# SYSTOLIC GROWTH OF LINEAR GROUPS

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ABSTRACT. We prove that the residual girth of any finitely generated linear group is at most exponential. This means that the smallest finite quotient in which the $n$-ball injects has at most exponential size. If the group is also not virtually nilpotent, it follows that the residual girth is precisely exponential.

## 1. INTRODUCTION

Let $\Gamma$ be a group with a finite generating subset $S$, and $|\cdot|_S$ the corresponding word length. We assume for convenience that $S$ is symmetric and contains the unit, so that $S^n$ is equal to the $n$-ball. The following three functions are attached to $(\Gamma, S)$:

- the growth: the cardinal $b_{\Gamma,S}(n)$ of $S^n$;
- the systolic growth: the function $\sigma_{\Gamma,S}$ mapping $n$ to the smallest $k$ such that some subgroup $H$ of index $k$ contains no nontrivial element of the $n$-ball; if no such $k$ exists, we define it as $+\infty$;
- the residual girth, or normal systolic growth $\sigma'_{\Gamma,S}$: same definition, with the additional requirement that $H$ is normal.

The growth is always defined and is at most exponential, while the systolic growth and residual girth take finite values if and only if $\Gamma$ is residually finite, and in this case they can be larger than exponential, as the example in [BSe] show. Furthermore, we have the obvious inequalities

$$b_{\Gamma,S}(n) \leq \sigma_{\Gamma,S}(2n+1) \leq \sigma^{\lhd}_{\Gamma,S}(2n+1).$$

The asymptotic behavior of these functions, for finitely generated groups, does not depend on the finite generating subset.

A simple example for the residual girth grows strictly faster than the systolic growth is the case of the integral Heisenberg group, for which the growth and systolic growth behaves as $n^4$ while the residual girth grows as $n^6$ (see [BSt, C]). Also the systolic growth may grow faster than the growth and actually can grow arbitrarily fast. We show here that in linear groups, this is not the case.

**Theorem 1.1.** *Assume that $\Gamma$ admits a faithful finite-dimensional representation over a field (or a product of fields). Then the residual girth (and hence the systolic*

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*Date*: August 21, 2014.

K.B. is supported in part by NSF DMS-1405609; Y.C. is supported in part by ANR GSG 12-BS01-0003-01.

*growth) of* $\Gamma$ *are at most exponential. In particular, if* $\Gamma$ *is not virtually nilpotent, then its residual girth and its systolic growth are exponential.*

Such a result was asserted by Gromov [G, p.334] for subgroups of $\mathrm{SL}_d(\mathbf{Z})$, under some technical superfluous additional assumption (non-existence of nontrivial unipotent elements).

The proof of Theorem 1.1 consists in finding small enough quotient fields of the ring of entries, while ensuring that the $n$-ball is mapped injectively. The argument can be simplified in case $\Gamma \subset \mathrm{GL}_d(\mathbf{Q})$, since then reduction modulo $p$ for all $p$ large enough work with no further effort; in this case the finite quotients are explicit, while in the general case we only find a suitable quotient field using a counting argument.

*Example* 1.2. The group $\mathbf{Z}\wr\mathbf{Z}$ has an exponential residual girth. Another example is $(\mathbf{Z}/6\mathbf{Z})\wr\mathbf{Z}$, which is linear over a product of 2 fields, but not over a single field.

*Remark* 1.3. Closely related functions are the residual finiteness growth, which maps $n$ to the smallest number $s_{\Gamma,S}(n)$ such that for every $g \in S^n \smallsetminus \{1\}$, there is a finite index subgroup of $\Gamma$ avoiding $g$, and $s^{\triangleleft}_{\Gamma,S}(n)$ defined in the same way with only normal finite index subgroups. For finitely generated group that are linear over a field, a polynomial upper bound for these functions is established in [BM], and in the case of higher rank arithmetic groups, the precise behavior is obtained in [BK]: for instance, for $\mathrm{SL}_d(\mathbf{Z})$ for $d \geq 3$, the normal residual finiteness growth grows as $n^{d^2-1}$.

## 2. Preliminaries on polynomials over finite fields

**Lemma 2.1.** *Let $F$ be a finite field with $q$ elements. Given an integer $n \geq 1$, the number of irreducible monic polynomials of degree $n$ in $F[t]$ is $\leq q^n/n$ and $\geq (q^n - q^{n-1})/n$.*

*Proof.* The case $n = 1$ being trivial, we can assume $n \geq 1$. By Gauss' formula this number $N_q(n)$ is equal to $(1/n)\sum_{d|n}\mu(n/d)q^d$, where $\mu$ is Möbius' function. Let $p > 1$ be the smallest prime divisor of $n$. Then

$$\sum_{d|n}\mu(n/d)q^d = q^n - q^{n/p} + \sum_{d|n,d>p}\mu(n/d)q^d \leq q^n - q^{n/p} + \sum_{d|n,d>p} q^d$$

$$\leq q^n - q^{n/p} + \sum_{k=0}^{n/p-1} q^k \leq q^n$$

A similar argument shows that $nN_q(n) \geq q^n - q^{1+n/p}$, which is $\geq q^n - q^{n-1}$ if $n \geq 3$; the cases $n \leq 2$ being trivial. ☐

**Lemma 2.2.** *Let $F$ be a field with $q$ elements. Let $P \in F[t]$ be a nonzero polynomial of degree $\leq n$. Then $P$ survives in a quotient field of $F[t]$ of cardinal $\leq 2nq$.*

*Proof.* Let $m \geq 1$ be the largest number such that every irreducible polynomial of degree $m-1$ divides $P$. Let us check that $q^m \leq 2nq$; the case $m = 1$ being trivial, we assume $m \geq 2$. By Lemma 2.1, there are $\geq (q^{m-1} - q^{m-2})/(m-1)$ monic irreducible polynomials of degree $m-1$. Hence their product, which has degree $\geq q^{m-1} - q^{m-2}$, divides $P$. Thus $q^{m-1} - q^{m-2} \leq n$. We have $1 - q^{-1} \geq 1/2$; thus $\frac{1}{2}q^m q^{-1} \leq n$, that is $q^m \leq 2nq$.

Some irreducible polynomial of degree $m$ does not divide $P$, hence the quotient provides a field quotient of cardinal $q^m \leq 2nq$ in which $P$ survives. □

**Corollary 2.3.** *Let $F$ be a field with $q$ elements and $P$ a nonzero polynomial in $F[t_1, \dots, t_k]$, of degree $\leq n$ with respect to each indeterminate. Then $P$ survives in a quotient field of cardinal $\leq (2n)^k q$.*

*Proof.* Induction on $k$. The result is trivial for $k = 0$. Write

$$P = \sum_{i=0}^{n} P_i(t_1, \dots, t_{k-1}) t_k^i.$$

Some $P_i$ is nonzero; fix such $i$. Then there exists, by induction, some quotient field $L$ of $F[t_1, \dots, t_{k-1}]$ of cardinal $\leq (2n)^{k-1}q$ in which $P_i$ survives. Then the image of $P$ in $L[t_k]$ has degree $\leq n$ and is nonzero; hence by Lemma 2.2, it survives in a quotient field of cardinal $2n((2n)^{k-1}q) = (2n)^k q$. □

## 3. Conclusion of the proof

**Proposition 3.1.** *Every finitely generated group that is linear over a field of characteristic $p$ has at most exponential residual girth.*

*Proof.* Such a group embeds into $\mathrm{GL}_d(K)$ where $K$ is an extension of degree $b$ of some field $K' = F_q(t_1, \dots, t_k)$, and hence embeds into $\mathrm{GL}_{bd}(K')$. Hence it is no restriction to assume that the group is contained in $\mathrm{GL}_d(F_q(t_1, \dots, t_k))$. We let $S$ be a finite symmetric generating subset with 1; it is actually contained in $\mathrm{GL}_d(F_q[t_1, \dots, t_k][Q^{-1}])$ for some nonzero polynomial $Q$.

Write $S = Q^{-\lambda}T$ with $\lambda$ a non-negative integer and $T \subset \mathrm{Mat}_d(F_q[t_1, \dots, t_k])$; write $s = \#(S) = \#(T)$. If $x$ is a matrix, let $b(x)$ be the product of all its nonzero entries (thus $b(0) = 1$). Let $m$ be such that every entry of every element of $T$ has degree $\leq m$ with respect to each variable. Then in $T^{2n}$, every entry of every element has degree $\leq 2nm$ with respect to each variable. Define $x_n = \prod_{y \in T^{2n}} b(y-1)$. Thus $x_n$ is a product of at most $d^2 s^{2n}$ polynomials of degree $\leq 2nm$ with respect to each variable. Define $x'_n = x_n Q$; assume that $Q$ has degree $\leq \delta$ with respect to each variable, so that $x'_n$ has degree $\leq 2d^2 mns^{2n} + \delta$ with respect to each variable.

Then, by Corollary 2.3, $x'_n$ survives in a finite field $F_n$ of cardinal $q_1 \leq q(4d^2 mns^{2n} + 2\delta)^k$. Thus $S^n$ is mapped injectively into $\mathrm{GL}_d(F_n)$, which has cardinal

$$\leq q_1^{d^2} \leq q^{d^2}(4d^2 mns^{2n} + 2\delta)^{kd^2}.$$

Since $m, d, k, s, q$ are fixed, this grows at most exponentially with respect to $n$. □

**Proposition 3.2.** *Every finitely generated group that is linear over a field of characteristic* 0 *has at most exponential residual girth.*

*Proof.* Similarly as in the proof of Proposition 3.1, we can suppose that the group is contained in $\mathrm{GL}_d(\mathbf{Q}(t_1,\dots,t_k))$. We let $S$ be a finite symmetric generating subset with 1; it is actually contained in $\mathrm{GL}_d(\mathbf{Z}[t_1,\dots,t_k][r^{-1}Q^{-1}])$ for some nonzero integer $r \geq 1$ and nonzero polynomial $Q$ with coprime coefficients.

Write $S = (Qr)^{-\lambda}T$ with $\lambda$ a non-negative integer and $T \subset \mathrm{Mat}_d(\mathbf{Z}[t_1,\dots,t_k])$; write $s = \#(S) = \#(T)$. Let $R$ be an upper bound on coefficients of entries of elements of $T$, and let $M$ be an upper bound on the number of nonzero coefficients of entries of elements of $T$. Then any product of $2n$ elements of $T$ is a sum of $\leq M^{2n}$ monomials, each with a coefficient of absolute value $\leq R^{2n}$. Since any entry of an element in $T^{2n}$ is a sum of at most $d^{2n-1}$ such products, we deduce that the coefficients of entries of elements of $T^{2n}$ are $\leq d^{2n-1}R^{2n}M^{2n}$. There exists a prime $p_n \in [2d^{2n-1}(RM)^2n, 4d^{2n-1}(RM)^{2n}]$. There exists $n_0$ such that for every $n \geq n_0$, $2d^{2n-1}(RM)^2n$ is greater than any prime divisor of $r$, and $2d^{2n-1}(RM)^2n$ is greater than the lowest absolute value of a nonzero coefficient of $Q$. Now we always assume $n \geq n_0$. Then $S^{2n}$ is mapped injectively into $\mathrm{GL}_d((\mathbf{Z}/p_n\mathbf{Z})[t_1,\dots,t_k][Q^{-1}])$.

Let $m$ be such that every entry of any element of $T$ has degree $\leq m$ with respect to each variable. The previous proof provides a quotient $\mathrm{GL}_d(F_n)$ of $\mathrm{GL}_d((\mathbf{Z}/p_n\mathbf{Z})[t_1,\dots,t_k][Q^{-1}])$ in which $S^n$ is mapped injectively, such that $\mathrm{GL}_d(F_n)$ has cardinal

$$\leq {p_n}^{d^2}(4d^2mns^{2n} + 2\delta)^{kd^2}$$

Here $m, d, s, k$ are independent of $n$. The latter number is

$$\leq \left(4d^{-1}(dRM)^{2n}\right)^{d^2}(4d^2mns^{2n} + 2\delta)^{kd^2},$$

which grows at most exponentially with respect to $n$. □

*Proof of Theorem 1.1.* First assume that $\Gamma$ is linear over some field. By Propositions 3.1 and 3.2, the residual girth, and hence the systolic growth, is at most exponential. If $\Gamma$ is not virtually nilpotent, then by the Tits-Rosenblatt alternative, it contains a free subsemigroup on 2 generators and hence has exponential growth, and therefore has at least exponential systolic growth and residual girth.

Now assume that $\Gamma$ is linear over some product of fields. Let $A$ be the ring generated by entries of $\Gamma$. This is a finitely generated reduced commutative ring; hence it has finitely many minimal prime ideals, whose intersection equals the set of nilpotent elements and hence is reduced to zero. Therefore $\Gamma$ embeds into a finite product of matrix group over various fields. We conclude that $\Gamma$ has at most exponential residual girth, using the following two general facts:

- suppose that $\Gamma_1, \dots, \Gamma_k$ are finitely generated groups and $\Gamma_i$ has residual girth asymptotically bounded above by some function $u_i \geq 1$, then the residual girth of $\prod_{i=1}^k \Gamma_i$ is asymptotically bounded above by $\prod u_i$;
- if $\Lambda_1 \subset \Lambda_2$ are finitely generated groups then the residual girth of $\Lambda_1$ is asymptotically bounded above by that of $\Lambda_2$.

□

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