On Kazhdan's Property T
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Yves Cornulier

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1 Preliminaries

1.1 Positive definite and conditionally negative definite kernels

An $\mathbf{R}$-valued (or $\mathbf{C}$-valued) *kernel*[1] on a set X is a function $\kappa : X \times X \to \mathbf{R}$ (or $X \times X \to \mathbf{C}$). We can think of it as a matrix, whose rows and columns are indexed by X. In particular, the kernel is *symmetric* if $\kappa(j,i) = \kappa(i,j)$ for all i,j, and is *hermitian* if $\kappa(j,i) = \overline{\kappa(i,j)}$ for all i,j, where $z \mapsto \bar{z}$ denotes complex conjugation.

Denote by $\mathbf{R}^{(X)}$ the real vector space with basis X. It can be interpreted as the space of finitely supported functions $X \to \mathbf{R}$ (i.e., functions $f : X \to \mathbf{R}$ such that $\{x : f(x) \neq 0\}$ is finite, which admits, as a basis, the family of Dirac functions $(\delta_x)_{x \in X}$, defined by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for all $y \notin x$. (**Exercise**: check it is indeed a basis.) Define similarly $\mathbf{C}^{(X)}$, complex vector space with basis X.

Each real kernel κ defines a bilinear form B_κ on $\mathbf{R}^{(X)}$ by

$$B_\kappa(f,g) = \sum_{x,y \in X} \kappa(x,y) f(x) g(y).$$

Exercise 1.1. Check that $\kappa \to B_\kappa$ is a linear isomorphism from the set of real kernels on X, to the set of bilinear forms on $\mathbf{R}^{(X)}$, and restricts to a linear

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[1] This is one of the analytic definitions of "kernel". It is unrelated to the (more recent) algebraic notion of kernel $\mathrm{Ker}(f)$ of a homomorphism or operator f.

isomorphism from the set of real symmetric kernels on X, to the set of symmetric bilinear forms on $\mathbf{R}^{(X)}$.

Similarly, every complex kernel κ defines a sesquilinear form on $\mathbf{C}^{(X)}$ by

$$B_\kappa(f,g) = \sum_{x,y\in X} \kappa(x,y) f(x)\overline{g(y)}.$$

We say that the symmetric (or hermitian) kernel κ is *positive definite*[2] if $B_\kappa(f,f) \geq 0$ for all $f \in \mathbf{R}^{(X)}$ (or all $f \in \mathbf{C}^{(X)}$).

For a real-valued kernel, can be positive definite as real-valued or complex-valued kernel. The following result shows that the two definitions match.

Proposition 1.2. *A real-valued kernel on the set X is positive definite as real kernel if and only if it is positive definite as complex-valued kernel.*

Proof. The "if" direction is trivial. So assume that κ is positive as real kernel. We have to show that $B_\kappa(f,f) \geq 0$ for all $f \in C^{(X)}$. Write $f = u + iv$, where $u, v \in \mathbf{R}^{(X)}$. Then

$$\begin{aligned} B_\kappa(f,f) =& B_\kappa(u+iv, u+iv) \\ =& B_\kappa(u,u) - iB_\kappa(u,v) + iB_\kappa(v,u) - i^2 B_\kappa(v,v) \\ =& B_\kappa(u,u) + B_\kappa(v,v) \geq 0. \end{aligned}$$

□

Endow the set of kernels with the pointwise convergence topology, so that $\kappa_i \to \kappa$ if and only if $\kappa_i(x) \to \kappa(x)$ for all x.

Proposition 1.3. *In the space of all (real-valued or complex-valued) kernels, the set of positive definite kernels is closed. Besides, it is stable under addition and multiplication by non-negative real numbers.*

Proof. Suppose that $\kappa_i \to \kappa$. If $f \in \mathbf{C}^{(X)}$, then since the support $\mathrm{Supp}(f)$ is finite, the convergence of κ_i to κ is uniform on $\mathrm{Supp}(f) \times \mathrm{Supp}(f)$. Therefore

$$\lim_i \sum_{x,y\in X} \kappa_i(x,y) f(x) f(y) = \sum_{x,y\in X} \kappa(x,y) f(x) f(y);$$

since the left-hand term is non-negative for all i, so is the right-hand term.

The other verifications are left to the reader as an exercise. □

Exercise 1.4. Show that every kernel of the form $\kappa(x) = \ell(x)\overline{\ell(y)}$, where $\ell : X \to \mathbf{C}$ is any function, is positive definite.

[2]Since it is *not* required that $B_\kappa(f,f) > 0$ for $f \neq 0$, it would be more consistent to call such kernels *positive semidefinite*, but this is, unfortunately, the established terminology.

Exercise 1.5. Here $\#(X) = d$. Check that the real matrix

$$\begin{pmatrix} \lambda & -1 & \cdots & \cdots & -1 \\ -1 & \lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda & -1 \\ -1 & \cdots & \cdots & -1 & \lambda \end{pmatrix}$$

(λ on the diagonal, -1 everywhere else) is positive definite if and only if $\lambda \geq d-1$.

Now define $\mathbf{R}_0^{(X)}$ as the hyperplane of $\mathbf{R}^{(X)}$ consisting of functions f such that $\sum_{x \in X} f(x) = 0$. We say that the symmetric kernel κ is *conditionally negative definite* if $B_\kappa(f, f) \leq 0$ for all $f \in \mathbf{R}_0^{(X)}$.

Exercise 1.6. Check that the matrix $\begin{pmatrix} 0 & \alpha & \gamma \\ \alpha & 0 & \beta \\ \gamma & \beta & 0 \end{pmatrix}$ is conditionally negative definite if and only if $\alpha, \beta, \gamma \geq 0$ and

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha \leq 0.$$

Exercise 1.7. Show that in the space of all real-valued kernels, the set of conditionally negative definite kernels is closed, and stable under addition and multiplication by non-negative real numbers.

Proposition 1.8. *If κ is a positive definite and real-valued kernel, then $\nu(x, y) = (\kappa(x, x) + \kappa(y, y))/2 - \kappa(x, y)$ is a conditionally negative definite kernel.*

Proof. If $f \in \mathbf{R}_0^{(X)}$ then

$$\sum_{x,y \in X} \nu(x, y) f(x) f(y) = \sum_{x,y \in X} (\kappa(x, x) + \kappa(y, y)) f(x) f(y)/2 - \sum_{x,y \in X} \kappa(x, y) f(x) f(y);$$

since

$$\sum_{x,y \in X} \kappa(x, x) f(x) f(y) = \sum_{x \in X} \kappa(x, x) f(x) \sum_{y \in X} f(x) = 0,$$

we deduce

$$\sum_{x,y \in X} \nu(x, y) f(x) f(y) = - \sum_{x,y \in X} \kappa(x, y) f(x) f(y) \leq 0. \quad \square$$

1.2 GNS constructions for kernels

We present here the so called GNS-construction (GNS stands for Gelfand-Naimark-Segal).

Proposition 1.9. *Let X be a set. Let $\mathcal{H}$ be a real (resp. complex) Hilbert space, with scalar product denoted by $\langle \cdot, \cdot \rangle$. Consider a function $u : X \to \mathcal{H}$. Then the kernel κ_u on X defined by $\kappa_u(x,y) = \langle u(x), u(y) \rangle$ is positive definite.*

Proof. We can suppose that the Hilbert space is complex, since the real case is then a particular case (by embedding a real Hilbert space into its *complexification*, details are left as an exercise). Write $B_u = B_{\kappa_u}$. Clearly B_u is hermitian. If $f \in \mathbf{C}^{(X)}$ then

$$\begin{aligned} B_u(f,f) &= \sum_{x \in X} \sum_{y \in X} \langle u(x), u(y) \rangle f(x) \overline{f(y)} \\ &= \left\langle \sum_{x \in X} u(x) f(x), \sum_{y \in X} u(y) f(y) \right\rangle \geq 0 \end{aligned}$$

□

The kernel κ_u is often called the *Gram* (or *Gramian*) kernel of the family of vectors $(u(x))_{x \in X}$. In applications, X is often finite and this is called *Gram matrix*.

Proposition 1.10. *Let X be a set. Let $\mathcal{H}$ be a real Hilbert space, with scalar product denoted by $\langle \cdot, \cdot \rangle$. Consider a function $u : X \to \mathcal{H}$. Then the kernel ν_u on X defined by $\nu_u(x,y) = \|u(x) - u(y)\|^2$ is conditionally negative definite.*

Proof. Write $B_u = B_{\nu_u}$. Clearly B_u is symmetric. If $f \in \mathbf{R}_0^{(X)}$ then

$$\begin{aligned} B_u(f,f) &= \sum_{x \in X} \sum_{y \in X} \langle u(x) - u(y), u(x) - u(y) \rangle f(x) f(y) \qquad (1) \\ &= \sum_{x \in X} \sum_{y \in X} \langle u(x), u(x) \rangle f(x) f(y) \\ &\quad - 2 \sum_{x \in X} \sum_{y \in X} \langle u(x), u(y) \rangle f(x) f(y) \\ &\quad + \sum_{x \in X} \sum_{y \in X} \langle u(y), u(y) \rangle f(x) f(y). \end{aligned}$$

□

Note that

$$\sum_{x \in X} \sum_{y \in X} \langle u(x), u(x) \rangle f(x) f(y) = \left(\sum_{x \in X} \langle u(x), u(x) \rangle f(x) \right) \left(\sum_{y \in X} f(y) \right),$$

which is equal to zero, because $\sum_{y\in X} f(y) = 0$ by definition of $\mathbf{R}_0^{(X)}$. So the first term in (1) vanishes, and similarly the third term in (1) vanishes. Therefore

$$\begin{aligned} B_u(f,f) &= -2\sum_{x\in X}\sum_{y\in X}\langle u(x), u(y)\rangle f(x)f(y) \\ &= -2\left\langle \sum_{x\in X} u(x)f(x), \sum_{y\in X} u(y)f(y)\right\rangle \le 0 \end{aligned}$$

Proposition 1.9 gives a lot of instances of positive definite kernels. The first GNS construction is the remarkable fact that these are actually the only ones.

Theorem 1.11. *Let X be a set, and let κ be a complex-valued positive definite kernel on X. Then there exists a complex Hilbert space $\mathcal{H}$ and a map $u : X \to \mathcal{H}$ such that $\kappa = \kappa_u$. Moreover, if $\kappa_u(X)$ generates a dense subspace in $\mathcal{H}$, then it satisfies the universal property that for every Hilbert space $\mathcal{H}'$ and $u' : X \to \mathcal{H}'$ such that $\kappa_{u'} = \kappa$, there exists a unique linear isometry $v : \mathcal{H} \to \mathcal{H}'$ such that $u' = v \circ u$.*

For real-valued positive definite kernels, the same construction holds (with real Hilbert spaces).

Proof. We only do the complex case, since the proof in the real case is essentially the same.

Start with κ as in the statement. First define on $\mathbf{C}^{(X)}$ a sesquilinear form B by

$$B(f,g) = \sum_{x\in X}\sum_{y\in X}\kappa(x,y)f(x)\overline{g(y)}.$$

Clearly, B is hermitian; in particular $B(f,f)$ is real for all $f \in \mathbf{C}^{(X)}$. It follows from the definition of positive definiteness that $B(f,f) \ge 0$ for all f.

We can then define a *completion* of $(\mathbf{C}^{(X)}, B)$ as follows: first consider the set C of Cauchy sequences in $\mathbf{C}^{(X)}$, namely those sequences (v_n) such that $\lim_{n,m\to+\infty} B(v_n - v_m, v_n - v_m) = 0$. Clearly, C is a complex vector subspace of the set of all sequences. On C, define a sesquilinear form B_0 by $B_0((v_n),(w_n)) = \lim_n B(v_n, w_n)$ (exercise: show that this limit indeed exists, by checking that $(B(v_n,w_n))_n$ is Cauchy). Passing to the limit, we see that B_0 is hermitian and $B_0((v_n),(v_n)) \ge 0$ for all $(v_n) \in C$. So the set C_0 of (v_n) such that $B_0((v_n),(v_n)) = 0$ is a complex subspace and $B_0((v_n),(w_n)) = 0$ for all $(w_n) \in C$ and $(v_n) \in C_0$. Define $C' = C/C_0$. Then B_0 factors through a bilinear hermitian form B' on C', which is a scalar product (i.e., $B'(v,v) > 0$ for all v). There is a natural map from $\mathbf{C}^{(X)}$ to C', mapping f to the class of the constant sequence (f). This map is an isometry from $(\mathbf{C}^{(X)}, B)$ to (C', B'), which has dense image (exercise: check it). To check that (C', B') is complete, it is enough to check that every Cauchy sequence in some dense subset, namely the image of $\mathbf{C}^{(X)}$, is convergent; this can be checked as an exercise as well.

Now if $u' : X \to \mathcal{H}'$ is as in the statement of the theorem, then u' extends to a unique $\mathbf{C}$-linear map $\mathbf{C}^{(X)} \to \mathcal{H}'$. This map is an isometry from $(\mathbf{C}^{(X)}, B)$ to $\mathcal{H}'$. By evaluation on Cauchy sequences, we see that it extends to C, and being an isometry, it vanishes on C_0 and thus factors through a linear isometry $C' \to \mathcal{H}'$. This shows the existence, the uniqueness is clear by linearity and density. □

Corollary 1.12. *If $u : X \to \mathcal{H}$, $u' : X \to \mathcal{H}'$ are maps whose image generate dense subspaces and $\kappa_u = \kappa_{u'}$, then there is a unique linear bijective isometry $v : \mathcal{H} \to \mathcal{H}'$ such that $v \circ u = u'$.*

Proof. By the theorem, there exists a unique linear isometry v such that $v \circ u = u'$ and there exists a linear isometry v' such that $v' \circ u' = u$. Also by the theorem and uniqueness, we have $v' \circ v = \mathrm{Id}_{\mathcal{H}}$ and similarly $v \circ v' = \mathrm{Id}_{\mathcal{H}'}$. Thus v is bijective. □

An application of the GNS-construction is a nice proof of the following

Proposition 1.13. *The product of two positive definite kernels is positive definite.*

Proof. Let κ_1, κ_2 be positive definite and let us show that the kernel κ defined by $\kappa(x, y) = \kappa_1(x, y)\kappa_2(x, y)$ is positive definite. By the GNS construction, there are Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and maps $u_i : X \to \mathcal{H}_i$ such that $\kappa_i(x, y) = \langle u_i(x), u_i(y)\rangle$ for all $x, y \in X$ and $i = 1, 2$. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (if $\mathcal{H}_i = \ell^2(Y_i)$ where Y_i is a discrete set endowed with the counting measure, $\mathcal{H} = \ell^2(Y_1 \times Y_2)$). Define $u(x) = u_1(x) \otimes u_2(x)$. Then $\langle u(x), u(y)\rangle = \kappa(x, y)$ for all x. □

Exercise. Find a direct proof of Proposition 1.13 not relying on the GNS construction. Hint: reduce to the case when X is finite so as to interpret κ_2 as a matrix, and write $\kappa_2 = MM^*$ (matrix product, M denoting the conjugate of the transpose of M).

Corollary 1.14. *If κ is positive definite, then so is e^κ.*

Proof. Since the set of positive kernels is stable by multiplication by non-negative scalars and by taking products (and thus positive powers), for every n, the kernel $\sum_{k=0}^n \kappa^n/n!$ is positive definite. Passing to the pointwise limit, we deduce that e^κ is positive definite. □

There is also a GNS-construction for conditionally negative definite kernels.

Theorem 1.15. *Let ν be a conditionally definite kernel on the set X, such that $\nu(x, x) = 0$ for all $x \in X$. Then there exists a real Hilbert space $\mathcal{H}$ and a map $u : X \to \mathcal{H}$ such that $\nu(x, y) = \|u(x) - u(y)\|^2$ for all $x, y \in X$.*

If moreover the affine subspace generated by $u(X)$ is dense in $\mathcal{H}$, and if $\mathcal{H}'$ is another Hilbert space and $u' : X \to \mathcal{H}'$ satisfies $\nu(x, y) = \|u'(x) - u(y)'\|^2$ for all $x, y \in X$, then there is a unique affine isometry $v : \mathcal{H} \to \mathcal{H}'$ such that $u' = v \circ u$.

Lemma 1.16. *Let X be a set and $x_0 \in X$. Let ν be a conditionally definite kernel on the set X, such that $\nu(x,x) = 0$ for all $x \in X$. Define*

$$\kappa(x,y) = \frac{1}{2}\left(\nu(x,x_0) + \nu(y,x_0) - \nu(x,y)\right).$$

Then κ is positive definite.

Proof. We have to check that for all $f \in \mathbf{R}^{(X)}$ we have $B_\kappa(f,f) \geq 0$. First observe that $B_\kappa(\delta_{x_0}, f) = 0$ for all f, where δ_{x_0} is the Dirac function at x_0. Indeed,

$$\begin{aligned}
2B_\kappa(\delta_{x_0}, f) = & \sum_{x,y\in X} \nu(x,x_0)\delta_{x_0}(x)f(y) + \sum_{x,y\in X} \nu(y,x_0)\delta_{x_0}(x)f(y) \\
& - \sum_{x,y\in X} \nu(x,y)\delta_{x_0}(x)f(y) \\
= & 0 + \sum_y \nu(y,x_0)f(y) - \sum_y \nu(x_0,y)f(y) = 0.
\end{aligned}$$

Write $f = f_0 + c\delta_{x_0}$, where $f_0 \in \mathbf{R}_0^{(X)}$, δ_{x_0} is the Dirac function at x_0 and $c \in \mathbf{R}$. Then

$$\begin{aligned}
2B_\kappa(f,f) = & 2B_\kappa(f_0,f_0) \\
= & 2\sum_{x,y\in X} \kappa(x,y)f_0(x)f_0(y) \\
= & \sum_{x,y\in X} \nu(x,x_0)f_0(x)f_0(y) + \sum_{x,y\in X} \nu(y,x_0)f_0(x)f_0(y) \\
& - \sum_{x,y\in X} \nu(x,y)f_0(x)f_0(y) \\
= & 0 + 0 - \sum_{x,y\in X} \nu(x,y)f_0(x)f_0(y) \geq 0
\end{aligned}$$

because ν is conditionally negative definite. $\square$

Proof of Theorem 1.15. If $X = \emptyset$ there is nothing to prove, so fix $x_0 \in X$. Define

$$\kappa(x,y) = \frac{1}{2}\left(\nu(x,x_0) + \nu(y,x_0) - \nu(x,y)\right).$$

By Lemma 1.16, κ is positive definite. By the GNS construction, there exists a real Hilbert space $\mathcal{H}$ and a map $u : X \to \mathcal{H}$ such that $\kappa(x,y) = \langle u(x), u(y)\rangle$ for all x,y and $u(X)$ generates a dense subspace of $\mathcal{H}$. Note that since $\kappa(x_0,x_0) = 0$, we have $u(x_0) = 0$. It follows that the affine subspace generated by $u(X)$ is also dense. Finally, we have

$$\begin{aligned}
\|u(x)-u(y)\|^2 &= \langle u(x)-u(y), u(x)-u(y)\rangle \\
&= \langle u(x), u(x)\rangle + \langle u(y), u(y)\rangle - 2\langle u(x), u(y)\rangle \\
&= \kappa(x,x) + \kappa(y,y) - 2\kappa(x,y) \\
&= 2\nu(x,x_0)/2 + 2\nu(y,x_0)/2 - 2(\nu(x,x_0)+\nu(y,x_0)-\nu(x,y))/2 \\
&= \nu(x,y).
\end{aligned}$$

Given u' as in the statement of the theorem, define $u''(x) = u'(x) - u(x_0)$. The "moreover" statement in the GNS construction (Theorem 1.11) implies that there is a linear isometry $q : \mathcal{H} \to \mathcal{H}'$ such that $u'' = q \circ u$. If $v = q + u'(x_0)$, then v is an affine isometry and $u' = v \circ u$. The uniqueness is clear. ☐

Exercise. Using the GNS construction, show that the matrix $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 4 & 4 \\ 1 & 4 & 0 & 4 \\ 1 & 4 & 4 & 0 \end{pmatrix}$ is not conditionally negative definite.

Theorem 1.17 (Schoenberg, 1938)**.** *Let ν be a real-valued symmetric kernel such that $\nu(x,x) = 0$ for all x. Then ν is conditionally negative definite if and only if $e^{-t\nu}$ is positive definite for all $t \geq 0$.*

Proof. Suppose that $e^{-t\nu}$ is positive definite for all $t \geq 0$. Since its value on the diagonal is 1, by Proposition 1.8 we deduce that $1 - e^{-t\nu}$ is conditionally negative definite. So for $t > 0$, $(1-e^{-t\nu})/t$ is conditionally negative definite as well. Since for t tending to zero this tends pointwise to ν, we deduce that ν is conditionally negative definite.

Conversely assume that ν is conditionally negative definite. Observe that the constant kernel equal to 1 is positive definite (this is for instance a particular case of Exercise 1.4). The case of $t > 0$ boils down to $t = 1$ (replacing ν by $t\nu$). So let us prove that $e^{-\nu}$ is positive definite. We can suppose that $X \neq \emptyset$; let us fix $x_0 \in X$. Define

$$\kappa(x,y) = (\nu(x,x_0) + \nu(y,x_0) - \nu(x,y)).$$

By Lemma 1.16, κ is positive definite and thus e^{κ} is positive definite by Corollary 1.14. We have

$$e^{-\nu(x,y)} = e^{\kappa(x,y)} e^{-\nu(x,x_0)} e^{-\nu(y,x_0)}. \tag{2}$$

The kernel $(x,y) \mapsto e^{-\nu(x,x_0)}e^{-\nu(y,x_0)}$ is also positive definite, by Exercise 1.4. Since a product of positive definite kernels is positive definite (Proposition 1.13), we deduce from (2) that κ is positive definite. ☐

1.3 Functions on groups

Given a complex or real-valued function φ on a group G, we can associate the kernel

$$\kappa_\varphi(g,h) = \varphi(g^{-1}h).$$

It is left-invariant, in the sense that $\kappa_\varphi(gh,gk) = \kappa_\varphi(h,k)$ for all $g,h,k \in G$. Conversely, if κ is a left-invariant kernel on G, then $\kappa = \kappa_\varphi$, where $\varphi(g) = \kappa(1,g)$.

Note that κ_φ is symmetric if and only if $\varphi(g) = \varphi(g^{-1})$ for all g, and hermitian if and only if $\varphi(g) = \overline{\varphi(g^{-1})}$ for all g (we then say that φ is symmetric, resp. hermitian).

The function φ is defined to be *positive definite* if the kernel κ_φ is positive definite, and (if real-valued), is said to be *conditionally negative definite* if κ_φ is conditionally negative definite.

Lemma 1.18. *Let H be a subgroup of the group G and φ the indicator function of H. Then φ is positive definite.*

Proof. If $\kappa(g,h) = \varphi(g^{-1}h)$, let us show that κ is a positive definite kernel. For $g \in G$, define u_g as the Dirac function at $g \in G/H$ in $\ell^2(G/H)$. Then $\langle u_g, u_h\rangle = 1$ if $g^{-1}h \in H$ and 0 otherwise. Thus $\langle u_g, u_h\rangle = \kappa(g,h)$. So κ is positive definite by Proposition 1.9. $\square$

The GNS constructions can be nicely interpreted in terms of groups representations and actions. Let us begin by the easy part

Proposition 1.19. *Consider a unitary representation π of G on a complex Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$. (This means that π is a homomorphism from G to the unitary group of $\mathcal{H}$.) Then for every $\xi \in \mathcal{H}$, the complex-valued function*

$$\varphi_\xi(g) = \langle \xi, \pi(g)\xi\rangle$$

is positive definite.

Similarly, consider an orthogonal representation π of G on a real Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$. Then for every $\xi \in \mathcal{H}$, the real-valued function

$$\varphi_\xi(g) = \langle \xi, \pi(g)\xi\rangle$$

is positive definite.

Proof. If $\varphi = \varphi_\xi$, then

$$\kappa_\varphi(g,h) = \langle \xi, \pi(g^{-1}h)\xi\rangle = \langle \pi(g)\xi, \pi(h)\xi\rangle,$$

which is a positive definite kernel by Proposition 1.9. $\square$

The GNS construction then provides a converse. Given a representation of G in a Hilbert space $\mathcal{H}$ by continuous operators and $\xi \in \mathcal{H}$, the vector ξ is called a *cyclic vector* for the representation if $\pi(G)\xi$ generates a dense subspace of $\mathcal{H}$. Note that every $\xi \in \mathcal{H}$ is cyclic inside the closure of the subspace generated by $\pi(G)\xi$.

Theorem 1.20. *Let φ be a complex-valued, positive definite function on G. Then there exists a unitary representation π of G in a complex Hilbert space, a cyclic vector $\xi \in \mathcal{H}$ such that $\varphi = \varphi_\xi$. Moreover, π is essentially unique, in the sense that if π' is another unitary representation with a cyclic vector ξ' such that $\varphi = \varphi_{\xi'}$ then there exists a (unique) linear isometry $f : \mathcal{H} \to \mathcal{H}'$, mapping ξ to ξ', and intertwining the representations, in the sense that $f(\pi(g)v) = \pi'(g)f(v)$ for all $v \in \mathcal{H}$ and $g \in G$.*

If φ is real-valued, the same statement holds with unitary replaced by orthogonal, and complex Hilbert replaced by real Hilbert.

Proof. We only do the complex case, since the real case is a straightforward adaptation. By the GNS construction for kernels, there exists a Hilbert space $\mathcal{H}$ and a map $u : G \to \mathcal{H}$ such that $\langle u(g), u(h)\rangle = \varphi(g^{-1}h)$ for all g, h, and $u(G)$ generates a dense subspace of $\mathcal{H}$. If $g, h \in G$, define $u_g(h) = u(gh)$. Then $\langle u_g(h), \rangle u_g(k)\rangle = \varphi(g^{-1}h)$. The "moreover" statement in Theorem 1.11 implies that there exists a unique linear isometry $j_g : \mathcal{H} \to \mathcal{H}$ such that $u_g = j_g \circ u$. By uniqueness, $u_1 = u$. Also, we have, for $g, h, k \in G$

$$j_{gh}(u(k)) = u_{gh}(k) = u(ghk) = u_g(hk) = j_g(u_h(k)) = j_g \circ j_h(u(k)).$$

Since j_{gh} and $j_g \circ j_h$ are bounded operators and coincide on $u(G)$, they are equal. Thus $\pi(g) = j_g$ defines a unitary representation, and by construction, if $\xi = u(1)$, we have $\varphi = \varphi_\xi$.

Now if $\varphi = \varphi_\xi = \varphi_{\xi'}$, where ξ, ξ' are cyclic vectors for unitary representations of G into Hilbert spaces $\mathcal{H}$, $\mathcal{H}'$, define $u(g) = \pi(g)\xi$ and $u'(g) = \pi'(g)\xi'$. Then by the "moreover" statement in Theorem 1.11, there exists a unique bijective linear isometry $f : \mathcal{H} \to \mathcal{H}'$ such that $u' = f \circ u$, i.e.

$$\pi'(g)\xi' = f(\pi(g)\xi)$$

for all $g \in G$. So

$$f(\pi(g)(\pi(h)\xi)) = \pi'(gh)\xi' = \pi'(g)\pi'(h)\xi' = \pi'(g)f(\pi(h)\xi);$$

since the closure of the linear span of $\{\pi(h)\xi : h \in G\}$ is all of $\mathcal{H}$, we deduce that for all $v \in \mathcal{H}$ we have

$$f(\pi(g)(v) = \pi'(gh)\xi' = \pi'(g)\pi'(h)\xi' = \pi'(g)f(v);$$

i.e., $f \circ \pi(g) = \pi'(g) \circ f$. Thus f intertwines π and π'. $\square$

If X is a discrete set and $f : X \to \mathbf{C}$ a function, recall that f is *proper* if $f^{-1}(B)$ is finite for every bounded subset B of $\mathbf{C}$. Intuitively, this means that f tends to infinity at "infinity of X". Also, we say that f is C^0 on X if for every subset B of $\mathbf{C}$ whose closure does not contain 0, we have $f^{-1}(B)$ finite.

Exercise 1.21. Show that $f : X \to \mathbf{C}$ is proper if and only if $1/(1 + |f|)$ is C^0.

Theorem 1.22. *Given a subset X of the countable group G, we have equivalences:*

(1) The set of real-valued positive definite functions on G vanishing at F, endowed with the pointwise convergence topology, admits the constant function 1 as a limit point;

(2) There exists a conditionally negative definite function on G that is proper at F.

Proof. Assume (2) and let ψ be a conditionally negative definite function on G that is proper on X. By Schoenberg's Theorem (Theorem 1.17), $e^{-t\psi}$ is positive definite; clearly they vanish at 0 for $t > 0$ and for t tending to 0, they tend pointwise to 1.

Conversely assume (1). Let D_n be an increasing sequence of finite sets in G, whose union is G. By assumption, there exists a real-valued positive definite function φ_n on G, which is C^0 on X, such that $|\varphi_n - 1| \leq 2^{-n}$ on D_n; by a normalization we can suppose that $\varphi_n(1) = 1$. It follows from Proposition 1.8 that $\psi_n = 1 - \varphi_n$ is conditionally negative definite. By construction, we have $|\psi_n| \leq 2^{-n}$ on D_n and there exists $Y_n \subset X$ with $X \smallsetminus Y_n$ finite, such that $|\psi_n| \geq 1/2$ on Y_n. Also $\psi_n \geq 0$, because every conditionally negative kernel vanishing on the diagonal has non-negative values.

Define $\psi = \sum_n \psi_n$. This series is pointwise absolutely convergent: if $g \in G$, then for some k, we have $g \in D_n$ for all $n \geq k$ and thus $|\psi_n(g)| \leq 2^{-n}$ for all $n \geq k$. On $Y = \bigcap_{k=1}^{2n} Y_k$, we have $\psi \geq n$, and $X \smallsetminus Y$ is finite. Thus ψ is proper on X. $\square$

Theorem 1.23. *Given a subset L of the countable group G, we have equivalences:*

(1) For every net (φ_i) of real-valued positive definite functions on G converging to 1, the convergence is uniform on L;

(2) Every conditionally negative definite function on G is bounded on L.

Proof. Suppose (1). Let ψ be a conditionally negative definite function on G. By Schoenberg's Theorem (Theorem 1.17), $e^{-t\psi}$ is positive definite for $t > 0$ and for $t \to 0$ they tend pointwise to 1. By (1), the convergence is uniform on L and in particular there exists $t > 0$ such that we have $e^{-t\psi} \geq 1/2$ on L. So $\psi \leq \log(2)/t$ on L.

Conversely suppose (1) fails. Let D_n be an increasing sequence of finite sets in G, whose union is G. Let P_n be the set of real-valued positive definite functions φ_n on G such that $|\varphi_n - 1| \leq 3^{-n}$ on D_n and such that $\varphi(1) = 1$. Define $\lambda_n = \inf\{|\varphi(g)| : \varphi \in P_n, g \in L\}$. Clearly (λ_n) is non-decreasing. If $\lambda_n \to 1$ then (1) follows, so let $\lambda < 1$ be the limit of (λ_n).

Pick $\varphi_n \in P_n$ with $\inf_L |\varphi_n| \leq (1+\lambda)/2$. Define $\psi_n = 1 - \varphi_n$, so $\sup_L |\psi_n| \geq (1-\lambda)/2$. Define $\psi = \sum 2^n \psi_n$. Similarly as in the proof of Theorem 1.22, the series is absolutely convergent. By construction, we have $\sup_L |\psi| \geq 2^n(1-\lambda)/2$ for all n and thus ψ is unbounded on L. □

Definition 1.24. If G is a countable discrete group and L a subset, we say that (G, L) has *relative Property T* (or *relative Kazhdan Property T*) if it satisfies the equivalent conditions of Theorem 1.23. If $L = G$, we simply say that G has *Property T*.

Remark 1.25. Clearly, if L is a finite subset of G, then (G, L) has relative Property T. These are the trivial examples. There are no obvious other examples. We will see in the sequel that $(\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ has relative Property T (although $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ does not have Property T) and that $\mathrm{SL}_3(\mathbf{Z})$ has Property T.

Exercise 1.26. If $f : G \to H$ is a homomorphism and (G, L) has relative Property T then show that $(H, f(L))$ also has relative Property T.

Exercise 1.27. Show that $\mathbf{Z}$ does *not* have Property T in two different ways:

- using (1) of Theorem 1.23;
- using (2) of Theorem 1.23.

If G is a group and π a unitary representation of G in a Hilbert space. If $L \subset G$ and $\varepsilon \geq 0$, we call ξ a (L, ε)-invariant vector if $\|\pi(g)\xi - \xi\| \leq \varepsilon$ for all $g \in X$. We say that π *almost has invariant vectors* if for every $\varepsilon > 0$ and K finite subset of G, there exists a (K, ε)-invariant unit vector.

Exercise 1.28. If G is countable and π is a unitary representation, then show that π almost has invariant vectors if and only if there exists a sequence (ξ_n) of unit vectors such that for every $g \in G$, we have

$$\lim_{n\to\infty} \|\pi(g)\xi_n - \xi_n\| = 0.$$

Theorem 1.29. *For a discrete countable group G and $L \subset G$, relative Property T for (G, L) is also equivalent to each of:*

(3) For every unitary representation π of G with almost invariant vectors and $\varepsilon > 0$, there is a (L, ε)-invariant vector;

(4) For every affine isometric action α of G on a Hilbert space $\mathcal{H}$ and $v \in \mathcal{H}$, the set $\alpha(X)$ is bounded.

If $X = H$ is a subgroup, it is also equivalent to:

(3)′ For every unitary representation π of G with almost invariant vectors, there is an L-invariant vector;

(4)′ For every affine isometric action α of G on a Hilbert space $\mathcal{H}$ and $v \in \mathcal{H}$, L has a fixed point.

Proof. Let us first show that (1)⇒(3). Let (ξ_n) be a sequence of invariant vectors and $\varphi_n(g) = \langle \pi(g)\xi_n, \xi_n$ the corresponding positive definite function. Then φ_n converges pointwise to 1, so the convergence is uniform on X. Thus, for n large enough, ξ_n is (X, ε)-invariant.

Suppose (3) and let us show (1). Let (φ_n) be a sequence of positive definite functions converging pointwise to 1; we can suppose that $\varphi_n(1) = 1$ for all n. By the GNS construction, there exists a unitary representation π_n of G in a Hilbert space $\mathcal{H}_n$ and a unit vector ξ_n such that $\langle \pi(g)\xi_n, \xi_n\rangle = \varphi_n(g)$ for all $g \in G$. Consider the representation $\bigoplus \pi_n$ of G into the Hilbert space

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n = \{(x_n) : \ x_n \in \mathcal{H}_n, \sum_n \|x_n\|^2 < \infty.$$

Then the (ξ_n) are almost invariant vectors in $\mathcal{H}$. So we have

$$\lim_{n\to\infty} \sup_{g\in L} \|\pi(g)\xi_n - \xi_n\| = 0.$$

We have

$$\begin{aligned} |1 - \varphi_n(g)| =& |\langle \xi_n, \xi_n\rangle - \langle \pi_n(g)\xi_n, \xi_n\rangle| \\ =& |\langle \xi_n - \pi_n(g)\xi_n, \xi_n\rangle| \\ \leq& \|\xi_n - \pi_n(g)\xi_n\|, \end{aligned}$$

thus

$$\lim_{n\to\infty} \sup_{g\in L} |1 - \varphi_n(g)| = 0,$$

which means that the convergence of φ_n to 1 is uniform on L.

For the equivalences with (3)′ and (4)′] (which we do not use here), we refer to Chapter 2 in [BHV]. It makes use of the "center lemma": any non-empty bounded subset of a Hilbert space is contained in a unique ball of minimal radius. □

Lemma 1.30. *Suppose that G is a discrete countable group, $L \subset G$ and (G, L) has relative Property T. Then L is contained in a finitely generated subgroup of G. In particular, if G has Property T then it is finitely generated.*

Proof. Write $G = \{g_1, g_2, \dots, \}$ and let H_n be the subgroup generated by $\{g_1, \dots, g_n\}$, it is finitely generated. Let φ_n be the indicator function of H_n. Then φ_n is positive definite by Lemma 1.18 and φ_n tends to 1 pointwise. So the convergence is uniform on X. So there exists n such that $\varphi_n \geq 1/2$ on X. So $\varphi_n = 1$ on X; this means that $X \subset H_n$. $\square$

1.4 Representations of abelian groups

Let V be a discrete abelian group. Its Pontryagin dual $\hat{V}$ is by definition the group of homomorphisms $V \to \mathbf{R}/\mathbf{Z}$, endowed with the addition law $(f+g)(v) = f(v) + g(v)$, and with the topology of pointwise convergence.

Lemma 1.31. *The Pontryagin dual $\hat{V}$ of the discrete abelian group V, is a compact (Hausdorff) topological group.*

Proof. $\hat{V}$ is a subgroup of the group H of all functions $V \to \mathbf{R}/\mathbf{Z}$ and thus is a Hausdorff topological group. Since $H = (\mathbf{R}/\mathbf{Z})^V$ is compact by the Tychonoff Theorem, to show that $\hat{V}$ is compact it is enough to check that it is closed in H. To check that it is closed, for $v, w \in V$, define $H_{v,w} = \{f : f(v+w) = f(v) + f(w)\}$; it is closed by definition of the product topology and $\hat{V} = \bigcap_{v,w} H_{v,w}$. $\square$

Exercise 1.32. Let $(e_i)_{1 \leq i \leq k}$ be the basis of $\mathbf{Z}^k$. Show that the mapping

$$\begin{array}{ccc} \widehat{\mathbf{Z}^k} & \to & (\mathbf{R}/\mathbf{Z})^k \\ f & \mapsto & (f(e_i))_{1 \leq i \leq k} \end{array}$$

is an isomorphism of topological groups (thus the Pontryagin dual of $\mathbf{Z}^k$ is the k-torus).

Proposition 1.33. *Let V be a discrete abelian group. For every $\chi \in \hat{V}$, consider the unitary representation π_χ of V in $\mathbf{C}$ defined by*

$$\pi_\chi(v)z = e^{2i\pi\chi(v)}z.$$

Then π_χ is a 1-dimensional irreducible representation of V; the π_χ for $\chi \in \hat{V}$ are pairwise non-isomorphic, and every 1-dimensional unitary representation of V is isomorphic to some π_χ.

Proof. The proof is left as an exercise. $\square$

A less trivial result is that every irreducible representation of V has this form. In turn, this follows from a considerably more general result, describing all unitary representation in terms of irreducible representations.

Here we have to be cautious. It is *not true* that any unitary representation of a discrete abelian splits as a direct sum of irreducible subrepresentations (necessarily one-dimensional). For instance, the regular representation of $\mathbf{Z}$ on $\ell^2(\mathbf{Z})$ given by $n \cdot f(m) = f(n-m)$ admits no one-dimensional sub-representation (exercise). For this reason, we need to introduce the following formalism.

Let $\mathcal{H}$ be a Hilbert space and $\mathrm{Proj}(\mathcal{H})$ the set of orthogonal projections of $\mathcal{H}$. If X is a topological space and $\mathcal{B}(X)$ the set of Borel subsets of X, a *projection-valued probability measure* in X, into $\mathcal{H}$, is a mapping

$$E : B(X) \to \mathrm{Proj}(\mathcal{H})$$

satisfying

1. $E(\emptyset) = 0$, $E(X) = \mathrm{Id}_{\mathcal{H}}$;
2. $E(B \cap B') = E(B)E(B')$ for all $B, B' \in \mathcal{B}(X)$;
3. $E\left(\bigsqcup_n B_n\right) = \sum_n E(B_n)$ for every sequence (B_n) of disjoint Borel subsets in X.

If $f : X \to \mathbf{C}$ is a bounded Borel function, we can define, in a natural way, a continuous operator $\mathcal{H} \to \mathcal{H}$

$$\int_{x\in X} f(x)dE(x)$$

as follows: for $\xi, \eta \in \mathcal{H}$, the function $B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta\rangle$ is a complex-valued Borel measure on X; the sesquilinear form $(\xi, \eta) \mapsto \int_{x\in X} f(x)dE_{\xi,\eta}(x)$ is continuous and therefore has the form $(\xi, \eta) \mapsto \langle \Phi\xi, \eta\rangle$ for some unique continuous operator $\Phi : \mathcal{H} \to \mathcal{H}$; by definition $\int_{x\in X} f(x)dE(x) = \Phi$.

Theorem 1.34. *Let V be a discrete abelian group and π an unitary representation of V into a Hilbert space $\mathcal{H}$. Then there exists a projection-valued probability measure $\hat{V} \to \mathrm{Proj}(\mathcal{H})$ such that for every $v \in V$ we have*

$$\pi(v) = \int_{\chi\in\hat{V}} \overline{\chi(v)}dE(\chi).$$

See [BHV, Appendix D] for the proof.

Corollary 1.35. *If π is irreducible then it is 1-dimensional.*

Proof. Define the support of E as the set of $\chi \in \hat{V}$ such that every neighbourhood N of χ satisfies $E(\chi) \neq 0$. Clearly this is a closed subset, and is not empty if $\mathcal{H} \neq 0$. If reduced to a point $\{\chi\}$, then the definition of integral implies that π is the scalar multiplication by χ and thus $\mathcal{H}$ is 1-dimensional by irreducibility, so π is equivalent to π_χ.

If the support of χ contains at least two points, then there exists a partition $\hat{V} = X_1 \sqcup X_2$ such that both $E(X_1)$ and $E(X_2)$ are nonzero. It follows that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (orthogonal sum), where $E(X_i)$ is the orthogonal projection on $\mathcal{H}_i$, and $\pi(V)$ stabilizes both $\mathcal{H}_1$ and $\mathcal{H}_2$. This contradicts the irreducibility. $\square$

Corollary 1.36. *Let V be a discrete abelian group. If $\hat{V}$ has an isolated point then V is finite.*

Proof. Since $\hat{V}$ is a compact topological group, this would imply that $\hat{V}$ is finite (say with k elements), and by Theorem 1.34 this would imply that every unitary representation of V is an orthogonal sum of k scalar representations. In particular, V contains a one-dimensional subrepresentation, namely there exists $f \in \ell^2(V) \smallsetminus \{0\}$ such that $v \cdot f = \chi(v) f$ for all $v \in V$. So $\langle v \cdot f, f \rangle = e^{2i\pi\chi(v)}$ (here $\underline{\pi} = 3.14\dots$). But $\langle v \cdot f, f \rangle$ tends to 0 when v leaves compact subsets, by an easy verification, while if V is infinite it is not possible that $e^{2i\pi\chi(v)}$ tend to zero: indeed, this means that $\chi(v)$ tends to $1/2$ in $\mathbf{R}/\mathbf{Z}$; this is absurd becausd fixing v_0, this would imply that $\chi(v+v_0) = \chi(v)+\chi(v_0)$ tends to zero, implying that $\chi(v_0) = 0$ for all v_0. $\square$

Exercise 1.37. Prove directly Corollary 1.36, without the use of unitary representations.

2 $\mathrm{SL}_2(\mathbf{R})$

Let $\mathrm{SL}_2(\mathbf{R})$ act on the projective line $P = \mathbf{P}^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

Exercise 2.1. Show that the action of SL_2 on P does not preserve any finite Borel measure.

Show that the diagonal action of SL_2 on $P \times P$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x,y) = \left(\frac{ax+b}{cx+d}, \frac{ay+b}{cy+d} \right)$$

preserves, in restriction to the complement of the diagonal $W = (P \times P) \smallsetminus \mathrm{Diag}(P)$, the Borel measure μ with density given by $1/(x-y)^2$.
Hint. If Ω_1, Ω_2 are open subsets in $\mathbf{R}^k$ and $\Phi : \Omega_1 \to \Omega_2$ is a diffeomorphism, and f is the density, with respect to the Lebesgue measure, of a measure μ_f, then $\Phi_* \mu_f$ also has a density g, given by

$$g(x) = \det(d\Phi_{\Phi^{-1}(x)})^{-1} f(\Phi^{-1}(x))$$

Show that μ takes finite values on compact subsets of W. *Hint.* Check that for every compact subset K of W there exists $\varepsilon > 0$ such that K is contained in $C_\varepsilon = \{(x,y) \in W : \ |x-y| \geq \varepsilon, \min(x,y) \leq 1/\varepsilon$ (draw a picture of this subset and integrate g on it).

Define the *hyperbolic plane* $\mathbf{H}^2$ as the open upper-half space $\{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ and the *compactified hyperbolic plane* $\overline{\mathbf{H}^2}$ as the one-point compactification of the closed upper-half-space $\{z \in \mathbf{C} : \mathrm{Im}(z) \geq 0\}$, and $P = \mathbf{R} \cup \{\infty\}$ is interpreted as the boundary of $\mathbf{H}^2$. The group $\mathrm{SL}_2(\mathbf{R})$ acts on $\overline{\mathbf{H}^2}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

preserving $\mathbf{H}^2$ and its boundary.

Lemma 2.2. *The action of* $\mathrm{SL}_2(\mathbf{R})$ *on* $\mathbf{H}^2$ *is proper, in the sense that the function*

$$\begin{array}{rcl} \mathrm{SL}_2(\mathbf{R}) \times \mathbf{H}^2 & \to & \mathbf{H}^2 \times \mathbf{H}^2 \\ (g, z) & \mapsto & (z, gz) \end{array}$$

is proper (the inverse image of any compact subset is compact).

Proof. This amounts to proving that if (g_n, z_n) is a sequence in $\mathrm{SL}_2(\mathbf{R}) \times \mathbf{H}^2$ such that both (z_n) and $(g_n z_n)$ are bounded, then (g_n) is bounded.

First consider the group T of upper triangular matrices; such a matrix acts as

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot z = a(az+b).$$

A simple verification shows that the orbital map

$$\begin{array}{rcl} w : \mathbf{R}_{>0} \times \mathbf{R} \simeq T & \to & \mathbf{H}^2 \\ (a,b) \simeq \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} & \mapsto & \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot i = a(ai+b) \end{array}$$

is a homeomorphism, with inverse given by $z \mapsto (\sqrt{\mathrm{Im}(z)}, \mathrm{Re}(z)/\sqrt{\mathrm{Im}(z)})$.

Define $h_n = w(z_n)$ and $k_n = w(g_n z_n)$, so that (h_n) and (k_n) are bounded. We have

$$i = k_n^{-1} g_n z_n = k_n^{-1} g_n h_n i,$$

so $s_n = k_n^{-1} g_n h_n$ belongs to the stabilizer of i, which is equal (exercise) to the compact group $\mathrm{SO}_2(\mathbf{R})$. Thus $g_n = k_n s_n h_n^{-1}$ is a product of three bounded elements and thus is bounded. □

If $\sigma = (x, y) \in W$, define D_σ as the unique (oriented) half circle in $\mathbf{H}^2$ joining x to y if $x, y \neq \infty$, and call it hyperbolic line joining x and y. If we consider, by extension, vertical half-lines to be half-circles, this definition extends to arbitrary $\sigma \in W$. Define H_σ to be the closed half-subspace of the hyperbolic plane $\mathbf{H}^2$, located on the *right* of the oriented line (xy).

Denote, for $z \in \mathbf{H}^2$,

$$W_z = \{\sigma \in W : z \in H_\sigma\};$$

this is a closed subset of W.

Lemma 2.3. *For all $z, z' \in \mathbf{H}^2$, the symmetric difference $W_z \Delta W_{z'}$ has compact closure. In particular, $W_z \Delta W_{z'} < \infty$, where μ is the measure given in Lemma 2.1.*

Proof. If C_ε is given in Exercise 2.1 and I_ε is its complement, then every D_σ with $\sigma \in I_\varepsilon$ lies either on the band $\{w : \mathrm{Im}(w) \leq \varepsilon/2\}$ or outside the disc of radius ε^{-1}. In particular, if $\varepsilon/2 < \min(\mathrm{Im}(z), \mathrm{Im}(z'), 1/|z|, 1/|z'|)$, then H_σ does not separate z and z' and thus $W_z \Delta W_{z'} \subset C_\varepsilon$. □

Thanks to the lemma, we have a well-defined map

$$\begin{array}{rcl} F : \mathbf{H}^2 \times \mathbf{H}^2 & \to & L^2(W, \mu) \\ (z, z') & \mapsto & 1_{W_z} - 1_{W_{z'}}. \end{array}$$

Proposition 2.4. *The kernel*

$$\begin{array}{rcl} \kappa : \mathbf{H}^2 \times \mathbf{H}^2 & \to & \mathbf{R} \\ (z, z') & \mapsto & \mu(W_z \Delta W_{z'}) \end{array}$$

is conditionally negative definite.

Proof. We have $\kappa(z, z') = \|F(z, z')\|^2$. Write W as an increasing union of a sequence of compact subsets $W[n]$. Then

$$F(z, z') = \lim_n 1_{W_z \cap W[n]} - 1_{W_{z'} \cap W[n]},$$

so

$$\kappa(z, z') = \|F(z, z')\|^2 = \lim_n \|1_{W_z \cap W[n]} - 1_{W_{z'} \cap W[n]}\|^2;$$

by Proposition 1.10, $(z, z') \mapsto \|1_{W_z \cap W[n]} - 1_{W_{z'} \cap W[n]}\|^2$ is conditionally negative definite. Passing to the limit, we deduce that κ is conditionally negative definite. □

Lemma 2.5. *κ is unbounded on $\mathbf{H}^2 \times \mathbf{H}^2$*

Proof. If $\lambda \in \mathbf{R}$, let us show that $\kappa(\lambda i, \lambda^{-1} i)$ tends to infinity when λ tends to $+\infty$. It is easy to check that it is a non-decreasing function of $\lambda \geq 1$, because the set $W_{\lambda i} \Delta W_{\lambda^{-1} i}$ itself grows with λ. Its union, when λ ranges over $\mathbf{R}$, is the set of pairs $(x, y) \in \mathbf{R}^2$ such that $x < 0 < y$ or $y < 0 < x$. The integral of $1/(x-y)^2$ over this domain is easily checked to be infinite (check it as an exercise!). □

Corollary 2.6 (Faraut–Harzallah, 1974)**.** *There exists on $\mathrm{SL}_2(\mathbf{R})$ a continuous proper conditionally negative definite function ψ (proper in the topological sense: the $\psi^{-1}(K)$ is compact for every compact subset K (it is then said that $\mathrm{SL}_2(\mathbf{R})$ has the Haagerup Property as a topological group).*

Corollary 2.7. *Discrete subgroups of $\mathrm{SL}_2(\mathbf{R})$ have the Haagerup Property.* □

Proof of Corollary 2.6. If $z \in \mathbf{H}^2$, the function $\psi(g) = \kappa(z, gz)$ is conditionally negative definite and unbounded. To show it is proper, pick any sequence (g_n) tending to infinity in $\mathrm{SL}_2(\mathbf{R})$ and let us show that $\psi(g_n)$ tends to infinity. We use the fact (exercise!) that for all $x, y \in \mathbf{H}^2$ there exists $\lambda \geq 1$ and $g \in \mathrm{SL}_2(\mathbf{R})$ such that $gx = \lambda^{-1}i$ and $gy = \lambda i$. Apply this to $(i, g_n i)$ to find a sequence h_n in $\mathrm{SL}_2(\mathbf{R})$ such that $h_n i = \lambda_n^{-1} i$ and $h_n g_n i = \lambda_n i$. By the properness of the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathbf{H}^2$, the sequence (λ_n) tends to infinity, since otherwise, after possible extraction, both (h_n) and $(h_n g_n)$ would be bounded and thus (g_n) would be bounded.

It follows that

$$\psi(g_n) = \kappa(i, g_n i) = \kappa(h_n i, h_n g_n i) = \kappa(\lambda_n^{-1} i, \lambda_n i)$$

tends to infinity. $\square$

Remark 2.8. It is possible to compute κ. For instance, let us compute $\kappa(\alpha i, \beta i)$ for $0 < \alpha < \beta$. If $x, y \in \mathbf{R} \smallsetminus \{0, \infty\}$, we can see that $D_{(xy)}$ separates x and y if and only if y is in the segment joining $-\alpha^2/x$ and $-\beta^2/x$ (thus $[-\alpha^2/x, -\beta^2/x]$ if $x < 0$ and $[-\beta^2/x, -\alpha^2/x]$. So

$$\begin{aligned}
\kappa(\alpha i, \beta i) = &\int_{x=-\infty}^{+\infty} \mathrm{sign(x)} \int_{y=-\beta^2/x}^{-\alpha^2/x} dxdy/(y-x)^2 \\
=&2\int_{x=0}^{+\infty} \int_{y=-\beta^2/x}^{-\alpha^2/x} \frac{dxdy}{(y-x)^2} \\
=&2\int_{x=0}^{+\infty} dx \left[-\frac{1}{y-x}\right]_{y=-\beta^2/x}^{-\alpha^2/x} \\
=&2\int_{x=0}^{+\infty} \left(\frac{x}{x^2+\alpha^2} - \frac{x}{x^2+\beta^2}\right) dx.
\end{aligned}$$

Now use that for $\lambda, u > 0$

$$\int_0^u \frac{tdt}{t^2+\lambda^2} = \frac{1}{2}\log(u^2/\lambda^2 + 1),$$

so

$$2\int_{x=0}^{u} \left(\frac{x}{x^2+\alpha^2} - \frac{x}{x^2+\beta^2}\right) dx = \log(u^2/\alpha^2+1) - \log(u^2/\beta^2+1)$$

now

$$\begin{aligned}
\log(u^2/\lambda^2+1) =&2\log(u) - 2\log(\lambda) + \log(1+\lambda^2/u^2) \\
=&2\log(u) - 2\log(\lambda) + o(1) \qquad (u \to +\infty),
\end{aligned}$$

thus

$$2\int_{x=0}^{u} \left(\frac{x}{x^2+\alpha^2} - \frac{x}{x^2+\beta^2}\right) dx = \log(\beta) - \log(\alpha) + o(1);$$

it follows that

$$\kappa(\alpha i, \beta i) = \log(\beta/\alpha).$$

Thus turns out to coincide with the so-called *hyperbolic distance* on $\mathbf{H}^2$. Using that every pair in $\mathbf{H}^2$ can be mapped by an element of $\mathrm{SL}_2(\mathbf{R})$ into the imaginary line $i\mathbf{R}_+$, it can be deduced that κ is actually *equal* to the hyperbolic distance on $\mathbf{H}^2 \times \mathbf{H}^2$. This is known as the *Crofton formula*.

Remark 2.9. The distance in a tree is conditionally negative definite. The proof is similar to the case of $\mathbf{H}^2$ and even simpler: the space W is now defined as the set of oriented edges, and for each oriented edge (x, y), the set $H_{(x,y)}$ is defined as the set of vertices that can be joined by a segment to y without passing through x.

The same approach as the one for $\mathbf{H}^2$ also extends to higher-dimensional real hyperbolic spaces $\mathbf{H}^n_{\mathbf{R}}$. However, the method of Faraut and Harzallah is different and also carries over the complex hyperbolic space $\mathbf{H}^n_{\mathbf{C}}$.

In contrast, it was proved by Kostant that in the quaternionic hyperbolic plane, no unbounded continuous function of the distance is conditionally negative definite.

3 Relative Property T with a normal abelian subgroup

Lemma 3.1. *Consider a countable group of the form $G = V \rtimes \Gamma$, where V is abelian. Suppose that there is a neighborhood N of 0 in $\hat{V}$ such that the only Γ-invariant mean μ on $\hat{V}$ satisfying $\mu(N) = 1$ is the Dirac measure at $\{0\}$. Then $(V \rtimes \Gamma, V)$ has relative Property T.*

Proof. We have to show that for every unitary representation π of G with almost invariant vectors (ξ_n), we have

$$\lim_{n\to\infty} \sup_{v\in V} \|\pi(g)\xi_n - \xi_n\| = 0.$$

Since N is a neighborhood of 0, there exists α and $g_1, \dots, g_m \in V$ such that

$$\{\chi \in \hat{V} :\ \forall k = 1, \dots, m,\ |1 - \chi'(g_k)| < \alpha\} \subset N.$$

(Here, for $\chi \in \hat{V} = \mathrm{Hom}(V, \mathbf{R}/\mathbf{Z})$, we write $\chi'(g) = e^{2i\pi\chi(g)}$.

Let (K_n) be an increasing sequence of finite subsets of G whose union is G, and let (ε_n) be a sequence of positive real numbers tending to 0. Let $\xi_n \in \mathcal{H}$ be a (K_n, ε_n)-invariant unit vector in $\mathcal{H}$. We can suppose that each K_n contains all g_k. We assume by contradiction (extracting if necessary) that $\sup_{v\in V} \|\sigma(v)\xi_n - \xi_n\| > \eta$, for some $\eta > 0$ and all n.

Let E be the projection-valued probability measure associated to $\sigma = \pi|_V$, so that $\sigma(v) = \int_{\chi\in\hat{V}} \chi'(v)dE(\chi)$ for all $v \in V$. For $\xi \in \mathcal{H}$ of norm one, let μ_ξ be the probability measure on $\hat{V}$ defined by $\mu_\xi(B) = \langle E(B)\xi, \xi\rangle$. We have, for every k

$$\|\sigma(g_k)\xi_n - \xi_n\|^2 = \int_{\chi\in\hat{V}} |1-\chi'(g_k)|^2 d\mu_{\xi_n}(\chi) \leq \varepsilon_n^2.$$

Define $A_k = \{\chi \in \hat{V} : \ |1-\chi'(g_k)| < \alpha\}$ and B_k its complement in $\hat{V}$ (so $\bigcap_k A_k \subset N$.

$$\varepsilon_n^2 \geq \int_{\chi\in B_k} |1-\chi'(g)|^2 d\mu_{\xi_n}(\chi) \geq \alpha^2 \mu_{\xi_n}(B_k),$$

thus

$$\mu_{\xi_n}\left(\bigcap_{k=1}^n A_k\right) \geq 1 - m\varepsilon_n^2/\alpha^2,$$

so

$$\mu_{\xi_n}(N) \geq 1 - m\varepsilon_n^2/\alpha^2.$$

We have, for some $v \in V$, $\|\sigma(v)\xi_n - \xi_n\| \geq \eta$. So

$$\begin{aligned}\|\sigma(v)\xi_n - \xi_n\|^2 &= \int_{\chi\in\hat{V}} |1-\chi'(v)|^2 d\mu_{\xi_n}(\chi) \\ &= \int_{\chi\in\hat{V}\smallsetminus\{0\}} |1-\chi'(v)|^2 d\mu_{\xi_n}(\chi) \\ &\leq 4\mu_{\xi_n}(\hat{V}\smallsetminus\{0\})\end{aligned}$$

(because $|1-\chi'|^2 \leq 4$), so we deduce that $\mu_{\xi_n}(\hat{V}\smallsetminus\{0\}) \geq \eta^2/4$. Viewing each μ_{ξ_n} as a function $\mathcal{B}(\hat{V}) \to [0,1]$, the set of such functions, with pointwise convergence, is compact by Tychonoff's Theorem, so that the sequence (μ_{ξ_n}) has, by compactness, a limit point μ, which is a mean. In particular, μ is a mean on the Borel subsets of $\hat{V}$, $\mu(N) = 1$ and $\mu(N \smallsetminus \{0\}) \geq \eta^2/4 > 0$.

For every g we have

$$\lim_{n\to\infty} \int_{\chi\in\hat{V}} |1-\chi'(g)|^2 d\mu_{\xi_n} = \lim_{n\to\infty} \|\sigma(g)\xi_n - \xi_n\|^2 = 0.$$

Let us now check that μ is Γ-invariant. This follows if we check that for every Borel set B and $g \in \Gamma$, we have $\lim_{n\to\infty} \mu_{\xi_n}(B) - \mu_{\xi_n}(gB) = 0$. Indeed, for $g \in \Gamma$ and $v \in V$

$$\begin{aligned}\sigma(gvg^{-1}) &= \int_{\chi\in\hat{V}} \chi'(gvg^{-1})dE(\chi) \\ &= \int_{\chi\in\hat{V}} (g^{-1}\cdot\chi)'(v)dE(\chi) \\ &= \int_{\chi\in\hat{V}} \chi'(v)dE(g\cdot\chi)\end{aligned}$$

and

$$\begin{aligned}\sigma(gvg^{-1}) =&\sigma(g)\sigma(v)\sigma(g^{-1})\\ =&\int_{\chi\in\hat{V}}\chi'(v)\sigma(g)dE(\chi)\sigma(g^{-1})\end{aligned}$$

by uniqueness of the projection-valued probability measure (Theorem 1.34), we deduce that $E(gB)=\sigma(g)E(B)\sigma(g)^{-1}$.

So we have, for $g\in\Gamma$, and writing $\sigma(g)^{-1}\xi_n=\xi_n+q_n$

$$\begin{aligned}\mu_{\xi_n}(gB) =&\langle E(gB)\xi_n,\xi_n\rangle\\ =&\langle\sigma(g)E(B)\sigma(g)^{-1}\xi_n,\xi_n\rangle\\ =&\langle E(B)\sigma(g)^{-1}\xi_n,\sigma(g)^{-1}\xi_n\rangle\\ =&\langle E(B)(\xi_n+q_n),\xi_n+q_n\rangle\\ =&\langle E(B)(\xi_n),\xi_n\rangle+\langle E(B)\xi_n,q_n\rangle+\langle E(B)q_n,\sigma(g)^{-1}\xi_n\rangle;\end{aligned}$$

so

$$\begin{aligned}\mu_{\xi_n}(gB)-\mu_{\xi_n}(B) =&\langle E(B)\xi_n,q_n\rangle+\langle E(B)q_n,\sigma(g)^{-1}\xi_n\rangle\\ \leq&\|E(B)\xi_n\|\cdot\|q_n\|+\|E(B)q_n\|\|\sigma(g)^{-1}\xi_n\|\\ \leq&2\|q_n\|=2\|\sigma(g^{-1})\xi_n-\xi_n\|,\end{aligned}$$

which tends to zero, for g fixed, when n tends to ∞. Thus μ is Γ-invariant.

So if we define $\mu'(B)=\mu(B\smallsetminus\{0\})/\mu(\hat{V}\smallsetminus\{0\})$, then μ' is a Γ-invariant mean on the Borel subsets of $\hat{V}$, $\mu'(\{0\})=0$ and $\mu'(N)=1$. □

Theorem 3.2 (Kazhdan)**.** *$(\mathrm{SL}_2(\mathbf{Z})\ltimes\mathbf{Z}^2,\mathbf{Z}^2)$ has relative Property T.*

Proof. If $V=\mathbf{Z}^2$, we identify $\hat{V}$ to the 2-torus $\mathbf{R}^2/\mathbf{Z}^2$. If by contradiction relative Property T fails, by Lemma 3.1 there exists a Γ-invariant mean μ supported by $[-1/5,1/5]^2$. So this mean is invariant by the two generators $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix}$; since each of this generators, for its action on $\mathbf{R}^2$, maps $[-1/5,1/5]^2$ into $[-2/5,2/5]^2$, which is mapped injectively into $\mathbf{R}^2/\mathbf{Z}^2$, we deduce that μ, as a mean on $\mathbf{R}^2$, is also invariant by the action of Γ on $\mathbf{R}^2$. Pushing forward, we obtain a Γ-invariant mean on the projective line, but this is not possible. □

Consider $d\times d$ matrices over any unital ring. Let $E_{ij}(a)$ be the matrix with all entries 0 except the (i,j) entry, equal to a. For $i\neq j$, define $e_{ij}(a)=I+E_{ij}(a)$. It is invertible, its inverse being given by $e_{ij}(-a)$ These are called *elementary matrices* in $\mathrm{GL}_d(A)$.

Theorem 3.3 (Carter-Keller)**.** *Every matrix in $\mathrm{SL}_3(\mathbf{Z})$ is a product of at most 60 elementary matrices.*

The proof is algebraic (including some arithmetic) and complicated, we omit it, referring to Chapter 4 in [BHV].

Corollary 3.4. $\mathrm{SL}_3(\mathbf{Z})$ *has Property T.*

Proof. Consider the subgroup Λ of $\Gamma = \mathrm{SL}_3(\mathbf{Z})$ consisting of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \qquad (x,y) \in \mathbf{Z}^2,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z});$$

there is an isomorphism $f : \mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2 \to \Lambda$ such that $f(\mathbf{Z}^2) = e_{12}(\mathbf{Z})e_{13}(\mathbf{Z})$.

Let ψ be a conditionally negative definite function on Γ. So $\psi \circ f$ is conditionally negative definite on $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ and thus is bounded on $\mathbf{Z}^2$, by Theorem 3.2. It follows that ψ is bounded on $f(\mathbf{Z}^2)$ and thus is bounded on $e_{12}(\mathbf{Z})$. The same proof (permuting the entries) shows that ψ is bounded on $e_{ij}(\mathbf{Z})$ for all $i \neq j$. Now observe that since $\ell = \sqrt{\psi}$ is a sub-additive (or length) function: $\ell(xy) \le \ell(x) + \ell(y)$, because the square root of any conditionally negative definite kernel is a pseudo-distance. By Theorem 3.3, it follows that $\ell = \sqrt{(\psi)}$ is bounded on $\mathrm{SL}_3(\mathbf{Z})$ and thus ψ is bounded. $\square$

Exercise 3.5. Using Lemma 1.30, show that if R is unital ring but is not finitely generated, then $\mathrm{EL}_3(R)$ does not have Property T and $(\mathrm{GL}_2(R) \ltimes R^2, R^2)$ does not have relative Property T.

Remark 3.6. Lemma 3.1 is a variant of a result of Shalom (1999, Transactions AMS). An elaboration on Theorem 3.2, based on Lemma 3.1, shows that $(\mathrm{GL}_2(R) \ltimes R^2, R^2)$ has relative Property T for every finitely generated unital (associative) ring R; this was established by Shalom for R commutative (Publications IHES, 1999) and Kassabov later observed that the argument extends to R not commutative.

The original proof by Kazhdan of Property T for $\mathrm{SL}_3(\mathbf{Z})$ (1967) used its embedding as a discrete subgroup of $\mathrm{SL}_3(\mathbf{R})$ with finite covolume (the bounded generation for $\mathrm{SL}_3(\mathbf{R})$ being much easier than the bounded generation for $\mathrm{SL}_3(\mathbf{Z})$). This approach cannot carry over general rings.

For Theorem 3.3, the natural statement consists in considering the group $\mathrm{EL}_3(R)$ generated by elementary matrices. For R commutative (so that the determinant makes sense), it is contained in $\mathrm{SL}_3(R)$ and not equal in many examples (related to the algebraic K-theory group $K_1(R)$) although for polynomial rings $\mathbf{Z}[t_1, \dots, t_k]$ they are equal, by a difficult result of Suslin. On the other hand, it is not even known whether $\mathrm{EL}_3(\mathbf{Z}[t])$ is boundedly generated by elementary matrices.

However, for a unital finitely generated ring R, it was established later that $\mathrm{EL}_3(R)$ has Property T: Shalom and Vaserstein (2006) in the commutative case (using a weak notion of bounded generation) and Ershov and Jaikin-Zapirain

(2010) in the general case (by other methods, still relying on the relative Property T for $\mathrm{EL}_2(R) \ltimes R^2$.

References

[BHV] B. Bekka, P. de la Harpe, A. Valette. Kazhdan's Property (T). New math. monographs 11, Cambridge Univ. Press 2008.