LARGE SCALE SIMPLE CONNECTEDNESS IN GEOMETRIC GROUP THEORY

1. Cayley Graph

Let G be a group and S a generating set. Define its Cayley graph $\Gamma(G,S)$ as the graph with G as set of vertices and $\{\{g, gs\} | g \in G, s \in S\}$ as set of (non-oriented) edges. This graph is connected and invariant under the left action of G. It is endowed with the path metric, whose restriction to G is known as the word metric:

$$d_S(g,h) = \inf\{n | \exists s_1, \dots, s_n \in S^{\pm 1} | g = hs_1 \dots s_n\}.$$

2. Bounded presentedness

Let F_S be the (abstract) group freely generated by the set S (not taking G into account). Then there is an obvious natural morphism of F_S onto G, mapping $s \in S \subset F_S$ to $s \in S \subset G$.

Define G as boundedly presented by S if there exists a subset $\mathcal{R}$ of elements of F_S, called *relators*, of bounded word length with respect to $S^{\pm}$, such that the kernel of the natural map $F_S \to G$ coincides with the normal subgroup of F_S generated $\mathcal{R}$, which is also the subgroup generated by conjugates of elements of $\mathcal{R}$ in F_S.

Remark 2.1. When S is finite, we say that G is *finitely* presented by S rather than boundedly presented by S. When G is a locally compact group (always assumed Hausdorff) and S is compact, we say that G is compactly presented by S.
Exercise 1.

1. Let G be generated by a finite subset S. Suppose that G is finitely presented by S. Show that G is finitely presented by any other finite generating subset T.
2. Find a subset $S \subset \mathbf{Z}$ such that $\mathbf{Z}$ is not boundedly presented by S. (In particular, we cannot drop the assumption T finite in 1.)
3. (Generalization of 1.) Let G be a locally compact group (always assumed Hausdorff) generated by a compact subset S. Suppose that G is compactly presented by S. Show that G is compactly presented by any other finite generating subset T.

Indications for Exercise 1.

1. Begin by the special cases $S \subset T$ and $T \subset S$.
2. Show that $\mathbf{Z}$ is not boundedly presented by the generating set $\{n! \, | n \in \mathbf{N}\}$.
3. Use the Baire category Theorem to show that $S \subset T^n$ for large n.

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3. Large scale category and quasi-isometries

Definition 3.1. Consider a map $f : X \to Y$ between metric spaces.

- f is *large-scale Lipschitz* if for some constants C_1, R_1, we have, for all $x, x' \in X$
$$d(f(x), f(x')) \leq C_1 d(x, x') + R_1;$$
- f is *large-scale expansive* if for some constants C_2, R_2, we have, for all $x, x' \in X$
$$d(f(x), f(x')) \geq C_2 d(x, x') - R_2;$$
- f is *essentially surjective*[1] if for some constant R_3, for all $y \in Y$,
$$d(y, f(X)) \leq R_3,$$
- f is a *quasi-isometric embedding* (or *large-scale bilipschitz embedding*) if it is both large-scale Lipschitz and large-scale expansive;
- f is a *quasi-isometry*[2] if it is a quasi-isometric embedding and is essentially surjective.

Definition 3.2. Two maps $f, f' : X \to Y$ between metric spaces are *at bounded distance* if for some constant R_4, for all $x \in X$
$$d(f(x), f'(x)) \leq R_4.$$

Exercise 2. Consider metric spaces X, Y, Z, maps $f, f' : X \to Y$ at bounded distance, and large-scale Lispchitz maps $g, g' : Y \to Z$ at bounded distance. Check that $g \circ f$ and $g' \circ f'$ are at bounded distance.

Definition 3.3. The *large scale category* is the category whose objects are metric spaces, and the objects between $X \to Y$ are large-scale Lipschitz maps modulo being at bounded distance.

Proposition 3.4. *A map $X \to Y$ is a quasi-isometry if and only if it defines an isomorphism in the large scale category.*

Exercise 3. Prove Proposition 3.4.

Proposition 3.4 implies that the existence of a quasi-isometry between two metric spaces defines an equivalence relation between them, called "being quasi-isometric".

For instance, non-empty bounded metric spaces constitute one quasi-isometry class of metric spaces.

Exercise 4. Let G be a locally compact, compactly generated group. If S_1, S_2 are compact generating sets, then the identity
$$(G, d_{S_1}) \to (G, d_{S_2})$$
is a quasi-isometry.

Exercise 5. Let G be a locally compact, compactly generated group and H a closed, cocompact subgroup. Show that H is compactly generated and that the embedding of H into G is a quasi-isometry.

[1]The reader might object that the inclusion of the empty set in a bounded metric space should be essentially surjective, but we shall not investigate further in this direction.

[2]This terminology is widely spread, although *quasi-similarity* would have been more accurate.

4. Large scale simple connectedness

A metric space X is *geodesic* if for every $x, y \in X$, there exists a isometric embedding of the segment $[0, d(x,y)]$ into X mapping 0 to x and $d(x,y)$ to y.

Definition 4.1. A metric space is *large scale simply connected* if it is quasi-isometric to some simply connected geodesic metric space[3].

Exercise 6.

1. Show that the metric subspace of $\mathbf{R}^2$ defined as
$$X = \{(n,x) | n \in \mathbf{N}, x \in \mathbf{R}\} \cup \{(x, n^2) | n \in \mathbf{N}, x \in \mathbf{R}\}$$
is large scale simply connected.
2. Show that the metric subspace of $\mathbf{R}^2$ defined as
$$Y = \{(n^2,x) | n \in \mathbf{N}, x \in \mathbf{R}\} \cup \{(x, n^2) | n \in \mathbf{N}, x \in \mathbf{R}\}$$
is *not* large scale simply connected.

5. Cayley complex

Let G be a group, presented by a set of generators S and a set of relations $\mathcal{R}$, i.e. a subset of the kernel of $F_S \to G$. The *Cayley complex* $\Gamma(G, S, \mathcal{R})$ is a polygonal complex structure whose 1-skeleton is the Cayley graph $\Gamma(G, \mathcal{R})$.

The polygonal structure is defined by adding polygons with vertices
$$g, gs_1, \dots, gs_1 \dots s_{n-1},$$
for $g \in G$ and $r = s_1 \dots s_n$ in $\mathcal{R}^{\pm 1}$ of length $n \geq 3$. Note that this polygonal structure is invariant under the left action of G.

Endow each such polygon with the usual Euclidean metric on the regular n-gon with edges of length 1, and endow this polygonal complex X with its geodesic metric.

Proposition 5.1. *The embedding of G, endowed with the word distance with respect to S, into any of its its Cayley complex is a quasi-isometric embedding. This is a quasi-isometry if and only if elements of $\mathcal{R}$ have bounded length.*

Proof. This is clearly a 1-Lipschitz map. Moreover, it is easy to check that it is bilipschitz with lower constant $1/\sqrt{2}$ (attained if and only if $\mathcal{R}$ contains relators of length 4). Moreover, if X is the Cayley complex,
$$\sup_{x \in X} d(x, G) = \sup\{\rho_k | \ \mathcal{R} \text{ contains elements of length } k\},$$
where ρ_k is the radius of the regular Euclidean k-gon of edge length 1, which proves the last statement. $\square$

We say that $(G, S, \mathcal{R})$ (usually denoted $\langle S | r = 1, r \in \mathcal{R}\rangle$ is a group presentation if $\mathcal{R}$ generates the kernel of $F_S \to G$ as a normal subgroup.

Exercise 7. Sketch out a picture of the Cayley complex for the following group presentations

- $$G_1 = \langle x, y | \ x^2 y^{-1} = 1\rangle$$

[3]Relevant only if the metric space is assumed large scale geodesic, i.e. quasi-isometric to some geodesic metric space.

- $$G_2 = \langle x, y |\ x^2 = y^3 = 1\rangle$$

Proposition 5.2. *The Cayley complex of every group presentation is simply connected.*

Proof. This follows from Lemmas 5.3, 5.4, and 5.5 below. ☐

Let X be a polygonal complex. A path: $\gamma : [0, r] \to X$ is called an *edge path* if there exist $0 = x_0 < x_1 < \cdots < x_n = r$ such that $\gamma(x_i)$ is a vertex for every $i = 0 \dots n$, and for every $i = 0 \dots n-1$, $\gamma(x_i)$ and $\gamma(x_{i+1})$ are joined by an edge (or equal) and γ maps linearly $[x_i, x_{i+1}]$ to the edge joining $\gamma(x_i)$ and $\gamma(x_{i+1})$.

Lemma 5.3. *In a polygonal complex, every path joining two vertices is homotopic, relatively to its endpoints, to an edge path.*

Proof (sketched). First step: γ is homotopic, relatively to its endpoints, to a path with values in the 1-skeleton X^1 of X. To prove this, let U be the complement of $\gamma^{-1}(X^1)$. Let $V =]a, b[$ be any connected component of U; then $[a, b]$ is mapped to a single polygon; by convexity, $\gamma_{|[a,b]}$ is homotopic to a path on the boundary of this polygon. Doing this on every connected component and fixing the rest, we define the desired homotopy.

Second step: by compactness, there exist $0 = x_0 < x_1 < \cdots < x_n$ such that each interval $[x_i, x_{i+1}]$ is mapped into a ball of radius < 2. Now, on a graph, every ball of radius < 2 is contractible. Therefore γ is homotopic (relatively to its endpoints) with a path with has constant speed in each interval $[x_i, x_{i+1}]$. Such a path is an edge path. ☐

On a polygonal complex, define a *combinatorial path* as a sequence of vertices $x_0, \dots, x_n$ with, for all i, $x_i\ x_{i+1}$ are either equal or connected by an edge.

When X^1 is the Cayley graph of a group G with generating set S, such a path can be described by its origin x_0 and the sequence labels $s_i \in S^{\pm 1} \cup \{1\}$ of the edge joining x_i to $x_i = x_{i-1}s_i$.

Let G be generated by a set S, with a set of relators $\mathcal{R}$, so that $G = F_S/\langle\mathcal{R}\rangle$. Consider the polygonal complex defined above.

There is a combinatorial notion of homotopy between closed paths defined as follows: this is the equivalence relation generated by the following equivalences $\gamma \sim \gamma'$, where

$$\gamma = (1 = x_0, x_1, \dots, x_n = 1);$$
$$\gamma' = (1 = x_0, \dots, x_k, y_1 \dots, y_{m-1}, x_\ell, \dots, x_n),$$

$x_k \dots, x_\ell, y_{m-1}, \dots, y_1, x_k$ bounds a polygon defined by $\mathcal{R}$, $0 \leq k \leq \ell \leq n$ (where we agree that any edge (x, y) bounds a 2-gon). A polygonal complex is said to be *combinatorially simply connected* if every closed combinatorial path is combinatorially homotopic to a constant combinatorial path.

Lemma 5.4. *Consider the Cayley complex $\Gamma(G, S, \mathcal{R})$. A closed combinatorial path based at 1, defined by a sequence of edges $s_1, \dots, s_n$ is combinatorially homotopic to the trivial combinatorial path* (1) *if and only if we can write, inside F_S,*

$$s_1 \dots s_n = \prod_{i=1}^{k} m_i r_i {m_i}^{-1},$$

for some k, $m_i \in F_S$, and $r_i \in \mathcal{R}^{\pm 1}$. In particular, it is combinatorially simply connected if and only if $\mathcal{R}$ generates the kernel of $F_S \to G$ as a normal subgroup.

Proof(sketched). Observe that the set $\mathcal{S}$ of words in the alphabet S defining a trivial combinatorial path can be viewed as a subset of F_S. It is stable under product and inversion, conjugation and contains all elements of the form r_i, and therefore contains all elements of the form $\prod_{i=1}^{k} m_i r_i {m_i}^{-1}$. Conversely, the set of those elements is closed under the equivalence defined above. □

Any edge path defines a combinatorial path as follows: let $\gamma : [0, r] \to X$ be an edge path. Then exists minimal n and elements $0 = x_0 < x_1 < \cdots < x_n = r$ such that for each i, $v_i = \gamma(x_i)$ is a vertex, and, if $i < n$, γ is either constant on $[x_i, x_{i+1}]$ or maps it linearly to the edge joining v_i to v_{i+1}. We say that the combinatorial path $(v_0, \ldots, v_n)$ is *associated* to the edge path γ.

Lemma 5.5. *Let $(v_0, \ldots, v_n)$ and $(w_0, \ldots, w_n)$ be combinatorial paths associated to edges paths γ and γ'. Suppose that $(v_0, \ldots, v_n)$ and $(w_0, \ldots, w_n)$ are combinatorially homotopic. Then γ and γ' are homotopic.*

Proof. Left to the reader. □

6. Bounded presentations and simple connectedness

Proposition 6.1. *Let G be a group generated by S. Then (G, S) is boundedly presented if and only if $\Gamma(G, S)$ is large scale simply connected.*

Proof. Suppose that G is boundedly presented, with a set $\mathcal{R}$ of relators of length $\leq k$. Then G is quasi-isometric to its Cayley complex by Proposition 5.1, which is geodesic by definition, and which is simply connected by Proposition 5.2.

Conversely suppose that G is quasi-isometric to a geodesic simply connected space X. Let G_n be the 2-complex with 1-skeleton the Cayley graph of G and with all combinatorial closed paths of length $\leq n$ bounding a polygon. Let us show that G_n is combinatorially simply connected for large n.

Let $i : G \to X$ be the quasi-isometry. We can suppose that i is 1-Lipschitz, and therefore we can extend i to a 1-Lipschitz map from the 1-skeleton of G to X. Let $j : X \to G$ be an inverse quasi-isometry (so that $j \circ i$ and $i \circ j$ are at bounded distance to identity). We can suppose that $j(i(1)) = 1$. Let $M, W < \infty$ be constants such that any two points in X at distance ≤ 1 are mapped to points at distance $\leq M$, and $j \circ i$ is at distance W to the identity.

Let us show that G_{4M} is combinatorially simply connected.

Let $(1 = v_0, v_1, \ldots, v_n = 1)$ be a closed combinatorial path. By geodesicity of X, we can extend it to a 1-Lipschitz map γ from $[0, n]$ to X, mapping i to v_i. Let $(t, x) \mapsto \gamma_t(x)$ be a homotopy $[0, n] \times [0, 1] \to X$, where $\gamma_0 = \gamma$ and γ_1 is constant equal to 1. By uniform continuity, there exists m such that for al t_1, t_2, x_1, x_2,

$$|t_1 - t_2| \leq 1/m, |x_1 - x_2| \leq 1/m \Rightarrow d(\gamma_{t_1}(x_1), \gamma_{t_2}(x_2)) \leq 1.$$

We need to introduce the following definition: on a graph, a p-combinatorial path through a sequence of vertices $(w_0, \ldots, w_n)$ is a combinatorial path $v_0, \ldots, v_m$ such that for some sequence $0 = i_0 < i_1 < \cdots < i_n = m$ with $|i_{j+1} - i_j| \leq p$, $v_{i_j} = w_j$. (In particular, it exists only if $d(w_i, w_{i+1}) \leq p$ for all i).

First step: the combinatorial path $(v_i)_{0 \leq i \leq n}$ is combinatorially homotopic to any M-combinatorial path through $(j \circ \gamma(i))_{0 \leq i \leq n}$. Indeed, pick combinatorial paths of length $\le W$ between v_i and $j\circ\gamma(i)$. This defines "squares" of length $\le 2W+M+1$ through v_i, $j\circ\gamma(i)$, $j\circ\gamma(i+1)$ and v_{i+1}. Such squares bound polygons in G_{2W+M+1}.

Second step: any M-combinatorial path $(j\circ\gamma(i))_{0\le i\le n}$ is combinatorially homotopic to any M-combinatorial path through $(j\circ\gamma(i/m)_{0\le i\le mn}$. This is checked by inserting all $j\circ\gamma(i/m)$ one by one: so we are reduced to check that a path of the form $j\circ\gamma(x_0),\dots,j\circ\gamma(x_n)$, with $0=x_0<x_1\cdots<x_n=1$, is combinatorially homotopic any M-combinatorial path through a sequence of the form

$$j\circ\gamma(x_0),\dots,j\circ\gamma(x_i),j\circ\gamma(y),j\circ\gamma(x_{i+1}),j\circ\gamma(x_n),$$

when $x_i\le y\le x_{i+1}\le x_i+1$. Now each edge of the "triangle" $j\circ\gamma(x_i),j\circ\gamma(y),j\circ\gamma(x_{i+1})$ has length $\le M$. So there exists a closed path of length $\le 3M$ passing through these three points. By assumption, it bounds a polygon in G_{3M}.

Third step: any combinatorial path through $(j\circ\gamma(i/m)_{0\le i\le mn}$ is combinatorially homotopic to the constant path $(1)_{0\le i\le mn}$. To see this, it suffices to check that for every t, any M-combinatorial path through $(j\circ\gamma_x(i/m))_{0\le i\le mn}$ is combinatorially homotopic to any M-combinatorial path through $(j\circ\gamma_{x+1/m}(i/m))_{0\le i\le mn}$.

Indeed, pick combinatorial paths of length $\le M$ between $j\circ\gamma_x(i/m)$ and $j\circ\gamma_{x+1/m}(i/m)$ for each i. This defines "squares" of length $\le 4M$ through $j\circ\gamma_x(i/m)$, $j\circ\gamma_{x+1/m}(i/m)$ $j\circ\gamma_x((i+1)/m)$ and $j\circ\gamma_{x+1/m}((i+1)/m)$. Such "squares" bound polygons in G_{4M}.

Combining the three steps, we obtain that G_k is combinatorially simply connected for $k=\max(4M,2W+M+1)$. Now if $\mathcal{R}$ is the set of elements of length $\le k$ in the kernel of $F_S\to G$, then G_k is the Cayley complex of $(G,S,\mathcal{R})$. It then follows from Lemma 5.4 that $(G,S,\mathcal{R})$ is a group presentation, so that (G,S) is boundedly presented. □

7. Quasi-geodesic spaces

On a metric space X, a C-path of length n between two points x,y is a sequence $x=x_0,x_1,\dots,x_n=y$ such that $d(x_i,x_{i+1})\le C$ for all i.

We say that a metric space is quasi-geodesic if for some constants C,C', for every $n\in\mathbf{N}$ and any two points x,y at distance $\le n$ there exists a C-path between x and y of length $\le C'n$.

Exercise 8. Show that a space is quasi-geodesic if and only if it is quasi-isometric to some geodesic metric space. (In particular, being quasi-geodesic is closed under quasi-isometries.)

A map $f:X\to Y$ between metric spaces is called *uniform* if for every n, we have $\sup_{d(x,y)\le n} d(f(x),f(y))<\infty$. It is called *uniformly proper* if $\inf_{d(x,y)\ge n} d(f(x),f(y))$ goes to infinity when $n\to\infty$ (where $\sup\emptyset=0$ and $\inf\emptyset=\infty$). It is a *uniform embedding* if it is both uniform and uniformly proper.

Lemma 7.1. *Let $f:X\to Y$ be a uniformly proper map which is cobounded. Suppose that X is quasi-geodesic. Then for some constant C, $d(f(x),f(y))\ge Cd(x,y)-C$ for all x,y.*

Proof. By assumption, there exists $M<\infty$ such that every point in Y is at distance $\le M$ to the image. Now fix $x,y\in X$. By geodesicity, there exists a C-path $f(x)=u_0,u_1,\dots,u_n=f(y)$, with $n\le C'(d(f(x),f(y))+1$. Pick $x_i\in X$ such that $d(f(x_i),u_i)\le M$; we can choose $x_0=x$ and $x_n=y$. In particular, $d(f(x_i),f(x_{i+1})\le$

$2M + C$. Now there by uniform properness, there exists a constant D, depending only on f (not on x, y) such that $d(f(z), f(w)) \leq 2M + C$ implies $d(z, w) \leq D$. Therefore $d(x_i, x_{i+1}) \leq D$. Therefore $d(x, y) \leq Dn \leq DC'(d(f(x), f(y)) + 1)$. □

8. Lie groups

Lemma 8.1. *Let G be a connected Lie group. Then G has a compact subgroup K such that G/K is diffeomorphic to a Euclidean space.*

Exercise 9. Prove the lemma for $G = \mathrm{GL}_n(\mathbf{R})$.

Indication: use the polar decomposition.

Let G be a group generated by a set S. Let G act by isometries on a metric space X. We say that the action is *regular* if for any $x \in X$, the set $\{sx | s \in S\}$ is bounded. We say that the action is *proper* if for every unbounded subset W of G (endowed with the S-word metric) and every $x \in X$, the set $\{gx | g \in W\}$ is unbounded. We say that the action is *cobounded* if there exists $M < \infty$ such that for every $x, y \in X$ there exists $g \in G$ such that $d(gx, y) \leq M$.

Lemma 8.2. *Let G be a group generated by a subset S, acting by isometries on a quasi-geodesic metric space X. Suppose that the action is regular, proper and cobounded. Then for every $x \in X$, the map $g \mapsto gx$ is a quasi-isometry.*

Proof. Set $L(g) = d(x, gx)$. Then L is a length function on G, i.e. satisfies $L(1) = 0$, $L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$. Therefore it satisfies $L(g) \leq M|g|_S$, where $M = \sup_{s \in S} L(s)$, which is bounded as the action is regular. Therefore, for all g, h, $d(gx, hx) = L(g^{-1}h) \leq M|g^{-1}h|_S$.

On the other hand, the map $g \mapsto gx$ satisfies the assumptions of Lemma 7.1, so there exists C such that for all g, h, $d(gx, hx) \geq C|g^{-1}h|_S - C$.

Since moreover every point is at distance $\leq W$ to some gx, we get that the map $g \mapsto gx$ is a quasi-isometry. □

Exercise 10.

(1) Show that the statement of Lemma 8.2, where "cobounded" is dropped in the hypotheses and "is a quasi-isometry" is replaced by "is a quasi-isometric embedding" in the conclusion, is false.
(2) Show that the assumption that X is geodesic cannot be dropped in Lemma 8.2.

Indications:

(1) Consider the action of a parabolic isometry on the hyperbolic plane.
(2) Consider the action by translations of $\mathbf{Z}$ on $\mathbf{R}$ endowed with the distance $d(x, y) = \sqrt{|x - y|}$.

Proposition 8.3. *Let G be any connected Lie group. Then G is compactly presented.*

Proof. By Lemma 8.1, there exists a compact subgroup K such that G/K is diffeomorphic to a Euclidean space. Consider the action of K on the tangent space at the basepoint x_0 of G/K. It preserves a Euclidean structure at the tangent space of G/K at x_0. Push forward this Euclidean structure on G/K by the left action of G. The K-invariance implies that the resulting Euclidean structure at $gK \in G/K$ does not depend on the choice of its representing element $g \in G$. This therefore defines a left-invariant Riemannian structure on G/K, on which G acts by isometries. By homogeneousness, is a complete Riemannian structure, and is therefore geodesic. Let S be any compact subset with non-empty interior. Then the subgroup generated by S is open and therefore is all of G. Endow G with this generating subset. By continuity, the action of G on G/K is regular. As closed balls in X are compact (by completeness), the action of G on G/K is proper. It is transitive and therefore cobounded. Accordingly, by Lemma 8.2, G is quasi-isometric to X. Therefore $\Gamma(G, S)$ is large scale simply connected, so that G is compactly presented. □

Lemma 8.4. *Let G be a locally compact, compactly generated group. Let $H \subset G$ be a cocompact closed subgroup. Then the embedding of H into G is a quasi-isometry.*

Proof. Let T be a compact generating subset of H, W a compact subset such that $G = HW$ and $S = T \cup W$, which is a compact generating subset of G. Then for every $h \in H$, $|h|_T \geq |h|_S$. To get a reverse inequality, it suffices in view of Lemma 7.1 to show that the embedding of H into G is uniformly proper. By equivariance, this means that this embedding is proper. In set-theoretic terms, this means that for every bounded subset of G, its intersection with H is bounded. This is obvious as a subset of G (resp. H) is bounded if and only if it has compact closure, and H is closed in G. □

Exercise 11. Let G be a group endowed with a left-invariant metric. Check that this metric is quasi-geodesic if and only if it is equivalent to the word metric with respect to some generating subset.

9. Topological HNN-extensions

Let G be a Hausdorff topological group in which open subgroups form a basis of neighbourhoods of 1. Consider open subgroups H_1, H_2 along with an isomorphism $\phi : H_1 \to H_2$. Consider the HNN-extension

$$\Gamma = HNN(G, H_1, H_2, \phi).$$

This is the group with presentation

$$\langle G, t | tgt^{-1} = \phi(g) \ \forall g \in h_1 \rangle.$$

Say that a net (g_i) in Γ converges to $g \in \Gamma$ if for every open subgroup L of G, eventually $g^{-1}g_i \in L$. This defines a topology on Γ.

Lemma 9.1. *This topology makes Γ a Hausdorff topological group, and the embedding of G into Γ is a homeomorphism onto an open subgroup. In particular, if G is locally compact then Γ is also locally compact.*

Proof. Let us show that the inversion map is continuous. Suppose that $g_i \to g$. Let L be any open subgroup of G. By Lemma 9.2, eventually $g^{-1}g_i \in g^{-1}Lg \cap G$. Therefore eventually $g_i g^{-1} \in L$. Thus eventually $gg_i^{-1} \in L$, i.e. $g_i^{-1} \to g^{-1}$.

Let us show that the law is continuous. Suppose that $g_i \to g$ and $h_i \to H$. Let L be any open subgroup of G. By Lemma 9.2, $hLh^{-1} \cap G$ is an open subgroup of G. As $h_i^{-1} \to h^{-1}$, this implies that eventually $g^{-1}g_i \in hLh^{-1}$ and $hh_i^{-1} \in hLh^{-1}$, i.e. $h_i h^{-1} \in hLh^{-1}$. Therefore eventually $g^{-1}g_i h_i h^{-1} \in hLh^{-1}$, i.e. $(gh)^{-1}g_i h_i \in L$. So $g_i h_i \to gh$.

Thus we have a topological group. It is Hausdorff if and only if $\{1\}$ is closed, i.e. $1 \to g$ implies $g = 1$, which is fulfilled because the intersection of all open subgroups is trivial.

It is straightforward that G is open in Γ and that the induced topology coincides with the original one. $\square$

Lemma 9.2. *For all open subgroups L, M of G and $x \in \Gamma$, $L \cap xMx^{-1}$ is an open subgroup of G.*

Proof. First case: $x = t$ (the case $x = t^{-1}$ being similar). Set $M' = M \cap H_1$. Then $tM't^{-1} \subset H_2$. So

$$L \cap tMt^{-1} \supset L \cap H_2,$$

which is an open subgroup of G.

Second case: $x \in G$. Then xMx^{-1} is an open subgroup of G and therefore so is $L \cap xMx^{-1}$.

General case: argue by induction of the word length $|x|$ of x with respect to the generating subset $G \cup \{t\}$ of Γ. The case $|x| \leq 1$ is already settled, so suppose $|x| \geq 2$. Write $x = wy$ with $|w| = 1$ and $|y| = |x| - 1$. Then

$$L \cap xMx^{-1} \supset L \cap xMx^{-1} \cap wLw^{-1} = L \cap w(yMy^{-1} \cap L)w^{-1}$$

By induction assumption, $yMy^{-1} \cap L$ is an open subgroup of G. By the case of length 1, $L \cap w(yMy^{-1} \cap L)w^{-1}$ is an open subgroup of G. $\square$

10. A few metabelian groups

Proposition 10.1. *The p-adic affine group $\mathbf{Q}_p^* \ltimes \mathbf{Q}_p$ is compactly presented.*

It contains as a cocompact subgroup $\mathbf{Z} \ltimes_p \mathbf{Q}_p$, on which we will focus. This group has the following "presentation"

$$\langle \mathbf{Z}_p, t |\ txt^{-1} = x^p\ \forall x \in \mathbf{Z}_p \rangle.$$

This means, more formally: a set of generators $\{t\} \cup \{s_x | x \in \mathbf{Z}_p\}$ along with the relations $ts_xt^{-1} = s_{x^p}$ and the (implicit) relations of length three $s_xs_y = s_xy$ for $x, y \in \mathbf{Z}_p$.

Exercise 12. Check that this is indeed a presentation of $\mathbf{Z} \ltimes_p \mathbf{Q}_p$.

Proposition 10.2. *The p-adic SOL-group $\mathbf{Q}_p^* \ltimes (\mathbf{Q}_p \times \mathbf{Q}_p)$, acting by diagonal matrices of determinant one, is not compactly presented.*

This group has the following presentation. Let Z_1, Z_2 denote two copies of $\mathbf{Z}_p$.

$$\langle Z_1, Z_2, t |\ txt^{-1} = x^p\ \forall x \in Z_1, t^{-1}yt = y^p\ \forall y \in Z_2,$$
$$[t^{-k}xt^k, t^\ell yt^{-\ell}] = 1\ \forall x \in Z_1, y \in Z_2, k, l \in \mathbf{N} \rangle.$$

The latter family has unbounded length; however this is not enough to prove that G is not compactly presented. To carry out this, let us consider the presentation with the same set of generators and the set of relators $\mathcal{R}_n$, which is the same, except that in the commuting relators, k, ℓ only range from 0 to n. Now those relators have bounded length. Denote by G_n the group defined by this "truncated" presentation.

Observe that $\mathcal{R}_n$, the relators for which $(k, \ell) \neq (n, n)$ are redundant. This gives readily

Lemma 10.3. *The mapping $t \mapsto t$, $Z_1 \ni x \mapsto p^{-n}x$, $Z_2 \ni y \mapsto p^n y$ is an isomorphism of G_0 onto G_n.*

Exercise 13. Prove in detail Lemma 10.3.

Indication: check that this is a well-defined group homomorphism, define similarly its expected inverse, and check that both composites are identity.

Therefore, the study of G_n is reduced to that of G_0. Observe that G_0 is a HNN-extension of $\mathbf{Z}_p \times \mathbf{Z}_p$, for the isomorphism $\mathrm{diag}(p, p^{-1})$ between its two open subgroups $\mathbf{Z}_p \times p\mathbf{Z}_p$ and $p\mathbf{Z}_p \times \mathbf{Z}_p$. This is a non-ascending HNN-extension (i.e. none of the two subgroups is contained in the other) and therefore G_0, and thus G_n contains a non-abelian free subgroup. In particular, the mapping $G_n \to G$ is not injective for any n. Now G_n is a locally compact, compactly generated group, and the kernel N of $G_0 \to G$ is discrete, because it has trivial intersection with the open subgroup $\mathbf{Z}_p \times \mathbf{Z}_p$. Let $N_n \subset N$ be the kernel of $G \to G_n$, so that (N_n) is an increasing sequence of proper subgroups of N with union N. The intersection of N_n with any bounded ball is finite; therefore for any k, there exists n_k such that the intersection of the k-ball of G_0 with N is contained in N_{n_k}. This proves that N is *not* generated, as a normal subgroup, by relations of bounded length. Thus G is not compactly generated.

Propositions 10.1 and 10.2 have the following generalization.

Proposition 10.4. *Fix $i_1, \dots, i_n \in \mathbf{Z}$. Let $A = diag(p^{i_1}, \dots, p^{i_n})$ and $G = \mathbf{Q}_p^* \ltimes_A \mathbf{Q}_p^n$. Then*

- *G is compactly generated if and only if all $i_k \neq 0$;*
- *G is compactly generated if either all $i_k > 0$ or all $i_k < 0$.*

The first statement is an easy observation: indeed if some $i_k = 0$, then G possesses $\mathbf{Q}_p$ as a quotient, which is not compactly generated[4].

Exercise 14. Prove Proposition 10.4.

Indication: for the second statement, adapt the proofs of Propositions 10.1 10.2.

11. Dehn function

The real analogues $\mathbf{R} \ltimes \mathbf{R}^n$ of the groups above are connected Lie groups and are therefore compactly presented. However, there is still an important difference between a group like $\mathbf{R} \ltimes \mathbf{R}$ and $\mathrm{SOL}(\mathbf{R})$, which is similar to large scale simple connectedness, but more subtle.

Let G be a group generated by a subset S, and let N be the kernel of $F_S \to G$, and suppose that N is generated by a subset $\mathcal{R}$ as a normal subgroup. Let T be the set of conjugates in G of elements of $\mathcal{R}$, so that N is generated by T as a group. Define the *area* $a(g)$ of $g \in G$ as the word length of g with respect to T. Define the Dehn function of $(G, S, \mathcal{R})$ as

$$\delta(n) = \sup\{a(g) | g \in N, |g|_S \leq n\}$$

(where $\sup \emptyset = 0$).

[4]This is essentially the only obstruction: by a theorem of Borel and Tits (1966), an algebraic group $G(\mathbf{Q}_p)$ is non-compactly generated if and only if it has a normal cocompact algebraic subgroup having $\mathbf{Q}_p$ as a quotient.

Exercise 15. Compute the (exact value of the) Dehn function $\delta_i(n)$ for the following presentations

$$G_1 = \langle x, y |\ y^2 = 1\rangle$$
$$G_2 = \langle x_n (n \in \mathbf{N}) |\ x_0 = 1, x_n x_{n+1}^{-1} = 1 (n \in \mathbf{N})\rangle$$

Let us define a similar notion for graphs. Let Γ be a connected graph with a base-vertex called 1. Consider the set $\mathcal{L}_\Gamma$ of all closed 1-paths based at 1, i.e. sequences of vertices $1 = v_0, v_1, \ldots, v_d = 1$ with $d(v_i, v_{i+1}) \leq 1$ for all i.

Fix an integer n. Define a structure of polygonal complex on Γ by adding i-gons at every closed 1-path of length $\leq n$. Define the incidence relations $\sim_0$ and $\sim_1$ as follows:

$$(v_0, \ldots, v_k, \ldots, v_d) \sim_0 (v_0, \ldots, \hat{v}_k, \ldots, v_d)$$

if $v_{k-1} = v_k$ (this means removing a redundant vertex);

$$(v_0, \ldots, v_{k-1}, v_k, \ldots, v_d) \sim_0 (v_0, \ldots, \hat{v}_{k-1}, \hat{v}_k, \ldots, v_d)$$

if $v_{k-1} = v_{k+1}$ (this means: removing a back-and-forth);

$$(v_0, \ldots, v_k, \ldots, v_\ell, \ldots, v_d) \sim_1 (v_0, \ldots, v_k, w_1 \ldots, w_{m-1}, v_\ell, \ldots, v_d)$$

if $(v_k, \ldots, v_\ell, w_{m-1} \ldots, w_1, v_k)$ is the boundary of a polygon (in particular $\ell - k + m \leq n$).

Let d be the greatest distance on $\mathcal{L}_\Gamma$ for which $d(u, v) \leq i$ whenever $u \sim_i v$. This is a geodesic distance. Define $\delta_{\Gamma,nn}(m) = \sup(d(u, 1))$, where u ranges over closed 1-paths of length $\leq m$ and 1 is the path (1).

Lemma 11.1.

$$\delta_{\Gamma,n}(m) \leq \delta_{\Gamma,k}(n)\delta_{\Gamma,k}(m).$$

In particular, if $\delta_{\Gamma,k}(n)$ and $\delta_{\Gamma,n}(k)$ is finite then $\delta_{\Gamma,n}$ and $\delta_{\Gamma,k}$ are equivalent.

Proof. This is left as an exercise. □

Lemma 11.2. *Suppose that Γ, Λ are quasi-isometric graphs. Then there exists C such that for every m, n*

$$\delta_{\Gamma,Cm}(n) \leq \delta_{\Lambda,m}(n) + n.$$

Proof (sketched). Consider $f : \Gamma \to \Lambda$ and $g : \Lambda \to \Gamma$ quasi-inverse M-Lipschitz quasi-isometries mapping the base-points to each other. Suppose that $g \circ f$ is at distance M to identity.

Let $v_0, \ldots, v_n$ be a closed 1-path. Interpolate its image by f by a 1-path $w_0, \ldots, w_{Mn}$ where $w_{Mi} = f(v_i)$. Interpolate in turn its image by g by a 1-path $V_0, \ldots, V_{M^2n}$, where $V_{M^2i} = g \circ f(v_i)$.

Then the distance of $(v_0, \ldots, v_n)$ to $(V_0, \ldots, V_{M^2n})$ in $\mathcal{L}_{\Gamma,(M+1)^2}$ is at most n (using a polygon through $v_i, v_{i+1}, V_{M^2(i+1)}, V_{M^2i}$).

Now if we have two closed paths $(a_0, \ldots, a_k) \simeq (b_1, \ldots, d_\ell)$, i.e. they differ "by one K-gon", then their images by g can be interpolated to paths differing by one MK-gon. Thus, if in $\mathcal{L}_{\Lambda,K}$, $(a_0, \ldots, a_k)$ is at distance at most n to the constant path 1, then $(g(a_0), \ldots, g(a_k))$ is at distance at most n to 1 in $\mathcal{L}_{\Lambda,MK}$.

Thus we get, if $M \geq 1$ and $K \geq M + 3$,

$$\delta_{\Gamma,MK}(n) \leq \delta_{\Lambda,K}(n) + n.$$

□

Proposition 11.3. *Let $(G, S, \mathcal{R})$ be a compactly presented group. Then $\delta(n) < \infty$.*

Proof. The set of closed 1-paths of length n, based at 1, is compact. Let $(v_0, \ldots, v_n)$ be one of them. If $(w_0, \ldots, w_n)$ is close to $(v_0, \ldots, v_n)$, then the two are at distance $\leq k$, where k is an integer such that the word length of (G, S) is bounded by k nearby 1. It follows that the area function is locally bounded. By compactness, it is bounded. □

12. Computation of some Dehn functions

Let G be a group, S a set of generators, and $\mathcal{R} \subset F_S$ a set of relators, so that G is the quotient of F_S by the normal subgroup N generated by $\mathcal{R}$. The area function can be redefined as follow: let $\mathcal{R}^c$ be the set of conjugates in F_S of elements of $\mathcal{R}$. Then, for $g \in F_S$, let $a(g)$ be the word length of g with respect to $\mathcal{R}^c$; if $g \in N$, agree that $a(g) = \infty$. Let $b(g, h) = a(g^{-1}h)$ be the corresponding right-invariant $(\mathbf{R}_+ \cup \{infty\})$-valued distance. As the "generating"-set is invariant under conjugation, this is also a right-invariant distance.

Lemma 12.1.

$$a([x, y]) = a([y, x]) \quad \textit{and}$$
$$a([x^n, y^m]) \leq |mn| a([x, y]) \quad \forall m, n \in \mathbf{Z}$$

Proof. For the first statement just observe that $[y, x] = [x, y]^{-1}$.

For the second statement, first check the cases when $|m|, |n| = 1$; also . Then prove the general inequality $a([xy, z]) \leq a([x, z]) + a([y, z])$ and deduce the general case. □

Exercise 16. Complete the proof of Lemma 12.1.

Proposition 12.2. *The group presentation*

$$G = \langle x, y |\ [x, y] = 1 \rangle$$

has Dehn function $\delta(n) \leq n(n-1)/2$.

Proof. Observe that $G \simeq \mathbf{Z}^2$; in particular every $g \in G$ can be written in a unique way, in G, as $x^\alpha y^\beta$ for $(\alpha, \beta) \in \mathbf{Z}^2$, where $|\alpha| + |\beta| \leq |g|$. In our language, where we prefer to work inside the free group F_S, this states that for every $g \in F_S$, there exists a unique $(\alpha, \beta) \in \mathbf{Z}^2$ such that $b(g, x^\alpha y^\beta) < \infty$. Let us show by induction on $n = |g|$ that $b(g, x^\alpha y^\beta) \leq u_n = n(n-1)/2$.

This is clear for $n = 0$. Suppose that this is proved for n, and suppose $|g| = n+1$. There are two cases. If $g = x^e m$ with $|e| = 1$ and $|m| = n$, then $b(g, x^{\alpha+e} y^\beta) = b(x^e m, x^{a+e} y) = b(m, x^\alpha y^\beta) \leq u_n$ by induction. Therefore $b(g, x^{\alpha+e} y) \leq u_n \leq u_{n+1}$. If $g = y^e m$ with $|e| = 1$ and $|m| = n$, then

$$\begin{aligned} b(g, x^\alpha y^{\beta+e}) &= b(y^e m, x^\alpha y^{\beta+e}) = b(m, y^{-e} x^\alpha y^e y^\beta) \\ &= a(y^{-e} x^\alpha y^e x^{-\alpha} x^\alpha y^\beta m^{-1}) \leq a([y^{-e}, x^\alpha]) + a(x^\alpha y^\beta m^{-1}) \\ &\leq |\alpha| + b(g, x^\alpha y^\beta) \leq n + u_n \leq u_{n+1}, \end{aligned}$$

by induction and by Lemma 12.1.

Now, if $a(g) < \infty$, we have $(\alpha, \beta) = (0, 0)$, so this gives $a(g) \leq u_{|g|}$, and therefore $\delta(n) \leq u_n$ for all n. □

Exercise 17. Generalize Proposition 12.2 as follows: if H_i has presentation

$$\langle S_i | \mathcal{R}_i \rangle$$

for $i = 1, 2$, if $G = H_1 \times H_2$ is presented as

$$\langle S_1 \cup S_2 | \ \mathcal{R}_1 \cup \mathcal{R}_2 \cup [S_1, S_2] \rangle$$

(where $[S_1, S_2] = \{[s_1, s_2] | \ s_1 \in S_1, s_2 \in S_2\}$), then

$$\max(\delta_{H_1}(n), \delta_{H_2}(n)) \leq \delta_G(n) \leq \delta_{H_1}(n) + \delta_{H_2}(n) + n(n-1)/2.$$

Proposition 12.3. *The group presentation*

$$H = \langle x, y | \ [x, [x, y]] = [y, [x, y]] = 1 \rangle$$

has Dehn function $\delta(n) \leq (4n^3 + 12n^2 + 11)$.

Proof. We shall show that every element is equal in G a a unique element of the form $x^\alpha y^\beta z^\gamma$, where we denote $z = [x, y]$.

Looking in the abelianization, we get the uniqueness of (α, β). So the uniqueness of γ is equivalent to say that z is torsion-free, as we see by using the usual representation as upper unipotent 3×3 matrices.

Let us prove something stronger than the existence, namely: for every $m \in F_S$ with $|m| = n$, there exists (α, β, γ) in $\mathbf{Z}^3$ such that $b(m, x^\alpha y^\beta z^\gamma) \leq v_n = 4n^3 + 12n^2 + 11$, with $|\alpha|, |\beta| \leq n$ and $|\gamma| \leq n(n-1)/2$.

This is clear for $n = 0$; suppose this is proved for n, and consider m' of length $n+1$. If $m' = x^e m$ with $|m| = n$ and $b(m, x^\alpha y^\beta z^\gamma) \leq v_n$, then $b(m', x^{\alpha+1} y^\beta z^\gamma) \leq v_n \leq v_{n+1}$ and $\gamma \leq u_n \leq u_{n+1}$. So suppose $m' = y^e m$ with $|m'| = n$. Then

$$\begin{aligned}
b(y^e m, x^\alpha y^{\beta+e} z^{\gamma-\alpha e}) &= a(y^{-e} x^\alpha y^{\beta+e} z^{\gamma-\alpha e} m^{-1}) \\
&= a(x^\alpha y^{-e} [y^e, x^{-\alpha}] y^{\beta+e} z^{\gamma-\alpha e} m^{-1}) \\
&= a([y^e, x^{-\alpha}] y^{\beta+e} z^{\gamma-\alpha e} m^{-1} x^\alpha y^{-e}) \\
&\leq a([y^e, x^{-\alpha}] z^{-e\alpha}) + a(z^{e\alpha} y^{\beta+e} z^{\gamma-\alpha e} m^{-1} x^\alpha y^{-e}) \\
&= a([y^e, x^{-\alpha}] z^{-e\alpha}) + a([z^{e\alpha}, y^{\beta+e}]) + a(y^{\beta+e} z^{e\alpha} z^{\gamma-\alpha e} m^{-1} x^\alpha y^{-e}).
\end{aligned}$$

Let us deal with each of these three terms:

(1) Let us check that

$$a([y^e, x^{-\alpha}] z^{-e\alpha}) \leq k_\alpha = |\alpha|(|\alpha| + 1)/2.$$

Let us prove it by induction for $\alpha \geq 0$ and $e = 1$, the case $\alpha \leq 0$ and/or $e = -1$ being similar. For $\alpha = 0$ this is obvious. Suppose that this is proved for $\alpha \geq 0$, i.e. $b(x^{-\alpha} y x^\alpha, y^{-1} z^{-\alpha}) \leq k_\alpha$. Then

$$\begin{aligned}
b(x^{-\alpha-1} y x^{\alpha+1}, y^{-1} z^{-\alpha-1}) &= b(x^{-\alpha} y x^\alpha, x y^{-1} z^{-\alpha-1} x^{-1}) \\
&\leq k_\alpha + b(y^{-1} z^{-\alpha}, x y^{-1} z^{-\alpha-1} x^{-1}) \\
&= k_\alpha + a(y x y^{-1} z^{-1} x^{-1} [x, z^{-\alpha}]) \\
&= k_\alpha + a([x, y] z^{-1}) + a([x, z^{-\alpha}]) \\
&\leq k_\alpha + 1 + \alpha \leq k_{\alpha+1}.
\end{aligned}$$

(2) By Lemma 12.1,

$$a([z^{e\alpha}, y^{\beta+e}]) \leq |\alpha|(|\beta| + 1)$$

(3)
$$a(y^{\beta+e}z^{e\alpha}z^{\gamma-\alpha e}m^{-1}x^{\alpha}y^{-e}) = a(x^{\alpha}y^{\beta+e}z^{\gamma}m^{-1}) \leq v_n.$$

We thus get

$$b(m', x^{\alpha}y^{\beta+e}z^{\gamma-\alpha e}) \leq v_n + |\alpha|(|\alpha|+1)/2 + |\alpha|(n-|\alpha|+1)$$
$$= v_n + (n+3/2)^2/2 - (\alpha - n - 3/2)^2/2 \leq v_n + (n+3/2)^2/2 = v_{n+1}.$$

In particular, if $a(m) < \infty$, we get $a(m) \leq v_n$. □

Proposition 12.4. *The group presentation*

$$G = \langle x, y |\ [x,y] = 1\rangle$$

has Dehn function $\delta(n) \geq (n-3)^2/4$.

Proof. It suffices to show that $k = a(x^n y^n x^{-n} y^{-n})$, which is finite, is at least equal to n^2. Indeed, write in F_S

$$x^n y^n x^{-n} y^{-n} = \prod_{i=1}^{k} c_i [x,y]^{\varepsilon_i} c_i^{-1},$$

where $\varepsilon_i \in \{-1, 0, 1\}$. Take the image inside the Heisenberg H group of Proposition 12.3. This gives the inequality in H

$$x^n y^n x^{-n} y^{-n} = [x,y]^{\sigma},$$

where $\sigma = \sum_{i=1}^{k} \varepsilon_i$. On the other hand, in H we have $x^n y^n x^{-n} y^{-n} = [x,y]^{n^2}$, and $[x,y]$ is torsion-free in H. Therefore we get $k \geq n^2$.

So we get $\delta(4n) \geq n^2$ for all n, which implies the desired inequality. □

Proposition 12.5. *The group presentation*

$$H = \langle x, y, z |\ [x,[x,y]] = [y,[x,y]] = 1\rangle$$

has Dehn function $\delta(n) \geq ((n-11)/12)^3$.

Proof. We only skecth the proof which is, to a certain extent, similar to that of Proposition 12.4. It consists in showing that $\delta([[x^n, y^n], x^n]) \geq n^3$. To see this, write $[[x^n, y^n], x^n]$ as a product of $a = a([[x^n, y^n], x^n])$ conjugates of relators, and then take the image in the group

$$N = \langle x, y, z, t | [x,y] = z, [x,z] = t, [x,t] = [y,t] = [y,z] = 1\rangle$$

This group can be viewed as the semidirect product $\mathbf{Z} \ltimes (\mathbf{Z}[X]/X^3)$, where the left-hand $\mathbf{Z} = \langle x \rangle$ acts by multiplication by the invertible element $1 + X$, and y, z, t correspond to the elements $1, X, X^2$ of $\mathbf{Z}[X]/X^3$. Then a direct calculation shows that, in N, we have $[[x^n, y^n], x^n] = t^{n^3}$. Now the relators of H can be computed in N: $[x,[x,y]] = t$ and $[y,[x,y]] = 1$; in particular they are central in N. In particular, the product of a conjugates of relators is actually a genuine product of a elements among 1 and $t^{\pm 1}$; as this product in N is equal to t^{n^3} and t is torsion-free in N (as follows from this very representation as $\mathbf{Z} \ltimes \mathbf{Z}^3$), we get that $a \geq n^3$). □

Exercise 18. Fill in the details in the proof of Proposition 12.5. In particular, check that we have indeed an isomorphism from N to this $\mathbf{Z} \ltimes_{1+X} \mathbf{Z}[X]/X^3$; and check the equality $[[x^n, y^n], x^n] = t^{n^3}$ in the latter group.

Gersten, Holt and Riley proved (2003, GAFA) that every finitely generated group of nilpotency length c has Dehn function bounded by a polynomial of degree $\leq c+1$. This is likely to hold more generally for all simply connected nilpotent Lie groups. This is not optimal: the higher Heisenberg groups, of dimension $2n+1 \geq 5$, have quadratic Dehn function (Allcock, 1998, previously stated by Thurston and sketched by Gromov).

Gromov (1987) proved that a f.g. group has linear Dehn function if and only if it is word hyperbolic. The converse was improved by Olshanskii (1991, IJAC), who proved that a f.g. group with subquadratic Dehn function is actually word hyperbolic. All these results are likely to hold for locally compact groups, but unfortunately, so far, most people in geometric group theory seem reluctant to go into general locally compact groups.