# QUASI-ISOMETRICALLY EMBEDDED FREE SUB-SEMIGROUPS

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Abstract. If $G$ is either a connected Lie group, or a finitely generated solvable group with exponential growth, we show that $G$ contains a quasi-isometrically embedded free sub-semigroup on 2 generators.

## 1. Introduction

Let $G$ be a locally compact group generated by a compact subset $S$. The word length function is defined in the usual way: $|g|_S \le n$ if and only if $g$ is the product of $n$ elements of $S^{\pm 1}$. If $G$ is a connected Lie group, then it can also be endowed with the distance defined by a left invariant Riemannian metric. All corresponding lengths are equivalent, where two lengths $\ell, \ell'$ are called equivalent if $\ell \preceq \ell' \preceq \ell$ and $\ell \preceq \ell'$ means that there exists a constant $\alpha > 0$ such that for all $g$ in the group, $\ell(g) \le \alpha \ell'(g) + \alpha$. In the sequel, compactly generated groups will be endowed with a length equivalent to one of these, the choice of which has no importance.

We are especially interested in quasi-isometric homomorphic embeddings of a free semigroup into some compactly generated groups.

Recall that a map $f : X \to Y$ between two metric spaces is *large scale Lipschitz* if

$$\exists \alpha > 0, \forall x, y \in X, d(f(x), f(y)) \le \alpha d(x, y) + \alpha.$$

It is called a *quasi-isometric embedding* if moreover it satisfies

$$\exists \beta > 0, \forall x, y \in X, \beta d(x, y) - \beta \le d(f(x), f(y)).$$

In the sequel, $X = \Gamma$ will mainly denote a free (non-abelian) semigroup on two generators, that we simply call "the" free semigroup.

It is then straightforward that every homomorphism $f$ of $\Gamma$ into any compactly generated group is large scale Lipschitz. On the other hand, $f$ is not necessarily a quasi-isometric embedding: for instance this is obvious when $f$ is taken as the trivial homomorphism, or even if it is not injective. Also, there are many proper homomorphisms of free (semi)groups that are not quasi-isometric embeddings, e.g. the embedding of a finite index free subgroup of $\mathrm{SL}_2(\mathbf{Z})$ into $\mathrm{SL}_2(\mathbf{R})$ (or into $\mathrm{SL}_2(\mathbf{Z}[1/2])$). The lack of being large scale dilating can be encoded in the *compression function*

$$\rho(r) = \inf\{d(f(x), f(y)) : x, y \in X, d(x, y) \ge r\}.$$

Then $f$ is large scale dilating if and only if $\rho(r) \succeq r$, and this the maximal possible behaviour for $\rho$ when $f$ is assumed large scale Lipschitz (excluding the trivial case when $X$ is bounded). There is much work concerning the compression function (and also the related *distortion* function), see for instance [Grom, Farb, Ols, Osin].

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Our interest for the existence of quasi-isometrically embedded free subsemigroups comes from the following result of Bourgain [Boug]: a regular tree of degree at least three has no quasi-isometric embedding into a Hilbert space (nor into any uniformly convex Banach space); note that a free semigroup is quasi-isometric to such a tree.

Existence of a quasi-isometrically embedded free subsemigroup is also a strengthening of the existence of a free subsemigroup. We prove for instance the following theorem, which improves the Rosenblatt alternative [Ros].

**Theorem 1.1.** *Let $\Gamma$ be a finitely generated solvable group. Suppose that $\Gamma$ is not virtually nilpotent. Then $\Gamma$ has a quasi-isometrically embedded free subsemigroup.*

We give two related results for non-discrete groups.

**Theorem 1.2.** *Let $G$ be a a compactly generated group that is not of polynomial growth. Suppose that either*

(1) *$G$ is a connected Lie group, or*
(2) *$G$ is a linear algebraic group over a local field of characteristic zero.*

*Then $G$ has a quasi-isometrically embedded free semigroup.*

This theorem is not true for general compactly generated groups, e.g. the free Burnside groups of odd large exponent, that have exponential growth [Adi] but do not even contain any homomorphically embedded copy of $\mathbf{Z}$. On the other hand, it is not known if every locally compact compactly generated group with exponential growth $G$ contains a quasi-isometrically embedded regular ternary tree. This is known to hold for non-amenable $G$ and non-unimodular $G$: more precisely, it is known [BeSc, Corollary 1.6] that the regular ternary tree quasi-isometrically embeds into every graph with positive Cheeger constant. Moreover, by [Te2, Theorem 2], if $G$ is non-amenable or non-unimodular, then $G$ is quasi-isometric to a graph with positive Cheeger constant.

From our results we get, relying on [Boug], the following corollary.

**Corollary 1.3.** *Let $G$ be given as in Theorem 1.1 or 1.2. Then $G$ has a quasi-isometrically embedded tree, and in particular $G$ does not quasi-isometrically embed into any uniformly convex (or superreflexive) Banach space.*

Further consequences on the compression of such an embedding can be found in [Te1]

## 2. The quasi-isometric ping-pong Lemma

We call here a metric space 1-quasi-geodesic if there exists constants $c_1, c_2$ such that every pair of points $x, y$ are joined by a "path" $x = x_0, \dots, x_n = y$ with $n \leq c_1 d(x, y) + c_2$ and $d(x_i, x_{i+1}) \leq 1$ for all $i$.

In any metric space $X$, denote by $B_r(x)$ the closed ball of centre $x$ and radius $r$, and, for $Y \subset X$, define

$$\mathcal{B}_r(Y) = \bigcup_{y \in Y} B_r(y).$$

**Lemma 2.1** (Quasi-isometric ping-pong Lemma for semigroups)**.** *Let $X$ be a 1-quasi-geodesic space and let $(a_i)_{i \in I}$, $I = \{1 \dots d\}$ $(d \geq 2)$ be a family of surjective* *isometries of $X$. Assume that there exists a family of nonempty disjoint subsets $(A_i)_{i\in I}$ such that, setting $C = \bigcup_i A_i$*

$$\forall i \in I, \quad \mathcal{B}_1(a_i C) \subset A_i.$$

*Then for every $x \in X$, the mapping $m \to m(a_1, \dots, a_d)x$ defines a quasi-isometric embedding of the free semigroup of rank $d$ into $X$.*

*In particular, if $G$ is a locally compact compactly generated group, with elements $x_1, \dots, x_d$ and if $G$ acts by isometries on $X$ in such a way that $x_i$ acts by $a_i$, then $x_1, \dots, x_d$ freely generate a quasi-isometrically embedded subsemigroup of $G$.*

*Proof.* View $\mathcal{B}_1$ as a self-map of $2^X$, and denote by $\mathcal{B}_1^n$ its $n$-th iterate. The triangle inequality yields, for all $Y \subset X$ and $n \geq 0$ the inclusion $\mathcal{B}_1^n(Y) \subset \mathcal{B}_n(Y)$. On the other hand, the 1-quasi-geodesic condition is easily seen to be equivalent to the existence of a constant $c > 0$ such that for all $Y \subset X$ and $n \in \mathbf{N}^*$, we have

$$\mathcal{B}_1^n(Y) \supset \mathcal{B}_{cn}(Y).$$

**Claim 2.2.** Let $m = a_{j_1} \dots a_{j_n}$ be a word of length $n \geq 1$ in $(a_i)_{i\in I}$. Then

$$\mathcal{B}_1^n(m(C)) \subset A_{j_1}.$$

*Proof.* First note that since every $a_j$ is a surjective isometry, it commutes with the map $\mathcal{B}_1$ and its iterates. Argue by induction on $n$. For $n = 1$, the claim is contained in the hypotheses of the theorem. Fix $n \geq 2$, and assume that the lemma holds for $n - 1$. Write $m' = a_{j_1} \dots a_{j_n}$. We have

$$\begin{aligned} \mathcal{B}_1^n(m(C)) &= \mathcal{B}_1 \circ \mathcal{B}_1^{n-1}(a_{j_1} m'(C)) \\ &= \mathcal{B}_1(a_j \mathcal{B}_1^{n-1}(m'(C))) \\ &\subset \mathcal{B}_1(a_{j_1} A_{j_2}) \quad \text{(by induction assumption)} \\ &\subset A_{j_1}. \qquad \square \end{aligned}$$

Denote by $\delta$ the word distance in the free semigroup generated by the letters $(a_i)_{i\in I}$. Fix $x \in C$.

Consider two words $m, m'$. Denoting by $v$ their greatest common prefix, we can write $m = va_{i_1} \dots a_{i_n}$ and $m' = va_{j_1} \dots a_{j_k}$, where $n+k = \delta(m, m')$. If $k, n > 0$ then the claim implies that

$$\mathcal{B}_1^n(v^{-1}m(C)) \subset A_{i_1}, \quad \mathcal{B}_1^k(v^{-1}m'(C)) \subset A_{j_1}.$$

As $A_{i_1} \cap A_{j_1} = \emptyset$, we deduce that $\mathcal{B}_1^n(v^{-1}m(C)) \cap \mathcal{B}_1^k(v^{-1}m'(C)) = \emptyset$ and therefore

$$\mathcal{B}_{cn}(v^{-1}m(C)) \cap \mathcal{B}_{ck}(v^{-1}m'(C)) = \emptyset.$$

Now fix $x_0$ in $C$, which is not empty. From the previous formula we get

$$d(mx_0, m'x_0) = d(v^{-1}mx_0, v^{-1}m'x_0) \geq c(n + k).$$

It remains the case when $kn = 0$. The case $k = n = 0$ being obvious, we can suppose $k = 0 \neq n$. Fix $j \neq i_1$. Then

$$d(mx_0, m'x_0) \geq d(mx_0, va_j x_0) - d(va_j x_0, m'x_0)$$
$$= d(mx_0, va_j x_0) - d(a_j x_0, x_0) \geq c(k + n + 1) - d(a_j x_0, x_0) \geq c\delta(m, m') - \alpha,$$

where $\alpha = \sup_j d(a_j x_0, x_0)$.

We immediately deduce, for every $x \in X$ and all words $m, m'$,

$$d(mx_0, m'x_0) \geq c\delta(m, m') - \alpha',$$

where $\alpha' = \alpha + 2d(x, x_0)$. $\square$

**Lemma 2.3** (Quasi-isometric ping-pong Lemma for groups)**.** *Let $X$ be a 1-quasi-geodesic space and let $(a_i)_{i\in I}$, $I = \{1, \dots d\}$ $(d \geq 1)$ be a family of surjective isometries of $X$. Consider a family of $2d$ disjoint nonempty subsets $(A_i^{\pm})_{i\in I}$. For every $i \in I$, write*

$$C_i^+ = \left(\bigcup_{j\in I} A_j^+\right) \cup \left(\bigcup_{j\neq i} A_j^-\right) \qquad \textit{and} \qquad C_i^- = \left(\bigcup_{j\in I} A_j^-\right) \cup \left(\bigcup_{j\neq i} A_j^+\right).$$

*Assume that, for every $i \in I$,*

$$\mathcal{B}_1(a_i^+(C_i^+)) \subset A_i^+ \qquad \textit{and} \qquad \mathcal{B}_1(a_i^-(C_i^-)) \subset A_i^-.$$

*Then for every $x \in X$, the mapping $m \to m(a_1, \dots, a_d)x$ defines a quasi-isometric embedding of the free group of rank $d$ into $X$.*

*In particular, if $G$ is a locally compact compactly generated group, with elements $x_1, \dots, x_d$ and if $G$ acts by isometries on $X$ in such a way that $x_i$ acts by $a_i$, then $x_1, \dots, x_d$ are free generators of a quasi-isometrically embedded subgroup of $G$.*

*Proof.* The following claim is similar to Claim 2.2.

**Claim 2.4.** Let $m = a_{j_1}^{\varepsilon_1} \dots a_{j_n}^{\varepsilon_n}$ be a reduced word of length $n \geq 1$ in $(a_i^{\pm})$ $(i \in \{1, \dots, n\}, \varepsilon_i \in \{\pm\})$. Then

$$\mathcal{B}_1^n(m(C_{j_n}^{\varepsilon_n})) \subset A_{j_1}^{\varepsilon_1}.$$

*Proof.* Argue by induction on $n$, the case $n = 1$ being obvious. If $n \geq 2$ and the claim is proved for $n-1$, then $\mathcal{B}_1^{n-1}(a_{j_2}^{\varepsilon_1} \dots a_{j_n}^{\varepsilon_n} C_{j_n}^{\varepsilon_n})$ is contained in $A_{j_2}^{\varepsilon_2}$, so that $\mathcal{B}_1^n(m(C_{j_n}^{\varepsilon_n}))$ is contained in $\mathcal{B}_1(a_{j_1} A_{j_2}^{\varepsilon_2})$, which is contained in $A_{j_1}^{\varepsilon_1}$ because the $(j_1, \varepsilon_1) \neq (j_2, -\varepsilon_2)$ as the word is reduced. $\square$

Denote by $\delta$ the word distance in the free group generated by the letters $(a_i)_{i\in I}$. For each $m$ as in the claim, we can find $x_m \in A_j^{\varepsilon}$ where $(j, \varepsilon) \notin \{(j_1, \varepsilon_1), (j_n, -\varepsilon_n)\}$; the mapping $m \mapsto x_m$ can be chosen so that it has finite image (of cardinality 3 for instance). By construction $x_m \in C_{j_n}^{\varepsilon_n} - A_{j_1}^{\varepsilon_1}$. It follows from the claim that $\mathcal{B}_{cn}(m(C_{j_n}^{\varepsilon_n})) \subset A_{j_1}^{\varepsilon_1}$, so that

$$d(mx_m, x_m) \geq cn = c\delta(m, 1).$$

Accordingly, if $x \in X$, we have, for every word $m$

$$d(mx, x) \geq c\delta(m, 1) - 2\alpha,$$

where $\alpha = \sup\{d(x_m, x) : m \text{ reduced word}\} < \infty$. This shows that the mapping $m \mapsto mx$ is a quasi-isometric embedding of the free group. $\square$

## 3. Ping-pong in dimension two

The following lemma is used both in the case of amenable Lie groups and of finitely generated virtually solvable groups.

**Lemma 3.1.** *Let* $\mathbf{K}$ *be a local field. Consider the* $(ax+b)$*-group*

$$G = \mathbf{K} \rtimes \mathbf{K}^* \simeq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{K}^*, b \in \mathbf{K} \right\}.$$

*Let*

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}$$

*be non-commuting elements in* $\mathbf{K} \rtimes \mathbf{K}^*$ *and suppose that*

- $\mathbf{K}$ *is ultrametric and* $|a_1|, |a_2| < 1$*, or*
- $\mathbf{K}$ *is Archimedean and* $|a_1| + |a_2| + 3|a_1||a_2| < 1$ *(e.g.* $|a_1|, |a_2| < 1/3$*).*

*Then the sub-semigroup generated by* $(v,a)$ $(w,b)$ *is free and quasi-isometrically embedded in* $\mathbf{K} \rtimes \mathbf{K}^*$.

*Proof.* In the Archimedean case, we can suppose that $\mathbf{K} = \mathbf{C}$. We make $G$ act in the standard way by isometries on the hyperbolic 3-space $\mathbf{H}^3_{\mathbf{R}}$ and on its boundary $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ as the stabilizer of $\infty$.

We are going to apply the quasi-isometric ping-pong lemma by choosing two half-spaces. A half-space that does not contain $\infty$ in its closure is defined by a (Euclidean) disc in $\mathbf{C}$.

For $i \in \{1,2\}$, $A_i$ is a hyperbolic element with repulsive point $\infty$ and attractive point $x_i \in \mathbf{C}$. Set $d = |x_2 - x_1|$. As $A_1$ and $A_2$ are suppose not to commute, we have $d < 0$. If $\{1,2\} = \{i,j\}$, set

$$r_i = \frac{d|a_i|(1+|a_j|)}{1 - |a_1||a_2|}.$$

The assumption $|a_1| + |a_2| + 3|a_1||a_2| < 1$ exactly states that $r_1 + r_2 < d$, i.e. the closed discs $D(x_i, r_i)$, $i = 1,2$, are disjoint. Fix $\varepsilon > 0$ so small that $r_1 + r_2 + 2\varepsilon < d$, and set $\rho_i = r_i + \varepsilon$.

Obviously, the image of $D_i = D(x_i, \rho_i)$ by $A_i$ is contained in the interior of $D_i$. Now let us look at the image of $D_2$ by $A_1$. Take $x \in D_2$. Then

$$\begin{aligned} |A_1 x - x_1| &\leq |A_1 x - A_1 x_2| + |A_1 x_2 - x_1| \\ &\leq |a_1||x - x_2| + |A_1 x_2 - A_1 x_1| \\ &\leq |a_1|(r_2 + \varepsilon) + |a_1| d = r_1 + |a_1|\varepsilon < \rho_1. \end{aligned}$$

A similar inequality is obtained by permuting 1 and 2.

Thus $A_i(D_1 \cup D_2)$ is contained in the interior of $D_i$. Let now $B_i$ denote the closed half space defined by $D_i$. Then, multiplying the metric on $\mathbf{H}^3_{\mathbf{R}}$ by a suitable constant, $B_1$ and $B_2$ satisfy all the hypotheses of Lemma 2.1.

The ultrametric case is analog and even simpler. Consider the attractive point $x_1$ of $A_i$ in $\mathbf{P}^1(\mathbf{K})$, and set $d = d(x_1, x_2)$. Take for $D_i$ the open ball of centre $x_i$ and radius $d$. This is actually a closed ball of strictly smaller radius (because the metric takes a discrete set of values) and $D_1 \cap D_2 = \emptyset$ (because $d$ is an ultrametric). Now $D_1$ and $D_2$ are the boundaries of two disjoint half-trees $B_1$ and $B_2$ in the tree of $\mathrm{PGL}_2(\mathbf{K})$, satisfying all the hypotheses of Lemma 2.1. $\square$

*Remark* 3.2. In the ultrametric case, it is not hard to show that the condition $\max(|a_1|, |a_2|) < 1$ is optimal so as to have a quasi-isometrically embedded free semigroup.

The Archimedean case is more involved. When $\mathbf{K} = \mathbf{R}$, the condition $|a_1| + |a_2| + 3|a_1||a_2| < 1$ is not optimal and can be replaced by the weaker condition $|a_1| + |a_2| < 1$; this can be shown directly by applying the quasi-isometric ping-pong Lemma to suitable intervals that are not centered at the attractive points; moreover the usual ping-pong Lemma applies when $|a_1|+|a_2| \le 1$ (always supposing $\max(|a_1|, |a_2|) < 1$) but the free semigroup can be non-quasi-isometrically embedded. For instance, if

$$A_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1/2 & 1 \\ 0 & 1 \end{pmatrix}$$

then $A_1$ and $A_2$ generate a free semigroup, but a direct calculation shows that $A_1 A_2^{n-1}$ and $A_2 A_1^{n-1}$ remain at bounded distance.

Finally, the complex case seem even much more tricky, as for instance there is no simple characterization of the elements $a \in \mathbf{C}$ such that the matrices

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$$

generate a quasi-isometrically embedded semigroup; it easily implies $|a| < 1$ and can be shown to be true if $|a| < 1/2$, but it is also true for some other values of modulus between $1/2$ and $1$.

## 4. Free sub-semigroups in Lie groups and $p$-adic groups

We make use of the following unpublished lemma of Guivarc'h.

**Lemma 4.1.** *Let $G$ be a connected solvable Lie group. Suppose that $G$ has exponential growth. Then $G$ has a quotient isomorphic to a closed (in the ordinary topology) subgroup of the affine complex group $\mathbf{C}^* \ltimes \mathbf{C}$ (which is also isomorphic to the group of orientation-preserving similarities of $\mathbf{R}^2$). More precisely, $G$ is isomorphic to one of the following groups*

- $\mathbf{R}_+^* \ltimes \mathbf{R}$
- $\mathbf{C}^* \ltimes \mathbf{C}$
- $\mathbf{R}_+^* \ltimes \mathbf{C}$, $\mathbf{R}$ *acting by a one parameter subgroup of similarities that is neither contain in the reals, nor in the complexes of modulus one.*

*Proof.* Let $\mathfrak{g}_0$ be the Lie algebra of $G$. Then $\mathfrak{g}_0$ has a quotient $\mathfrak{g}$ that is not of type R, but all of whose proper quotients are of type R.

Denote by $\mathfrak{n}$ the nilpotent radical of $\mathfrak{g}$. It can be decomposed as a sum of characteristic subspaces under the adjoint action, denoted $\mathfrak{n}_\alpha$ for $\alpha \in \mathrm{Hom}(\mathfrak{g}, \mathbf{C})$. We have $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] = \mathfrak{n}_{\alpha+\beta}$ for all $\alpha, \beta$. Taking a similar decomposition for the action on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, we see that if $\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]$ is of type R, then $\mathfrak{g}$ is also of type R (i.e. $\mathfrak{n}_\alpha \neq 0$ only if $\alpha \in i\mathrm{Hom}(\mathfrak{g}, \mathbf{R})$). Therefore $[\mathfrak{n}, \mathfrak{n}] = 0$, i.e. $\mathfrak{n}$ is abelian.

By [Bouk, Chap. 6, p.19-20], there exists a Cartan subalgebra, that is a nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$. Let $\mathfrak{z}$ be the centre of $\mathfrak{h}$. Then $\mathfrak{z} \cap \mathfrak{n}$ centralizes both $\mathfrak{h}$ and $\mathfrak{n}$, hence is central in $\mathfrak{g}$. As being of type R is preserved by central extensions, $\mathfrak{g}/(\mathfrak{z} \cap \mathfrak{n})$ is not of type R, so that $\mathfrak{z} \cap \mathfrak{n} = 0$. Therefore $\mathfrak{h} \cap \mathfrak{n}$ is an ideal in $\mathfrak{h}$ having trivial intersection with the centre of $\mathfrak{h}$; as $\mathfrak{h}$ is nilpotent, this implies that $\mathfrak{h} \cap \mathfrak{n} = 0$, so that $\mathfrak{g} \simeq \mathfrak{n} \rtimes \mathfrak{h}$.

So $\mathfrak{h} \simeq \mathfrak{g}/\mathfrak{n}$ is abelian, and $\mathfrak{n}$ is an irreducible $\mathfrak{h}$-module; in particular it has dimension at most 2. Moreover as $\mathfrak{n}$ is the nilpotent radical, the action of $\mathfrak{h}$ is faithful. This also implies that $\mathfrak{h}$ has dimension at most 2. Looking more closely at irreducible abelian subalgebras in $\mathfrak{gl}_n(\mathbf{R})$ for $n = 1, 2$, we get the desired list.

This proves that if $\tilde{G}$ is the universal covering of $G$, then it has a closed connected normal subgroup $W$ such that $L = \tilde{G}/W$ is isomorphic to one of the groups given in the lemma. Now the image of the centre of $\tilde{G}$ maps to the centre of $L$, which is trivial in all cases. This means that the centre of $\tilde{G}$ is contained in $W$, and in particular $L$ is a quotient of $G$. □

**Theorem 4.2.** *Let $G$ be a connected amenable Lie group with non-polynomial growth. Then $G$ has a quasi-isometrically embedded free sub-semigroup on 2 generators.*

*Proof.* A connected amenable Lie group has cocompact radical. As the embedding of a cocompact subgroup is obviously quasi-isometric, we can suppose $G$ solvable. By Lemma 4.1, $G$ has a quotient which is a closed subgroup of $\mathbf{C}^* \ltimes \mathbf{C}$. As we can lift a quasi-isometrically embedded semigroup from any quotient, we can suppose that $G$ is a closed subgroup of $\mathbf{C}^* \ltimes \mathbf{C}$ given as in Lemma 4.1. Now $\mathbf{C}^* \ltimes \mathbf{C}$ is the stabilizer of the point at infinity $\infty \in \mathbf{P}^1(\mathbf{C})$, for the isometric action of $\mathrm{PGL}_2(\mathbf{C})$ on the hyperbolic 3-space $\mathbf{H}^3_{\mathbf{R}}$. Hyperbolic elements inside $\mathbf{C}^* \ltimes \mathbf{C}$ are elements $(a, b)$ with $|a| \neq 1$, they have origin $\infty$ if and only if $|a| < 1$. Two such hyperbolic elements have the same target if and only if they commute. We thus immediately see that in all cases given by Lemma 4.1, there are two hyperbolic elements with origin $x_0$ and distinct targets; taking suitable powers we get by Lemma 3.1 two generators of a quasi-isometrically embedded semigroup. □

Let $\mathbf{K}$ be a local field of characteristic zero, and $G$ a solvable connected linear algebraic $\mathbf{K}$-group. Decompose it as $G = DKU$, where $U$ is the unipotent radical, $D$ is a maximal split torus, and $K$ is compact. We say that $G$ is of type R if $[D, U] = 1$. This is equivalent to say that $KU$ (which does not depend on the choice of the Levi factor) is almost a direct factor in $G$. In this case $G$ has a cocompact nilpotent subgroup, namely $DU$.

**Lemma 4.3.** *Let $\mathbf{K}$ be a local field of characteristic zero, and $G$ a solvable linear algebraic $\mathbf{K}$-group that is not of type R. Then $G$ has a cocompact subgroup (namely, $DU$ as above) having the affine group $\mathbf{K}^* \ltimes \mathbf{K}$ as a quotient.*

*Proof.* We can diagonalize the action of $D$ on $\mathfrak{u}$, writing $\mathfrak{u} = \bigoplus \mathfrak{u}_\alpha$, where $\alpha \in \mathrm{Hom}(D, \mathbf{K}^*) \simeq \mathbf{Z}^n$ ($n$ being the rank of $D$). We have $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = \mathfrak{u}_{\alpha\beta}$ for all $\alpha, \beta$. It follows that $\mathfrak{u}_1$ is a Lie subalgebra. By assumption $\mathfrak{u}_1 \neq \mathfrak{u}$. Therefore, as $\mathfrak{u}$ is nilpotent, we have $\mathfrak{u}_1 + [\mathfrak{u}, \mathfrak{u}] \neq \mathfrak{u}$. It follows that $G/[U, U]$ is not of type R, so that we are reduced to the case when $U$ is abelian. As the action of $D$ on $U$ is diagonalizable, taking again a quotient if necessary we can suppose that $U$ is one-dimensional and that $D$ acts faithfully on $U$. In this latter case, $G$ is isomorphic to the affine group. □

**Theorem 4.4.** *Let $G$ be a linear algebraic group over a field of characteristic zero suppose that $G$ is compactly generated of non-polynomial growth. Then $G$ has a quasi-isometrically embedded free sub-semigroup on 2 generators.*

The proof, which makes use of Lemma 4.3, is similar to that of Theorem 4 and we omit it.

## 5. Free sub-semigroups in finitely generated solvable groups

**Theorem 5.1.** *Let $\Gamma$ be a finitely generated solvable, non-virtually-nilpotent group. Then $\Gamma$ has a quasi-isometrically embedded free sub-semigroup on 2 generators.*

*Proof.* By a result of Groves [Grov], taking a finite subgroup of finite index if necessary there exists an infinite field $K$ and a homomorphism $\Gamma \to K^* \ltimes K$ with Zariski dense image.

Then the image contains two non-commuting elements $A_1 = (a_1, b_1)$ and $A'_2 = (a'_2, b'_2)$ where $a_1, a_2$ are not roots of unity. Indeed, the image of $\Gamma$ in $K^*$ is finitely generated, and therefore its torsion subgroup $T$ is finite. Then the preimage in $K^* \ltimes K$ of $K^* - T$ is an open Zariski dense subset, so that its intersection with $\Gamma$ is also Zariski dense, and therefore contains two non-commuting elements.

By a lemma of Tits, there exists a local field $\mathbf{K}$ and an embedding of $K$ in $\mathbf{K}$ so that $|a_1| < 1$ in $\mathbf{K}$. Then for sufficiently large $n \geq 0$, we have, setting $A_2 = A'_2 A_1^n$ and $A_2 = (a_2, b_2)$, the inequality $a_1 + a_2 + 3a_1a_2 < 1$. Therefore by Lemma 3.1, $A_1$ and $A_2$ generate a quasi-isometrically embedded free semigroup in $\mathbf{K}^* \ltimes \mathbf{K}$. Therefore if we lift them in $\Gamma$, we get a quasi-isometrically embedded free semigroup. $\square$

The following (classical) remark shows that all cases above ($\mathbf{K}$ of all characteristics, and, in characteristic zero, both ultrametric and Archimedean cases) were necessary to consider in the proof of Theorem 5.1.

*Remark* 5.2.

(1) Let $P$ be the polynomial $X^2 - \frac{6}{5}X + 1$. This is the minimal polynomial over $\mathbf{Q}$ of the element $(3+4i)/5$ of $\mathbf{C}$ (which has modulus 1). Set $G = \mathbf{Z} \ltimes \mathbf{Z}[1/5]^2$, where $\mathbf{Z}$ acts by the companion matrix of $P$. Then $G$ is a finitely generated metabelian group with exponential growth, but for every representation of $G$ in $\mathrm{GL}_n(\mathbf{R})$ for any $n$, the Zariski closure of the image has polynomial growth.

(2) Let $Q$ be the polynomial $X^3 + X + 1$. It is straightforward that if $\alpha$ is a root of $Q$ in an ultrametric local field, then $|\alpha| = 1$. Set $G = \mathbf{Z} \ltimes \mathbf{Z}^3$, where $\mathbf{Z}$ acts by the companion matrix of $Q$. Then $G$ is a polycyclic metabelian group with exponential growth, but for every representation of $G$ in an ultrametric local field, the closure of the image has polynomial growth.

(3) If $G = \mathbf{Z}/p\mathbf{Z} \wr \mathbf{Z}$, any linear representation of $G$ in characteristic $\neq p$ has virtually abelian image.

(4) If $G$ is polycyclic, or is the Baumslag-Solitar group $\mathbf{Z}[1/n] \rtimes_n \mathbf{Z}$, then any linear representation of $G$ in characteristic $p > 0$ has virtually abelian image.

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