN = G$ and $(L/M) \cap (U/M)^\circ$ is the distal part of $(U/M)^\circ$. If Z is distal as A-module, then L also has polynomial growth and we are done with $F = L$. Otherwise, M is irreducible non-distal and L contains the distal part of U°. We have the extension $1 \to L \cap U \to L \to L/L \cap U \to 1$. Note that $L \cap U$ is sub-unipotent, making use that $L \cap U^\circ$ is connected. If $\dim(L) < \dim(U)$, we can argue once more by induction within L to find the desired subgroup. Otherwise, $L \cap U$ is open in U. This means that $(L \cap U)/M$ is open in U/M, and if this happens, G/M has polynomial growth. If U is not abelian, this forces G to have polynomial growth, and then we are done with $F = L$. Otherwise U is abelian. In this case, if the distal part is nontrivial, we can argue in the same way with $M = D$. If the distal part is trivial, and we choose M irreducible as above, the previous argument works as soon as U/M is nontrivial. So the remaining case is when U is irreducible and non-distal; in particular the distal part of U° is trivial. In this case, there exists an element acting as a contraction (if g acts non-distally, the contraction part of either g or g^{-1} is a nonzero submodule, hence is all of M), so we can invoke Lemma 8.7.

Now suppose that G is compactly generated and let us prove the last statement. It is enough to show that the subgroup of U generated by $(F \cap U) \cup \mathrm{Contr}(F)$ is cocompact, i.e., contains U_{div}. First, it contains U°, because we have ensured that $F \cap U$ contains the distal part. For the non-Archimedean part, that G is compactly generated implies that U is compactly generated as normal subgroup (because G/U is compactly presented, being abelian), and hence it follows that the non-Archimedean part of U_{div} is contained in the subgroup generated by $\mathrm{Contr}(F)$ (indeed, otherwise we would obtain an A-equivariant quotient of U isomorphic to $\mathbf{Q}_p^k$ with an irreducible distal action, for some p, k and get a contradiction). $\square$

Proposition 8.8. Let Γ be a finitely generated amenable subgroup of $\mathrm{GL}_m(\mathbf{Q})$, and let $G = \Gamma V \subset \mathrm{GL}_m(\mathbf{A} \times \mathbf{R})$ with $V = \mathcal{C}(U)$ as defined as before Proposition 8.4. Then G has an open normal finite index subgroup G' of the form VF with F compactly generated of polynomial growth, such that the subgroup of G' generated by $F \cup \mathrm{Contr}(F)$ is cocompact. In particular, G' belongs to the class $\mathfrak{C}''$ (Definition 14).

Proof. Γ has a finite index subgroup Λ whose Zariski closure is unipotent-by-abelian. It follows that $G' = \Lambda V$ is open normal of finite index in G, and G'/V is abelian. So Lemma 8.7 applies. The last statement immediately follows. □

Corollary 8.9. Every finitely generated amenable subgroup of $\mathrm{GL}_m(\mathbf{Q})$ embeds as a cocompact lattice into a locally compact group G with an open subgroup of finite index G' in the class $\mathfrak{C}''$. □

Corollary 8.10. Every finitely generated amenable subgroup of $\mathrm{GL}_m(\mathbf{Q})$ has Property $\mathcal{WAP}_{\mathrm{fd}}$, and hence has Property $\mathcal{H}_{\mathrm{fd}}$.

Proof. Use the notation of Corollary 8.9, so that G' belongs to the class $\mathfrak{C}''$. By Theorem 7.4, we deduce that G has Property $\mathcal{WAP}_{\mathrm{fd}}$. Hence by Theorem 4.14, Γ has Property $\mathcal{WAP}_{\mathrm{fd}}$: □

Using that every finitely generated VSP group is quotient of a virtually torsion-free finitely generated VSP group [KL17], and the fact that $\mathcal{WAP}_{\mathrm{fd}}$ passes to quotients, this can be improved to

Corollary 8.11. Every finitely generated amenable VSP group has Property $\mathcal{WAP}_{\mathrm{fd}}$.

Corollary 8.12. For every finitely generated VSP group G equipped with a finite generating subset S, there exists $c > 0$ such that the L^p-isoperimetric profile inside balls (see §1.3) satisfies

$$J^b_{G,p}(n) \geq cn. \tag{8.13}$$

Proof. By Lemma 6.4, groups of the class $\mathfrak{C}''$ have a strong controlled Følner sequence (called a controlled Følner pair in [T11]). By [T11, Proposition 4.9], this implies that groups of the class $\mathfrak{C}''$ satisfy (8.13). On the other hand, G being quasi-isometric to its cocompact hull, we deduce from Theorem 15 and [T08, Theorem 1] that every amenable finitely generated subgroup of $\mathrm{GL}_m(\mathbf{Q})$ satisfies (8.13). Now, we once again apply the main result of Kropholler and Lorensen [KL17] and the fact that $J^b_{*,p}$ behaves well under taking quotients [T13, Theorem 1] to conclude. □

9. Mean ergodic theorem and Bourgain's theorem

9.1. **Proof of Proposition 19.** The "if" part was already addressed in the introduction. For the other direction, suppose that G does not have $\mathcal{WAP}_{\mathrm{t}}$, but has $\mathcal{WAP}_{\mathrm{fd}}$. This means that G has a finite-dimensional orthogonal representation π with nonzero first reduced cohomology and no non-zero invariant vector. Now, recall that by the standard Gaussian construction (see [BHV08, Corollary A.7.15]), one can assume that π is a subrepresentation of some orthogonal representation π' of G coming from a measure-preserving ergodic action on some probability space X. Let b be a 1-cocycle for π that is not an almost coboundary, and let b' be the corresponding cocycle for π': note that since π has no invariant vectors, b' is orthogonal to the space of constant functions. One therefore has $b'(g)(x) = c(g)(x)$ where c is a square-integrable cocycle $c : X \times G \to \mathbf{R}$ of zero average. Using that b' is not an almost coboundary, one easily checks that $\frac{1}{|g|}(c(g)(x))$ does not tend to zero in L^2-norm as $|g| \to \infty$. In particular the ergodic theorem for G in L^2 (defined similarly) fails; since the inclusion $L^2(X) \to L^1(X)$ is continuous, it also fails in L^1.

Remark 9.1. The above proof works with no change if G has Property $\mathcal{H}_{\mathrm{fd}}$ and not $\mathcal{H}_{\mathrm{t}}$. This is actually more general, since the reader can easily check that if G has Property $\mathcal{WAP}_{\mathrm{fd}}$ but not $\mathcal{WAP}_{\mathrm{t}}$, then it has Property $\mathcal{H}_{\mathrm{fd}}$ but not $\mathcal{H}_{\mathrm{t}}$.

9.2. **Proof of Corollary 21.** For every commutative unital ring R and $t \in R^\times$, consider the group

$$A(R,t) = \left\{ \begin{pmatrix} t^n & x \\ 0 & t^{-n} \end{pmatrix}; x \in R, n \in \mathbf{Z} \right\} \subset \mathrm{GL}_2(R).$$

Let us fix some prime p. Note that the ring $\mathbf{F}_p[t,t^{-1}]$ embeds densely in $\mathbf{F}_p(\!(t)\!)$, but the diagonal embedding $\mathbf{F}_p[t,t^{-1}] \to \mathbf{F}_p(\!(t)\!) \oplus \mathbf{F}_p(\!(t)\!)$ sending t to (t,t^{-1}) is easily seen to be discrete and cocompact. The lamplighter group $L_p = \mathbf{F}_p \wr \mathbf{Z}$ can be described as $A(\mathbf{F}_p[t,t^{-1}],t)$, and therefore embeds as a cocompact lattice in $G = (\mathbf{F}_p(\!(t)\!))^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by t on the first factor and by t^{-1} on the second factor. First, we note that this implies that L_p has $\mathcal{WAP}_{\mathrm{t}}$. On the other hand the group G, and therefore L_p quasi-isometrically embeds as a subgroup of $\left(A(\mathbf{F}_p(\!(t)\!),t)\right)^2$. Observe that $A(\mathbf{F}_p(\!(t)\!),t)$ acts properly and cocompactly on the Bass-Serre tree of $\mathrm{SL}(2,\mathbf{F}_p(\!(t)\!))$. It follows that L_p embeds quasi-isometrically into a product of two $(p+1)$-regular trees. Therefore, in order to show Bourgain's theorem, it is enough to prove that L_p does not quasi-isometrically embed into any superreflexive Banach space. Since L_p is amenable, by [NP, Theorem 9.1] it is enough to prove that L_p does not admit any affine isometric action on some superreflexive Banach space E whose orbits are quasi-isometrically embedded. Consider the 1-cocycle b associated to such an action. Since L_p has Property $\mathcal{WAP}_{\mathrm{t}}$, this cocycle decomposes as $b = b_1 + b_2$, where b_1 is a group homomorphism to E, and b_2 is an almost coboundary. Approximating b_2 by coboundaries, one easily checks that it is sublinear, namely $\|b_2(g)\|/|g| \to 0$ as $|g| \to \infty$ (where $|\cdot|$ is some arbitrary word metric on L_p). This clearly implies that b cannot be a quasi-isometric embedding. So the corollary is proved.

Remark 9.2. Note that in Corollary 21, we only recover the qualitative part of Bourgain's theorem. Indeed, the latter also provides optimal quantitative estimates on the distortion (as in [T11]), which do not follow from the approach here.

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CNRS and Univ Lyon, Univ Claude Bernard Lyon 1, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne

E-mail address: `cornulier@math.univ-lyon1.fr`

Laboratoire de Mathématiques, Bâtiment 425, Université Paris-Sud 11, 91405 Orsay, FRANCE

E-mail address: `romain.tessera@math.u-psud.fr`