INVARIANT PROBABILITIES ON PROJECTIVE SPACES

YVES DE CORNULIER

ABSTRACT. Let $\mathbf{K}$ be a local field. We classify the linear groups $G \subset GL(V)$ that preserve an probability on the Borel subsets of the projective space $P(V)$.

Warning: we know that the main theorem of this paper is not new, but we are not aware of the existence of a similar proof in the literature.

1. Introduction

If $F = P(V)$ is a projective space over the field K, we denote by $\mathrm{Aut}(F)$ the group of automorphisms of F (so that $\mathrm{Aut}(F) = PGL(V)$).

If $G \subset GL(V)$ is any subgroup, denote by G^{0Z} the intersection of G with the Zariski connected component of its Zariski closure. Then G^{0Z} is open in G (for the topology induced by the Zariski topology of $GL(V)$), and has finite index in G.

Here is the main theorem.

Theorem 1.1. *Let G be a subgroup of $GL(n, \mathbf{K})$, $\mathbf{K}$ a local field. Then the following conditions are equivalent.*

- *(i) G preserves a probability on the Borel subsets of $P^n(\mathbf{K})$.*
- *(ii) G preserves an invariant mean on the Borel subsets of $P^n(\mathbf{K})$.*
- *(ii') G preserves an invariant mean on the Borel subsets of $\mathbf{K}^n \backslash \{0\}$.*
- *(iii) There exists a G^{0Z}-stable subspace $F \subset P^n(\mathbf{K})$ such that the closure of the image of $G^{0Z} \to \mathrm{Aut}(F)$ is amenable.*
- *(iv) There exists a G^{0Z}-stable subspace $F \subset P^n(\mathbf{K})$ such that the closure of the image of $G^{0Z} \to \mathrm{Aut}(F)$ is compact.*

The implications (iv)⇒(iii) and (ii')⇒(ii) are obvious, and (iii)⇒(ii') uses standard properties of amenability.

The implication (ii)⇒(i) follows from

Lemma 1.2. *If G acts by homeomorphisms on a* compact *space X and preserves a mean on the Borel subsets of X, then it preserves a probability on the Borel subsets of X.*

Proof: We shall define a natural projection from the set of all means on Borel subsets of X onto the set of all Borel probabilities on X. The naturality of this projection implies that it maps invariant means to invariant probabilities.

To carry out the construction, we first only assume that X is locally compact. Let m be a mean on the Borel subsets of X. Let δ be the counting measure on the Borel subsets of X. By an easy classical result ([HR] (20.35) Theorem), m defines a unique linear form M on the space $L^\infty(\delta)$ of measurable bounded functions on X, satisfying $M(1_Y) = m(Y)$ for all Borel subsets $Y \subset X$. By restriction, M defines a positive linear continuous form on $C_c(X)$, the fonctions with compact support. By the Riesz representation theorem, since X is locally compact, there exist a unique (σ-additive) measure μ on the Borel subsets of X such that $\langle M, f\rangle = \int f \, d\mu$ for all $f \in C_c(X)$. Now we use that X is compact to say that $\mu(X) = 1$ (if X is not compact, this construction may map means to the zero measure). ■

Our purpose is to prove (i)⇒(iv). We shall see that the weaker implication (i)⇒(iii) is easier to prove when we deal with distal groups.

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We will also use some facts about distal groups with seem known to specialists, but were not found by the author in the literature. We thank Robinson Edward Raja, Yves Guivarc'h and Bertrand Rémy for their explanations on distal groups, Olivier Wittenberg for his explanations on extensions of norms to field extensions, and Yehuda Shalom for pointing me out lemma 1.2.

2. Preliminaries

2.1. The quasi-linear topology.

Let K be a field.

We define a topology $\mathscr{L}$ on $P^n(K)$ (or $K^{n+1}\backslash\{0\}$). The closed subsets of $\mathscr{L}$ are the finite unions of linear subspaces. It is immediate that this family is closed under finite unions and finite intersection. But, since these are Zariski subsets, any decreasing family in eventually stationnary, so that $\mathscr{L}$ must also be closed under infinite intersections.

If a group G acts linearly on a nonzero finite-dimensional K-vector space V, we say that the action is (linearly) primitive if there are no nontrivial $\mathscr{L}$-closed subsets invariant by G. Of course, this implies irreducibility, but the converse is false: take a basis (e_i) of V, and $F = \bigcup Ke_i$. Then the stabilizer of F in $GL(V)$ acts irreducibly, but not linearly primitively if $\dim(V) \geq 2$.

Nevertheless, there is almost a converse.

Lemma 2.1. *Let $G \subset GL(V)$. If G does not act (linearly) primitively on V, then G^{0Z} acts not irreducibly on V. The converse is true if K is infinite.*

Proof: Let G preserve nontrivial $F \in \mathscr{L}$. Let X be the set of all the maximal linear subspaces contained in F. Then X is finite, and G acts on X. By connectivity, G^{0Z} is contained in the kernel of this action. So G^{0Z} preserves any subspace $W \in X$, these are nontrivial since X is nontrivial.

For the converse, if G^{0Z} preserves a nontrivial subspace W, then we can take $F = \bigcup_{g\in G} gW$, and $F \in \mathscr{L}$ because G^{0Z} has finite index in G. If the ground field is infinite, $F \neq V$. ■

Note that for a finite field, $G^{0Z} = \{1\}$, hence never acts irreducibly if the dimension is ≥ 2.

We used implicitely, in the above arguments, several times the classical easy result, that we leave as an exercise: if K is infinite, V is a K-vector space and $V_1, \dots, V_n, W$ are linear subspaces such that $W \subset \bigcup_{1\leq i\leq n} V_i$, then $W \subset V_i$ for some i.

2.2. Dynamics.

Definition 2.2. Let X be a topological space, and $f : X \to X$ a homeomorphism. We say that $x \in X$ is a wandering point if there exists a open neighbourhood V of x such that the $f^n(V)$ are pairwise disjoint for infinitely $n \in \mathbf{N}$.

Lemma 2.3. *Let G act on a topological space X, preserving a probability μ. If $x \in X$ is wandering for some $g \in G$, then x in not in* $\mathrm{supp}(\mu)$.

Proof: There exists a open neighbourhood V of x such that the $g^{n_i}K$, $i, n_i \in \mathbf{N}$ are pairwise disjoint. But $\mu(g^{n_i}V) = \mu(V)$ for all i, so $\mu(\bigcup_{0\leq i<n} g^{n_i}V) = n\mu(V) \leq 1$ for all n. This implies $\mu(V) = 0$, hence $x \notin \mathrm{supp}(\mu)$. ■

3. Distal groups

Recall the following result from [DGS] (theorem 5.1, p. 17): if $(\mathbf{K}, |\cdot|)$ is a *complete* normed field, then the norm extends to every finite extension field in a unique way.

In the sequel, V we denote a finite dimensional vector space over the field $\mathbf{K}$.

Definition 3.1. Let $\mathbf{K}$ be a complete normed field.

We say that $g \in GL(V)$ is *distal* if every eigenvalue of g (in some finite extension) has modulus 1, and that is is *projectively distal* if all its eigenvalues have the same modulus.

If $G \subset GL(V)$ is a subgroup, it it said to be *distal* (resp. *projectively distal*) if every $g \in G$ is distal (resp. projectively distal).

Recall Burnside's density theorem: if V is an *absolutely irreducible* representation of G over a field K, then G spans $\mathrm{End}(V)$ as a K-vector space.

Lemma 3.2. *Let* $\mathbf{K}$ *be a local field. Equip* $\mathrm{End}(V)$ *with a norm of* $\mathbf{K}$*-vector space. Let* $G \subset GL(V)$ *be a subgroup. Then*

1) Suppose that G *is bounded. Then* G *is relatively compact in* $GL(V)$*.*

2) Suppose that G *is bounded modulo scalars, that is, there exists a family* $(\lambda_g)_{g\in G}$ *in* $\mathbf{K}^*$*, such that* $\{\lambda_g g,\ g \in G\}$ *is bounded, and* $\lambda_g\lambda_{g^{-1}} = 1$ *for all* $g \in G\}$*. Then the image of* G *in* $PGL(V)$ *is relatively compact.*

Proof: If G is bounded, if (g_i) is a sequence in G, then (g_i, g_i^{-1}) is bounded, so has a cluster point (g, h) in $\mathrm{End}(V) \times \mathrm{End}(V)$. Obviously, $gh = 1$, so that g is a cluster point of (g_i) and $g \in GL(V)$.

If G is projectively bounded, the argument is analogous. ■

Lemma 3.3. *Let* $\mathbf{K}$ *a complete normed field. Let* $G \subset GL(V)$ *be a subgroup. Suppose* G *acts absolutely irreducibly on* V*.*

If G *is distal, then* G *bounded. If, moreover,* $\mathbf{K}$ *is local, then* G *is relatively compact.*

If G *is projectively distal, then it is bounded modulo scalars (as defined in lemma 3.2). If, moreover,* $\mathbf{K}$ *is local, then the image of* G *in* $PGL(V)$ *is relatively compact.*

Proof: Case 1. First suppose that G is distal. The hypothesis implies that $\mathrm{Tr}(G)$ is bounded. Thus, for all $g \in G$, $\{Tr(gh),\ h \in G\}$ is bounded. So, by linearity and Burnside's density theorem, for all $x \in \mathrm{End}(V)$, $\{Tr(xh),\ h \in G\}$ is bounded. This implies, since $(x, y) \mapsto \mathrm{Tr}(xy)$ is nondegenerate, that G is bounded. If $\mathbf{K}$ is local, use lemma 3.2.

Case 2. Suppose now that G is only projectively distal. If $g \in G$, denote by $r(g)$ the common modulus of all its eigenvalues ($r(g) = |\det(g)|^{1/d}$, where $d = \dim(V)$, so that r is multiplicative). Let I be a compact neighbourhood of 1 in $\mathbf{R}_+^*$, symmetric by inversion $x \mapsto x^{-1}$, such that $|\mathbf{K}| \cap I \neq \{1\}$. For all $g \in G$, one can choose $\lambda_g \in \mathbf{K}^*$ such that $r(\lambda_g g) \in I$; arranging so that $\lambda_{g^{-1}} = \lambda_g^{-1}$ for all $g \in G$. Since $\{r((\lambda_g g)(\lambda_h h)),\ g, h \in G\}$ is bounded, $\{\mathrm{Tr}((\lambda_g g)(\lambda_h h)),\ g, h \in G\}$ is bounded, and we can argue as in case 1 to obtain that $\{\lambda_g g,\ g \in G\}$ is bounded. So G is bounded modulo scalars. Now use lemma 3.2. ■

Lemma 3.4. *Let* $\mathbf{K}$ *be a local field. Let* $G \subset GL(V)$ *be a distal (resp. projectively distal) subgroup. Then there exists a finite extension* $\mathbf{K} \subset \mathbf{L}$ *and a flag of* $\mathbf{L}$*-subspaces*

$$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_d = W = V \otimes \mathbf{L}$$

such that $G(W_i) = W_i$ *for all* i*, and the image of* $G \to GL(W_i/W_{i-1})$ *(resp.* $G \to PGL(W_i/W_{i-1})$*) is relatively compact.*

Proof: Let Ω be the algebraic closure of $\mathbf{K}$. Set $W' = V \otimes \Omega$, with the induced action of G. Let

$$0 = W'_0 \subsetneq W'_1 \subsetneq \cdots \subsetneq W'_d = W'$$

be a flag of G-invariant subspaces such that G acts irreducibly on W_i/W_{i-1} for all $i = 1 \ldots d$. There exists a finite extension $\mathbf{L}$ of $\mathbf{K}$ such that all subspaces W_i are defined over $\mathbf{L}$: $W'_i = W_i \otimes_L \Omega$.

We look at the action of G on W_i/W_{i-1}. Since it is absolutely irreducible, if G is distal (resp. projectively distal), lemma 3.3 implies that the image of $G \to GL(W_i/W_{i-1})$ (resp. $PGL(W_i/W_{i-1})$) is relatively compact. ■

Corollary 3.5. *Let* $\mathbf{K}$ *be a local field. If* $G \subset GL(V)$ *is (projectively) distal, then its closure is amenable.*

Proof: Indeed, it is a closed subgroup of a compact-by-unipotent, hence amenable locally compact group. ■

Note that this is sufficient for our purposes if one only want to prove (i)⇒(iii) of the main theorem.

Lemma 3.6 (Furstenberg)**.** *Let* $\mathbf{K}$ *be a local field, and* $G \subset GL(V)$ *be a subgroup. Suppose that* G *is not relatively compact in* $PGL(V)$*, and preserves a probability* μ *on the Borel subsets of* $P(V)$*. Then there exist two proper projective subspaces* $W, W' \subset P(V)$ *such that* $\mu(W \cup W') = 1$*.*

Proof: See [FUR] or [SHA]. ■

Corollary 3.7. *Let* $G \subset GL(V)$ *be a closed subgroup. Suppose that* G *is amenable, that its Zariski closure is Zariski-connected, and that it acts irreducibly on* V*. Then* G *is relatively compact in* $PGL(V)$*.*

Proof: Since G is amenable and $P(V)$ compact, G preserves a probability μ on $P(V)$. If G is not relatively compact in $PGL(V)$, then, using Furstenberg's lemma, the quasi-linear closure of $\mathrm{supp}(\mu)$ is proper. Since the Zariski closure of G is Zariski-connected, this implies that G preserves a proper subspace, a contradiction with irreducibility. ■

Corollary 3.8. *The conclusion of corollary 3.7 holds without assuming connectedness of the Zariski closure.*

Proof: Let $W \subset V$ be the sum of all G^{0Z}-irreducible subspaces of V. Then W is G-stable: indeed, G^{0Z} is normal in G, so that G permutes G^{0Z}-irreducible subspaces. So, since G acts irreducibly, $W = V$.

This proves that G^{0Z} acts completely reducibly on V. We decompose V into irreducible subspaces, and apply corollary 3.7 to all these subspaces, so that G^{0Z} is relatively compact. Since G^{0Z} has finite index in G, G is also relatively compact. ■

Corollary 3.9. *If $G \subset GL(V)$ is distal (resp. projectively distal) and acts irreducibly on V, then G is relatively compact.*

Proof: Follows immediately from corollaries 3.8 and 3.5 in the projectively distal case. Since the natural morphism $GL^1(V) \to PGL^1(V)$ ($GL^1(V)$ denoting matrices with determinant of norm one) has compact kernel, it is proper. This gives the distal case. ■

Theorem 3.10. *Let $\mathbf{K}$ be a local field. Let $G \subset GL(V)$ be a distal (resp. projectively distal) subgroup. Then there exists a flag of $\mathbf{K}$-subspaces*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

such that $G(V_i) = V_i$ for all i, and the image of $G \to GL(V_i/V_{i-1})$ (resp. $G \to PGL(V_i/V_{i-1})$) is relatively compact.

Proof: Follows immediately from corollary 3.9. ■

4. Action on the projective space

Let $\mathbf{K}$ be either a local field, or a normed algebraically closed field. Let T be an operator of V. For any real $r > 0$, define the weak characteristic subspace of T for r to be the direct sum of all characteristic subspaces of T for eigenvalues of modulus r. A point $x \in P(V)$ is said undistal (for T) if it belongs to none of the weak characteristic subspaces of T.

Proposition 4.1. *Let $\mathbf{K}$ be a local field. Let T be an operator of V. If $x \in P(V)$ is undistal for T, then x is wandering for T.*

We need the following lemma.

Lemma 4.2. *Let $\mathbf{K}$ be a nonarchimedean local field. Let $\|\cdot\|$ be the supremum norm on $\mathbf{K}^n$. Let T be an operator of $\mathbf{K}^n$ whose matrix in the standard canonical basis is upper triangular, with all coefficient of norm ≤ 1, and diagonal coefficients of norm 1. Then T preserves the norm $\|\cdot\|$.*

Proof of lemma 4.2. Using the the ultrametric inequality, we immediately get $\|T(x)\| \leq x$ for all x. Again with the ultrametric inequality, we obtain that T^{-1} satisfies the same hypothesis, so that T preserves the norm. ■

Proof of proposition 4.1. For the sake of simplicity, we shall assume that $\mathbf{K}$ is nonarchimedean (in the archimedean case, we have to deal with some polynomial terms, but they are dominated by exponential terms, so the proof can easily be modified).

Upon taking a finite extension, on can suppose that T is trigonalizable. Write $V = \bigoplus V_i$, V_i denoting weak characteristic subspaces for $\lambda_i > 0$, and equip V with the supremum norm. Decompose p in this direct sum: $p = \sum p_i$. Let $\mu_i \in \mathbf{K}$ with $|\mu_i| = \lambda_i$. Set $T_i = T|_{V_i}$. Putting $\mu_i^{-1}T_i$ in a Jordan form, it satisfies the hypotheses of the lemma. Let $i \neq j$ be such that $\lambda_i < \lambda_j$ and $p_i, p_j \neq 0$. Then

$$\frac{\|(T^n p)_j\|}{\|(T^n p)_i\|} = \left(\frac{\lambda_j}{\lambda_i}\right)^n \frac{\|p_j\|}{\|p_i\|}.$$

Setting

$$\Omega = \left\{ x = \sum x_i,\ \left| \log \frac{\|x_j\|}{\|x_i\|} - \log \frac{\|p_j\|}{\|p_i\|} \right| < \frac{1}{2} \log \frac{\lambda_j}{\lambda_i} \right\},$$

all $T^n(\Omega)$, $n \in \mathbf{Z}$ are pairwise disjoint, and Ω is a conic neighbourhood of p. ■

Proof of the implication $(i) \Rightarrow (iv)$ of theorem: we suppose that $G \subset GL(V)$ (V a finite-dimensional vector space over $\mathbf{K}$) preserves a probability μ on $P(V)$.

Case 1: G is projectively distal. By theorem 3.10, G acts compactly on some projective subspace, hence (iv) of theorem 1.1 is satisfied.

Case 2: G is not projectively distal. Let $g \in G$ be not distal. Then, by proposition 4.1 and lemma 2.3, the union of all its weak characteristic subspaces contains $\mathrm{supp}(\mu)$, hence its closure for the quasi-linear topology, which is G-invariant, and nontrivial. One of these characteristic subspaces, which we call W, must have a positive measure. Then W is preserved by G^{0Z}, so that we can argue by induction on the dimension. ■

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