NONLINEARITY OF SOME SUBGROUPS OF THE PLANAR CREMONA GROUP

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Abstract. We give some examples of non-nilpotent locally nilpotent, and hence nonlinear subgroups of the planar Cremona group.

0. Foreword

This note has been circulating since 2012 and has been posted to arXiv in 2017 (`arXiv:1701.00275`), *unchanged up to updates and minor corrections. Since it has been quoted at various places, I have finally wrote down a version aimed at publication. In addition to the existing material I have added a few remarks an additional section (Section 3), notably motivated by the follow-up paper* [Mat] *by O. Mathieu.*

1. Introduction

Let K be a field. The planar *Cremona group* $\mathrm{Cr}_2(K)$ of K is defined as the group of birational self-transformations of the 2-dimensional affine space over K. It can also be described as the group of K-automorphisms of the field of rational functions $K(t_1, t_2)$. More generally, one defines $\mathrm{Cr}_d(K)$.

We provide here two observations about the planar Cremona group. The first is an example of a non-linear finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$. The existence of such a subgroup was known to some experts: for instance it follows from an unpublished construction of S. Cantat (using superrigidity of lattices); our example has the additional feature of being 3-solvable. Its non-linearity follows from the fact it contains torsion-free nilpotent subgroups of arbitrary large nilpotency class. We also show here that $\mathrm{Cr}_2(K)$ has no nontrivial linear representation over any field, extending a result of Cerveau and Déserti (all representations below are assumed finite-dimensional).

The group of K-defined automorphisms $\mathrm{Aut}_K \mathbb{A}^2 \subset \mathrm{Cr}_2(K)$ of the affine plane (often denoted $\mathrm{Aut}(K^2)$ in the literature) will naturally appear in the discussion. We end this short introduction with a few questions.

(1) (Cantat) for $d \geq 2$, and any field K, is $\mathrm{Cr}_d(K)$ locally residually finite (i.e. is every finitely generated subgroup residually finite)?
(2) Does there exist a finitely generated subgroup of $\mathrm{Aut}_{\mathbf{C}} \mathbb{A}^2$ with no faithful linear representation? *(now answered positively by O. Mathieu* [Mat]*)*

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2. A nonlinear subgroup of the Cremona group

For a group G, define $G^1 = G$ and $G^{i+1} = [G, G^i]$. The group G is called n-step nilpotent if $G^{n+1} = \{1\}$; the smallest such n is called nilpotency class of G.

We provide in this section an example of a finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$ that is not linear over any field. It is 3-step solvable and actually lies in the Jonquières subgroup, that is, the group of birational transformations preserving the partition of $\mathbf{C}^2$ by horizontal lines.

If $f \in K(X)$ and $g \in K(X)^\times$, define $\alpha_f, \mu_g \in \mathrm{Cr}_2(K)$ by

$$\alpha_f(x, y) = (x, y + f(x)); \quad \mu_g(x, y) = (x, yg(x)).$$

We have

$$\alpha_{f+f'} = \alpha_f \alpha_{f'}; \quad \mu_{gg'} = \mu_g \mu_{g'}; \quad \mu_g \alpha_f \mu_g^{-1} = \alpha_{fg}.$$

Also for $t \in K$, define $s_t \in \mathrm{Cr}_2(K)$ by $(x, y) = (x + t, y)$, so that

$$s_t \alpha_{f(X)} s_t^{-1} = \alpha_{f(X-t)}; \quad s_t \mu_{g(X)} s_t^{-1} = \mu_{g(X-t)}.$$

For $n \geq 0$, consider the subgroup Γ_n of $\mathrm{Cr}_2(K)$ generated by s_1 and α_{X^n}; note that $\Gamma_n \subseteq \mathrm{Aut}_{\mathbf{Q}}(\mathbb{A}^2)$.

Lemma 2.1. *The finitely generated group Γ_n is nilpotent of class at most $n+1$; moreover if K has characteristic zero the nilpotency class of Γ_n is $n+1$, and Γ_n is torsion-free.*

Proof. We introduce the larger subgroup R_n, generated by s_1 and by the abelian subgroup A_n consisting of all α_P, where P ranges over polynomials of degree at most n. Then A_n is normalized by s_1 and $[s_1, A_n] \subset A_{n-1}$ for all $n \geq 1$, while $A_0 = \{1\}$. Therefore R_n is nilpotent of class at most $n+1$, and therefore so is Γ_n. Conversely, the n-iterated group commutator

$$[s_1, [s_1, \cdots, [s_1, \alpha_{X^n}] \cdots]] \in \Gamma_n^{n+1}$$

is equal to $\alpha_{\Delta^n X^n}$, where Δ is the discrete differential operator $\Delta P(X) = -P(X) + P(X-1)$. So if K has characteristic zero (or $p > n$) then $\Delta^n X^n \neq 0$ and hence Γ_n is not n-step nilpotent. In this case it is also clear that R_n is torsion-free. $\square$

Now assume that K has characteristic zero and consider the group $\Xi \subset \mathrm{Cr}_2(\mathbf{Q}) \subset \mathrm{Cr}_2(K)$ generated by $\{s_1, \alpha_1, \mu_X\}$.

Proposition 2.2. *The finitely generated group $\Xi \subset \mathrm{Cr}_2(\mathbf{Q})$ is solvable of length three; it is not linear over any field.*

The following lemma is most likely well-known.

Lemma 2.3. *Let K be a field and $n \geq 0$. Suppose that $\mathrm{GL}_n(K)$ has a torsion-free nilpotent subgroup of nilpotency class $c \geq 0$. Then $c \leq n$ (and if K has positive characteristic then $c \leq 1$).*

Proof. We can suppose that K is algebraically closed and $n \geq 1$. Since Γ is Zariski-dense in its Malcev completion, every finite index subgroup of Γ also has nilpotency class c. In particular, we can suppose that the Zariski closure of Γ in GL_n is connected, and hence assume that Γ consists of upper triangular matrices. If the characteristic is positive, the derived subgroup of Γ is torsion and we deduce that Γ is abelian.

Now assume that the characteristic is zero, and let us show by induction on $n \geq 0$ the given upper bound for Zariski-connected (rather than torsion-free) nilpotent subgroups. The case $n = 0$ is trivial; assume now $n \geq 1$. Let G be the Zariski closure, assumed connected, of Γ. Since G is nilpotent, we have $G = D \times U$ with D the unique maximal torus and U the unipotent radical. If D does not act by scalars, then G preserves a non-trivial direct product decomposition and by induction it follows that $c \leq n-1$. Otherwise D acts by scalars: then we directly see that the whole group of upper triangular matrices with constant diagonal has nilpotency class equal to n, and hence $c \leq n$. ☐

Proof of Proposition 2.2. From the conjugation relations above it is clear that the subgroup generated by s_1, all α_f and μ_g, is solvable of length at most three. If we restrict to those g of the form $\prod_{n \in \mathbf{Z}} (X-n)^{k_n}$ (where (k_n) is finitely supported), we obtain a subgroup containing Γ, that is clearly torsion-free.

Since $\mu_X^n \alpha_1 \mu_X^{-n} = \alpha_{X^n}$, we see that Ξ contains Γ_n for all n, which is torsion-free nilpotent of class $n+1$. Therefore, by Lemma 2.3 it has no linear representation over any field.

The fact that Ξ is not 2-step solvable (=metabelian) can be checked by hand, but also follows from the fact that every torsion-free finitely generated metabelian group is linear over a field of characteristic zero [Rem]. ☐

We easily see $\Gamma_n \subset \Gamma_{n+1}$ for all n. Denoting $\Gamma_\infty = \bigcup \Gamma_n$, we see that the torsion-free group Γ_∞ is locally nilpotent (that is, all its finitely generated subgroups are nilpotent) and the above argument works for it. Since Γ_∞ is contained in $\mathrm{Aut}_{\mathbf{Q}} \mathbb{A}^2$, we also get:

Proposition 2.4. *The group Γ_∞ is not linear over any field, and hence neither are $\mathrm{Aut}_{\mathbf{Q}} \mathbb{A}^2$ and $\mathrm{Aut}_{\mathbf{C}} \mathbb{A}^2$.*

With little further effort, it actually follows from the same argument that Γ_∞ (and hence Ξ) is not linear over any finite product of fields. Better, it is not linear over any product of fields (and therefore over any reduced commutative ring). This is based on the following improvement of Lemma 2.3 (which fixes an imprecise argument from the original unpublished version).

Lemma 2.5. *Let R be a reduced scalar (=commutative associative unital) ring and $n \geq 0$. Suppose that $\mathrm{GL}_n(R)$ has a torsion-free nilpotent subgroup of nilpotency class $c \geq 0$. Then $c \leq n$.*

Proof. We can suppose $c \geq 2$. Since every nilpotent group of class c has a finitely generated subgroup of class c (find a suitable nonzero iterated bracket and choose the subgroup generated by the group elements involved in the given bracket), we can suppose that this subgroup Γ is finitely generated, and hence, we can suppose that R is a finitely generated ring. Let $P_1, \dots, P_m$ be the minimal prime ideals in R (since R is noetherian, they are finitely many). Since R is reduced, $\bigcap_i P_i = \{0\}$. So $\mathrm{GL}_n(R)$ embeds into $\prod_i \mathrm{GL}_n(K_i)$, where K_i is the field of fractions of R/P_i. Let Γ_i be the projection of Γ in $\mathrm{GL}_n(K_i)$, and T_i the set of torsion elements in Γ_i. Since Γ_i is a finitely generated nilpotent group, T_i is a finite normal subgroup of Γ_i. Since Γ is torsion-free and its diagonal map into $\prod_i \Gamma_i$ is injective and the T_i are finite, the diagonal map of Γ into $\prod_i \Gamma_i/T_i$ is also injective. So there exists j such that Γ_j/T_j has nilpotency class c. Hence, if Λ_j is a torsion-free finite index subgroup of Γ_i, then Λ_j has nilpotency class c. Hence, by Lemma 2.3, we have $c \leq n$. $\square$

A nice observation by D. Cerveau and J. Déserti [CD, Lemme 5.2] is that the Cremona group has no faithful linear representation in characteristic zero. Actually, an easy refinement of the same argument provides a stronger result.

Proposition 2.6. *If K is an algebraically closed field, there is no nontrivial finite-dimensional linear representation of $\mathrm{Cr}_2(K)$ over any field.*

(Note that since the Cremona group is not simple by a result of S. Cantat and S. Lamy [CL], the non-existence of a faithful representation does not formally imply the non-existence of a nontrivial representation.)

Proof of Proposition 2.6. In $\mathrm{Cr}_2(K)$, there is a natural copy of $G = (K^\times)^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by the automorphism $\sigma(x, y) = (x, xy)$ of $(K^\times)^2$. Here, it corresponds, in affine coordinates, to the group of transformations of the form

$$(x_1, x_2) \mapsto (\lambda_1 x_1, x_1^n \lambda_2 x_2) \quad \text{for} \quad (\lambda_1, \lambda_2, n) \in (K^\times)^2 \times \mathbf{Z}.$$

Consider an linear representation $\rho : G \to \mathrm{GL}_n(F)$, where F is any field (here G is viewed as a discrete group). If p is a prime which is nonzero in K and if $\omega_p \in K$ is a primitive p-root of unity, set $\alpha_p(x_1, x_2) = (\omega_p x_1, \omega_p x_2)$ and $\beta_p(x_1, x_2) = (x_1, \omega_p x_2)$. Then $\sigma \alpha_p \sigma^{-1} \alpha_p^{-1} = \beta_p$ and commutes with both σ and α_p. An argument of Birkhoff [Bi, Lemma 1] shows that if $\rho(\alpha_p) \neq 1$ then $n > p$ (the short argument given in the proof of [CD, Lemme 5.2] for F of characteristic zero works if it is assumed that p is not the characteristic of F).

Picking p to be greater than n and the characteristics of K and F, this shows that if we have an arbitrary representation $\pi : \mathrm{Cr}_2(K) \to \mathrm{GL}_n(F)$, the restriction of π to $\mathrm{PGL}_3(K)$ is not faithful; since $\mathrm{PGL}_3(K)$ is simple, this implies that

π is trivial on $\mathrm{PGL}_3(K)$; since $\mathrm{Cr}_2(K)$ is generated by $\mathrm{PGL}_3(K)$ as a normal subgroup, this yields the conclusion. $\square$

Remark 2.7. The subsequent paper [Reg] by A. Regeta also uses the idea of considering suitable families of (solvable) algebraic subgroups to prove that $\mathrm{Aut}(X)$ is non-linear for various varieties X (in characteristic zero).

Remark 2.8. The subsequent paper [Mat] by O. Mathieu shows, by a more elaborate argument, that $\mathrm{Aut}_K(\mathbb{A}^2)$ is not linear over any field, regardless of the infinite field K. The case when K has positive characteristic is indeed significantly more difficult.

3. Additional comments

After this paper has been circulating, O. Mathieu [Mat] has answered positively Question (2) of the introduction. This provides a stronger obstruction to linearity for $\mathrm{Aut}_{\mathbf{Q}}\,\mathbb{A}^2$. Our proof of Proposition 2.4 was based on the fact that $\mathrm{Aut}_{\mathbf{Q}}\,\mathbb{A}^2$ contains torsion-free nilpotent subgroups of unbounded nilpotency class. These can be viewed as complementary results, and actually have no "common denominator", in view of the following fact:

Proposition 3.1. *Let K be a field of characteristic zero and let G be a finitely generated subgroup of $\mathrm{Aut}_K\,\mathbb{A}^2$. Then torsion-free nilpotent subgroups of G have bounded nilpotency class, i.e., have nilpotency class $\le c_G$, for some c_G depending only on G.*

(This is false without the restriction to torsion-free groups, simply because the finite dihedral groups of order 2^n have unbounded nilpotency class ($n-1$ for each $n \ge 2$), and are isomorphic to subgroups of $\mathrm{GL}_2(\mathbf{C})$.)

To prove the proposition, we can suppose that $K = \mathbf{C}$. Then we have the amalgam decomposition $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2 = A *_C B$ where A is the subgroup of affine automorphisms and B is the the subgroup of Jonquières automorphisms, i.e., of the form $(x, y) \mapsto (ax+b, cy+P(x))$ and $C = A \cap B$. This decomposition induces an inversion-free action on a tree with two vertex orbits, one edge orbit, and one edge with stabilizer C whose vertices have stabilizers A and B.

S. Lamy [Lam] has "classified" subgroups Γ of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$ with respect to the action on this tree. For torsion-free subgroups Γ, this classification simplifies as follows

Lemma 3.2 (Lamy)**.** *For a torsion-free subgroup of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$, one of the three mutually exclusive possibilities holds*

(a) Γ *has a fixed vertex, or equivalently is conjugate to a subgroup of either A or B;*
(b) Γ *is infinite cyclic, generated by a loxodromic element;*
(c) Γ *has two loxodromic elements with disjoint endpoints, and in particular contains a non-abelian free subgroup.*

On the proof. This is not summarized in this way in [Lam]. For each group acting inversion-free on a tree, in general we have exactly one of the five possibilities

(a) *(bounded)* there is a fixed vertex;
(a') *(horocyclic)* each element has a fixed vertex, there is no fixed vertex, there is a unique fixed point at infinity;
(b) *(axial)* there is a loxodromic element and an invariant geodesic;
(b') *(focal)* there loxodromic elements, all sharing a common endpoint, but no invariant geodesic;
(c) *(general type)* there are two loxodromics with no common endpoint.

It is a general standard fact that (c) implies the existence of a non-abelian free subgroup.

In the setting of subgroups of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$, [Lam, Cor. 4.2] excludes Case (b'). Also, [Lam, Prop. 3.3] says that elements with an unbounded fixed-point-set are torsion. This implies that every subgroup as in (a') is torsion and, in case (b), the kernel of the natural homomorphism to the automorphism group of the unique invariant geodesic is torsion. Hence, for a torsion-free subgroup of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$, (a') is impossible, and (b) implies that the group is isomorphic to either $\mathbf{Z}$ or the infinite dihedral group, the latter being again excluded since it is not torsion-free. □

Proof of Proposition 3.1. It follows that every torsion-free nilpotent subgroup of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$ is conjugate into either A or B. Since A embeds into $\mathrm{GL}_3(\mathbf{C})$, every torsion-free nilpotent subgroup of A has nilpotency class ≤ 2. Hence every torsion-free nilpotent subgroup of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$, of class ≥ 3, is conjugate into B.

Let B_n be the subgroup of B of such automorphisms with P of degree $\le n$. Then, for $n \ge 1$, we have $C \subset B_n$, and, denoting by G_n the subgroup generated by $A \cup B_n$, we have $G_n = A *_C B_n$. If Γ is a finitely generated subgroup of $\mathrm{Aut}_{\mathbf{C}}\,\mathbb{A}^2$, it is contained in G_n for some n. In turn, let N be a torsion-free nilpotent subgroup of G_n, of nilpotency class ≥ 3. By the previous discussion, it fixes, in the tree T associated to the amalgam decomposition $A *_C B$, a unique vertex, which is in the orbit of the vertex fixed by B. Since the tree T' associated to the amalgam decomposition can be viewed as a subtree of T, we deduce that this vertex belongs to T' and hence, in G_n, the subgroup N is conjugate to a subgroup of B_n. Since, by Lemma 2.3, torsion-free nilpotents subgroups of B_n have bounded nilpotency class, we deduce that torsion-free nilpotent subgroups of G_n (and hence of Γ) have bounded nilpotency class. □

Remark 3.3. Another obstruction to linearity is the existence of (not necessarily torsion-free) solvable subgroups of *unbounded* solvability class [FN]. However, this obstruction is not satisfied by the planar Cremona group $\mathrm{Cr}_2(\mathbf{C})$ (Urech [U, Theorem 1.8], essentially following Déserti [D]).

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