

# KAZHDAN AND HAAGERUP PROPERTIES IN ALGEBRAIC GROUPS OVER LOCAL FIELDS

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ABSTRACT. Given a Lie algebra  $\mathfrak{s}$ , we call Lie  $\mathfrak{s}$ -algebra a Lie algebra endowed with a reductive action of  $\mathfrak{s}$ . We characterize the minimal  $\mathfrak{s}$ -Lie algebras with a nontrivial action of  $\mathfrak{s}$ , in terms of irreducible representations of  $\mathfrak{s}$  and invariant alternating forms.

As a first application, we show that if  $\mathfrak{g}$  is a Lie algebra over a field of characteristic zero whose amenable radical is not a direct factor, then  $\mathfrak{g}$  contains a subalgebra which is isomorphic to the semidirect product of  $\mathfrak{sl}_2$  by either a nontrivial irreducible representation or a Heisenberg group (this was essentially due to Cowling, Dorofaeff, Seeger, and Wright). As a corollary, if  $G$  is an algebraic group over a local field  $\mathbf{K}$  of characteristic zero, and if its amenable radical is not, up to isogeny, a direct factor, then  $G(\mathbf{K})$  has Property (T) relative to a noncompact subgroup. In particular,  $G(\mathbf{K})$  does not have Haagerup's property. This extends a similar result of Cherix, Cowling and Valette for connected Lie groups, to which our method also applies.

We give some other applications. We provide a characterization of connected Lie groups all of whose countable subgroups have Haagerup's property. We give an example of an arithmetic lattice in a connected Lie group which does not have Haagerup's property, but has no infinite subgroup with relative Property (T). We also give a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), with pairwise non-isomorphic (resp. isomorphic) Lie algebras.

## 1. INTRODUCTION

In the sequel, all Lie algebras are finite-dimensional over a field of characteristic zero, denoted by  $K$ , or  $\mathbf{K}$  when it is a local field. If  $\mathfrak{g}$  is a Lie algebra, denote by  $\text{rad}(\mathfrak{g})$  its radical and  $Z(\mathfrak{g})$  its centre,  $D\mathfrak{g}$  its derived subalgebra, and  $\text{Der}(\mathfrak{g})$  the Lie algebra of all derivations of  $\mathfrak{g}$ . If  $\mathfrak{h}_1, \mathfrak{h}_2$  are Lie subalgebras of  $\mathfrak{g}$ ,  $[\mathfrak{h}_1, \mathfrak{h}_2]$  denotes the Lie subalgebra generated by the brackets  $[h_1, h_2]$ ,  $(h_1, h_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2$ .

Let  $\mathfrak{g}$  be a Lie algebra with radical  $\text{rad}(\mathfrak{g}) = \mathfrak{r}$  and semisimple Levi factor  $\mathfrak{s}$  (so that  $\mathfrak{g} \simeq \mathfrak{s} \ltimes \mathfrak{r}$ ). We focus here on aspects of  $\mathfrak{g}$  related to the action of  $\mathfrak{s}$ . This suggests the following definitions.

If  $\mathfrak{s}$  is a Lie algebra, we define a *Lie  $\mathfrak{s}$ -algebra* to be a Lie algebra  $\mathfrak{n}$  endowed with a morphism  $i : \mathfrak{s} \rightarrow \text{Der}(\mathfrak{n})$ , defining a *completely reducible* action of  $\mathfrak{s}$  on  $\mathfrak{n}$ . (This latter technical condition is empty if  $\mathfrak{s}$  is semisimple.)

A Lie  $\mathfrak{s}$ -algebra naturally embeds in the semidirect product  $\mathfrak{s} \ltimes \mathfrak{n}$ , so that we write  $i(s)(n) = [s, n]$  for  $s \in \mathfrak{s}$ ,  $n \in \mathfrak{n}$ .

By the *trivial irreducible module* of  $\mathfrak{s}$  we mean a one-dimensional space vector space endowed with a trivial action of  $\mathfrak{s}$ . We say that a module (over a Lie algebra or over a group) is *full* if it is completely reducible and does not contain the trivial irreducible module.

**Definition 1.1.** Let  $\mathfrak{s}$  be a Lie algebra. We say that a Lie  $\mathfrak{s}$ -algebra  $\mathfrak{n}$  is *minimal* if  $[\mathfrak{s}, \mathfrak{n}] \neq 0$ , and for every  $\mathfrak{s}$ -subalgebra  $\mathfrak{n}'$  of  $\mathfrak{n}$ , either  $\mathfrak{n}' = \mathfrak{n}$  or  $[\mathfrak{s}, \mathfrak{n}'] = 0$ .

It is clear that a Lie  $\mathfrak{s}$ -algebra  $\mathfrak{n}$  satisfying  $[\mathfrak{s}, \mathfrak{n}] \neq 0$  contains a minimal  $\mathfrak{s}$ -subalgebra. We begin by a characterization of minimal  $\mathfrak{s}$ -algebras:

**Theorem 1.2.** *Let  $\mathfrak{s}$  be a Lie algebra. A solvable Lie  $\mathfrak{s}$ -algebra  $\mathfrak{n}$  is minimal if and only if it satisfies the following conditions 1), 2), 3), and 4):*

- 1)  $\mathfrak{n}$  is 2-nilpotent (that is,  $[\mathfrak{n}, D\mathfrak{n}] = 0$ ).
- 2)  $[\mathfrak{s}, \mathfrak{n}] = \mathfrak{n}$ .
- 3)  $[\mathfrak{s}, D\mathfrak{n}] = 0$ .
- 4)  $\mathfrak{n}/D\mathfrak{n}$  is irreducible as a  $\mathfrak{s}$ -module.

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**Definition 1.3.** We call a solvable Lie  $\mathfrak{s}$ -algebra  $\mathfrak{n}$  *almost minimal* if it satisfies conditions 1), 2), and 3) of Theorem 1.2.

This definition has the advantage to be invariant under field extensions. Note that an almost minimal solvable Lie  $\mathfrak{s}$ -algebra  $\mathfrak{n}$  satisfies Condition 4'):  $\mathfrak{n}/D\mathfrak{n}$  is a full  $\mathfrak{s}$ -module.

The classification of (almost) minimal solvable Lie  $\mathfrak{s}$ -algebras can be deduced from the classification of irreducible  $\mathfrak{s}$ -modules. Let  $\mathfrak{v}$  be a full  $\mathfrak{s}$ -module (equivalently, an abelian Lie  $\mathfrak{s}$ -algebra satisfying  $[\mathfrak{s}, \mathfrak{v}] = \mathfrak{v}$ ). Recall that a bilinear form  $\varphi$  on  $\mathfrak{v}$  is called  $\mathfrak{s}$ -invariant if it satisfies  $\varphi([s, v], w) + \varphi(v, [s, w]) = 0$  for all  $s \in \mathfrak{s}$ ,  $v, w \in \mathfrak{v}$ . Let  $\text{Bil}_{\mathfrak{s}}(\mathfrak{v})$  (resp.  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v})$ ) denote the space of all  $\mathfrak{s}$ -invariant bilinear (resp. alternating bilinear) forms on  $\mathfrak{v}$ .

**Definition 1.4.** We define the Lie  $\mathfrak{s}$ -algebra  $\mathfrak{h}(\mathfrak{v})$  as follows: as a vector space,  $\mathfrak{h}(\mathfrak{v}) = \mathfrak{v} \oplus \text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ ; it is endowed with the following bracket:

$$(1.1) \quad [(x, z), (x', z')] = (0, e_{x, x'}) \quad x, x' \in \mathfrak{v} \quad z, z' \in \text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$$

where  $e_{x, x'} \in \text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$  is defined by  $e_{x, x'}(\varphi) = \varphi(x, x')$ .

This is a 2-nilpotent Lie  $\mathfrak{s}$ -algebra under the action  $[s, (x, z)] = ([s, x], 0)$ , which is almost minimal. Other almost minimal Lie  $\mathfrak{s}$ -algebras can be obtained by taking the quotient by a linear subspace of the centre. The following theorem states that this is the only way to construct almost minimal solvable Lie  $\mathfrak{s}$ -algebras.

**Theorem 1.5.** *If  $\mathfrak{n}$  is an almost minimal solvable Lie  $\mathfrak{s}$ -algebra, then it is isomorphic (as a  $\mathfrak{s}$ -algebra) to  $\mathfrak{h}(\mathfrak{v})/Z$ , for some full  $\mathfrak{s}$ -module  $\mathfrak{v}$  and some subspace  $Z$  of  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ . It is minimal if and only if  $\mathfrak{v}$  is irreducible.*

*Moreover, the almost minimal  $\mathfrak{s}$ -algebras  $\mathfrak{h}(\mathfrak{v})/Z$  and  $\mathfrak{h}(\mathfrak{v})/Z'$  are isomorphic if and only if  $Z'$  and  $Z$  are in the same orbit for the natural action of  $\text{Aut}_{\mathfrak{s}}(\mathfrak{v})$  on the Grassmannian of  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ .*

**Remark 1.6.** If  $\mathfrak{s}$  is semisimple,  $\mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})$  is the universal central extension of the perfect Lie algebra  $\mathfrak{s} \ltimes \mathfrak{v}$ .

The case of  $\mathfrak{sl}_2$  is essential, and there is a simple description for it. Recall that if  $\mathfrak{s} = \mathfrak{sl}_2$ , then, up to isomorphism, there exists exactly one irreducible  $\mathfrak{s}$ -module  $\mathfrak{v}_n$  of dimension  $n$  for every  $n \geq 1$ . If  $n = 2m$  is even, it has a central extension by a one-dimensional subspace, giving a Heisenberg Lie algebra, on which  $\mathfrak{sl}_2$  acts naturally (see 2.2 for details), denoted by  $\mathfrak{h}_{2m+1}$ . Theorem 1.5 thus reduces as:

**Proposition 1.7.** *Up to isomorphism, the minimal solvable Lie  $\mathfrak{sl}_2$ -algebras are  $\mathfrak{v}_n$  and  $\mathfrak{h}_{2n-1}$  ( $n \geq 2$ ).*

Let  $\mathfrak{g}$  be a Lie algebra,  $\mathfrak{r}$  its radical and  $\mathfrak{s}$  a semisimple factor. Write  $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$  by separating anisotropic and isotropic factors<sup>1</sup>. The ideal  $\mathfrak{s}_c \ltimes \mathfrak{r}$  is sometimes called the *amenable radical* of  $\mathfrak{g}$ .

**Definition 1.8.** We call  $\mathfrak{g}$  M-decomposed if  $[\mathfrak{s}_{nc}, \mathfrak{r}] = 0$ . Equivalently,  $\mathfrak{g}$  is M-decomposed if the amenable radical is a direct factor of  $\mathfrak{g}$ .

**Proposition 1.9.** *Let  $\mathfrak{g}$  be a Lie algebra, and keep notation as above. Suppose that  $\mathfrak{g}$  is not M-decomposed. Then there exists a Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  which is isomorphic to  $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$  for some  $n \geq 2$ .*

This result is essentially due to [CDSW], where it is not explicitly stated, but it is actually proved in the proof of Proposition 8.2 there (under the assumption  $K = \mathbf{R}$ , but their argument generalizes to any field of characteristic zero). This was a starting point for the present paper.

Let  $G$  be a locally compact,  $\sigma$ -compact group. Recall that  $G$  has the Haagerup Property if it has a metrically proper isometric action on a Hilbert space; in contrast,  $G$  has Kazhdan's Property (T) if every isometric action of  $G$  on a Hilbert space has a fixed point. See 4.1 for a short reminder about Haagerup and Kazhdan Properties.

We provide corresponding statements for Proposition 1.9 in the realm of algebraic groups and connected Lie groups. As a consequence, we get the following theorem, which was the initial

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<sup>1</sup> $c$  and  $nc$  respectively stand for “non-compact” and “compact”; this is related to the fact that if  $S$  is a simple algebraic group defined over the local field  $\mathbf{K}$ , then its Lie algebra is  $\mathbf{K}$ -isotropic if and only if  $S(\mathbf{K})$  is not compact.

motivation for the results above. It was already proved, in a different way, for connected Lie groups in [CCJJV, Chap. 4].

**Theorem 1.10.** *Let  $G$  be either a connected Lie group, or  $G = \mathbf{G}(\mathbf{K})$ , where  $\mathbf{G}$  is a linear algebraic group over the local field  $\mathbf{K}$  of characteristic zero. Let  $\mathfrak{g}$  be its Lie algebra. The following are equivalent.*

- (i)  $G$  has Haagerup's property.
- (ii) For every noncompact closed subgroup  $H$  of  $G$ ,  $(G, H)$  does not have relative Property (T).
- (iii) The following conditions are satisfied:
  - $\mathfrak{g}$  is  $M$ -decomposed.
  - All simple factors of  $\mathfrak{g}$  have  $\mathbf{K}$ -rank  $\leq 1$ .
  - (in the case of Lie groups or when  $\mathbf{K} = \mathbf{R}$ ) No simple factor of  $\mathfrak{g}$  is isomorphic to  $\mathfrak{sp}(n, 1)$  ( $n \geq 2$ ) or  $\mathfrak{f}_{4(-20)}$ .
- (iv)  $\mathfrak{g}$  contains no isomorphic copy of any one of the following Lie algebras
  - $\mathfrak{sl}_2 \times \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \times \mathfrak{h}_{2n-1}$  for some  $n \geq 2$ ,
  - (in the case of Lie groups or when  $\mathbf{K} = \mathbf{R}$ )  $\mathfrak{sp}(2, 1)$ .

*Remark 1.11.* The notion of  $M$ -decomposed (real) Lie algebras also appears in other contexts: heat kernel on Lie groups [Var], Rapid Decay Property [CPS], weak amenability [CDSW].

We derive some other results with the help of Theorem 1.5.

**Proposition 1.12.** *There exists a continuous family  $(\mathfrak{g}_t)$  of pairwise non-isomorphic real (or complex) Lie algebras satisfying the following properties:*

- (i)  $\mathfrak{g}_t$  is perfect, and
- (ii) the simply connected Lie group corresponding to  $\mathfrak{g}_t$  has Property (T).

Note that Proposition 1.12 with only (i) may be of independent interest; we do not know if it had already been observed. On the other hand, it is well-known that there exist continuously many pairwise non-isomorphic complex  $n$ -dimensional nilpotent Lie algebras if  $n \geq 7$ .

**Proposition 1.13.** *There exists a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), and with isomorphic Lie algebras.*

We also give the classification, when  $\mathbf{K} = \mathbf{R}$ , of the minimal  $\mathfrak{so}_3$ -algebras (Proposition 2.3). We use it to prove (ii) $\Rightarrow$ (i) in the following result (while the reverse implication is essentially due to [GHW, Theorem 5.1]).

**Theorem 1.14.** *Let  $G$  be a connected Lie group. Then the following are equivalent:*

- (i)  $G$  is locally isomorphic to  $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$ , for a solvable Lie group  $R$  and integers  $a, b, c$ .
- (ii) Every countable subgroup of  $G$  has Haagerup's property (when endowed with the discrete topology).

*Remark 1.15.* Assertion (i) of Theorem 1.14 is equivalent to: (ii') *The complexification  $\mathfrak{g}_{\mathbf{C}}$  of  $\mathfrak{g}$  is  $M$ -decomposed, and its semisimple part is isomorphic to  $\mathfrak{sl}_2(\mathbf{C})^n$  for some  $n$ .*

For instance,  $\mathrm{SO}_3(\mathbf{R}) \times \mathbf{R}^3$  has a countable subgroup which does not have Haagerup's property. An explicit example is given by  $\mathrm{SO}_3(\mathbf{Z}[1/p]) \times \mathbf{Z}[1/p]^3$ . It can also be shown that this group has no infinite subgroup with relative Property (T). This answers an open question in [CCJJV, Section 7.1]. This group is not finitely presented (this is a consequence of [Abe, Theorem 2.6.4]); we give a similar example in Remark 4.12 which is, in addition, finitely presented.

## 2. LIE ALGEBRAS

### 2.1. Minimal subalgebras.

**Proposition 2.1.** *Let  $\mathfrak{n}$  be a solvable Lie  $\mathfrak{s}$ -algebra.*

- 1) *The Lie  $\mathfrak{s}$ -subalgebra  $[\mathfrak{s}, \mathfrak{n}]$  is an ideal in  $\mathfrak{n}$  (and also in  $\mathfrak{s} \times \mathfrak{n}$ ), and  $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]] = [\mathfrak{s}, \mathfrak{n}]$ .*
- 2) *If, moreover,  $[\mathfrak{s}, D\mathfrak{n}] = 0$ , then  $[\mathfrak{s}, \mathfrak{n}]$  is an almost minimal Lie algebra (see Definition 1.3).*

**Proof:** 1) Let  $\mathfrak{v}$  be the subspace generated by the brackets  $[s, n]$ ,  $(s, n) \in \mathfrak{s} \times \mathfrak{n}$ . Since the action of  $\mathfrak{s}$  is completely reducible (see the definition of Lie  $\mathfrak{s}$ -algebra), it is immediate that  $[\mathfrak{s}, \mathfrak{n}]$  and  $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]]$  both coincide with the Lie subalgebra generated by  $\mathfrak{v}$ . Then, using Jacobi identity,

$$[\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]] = [\mathfrak{n}, [\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]]] \subseteq [\mathfrak{s}, [\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]]] + [[\mathfrak{s}, \mathfrak{n}], [\mathfrak{s}, \mathfrak{n}]] \subseteq [\mathfrak{s}, [\mathfrak{n}, \mathfrak{n}]] + [\mathfrak{s}, \mathfrak{n}] \subseteq [\mathfrak{s}, \mathfrak{n}].$$

2) Let  $\mathfrak{z}$  be the linear subspace generated by the commutators  $[v, w]$ ,  $v, w \in \mathfrak{v}$ . By Jacobi identity,

$$[\mathfrak{v}, \mathfrak{z}] = [[\mathfrak{s}, \mathfrak{v}], \mathfrak{z}] \subseteq [[\mathfrak{s}, \mathfrak{z}], \mathfrak{v}] + [\mathfrak{s}, [\mathfrak{v}, \mathfrak{z}]] \subseteq [[\mathfrak{s}, D\mathfrak{n}], \mathfrak{v}] + [\mathfrak{s}, D\mathfrak{n}] = 0.$$

Thus, the subspace  $\mathfrak{n}' = \mathfrak{v} \oplus \mathfrak{z}$  is a 2-nilpotent Lie  $\mathfrak{s}$ -subalgebra of  $\mathfrak{n}$ . The Lie subalgebra  $[\mathfrak{s}, \mathfrak{n}']$  contains  $\mathfrak{v}$ , hence also contains  $\mathfrak{z}$ , so  $[\mathfrak{s}, \mathfrak{n}']$  is equal to  $\mathfrak{n}'$ . Thus Conditions 1) and 2) of Definition 1.3 are satisfied, while Condition 3) follows immediately from the hypothesis  $[\mathfrak{s}, D\mathfrak{n}] = 0$ . ■

**Proof of Theorem 1.2.** Suppose (ii). Condition 4 implies  $\mathfrak{n} \neq 0$ . Then Condition 2 implies  $[\mathfrak{s}, \mathfrak{n}] = \mathfrak{n} \neq 0$ . Let  $\mathfrak{n}' \subseteq \mathfrak{n}$  be a  $\mathfrak{s}$ -subalgebra. Then, by irreducibility (Condition 4), either  $D\mathfrak{n} + \mathfrak{n}' = D\mathfrak{n}$  or  $D\mathfrak{n} + \mathfrak{n}' = \mathfrak{n}$ . In the first case,  $\mathfrak{n}'$  centralizes  $\mathfrak{s}$ . In the second case,  $\mathfrak{n} = [\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}' + D\mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}'] \subseteq \mathfrak{n}'$ , using Conditions 1 and 2, and the fact that  $\mathfrak{n}'$  is a  $\mathfrak{s}$ -subalgebra.

Conversely, suppose that (i) holds. Since  $\mathfrak{n}$  is solvable,  $D\mathfrak{n}$  is a proper  $\mathfrak{s}$ -subalgebra, so that, by minimality,  $[\mathfrak{s}, D\mathfrak{n}] = 0$ . By Proposition 2.1,  $[\mathfrak{s}, \mathfrak{n}]$  is a nonzero almost minimal Lie  $\mathfrak{s}$ -subalgebra of  $\mathfrak{n}$ , hence satisfies 1), 2), 3). The minimality implies that 4) is also satisfied. ■

**Proof of Theorem 1.5.** Let  $\mathfrak{n}$  be an almost minimal solvable Lie  $\mathfrak{s}$ -algebra. Let  $\mathfrak{v}$  be the subspace generated by the brackets  $[s, n]$ ,  $(s, n) \in \mathfrak{s} \times \mathfrak{n}$ . Since  $\mathfrak{n}$  is almost minimal,  $\mathfrak{v}$  is a complementary subspace of  $D\mathfrak{n}$ , and is a full  $\mathfrak{s}$ -module. If  $u \in D\mathfrak{n}^*$ , consider the alternating bilinear form  $\phi_u$  on  $\mathfrak{v}$  defined by  $\phi_u(x, y) = u([x, y])$ . This defines a mapping  $D\mathfrak{n}^* \rightarrow \text{Alt}_{\mathfrak{s}}(\mathfrak{v})$  which is immediately seen to be injective. By duality, this defines a surjective linear map  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v})^* \rightarrow D\mathfrak{n}$ , whose kernel we denote by  $Z$ . It is immediate from the definition of  $\mathfrak{h}(\mathfrak{v})$  that this map extends to a surjective morphism of Lie  $\mathfrak{s}$ -algebras  $\mathfrak{h}(\mathfrak{v}) \rightarrow \mathfrak{n}$  with kernel  $Z$ . This proves that  $\mathfrak{n}$  is isomorphic to  $\mathfrak{h}(\mathfrak{v})/Z$ .

The second assertion is immediate.

The third assertion follows from the proof of the first one, where we made no choice. Namely, take an isomorphism  $\psi : \mathfrak{h}(\mathfrak{v})/Z \rightarrow \mathfrak{h}(\mathfrak{v})/Z'$ . It gives by restriction an  $\mathfrak{s}$ -automorphism  $\varphi$  of  $\mathfrak{v}$ , which induces a unique automorphism  $\tilde{\varphi}$  of  $\mathfrak{h}(\mathfrak{v})$ . Let  $p$  and  $p'$  denote the natural projections in the following diagram of Lie  $\mathfrak{s}$ -algebras:

$$\begin{array}{ccc} \mathfrak{h}(\mathfrak{v}) & \xrightarrow{p} & \mathfrak{h}(\mathfrak{v})/Z \\ \tilde{\varphi} \downarrow & & \downarrow \psi \\ \mathfrak{h}(\mathfrak{v}) & \xrightarrow{p'} & \mathfrak{h}(\mathfrak{v})/Z' \end{array}$$

This diagram is commutative: indeed,  $p' \circ \tilde{\varphi}$  and  $\psi \circ p$  coincide in restriction to  $\mathfrak{v}$ , and  $\mathfrak{v}$  generates  $\mathfrak{h}(\mathfrak{v})$  as a Lie algebra. This implies  $Z = \text{Ker}(\psi \circ p) = \text{Ker}(p' \circ \tilde{\varphi}) = \tilde{\varphi}^{-1}(Z')$ . ■

**2.2. The example  $\mathfrak{sl}_2$ .** If  $\mathfrak{s} = \mathfrak{sl}_2(K)$ , then, up to isomorphism, there exists exactly one irreducible  $\mathfrak{s}$ -module  $\mathfrak{v}_n$  of dimension  $n$  for every  $n \geq 1$ .

Since  $\mathfrak{v}_n$  is absolutely irreducible for all  $n$ , by Schur's Lemma,  $\text{Bil}_{\mathfrak{s}}(\mathfrak{v}_n)$  is at most one dimensional for all  $n$ . In fact, it is one-dimensional. Indeed, take the usual basis  $(H, X, Y)$  of  $\mathfrak{sl}_2$  satisfying  $[H, X] = 2X$ ,  $[H, Y] = -2Y$ ,  $[X, Y] = H$ , and take the basis  $(e_0, \dots, e_{n-1})$  of  $\mathfrak{v}_n$  so that  $H.e_i = (n-1-2i)e_i$ ,  $X.e_i = (n-i)e_{i-1}$ , and  $Y.e_i = (i+1)e_{i+1}$ , with the convention  $e_{-1} = e_n = 0$ . Then  $\text{Bil}_{\mathfrak{s}}(\mathfrak{v}_n)$  is generated by the form  $\varphi_n$  defined by

$$\varphi_n(e_i, e_{n-1-i}) = (-1)^i \binom{i}{n-1}; \quad \varphi_n(e_i, e_j) = 0 \text{ if } i+j \neq n-1.$$

For odd  $n$ ,  $\varphi_n$  is symmetric so that  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v}_n) = 0$ ; for even  $n$ ,  $\varphi_n$  is symplectic and generates  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v}_n)$ . For even  $n$ , denote by  $\mathfrak{h}_{n+1}$  the one-dimensional central extension  $\mathfrak{h}(\mathfrak{v}_n)$ , well-known as the  $(n+1)$ -dimensional Heisenberg Lie algebra.

**Proof of Proposition 1.9.** Since  $\mathfrak{s}_{nc}$  is semisimple and isotropic, it is generated by its subalgebras  $K$ -isomorphic to  $\mathfrak{sl}_2$ . Since  $[\mathfrak{s}_{nc}, \mathfrak{v}] \neq 0$ , this implies that there exists some subalgebra  $\mathfrak{s}'$  of  $\mathfrak{s}_{nc}$  which

is  $K$ -isomorphic to  $\mathfrak{sl}_2$  and such that  $[\mathfrak{s}', \mathfrak{t}] \neq 0$ . Then the result is clear from Proposition 1.7. Notice that the proof gives the following slight refinement:  $\mathfrak{h}$  can be chosen so that  $\text{rad}(\mathfrak{h}) \subseteq \text{rad}(\mathfrak{g})$ . ■

**2.3. The example  $\mathfrak{so}_3$ .** We now study a more specific example. Let us deal with the field  $\mathbf{R}$  of real numbers, and with  $\mathfrak{s} = \mathfrak{so}_3$ .

Since the complexification of  $\mathfrak{so}_3$  is isomorphic to  $\mathfrak{sl}_2(\mathbf{C})$ , the irreducible complex  $\mathfrak{s}$ -modules make up a family  $(\mathbf{u}_n)$  ( $n \geq 1$ );  $\dim(\mathbf{u}_n) = n$ , which are the symmetric powers of the natural action of  $\mathfrak{su}_2 = \mathfrak{so}_3$  on  $\mathbf{C}^2$ .

If  $n = 2m + 1$  is odd, then this is the complexification of a real  $\mathfrak{so}_3$ -module  $\mathfrak{d}_{2m+1}^{\mathbf{R}}$  (of dimension  $n$ ). If  $n = 2m$  is even,  $\mathfrak{d}_n$  is irreducible as a  $4m$ -dimensional real  $\mathfrak{so}_3$ -module, we call it  $\mathbf{u}_{4m}$ .

These two families  $(\mathfrak{d}_{2n+1}^{\mathbf{R}})$  and  $(\mathbf{u}_{4n})$  make up all irreducible real  $\mathfrak{so}_3$ -modules.

**Proposition 2.2.** *The irreducible real  $\mathfrak{so}_3$ -modules make up two families: a family  $(\mathfrak{d}_{2n+1}^{\mathbf{R}})$  of  $(2n + 1)$ -dimensional modules ( $n \geq 0$ ), absolutely irreducible, and a family  $(\mathbf{u}_{4n})$  of  $4n$ -dimensional modules ( $n \geq 1$ ), not absolutely irreducible, preserving a quaternionic structure. ■*

Since  $(\mathfrak{d}_{2n+1}^{\mathbf{R}})$  is absolutely irreducible, the space of invariant bilinear forms on  $(\mathfrak{d}_{2n+1}^{\mathbf{R}})$  is generated by a scalar product, so that  $\text{Alt}_{\mathfrak{so}_3}(\mathfrak{d}_{2n+1}^{\mathbf{R}}) = 0$

On the other hand,  $\text{Alt}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  is three-dimensional, and is given by the imaginary part of an invariant quaternionic hermitian form.

In order to classify the minimal solvable  $\mathfrak{so}_3$ -algebras, we must determine the orbits of the natural action of  $\text{Aut}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  on  $\text{Alt}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$ . It is a standard fact that  $\text{Aut}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  is isomorphic to the group of nonzero quaternions, that  $\text{Alt}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  naturally identifies with the set of imaginary quaternions, and that the action of  $\text{Aut}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  on  $\text{Alt}_{\mathfrak{so}_3}(\mathbf{u}_{4n})$  is given by conjugation of quaternions. This implies that it acts transitively on each component of the Grassmannian.

For  $i = 0, 1, 2, 3$ , let  $Z_i$  be a fixed  $(3 - i)$ -dimensional linear subspace of  $\text{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ . Denote by  $\mathfrak{h}\mathbf{u}_{4n}^i$  the minimal Lie  $\mathfrak{so}_3$ -algebra  $\mathfrak{h}(\mathbf{u}_{4n})/Z_i$ ; of course,  $\mathfrak{h}\mathbf{u}_{4n}^0 = \mathbf{u}_{4n}$  and  $\mathfrak{h}\mathbf{u}_{4n}^3 = \mathfrak{h}(\mathbf{u}_{4n})$ .

**Proposition 2.3.** *Up to isomorphism, the minimal solvable Lie  $\mathfrak{so}_3(\mathbf{R})$ -algebras are  $\mathfrak{d}_{2n+1}^{\mathbf{R}}$  ( $n \geq 1$ ) and  $\mathfrak{h}\mathbf{u}_{4n}^i$  ( $n \geq 1, i = 0, 1, 2, 3$ ). ■*

There is an analogous statement to Proposition 1.9.

**Proposition 2.4.** *Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbf{R}$ . Suppose that  $[\mathfrak{s}_c, \mathfrak{t}] \neq 1$ . Then  $\mathfrak{g}$  has a Lie subalgebra which is isomorphic to either  $\mathfrak{so}_3 \times \mathfrak{d}_{2n+1}^{\mathbf{R}}$  or  $\mathfrak{so}_3 \times \mathfrak{h}\mathbf{u}_{4n}^i$  for some  $i = 0, 1, 2, 3$  and some  $n \geq 1$ . ■*

### 3. CORRESPONDING RESULTS FOR ALGEBRAIC GROUPS AND CONNECTED LIE GROUPS

**3.1. Minimal algebraic subgroups.** We now give the corresponding statements and results for algebraic groups.

Let  $S$  be a reductive  $K$ -group. A  $K$ - $S$ -group means a linear  $K$ -group endowed with a  $K$ -action of  $S$  by automorphisms.

Recall that the Lie algebra functor gives an equivalence of categories between the category of unipotent  $K$ -groups and the category of nilpotent Lie  $K$ -algebras. If  $S$  is semisimple and simply connected with Lie algebra  $\mathfrak{s}$ , it induces an equivalence of categories between the category of unipotent  $K$ - $S$ -groups and the category of nilpotent Lie  $S$ -algebras over  $K$ . If  $S$  is not simply connected (in particular, if  $S$  is not semisimple), this is no longer an essentially surjective functor, but it remains fully faithful.

A minimal (resp. almost minimal) solvable  $S$ -group  $N$  is defined similarly as in the case of Lie algebras; it is automatically unipotent (since it satisfies  $[S, N] = N$ ). Moreover,  $N$  is a minimal (resp. almost minimal) solvable  $K$ - $S$ -group if and only if its Lie algebra  $\mathfrak{n}$  is a minimal (resp. almost minimal) solvable Lie  $\mathfrak{s}$ -algebra. Proposition 2.1 and Theorem 1.2 also immediately carry over into the context of algebraic groups.

If  $S$  is reductive and  $V$  is a  $K$ - $S$ -module, we define the unipotent  $K$ - $S$ -group  $H(V)$  as follows: as a variety,  $H(V) = V \oplus \text{Alt}_S(V)^*$ ; it is endowed with the following group law:

$$(3.1) \quad (x, z)(x', z') = (x + x', z + z' + e_{x, x'}) \quad x, x' \in V \quad z, z' \in \text{Alt}_S(V)^*$$

where  $e_{x,x'} \in \text{Alt}_S(V)^*$  is defined by  $e_{x,x'}(\varphi) = \varphi(x, x')$ . This is a  $K$ - $S$ -group under the action  $s.(x, z) = (s.x, z)$ . It is clear that its Lie algebra is isomorphic as a Lie  $K$ - $S$ -algebra to  $\mathfrak{h}(\mathfrak{v})$ , where  $V = \mathfrak{v}$  viewed as a  $\mathfrak{s}$ -module. Here is the analog of Theorem 1.5.

**Theorem 3.1.** *If  $N$  is an almost minimal solvable  $K$ - $S$ -group, then it is isomorphic (as a  $K$ - $S$ -group) to  $H(V)/Z$ , for some full  $K$ - $S$ -module  $V$  and some  $K$ -subspace  $Z$  of  $\text{Alt}_S(V)^*$ . It is minimal if and only if  $V$  is irreducible.*

*Moreover, the almost minimal  $K$ - $S$ -groups  $H(V)/Z$  and  $H(V)/Z'$  are isomorphic if and only if  $Z'$  and  $Z$  are in the same orbit for the natural action of  $\text{Aut}_S(V)$  on the Grassmannian of  $\text{Alt}_S(V)^*$ .*

**3.2. The example  $\text{SL}_2$ .** The simply connected  $K$ -group with Lie algebra  $\mathfrak{sl}_2$  is  $\text{SL}_2$ . Denote by  $V_n$  and  $H_{2n-1}$  the  $\text{SL}_2$ -groups corresponding to  $\mathfrak{v}_n$  and  $\mathfrak{h}_{2n-1}$ . These are the solvable minimal  $\text{SL}_2$ -groups over  $K$ . The only non-simply connected  $K$ -group with Lie algebra  $\mathfrak{sl}_2$  is the adjoint group  $\text{PGL}_2$ ; thus the minimal solvable  $\text{PGL}_2$ -groups over  $K$  are  $V_{2n-1}$  for  $n \geq 2$ .

*Remark 3.2.* It is convenient, in algebraic groups, to deal with the unipotent radical rather than with the radical. It is straightforward to see that a reductive subgroup  $S$  of a linear algebraic group centralizes the radical if and only if it centralizes the unipotent radical. Indeed, suppose  $[S, R_u] = 1$ . We always have  $[S, R/R_u] = 1$  since  $R/R_u$  is central in  $G^0/R_u$  and  $S$  is connected ( $G^0$  denoting the unit component of  $G$ ). This easily implies that  $S$  acts trivially<sup>2</sup>.

Let  $G$  be a linear algebraic group over the field  $K$  of characteristic zero,  $R$  its radical,  $S$  a Levi factor, decomposed as  $S_{nc}S_c$  by separating  $K$ -isotropic and  $K$ -anisotropic factors.

**Proposition 3.3.** *Suppose that  $[S_{nc}, R] \neq 1$ . Then  $G$  has a  $K$ -subgroup which is  $K$ -isomorphic to either  $\text{SL}_2 \times V_n$ ,  $\text{PGL}_2 \times V_{2n-1}$ , or  $\text{SL}_2 \times H_{2n-1}$  for some  $n \geq 2$ .*

Let us mention the translation into the context of connected Lie groups, which is immediate from the Lie algebraic version.

**Proposition 3.4.** *Let  $G$  be a real Lie group. Suppose that  $[S_{nc}, R] \neq 1$ . Then there exists a Lie subgroup  $H$  of  $G$  which is locally isomorphic to  $\text{SL}_2(\mathbf{R}) \times V_n(\mathbf{R})$  or  $\text{SL}_2(\mathbf{R}) \times H_{2n-1}(\mathbf{R})$  for some  $n \geq 2$ .*

*Remark 3.5.* 1) An analogous result holds with complex Lie groups.

2) The Lie subgroup  $H$  is not necessarily closed; this is due to the fact that  $\widetilde{\text{SL}_2(\mathbf{R})}$  and  $H_{2n-1}(\mathbf{R})$  have noncompact centre. For instance, take an element  $z$  of the centre of  $H$  that generates an infinite discrete subgroup, and take the image of  $H$  in the quotient of  $H \times \mathbf{R}/\mathbf{Z}$  by  $(z, \alpha)$ , where  $\alpha$  is irrational.

3) It can be easily be shown that, if the Lie group  $G$  is linear, then the subgroup  $H$  is necessarily closed. In a few words, this is because the derived subgroup of the radical is unipotent, hence simply connected, and the centre of the semisimple part is finite.

**3.3. The example  $\text{SO}_3$ .** We go on with the notation of 2.3.

In the context of algebraic  $\mathbf{R}$ -groups as in the context of connected Lie groups, the simply connected group corresponding to  $\mathfrak{so}_3(\mathbf{R})$  is  $\text{SU}(2)$ . The only non-simply connected corresponding group is  $\text{SO}_3(\mathbf{R})$ .

The irreducible  $\text{SU}(2)$ -modules corresponding to  $\mathfrak{d}_{2m+1}^{\mathbf{R}}$  and  $\mathfrak{u}_{4n}$  are denoted by  $D_{2m+1}^{\mathbf{R}}$  and  $U_{4n}$ . Among those, only  $D_{2n+1}^{\mathbf{R}}$  provide  $\text{SO}_3(\mathbf{R})$ -modules.

Denote by  $HU_{4n}^i$  the unipotent  $\mathbf{R}$ -group corresponding to  $\mathfrak{h}\mathfrak{u}_{4n}^i$ ,  $i = 0, 1, 2, 3$ .

*Remark 3.6.* It can be shown that the maximal unipotent subgroups of  $\text{Sp}(n, 1)$  are isomorphic to  $HU_{4n}^3$ .

<sup>2</sup>Write, for  $s \in S$  and  $r \in R$ ,  $s.r = ru(s, r)$ , where  $u(s, r) \in R_u$  and  $u(s, r) = 1$  if  $r \in R_u$ . Then, for all  $s, t \in S$  and  $r \in R$   $st.r = s.ru(t, r) = (s.r)(s.u(t, r)) = ru(s, r)u(t, r)$ , so that  $u(st, r) = u(s, r)u(t, r)$ . This implies that if  $s \in D^n S = S$ , then  $u(s, r) \in D^n R_u$ . Taking  $n$  sufficiently large, we obtain  $u(s, r) = 1$  for all  $s \in S$  and  $r \in R$ , that is,  $[S, R] = 1$ .

**Proposition 3.7.** *Up to isomorphism, the minimal solvable Lie  $\mathrm{SO}_3(\mathbf{R})$ -algebras are  $D_{2n+1}^{\mathbf{R}}$  for  $n \geq 1$ ; the other minimal solvable Lie  $\mathrm{SU}(2)$ -algebras are  $HU_{4n}^i$ , for  $n \geq 1$ ,  $i = 0, 1, 2, 3$ . ■*

**Proposition 3.8.** *Let  $G$  be a linear algebraic  $\mathbf{R}$ -group. Suppose that  $[S_c, R] \neq 1$ . Then  $G$  has a  $\mathbf{R}$ -subgroup which is  $\mathbf{R}$ -isomorphic to either  $\mathrm{SU}(2) \times D_{2n+1}^{\mathbf{R}}$ ,  $\mathrm{SO}_3(\mathbf{R}) \times D_{2n+1}^{\mathbf{R}}$ , or  $\mathrm{SU}(2) \times HU_{4n}^i$  for some  $i = 0, 1, 2, 3$  and some  $n \geq 1$ .*

*Let  $G$  be a real Lie group. Suppose that  $[S_c, R] \neq 1$ . Then  $G$  has a Lie subgroup which is locally isomorphic to either  $\mathrm{SU}(2) \times D_{2n+1}^{\mathbf{R}}$  or  $\mathrm{SU}(2) \times HU_{4n}^i$  for some  $i = 0, 1, 2, 3$  and some  $n \geq 1$ . ■*

#### 4. APPLICATION TO HAAGERUP AND KAZHDAN PROPERTIES

**4.1. Reminder.** Recall [CCJJV, Chap. 1] that a locally compact,  $\sigma$ -compact group  $G$  has the *Haagerup Property* if there exists a metrically proper, isometric action of  $G$  on some affine Hilbert space.

If  $H$  is a subgroup of  $G$ , the pair  $(G, H)$  has *Kazhdan Property (T)*, or that  $H$  has Kazhdan's Property (T) relatively to  $G$ , if every isometric action of  $G$  on any affine Hilbert space has a fixed point in restriction to  $H$ . In the case when  $H = G$ ,  $G$  is said to have Property (T) (see [HV] or [BHV]).

As an immediate consequence of these definitions, if  $(G, H)$  has Property (T) and  $H$  is not relatively compact in  $G$ , then  $G$  does not have the Haagerup Property; this is a frequent obstruction to Haagerup Property, although it is not the only one (see Remark 4.12).

The class of groups with the Haagerup Property generalizes the class of amenable groups as a strong negation of Kazhdan's Property (T). For other motivations of the Haagerup Property and equivalent definitions, see [CCJJV].

In the following lemma, we summarize the hereditary properties of the Haagerup and Kazhdan Properties that we will use in the sequel.

**Lemma 4.1.** *The Haagerup Property for locally compact,  $\sigma$ -compact groups is closed under taking (H1) closed subgroups, (H2) finite direct products, (H3) direct limits [CCJJV, Proposition 6.1.1], (H4) extensions with amenable quotient [CCJJV, Example 6.1.6], and (H5) is inherited from lattices [CCJJV, Proposition 6.1.5].*

*Relative Property (T) is inherited by dense images: if  $(G, H)$  has Property (T) and  $f : G \rightarrow K$  is a continuous morphism, then  $(K, \overline{f(H)})$  has Property (T).*

#### 4.2. Continuous families of Lie groups with Property (T).

**Proof of Proposition 1.12.** We must construct a continuous family of connected Lie groups with Property (T) and with perfect and pairwise non-isomorphic Lie algebras.

Consider  $\mathfrak{s} = \mathfrak{sp}_{2n}(\mathbf{R})$  ( $n \geq 2$ ). Let  $\mathfrak{v}_i$ ,  $i = 1, 2, 3, 4$  be four nontrivial absolutely irreducible,  $\mathfrak{s}$ -modules which are pairwise non-isomorphic and all preserve a symplectic form<sup>3</sup>. Then  $\mathfrak{v} = \bigoplus_{i=1}^4 \mathfrak{v}_i$  is a full  $\mathfrak{s}$ -module and  $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v}) = \prod_{i=1}^4 \mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v}_i) \simeq (\mathbf{R}^*)^4$ . In particular,  $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^* \simeq \mathbf{R}^4$  and  $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$  acts diagonally on it. The action on the 2-Grassmannian, which is 4-dimensional, is trivial on the scalars, so that its orbits are at most 3-dimensional. So there exists a continuous family  $(P_t)$  of 2-planes in  $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$  which are in pairwise distinct orbits for the action of  $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$ . By Theorem 3.1, the Lie  $\mathfrak{s}$ -algebras  $\mathfrak{h}(\mathfrak{v})/P_t$  are pairwise non-isomorphic, and so the Lie algebras  $\mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})/P_t$  are pairwise non-isomorphic. The Lie algebras  $\mathfrak{g}_t$  are perfect, and the corresponding Lie groups  $G_t$  have Property (T): this immediately follows from Wang's classification [Wang, Theorem 1.9]. ■

*Remark 4.2.* These examples have 2-nilpotent radical. This is, in a certain sense, optimal, since there exist only countably many isomorphism classes of Lie algebras over  $\mathbf{R}$  with abelian radical, and only a finite number for each dimension.

**Proof of Proposition 1.13.** We must construct a continuous family of locally isomorphic, pairwise non-isomorphic connected Lie groups with Property (T). The proof is actually similar to that of Proposition 1.12. Use the same construction, but, instead of taking the quotient  $G_t$  by  $P_t$ , take the quotient  $H_t$  by a lattice  $\Gamma_t$  of  $P_t$ . If we take the quotient of  $H_t$  by its biggest compact normal subgroup  $P_t/\Gamma_t$ , we obtain  $G_t$ . Accordingly, the groups  $H_t$  are pairwise non-isomorphic. ■

<sup>3</sup>There exist infinitely many such modules, which can be obtained by taking large irreducible components of the odd tensor powers of the standard  $2n$ -dimensional  $\mathfrak{s}$ -module.

### 4.3. Characterization of groups with the Haagerup Property.

**Proposition 4.3.** *Let  $\mathbf{K}$  be a local field of characteristic zero and  $n \geq 1$ . Then the pairs  $(\mathrm{SL}_2(\mathbf{K}) \times V_n(\mathbf{K}), \widetilde{V_n(\mathbf{K})})$ ,  $(\mathrm{PGL}_2(\mathbf{K}) \times V_n(\mathbf{K}), V_n(\mathbf{K}))$ ,  $(\mathrm{SL}_2(\mathbf{K}) \times H_n(\mathbf{K}), H_n(\mathbf{K}))$ ,  $(\mathrm{SL}_2(\mathbf{R}) \times V_n(\mathbf{R}), V_n(\mathbf{R}))$ , and  $(\mathrm{SL}_2(\mathbf{R}) \times H_n(\mathbf{R}), H_n(\mathbf{R}))$  have Property (T).*

**Proof:** The first (and the fourth) case is well-known; it follows, for instance, from Furstenberg's theory [FUR] of invariant probabilities on projective spaces, which implies that  $\mathrm{SL}_2(\mathbf{K})$  does not preserve any probability on  $V_n(\mathbf{K})$  (more precisely, on its Pontryagin dual) other than the Dirac measure at zero. See, for instance, the proof [HV, Chap. 2, Proposition 2]. The second case is an immediate consequence of the first. For the third (resp. fifth) case, we invoke [CCJJV, Proposition 4.1.4], with  $S = \mathrm{SL}_2(\mathbf{K})$ ,  $N = H_n(\mathbf{K})$ , even if the hypotheses are slightly different (unless  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ ): the only modification is that, since here  $[N, S]$  is not necessarily connected, we must show that its image in the unitary group  $U_n$  is connected so as to justify Lie's Theorem. Otherwise, it would have a nontrivial finite quotient. This is a contradiction, since  $[N, S]$  is generated by divisible elements; this is clear, since, as the group of  $\mathbf{K}$ -points of a unipotent group, it has a well-defined logarithm. ■

**Corollary 4.4.** *Let  $G$  be either a connected Lie group, or  $G = \mathbf{G}(\mathbf{K})$ , where  $\mathbf{G}$  is a linear algebraic group over the local field  $\mathbf{K}$  of characteristic zero. Suppose that the Lie algebra  $\mathfrak{g}$  of  $G$  contains a subalgebra  $\mathfrak{h}$  isomorphic to either  $\mathfrak{sl}_2 \times \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \times \mathfrak{h}_{2n-1}$  for some  $n \geq 2$ . Then  $G$  has a noncompact closed subgroup with relative Property (T). In particular,  $G$  does not have Haagerup's property.*

**Proof:** Let us begin by the case of algebraic groups. By [Bor, Chap. II, Corollary 7.9], since  $\mathfrak{h}$  is perfect, it is the Lie algebra of a closed  $\mathbf{K}$ -subgroup  $H$  of  $G$ . Since  $H$  must be  $\mathbf{K}$ -isomorphic to either  $\mathrm{SL}_2 \times V_m$ ,  $\mathrm{PGL}_2 \times V_{2m-1}$ , or  $\mathrm{SL}_2 \times H_{2m-1}$  for some  $m \geq 2$ , Proposition 4.3 implies that  $G(\mathbf{K})$  has a noncompact closed subgroup with relative Property (T).

In the case of Lie groups, we obtain a Lie subgroup which is the image of an immersion  $i$  of  $\widetilde{\mathrm{SL}_2(\mathbf{R})} \times N$ , where  $N$  is either  $V_n(\mathbf{R})$  or  $H_{2n-1}(\mathbf{R})$ , for some  $n \geq 2$ , into  $G$ . By Proposition 4.3,  $(G, i(N))$  has Property (T). We claim that  $i(N)$  is not compact. Suppose the contrary. Then it is solvable and connected, hence it is a torus. It is normal in the closure  $H$  of  $i(G)$ . Since the automorphism group of a torus is totally disconnected, the action by conjugation of  $H$  on  $i(N)$  is trivial; that is,  $i(N)$  is central in  $H$ . This is a contradiction. ■

**Proof of Theorem 1.10.** As we already noticed in the reminder, (i) $\Rightarrow$ (ii) is immediate from the definition. We are going to prove (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

For the implication (iii) $\Rightarrow$ (i), in the algebraic case,  $G$  is isomorphic, up to a finite kernel, to  $S_{nc}(\mathbf{K}) \times \mathrm{Mr}(\mathbf{K})$ , where  $\mathrm{Mr}$  denotes the amenable radical of  $\mathbf{G}$ . The group  $\mathrm{Mr}(\mathbf{K})$  is amenable, hence has Haagerup's property. The group  $S_{nc}(\mathbf{K})$  also has Haagerup's property: if  $\mathbf{K}$  is Archimedean, it maps, with finite kernel, onto a product of groups isomorphic to  $\mathrm{PSO}_0(n, 1)$  or  $\mathrm{PSU}(n, 1)$  ( $n \geq 2$ ), and these groups have Haagerup's property, by a result of Faraut and Harzallah, see [BHV, Chap. 2]. If  $\mathbf{K}$  is non-Archimedean, then  $S_{nc}(\mathbf{K})$  acts properly on a product of trees (one for each simple factor) [BT], and this also implies that it has Haagerup's property [BHV, Chap. 2].

The same argument also works for connected Lie groups when the semisimple part has finite centre; in particular, this is fulfilled for linear Lie groups and their finite coverings. The case when the semisimple part has infinite centre is considerably more involved, see [CCJJV, Chap. 4].

(ii) $\Rightarrow$ (iv) Suppose that (iv) is not satisfied. If  $\mathfrak{g}$  contains a copy of  $\mathfrak{sl}_2 \times \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \times \mathfrak{h}_{2n-1}$  for some  $n \geq 2$ , then, by Corollary 4.4,  $G$  does not satisfy (ii). If  $\mathbf{K} = \mathbf{R}$ , we consider  $G$  as a Lie group with finitely many components. By a standard argument, since  $\mathrm{Sp}(2, 1)$  is simply connected with finite centre (of order 2), an embedding of  $\mathfrak{sp}(2, 1)$  into  $\mathfrak{g}$  corresponds to a closed embedding of  $\mathrm{Sp}(2, 1)$  or  $\mathrm{PSp}(2, 1)$  into  $G$ . Since  $\mathrm{Sp}(2, 1)$  has Property (T) [BHV, Chap. 3], this contradicts (ii).

(iv) $\Rightarrow$ (iii) If  $\mathfrak{g}$  is not M-decomposed, then, by Proposition 1.9, it contains a copy of  $\mathfrak{sl}_2 \times \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \times \mathfrak{h}_{2n-1}$  for some  $n \geq 2$ .

If  $\mathfrak{g}$  has a simple factor  $\mathfrak{s}$ , then  $\mathfrak{s}$  embeds in  $\mathfrak{g}$  through a Levi factor. If  $\mathfrak{s}$  has  $\mathbf{K}$ -rank  $\geq 2$ , then it contains a subalgebra isomorphic to either  $\mathfrak{sl}_3$  or  $\mathfrak{sp}_4$  [Mar, Chap I, (1.6.2)], and such a subalgebra contains a subalgebra isomorphic to  $\mathfrak{sl}_2 \times \mathfrak{v}_2$  (resp.  $\mathfrak{sl}_2 \times \mathfrak{v}_3$ ) [BHV, 1.4 and 1.5].

Finally, if  $\mathbf{K} = \mathbf{R}$  and  $\mathfrak{s}$  is isomorphic to either  $\mathfrak{sp}(n, 1)$  for some  $n \geq 2$  or  $\mathfrak{f}_{4(-20)}$ , then it contains a copy of  $\mathfrak{sp}(2, 1)$ . ■

*Remark 4.5.* Conversely,  $\mathfrak{sp}(n, 1)$  does not contain any subalgebra isomorphic to  $\mathfrak{sl}_2 \times \mathfrak{v}_n$  or  $\mathfrak{sl}_2 \times \mathfrak{h}_{2n-1}$  for any  $n \geq 2$ ; this can be shown using results of [CDSW] about weak amenability.

**4.4. Subgroups of Lie groups.** Let us exhibit some subgroups in the groups above.

*Observation 4.6.* Let  $G$  denote  $\mathrm{SL}_2 \times V_n$ ,  $\mathrm{PGL}_2 \times V_{2n-1}$ , or  $\mathrm{SL}_2 \times H_{2n-1}$  for some  $n \geq 2$ , and  $R$  its radical. Then, for every field  $K$  of characteristic zero,  $G(K)$  contains  $G(\mathbf{Z})$  as a subgroup. On the other hand, the pair  $(G(\mathbf{Z}), R(\mathbf{Z}))$  has Property (T), this is because  $G(\mathbf{Z})$  is a lattice in  $G(\mathbf{R})$ .

*Observation 4.7.* Now, let  $G$  denote  $\mathrm{SU}(2) \times D_{2n+1}^{\mathbf{R}}$ ,  $\mathrm{SO}_3(\mathbf{R}) \times D_{2n+1}^{\mathbf{R}}$ , or  $\mathrm{SU}(2) \times HU_{4n}^i$  for some  $i = 0, 1, 2, 3$ . These groups are all defined over  $\mathbf{Q}$ : this is obvious at least for all but  $\mathrm{SU}(2) \times HU_{4n}^i$  for  $i = 1, 2$ ; for these two, this is because the subspace  $Z_i$  can be chosen rational in the definition of  $HU_{4n}^i$ .

Let  $R$  be the radical of  $G$  and  $S$  a Levi factor defined over  $\mathbf{Q}$ . Let  $F$  be a number field of degree three over  $\mathbf{Q}$ , not totally real. Let  $\mathcal{O}$  be its ring of integers. Then  $G(\mathcal{O})$  embeds diagonally as an irreducible lattice in  $G(\mathbf{R}) \times G(\mathbf{C})$ . Its projection  $\Gamma$  in  $G(\mathbf{R})$  does not have Haagerup's property, since otherwise  $G(\mathbf{C})$  would also have Haagerup's property (by (H5) in Lemma 4.1), and this is excluded since it does not satisfy  $[S_{nc}, R] = 1$ , see Proposition 4.4 (the anisotropic Levi factor becomes isotropic after complexification).

**Proposition 4.8.** *Let  $G$  be a real Lie group,  $R$  its radical,  $S$  a semisimple factor. Suppose that  $[S, R] \neq 1$ . Then  $G$  has a countable subgroup without Haagerup's property.*

**Proof:** First case:  $[S_{nc}, R] \neq 1$ . Then, by Proposition 3.4,  $G$  has a Lie subgroup  $H$  isomorphic to a quotient of  $\tilde{H} = \widetilde{\mathrm{SL}_2(\mathbf{R})} \times R(\mathbf{R})$  by a discrete central subgroup, where  $R = V_n$  or  $H_{2n-1}$ , for some  $n \geq 2$ . Denote by  $\tilde{H}(\mathbf{Z})$  the inverse image of  $\mathrm{SL}_2(\mathbf{Z}) \times R(\mathbf{Z})$  in  $\tilde{H}$ . By the observation above,  $(\tilde{H}(\mathbf{Z}), R(\mathbf{Z}))$  has Property (T), so that its image in  $H$ , which we denote by  $H(\mathbf{Z})$ , satisfies  $(H(\mathbf{Z}), R_G(\mathbf{Z}))$  has Property (T), where  $R_G(\mathbf{Z})$  means the image of  $R(\mathbf{Z})$  in  $G$ . Observe that  $R_G(\mathbf{Z})$  is infinite: if  $R = V_n$ , this is  $V_n(\mathbf{Z})$ ; if  $R = H_{2n-1}$ , this is a quotient of  $H_{2n-1}(\mathbf{Z})$  by some central subgroup. Accordingly,  $H(\mathbf{Z})$  does not have Haagerup's property.

Second case:  $[S_c, R] \neq 1$ . By Proposition 3.8,  $G$  has a Lie subgroup  $H$  isomorphic to a central quotient of  $\mathrm{SU}(2)(\mathbf{R}) \times R$ , where  $R = D_{2n+1}^{\mathbf{R}}$  or  $HU_{4n}^i$ , for some  $n \geq 1$  and  $i = 0, 1, 2, 3$ .

First suppose that the radical of  $H$  is simply connected. Then, by Observation 4.7,  $H$  has a subgroup without the Haagerup property.

Now, let us deal with the case when  $H = \tilde{H}/Z$ , where  $Z$  is a discrete central subgroup. Then  $\tilde{H}$  has a subgroup  $\Gamma$  as above which does not have Haagerup's property. Let  $W$  denote the centre of  $\tilde{H}$ . The kernel of the projection of  $\Gamma$  to  $H$  is given by  $\Gamma \cap Z$ . We use the following trick: we apply an automorphism  $\alpha$  of  $\tilde{H}$  such that  $\alpha(\Gamma) \cap Z$  is finite. It follows that the image of  $\alpha(\Gamma)$  in  $H$  does not have Haagerup's property.

This allows to suppose that  $\Gamma \cap Z$  is finite, so that the image of  $\Gamma$  in  $H$  does not have Haagerup's property. Let us construct such an automorphism.

Observe that the representations of  $\mathrm{SU}(2)$  can be extended to the direct product  $\mathbf{R}^* \times \mathrm{SU}(2)$  by making  $\mathbf{R}^*$  act by scalar multiplication. This action lifts to an action of  $\mathbf{R}^* \times \mathrm{SU}(2)$  on  $HU_{4n}^i$ , where the scalar  $a$  acts on the derived subgroup of  $HU_{4n}^i$  by multiplication by  $a^2$ .

Now, working in the unit component of the centre  $W$  of  $\tilde{H}$ , which we treat as a vector space, we can take  $a$  so that  $a^2 \cdot (\Gamma \cap W)$  avoids  $Z \cap W$  ( $a$  clearly exists, since  $\Gamma$  and  $Z$  are countable). ■

**Definition 4.9.** Let  $G$  be a locally compact group. We say that  $G$  has Haagerup's property if every  $\sigma$ -compact open subgroup of  $G$  is.

*Remark 4.10.* In view of (H3) of Lemma 4.1, this is equivalent to: every compactly generated, open subgroup of  $G$  has Haagerup's property, and also equivalent to the existence of a  $C_0$ -representation with almost invariant vectors [CCJJV, Chap. 1]. In particular,  $G$  having Haagerup's property and  $(G, H)$  having Property (T) still imply  $H$  relatively compact.

All properties of the class of groups with Haagerup's property claimed in Lemma 4.1 also clearly remain true for general locally compact groups.

If  $G$  is a topological group, denote by  $G_d$  the group  $G$  endowed with the discrete topology.

**Proof of Theorem 1.14.** We remind that we must prove, for a connected Lie group  $G$ , the equivalence between

(i)  $G$  is locally isomorphic to  $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$ , with  $R$  solvable and integers  $a, b, c$ , and

(ii)  $G_d$  has Haagerup's property.

The implication (i)  $\Rightarrow$  (ii) is, essentially, a deep and recent result of Guentner, Higson, and Weinberger [GHW, Theorem 5.1], which implies that  $(\mathrm{PSL}_2(\mathbf{C}))_d$  has Haagerup's property. Let  $G$  be as in (i), and  $S$  its semisimple factor. Then  $G/S$  is solvable, so that, by (H4) of Lemma 4.1, we can reduce to the case when  $G = S$ . Now, let  $Z$  be the centre of the semisimple group  $G$ , and embed  $G_d$  in  $(G/Z)_d \times G$ , where  $G_d$  means  $G$  endowed with the discrete topology. This is a discrete embedding. Since  $G$  has Haagerup's property, this reduces the problem to the case when  $G$  has trivial centre. So, we are reduced to the cases of  $\mathrm{SO}_3(\mathbf{R})$ ,  $\mathrm{PSL}_2(\mathbf{R})$ , and  $\mathrm{PSL}_2(\mathbf{C})$ . The two first groups are contained in the third, so that the result follows from the Guentner-Higson-Weinberger Theorem.

Conversely, suppose that  $G$  does not satisfy (i).

If  $[S, R] \neq 1$ , then, by Proposition 4.8,  $G_d$  does not have Haagerup's property. Otherwise, observe that the simple factors allowed in (i) are exactly those of geometric rank one (viewing  $\mathrm{SL}_2(\mathbf{C})$  as a complex Lie group). Hence,  $S$  has a factor  $W$  which is not of geometric rank one. Then the result is provided by Lemma 4.11 below. ■

**Lemma 4.11.** *Let  $S$  be a simple Lie group which is not locally isomorphic to  $\mathrm{SO}_3(\mathbf{R})$ ,  $\mathrm{SL}_2(\mathbf{R})$  or  $\mathrm{SL}_2(\mathbf{C})$ . Then  $S_d$  does not have Haagerup's property.*

**Proof:** Let  $Z$  be the centre of  $S$ , so that  $S/Z \simeq G(\mathbf{R})$  for some  $\mathbf{R}$ -algebraic group  $G$ . By assumption,  $G(\mathbf{C})$  has factors of higher rank, hence does not have Haagerup's property. Let  $F$  be a number field of degree three over  $\mathbf{Q}$ , not totally real. Let  $\mathcal{O}$  be its ring of integers. Then  $G(\mathcal{O})$  embeds diagonally as an irreducible lattice in  $G(\mathbf{R}) \times G(\mathbf{C})$ , and is isomorphic to its projection in  $G(\mathbf{R})$ . Let  $\Gamma$  be the inverse image in  $S \times G(\mathbf{C})$  of  $G(\mathcal{O})$ . Then  $\Gamma$  is a lattice in  $S \times G(\mathbf{C})$ . Hence, by [CCJJV, Proposition 6.1.5],  $\Gamma$  does not have Haagerup's property. Note that the projection  $\Gamma'$  of  $\Gamma$  into  $S$  has finite kernel, contained in the centre of  $G(\mathbf{C})$ . So  $\Gamma'$  neither has Haagerup's property, and is a subgroup of  $S$ . ■

*Remark 4.12.* Theorem 1.14 is no longer true if we replace the statement “ $G_d$  has Haagerup's property” by “ $G_d$  has no infinite subgroup with relative Property (T)”. Indeed, let  $G = K \ltimes V$ , where  $K$  is locally isomorphic to  $\mathrm{SO}_3(\mathbf{R})^n$  and  $V$  is a vector space on which  $K$  acts nontrivially. Suppose that  $(G_d, H)$  has Property (T) for some subgroup  $H$ . Then  $(G_d/V, H/(H \cap V))$  has Property (T). In view of the Guentner-Higson-Weinberger Theorem (see the proof of Theorem 1.14),  $H/(H \cap V)$  is finite. On the other hand, since  $G$  has Haagerup's property,  $H \cap V$  must be relatively compact, and this implies that  $H \cap V = 1$ . Thus,  $H$  is finite.

Motivated by this example, it is easy to exhibit finitely generated groups without the Haagerup Property and do not have infinite subgroups with relative Property (T). For instance, let  $n \geq 3$ , and  $q$  be the quadratic form  $\sqrt{2}x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n-1}^2$ . Let  $G(R) = \mathrm{SO}(q)(R) \ltimes R^n$  and write, for any commutative  $\mathbf{Q}(\sqrt{2})$ -algebra  $R$ ,  $H(R) = \mathrm{SO}(q)(R)$ . Then  $\Gamma = G(\mathbf{Z}[\sqrt{2}])$  is such an example. The fact that  $\Gamma$  has no infinite subgroup  $\Lambda$  with relative Property (T) can be seen without making use of the Guentner-Higson-Weinberger Theorem: first observe that  $H(\mathbf{Z}[\sqrt{2}])$  is a cocompact lattice in  $\mathrm{SO}(n-1, 1)$ , hence has Haagerup's property. So the projection of  $\Lambda$  in  $H(\mathbf{Z}[\sqrt{2}])$  is finite. So, upon passing to a finite index subgroup, we can suppose that  $\Lambda$  is contained in the subgroup  $\mathbf{Z}[\sqrt{2}]^n$  of  $\Gamma = \mathrm{SO}(q)(\mathbf{Z}[\sqrt{2}]) \ltimes \mathbf{Z}[\sqrt{2}]^n$ . But then the closure  $L$  of  $\Lambda$  in the subgroup  $\mathbf{R}^n$  of the amenable group  $G(\mathbf{R}) = \mathrm{SO}(q)(\mathbf{R}) \ltimes \mathbf{R}^n$  is not compact, and  $(G(\mathbf{R}), L)$  has Property (T). This is a contradiction.

On the other hand,  $\Gamma$  does not have Haagerup's property, since it is a lattice in  $G(\mathbf{R}) \times G^\sigma(\mathbf{R})$  (use (H5) of Lemma 4.1), where  $\sigma$  is the nontrivial automorphism of  $\mathbf{Q}(\sqrt{2})$ , and  $G^\sigma(\mathbf{R}) \simeq \mathrm{SO}(n-1, 1) \ltimes \mathbf{R}^n$  does not have Haagerup's property, by Corollary 4.4. Note that  $\Gamma$ , as a cocompact lattice in a connected Lie group, is finitely presented.

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