

SEMISIMPLE ZARISKI CLOSURE OF COXETER GROUPS

YVES DE CORNULIER

ABSTRACT. Let W be an irreducible, finitely generated Coxeter group. The geometric representation provides an discrete embedding in the orthogonal group of the so-called Tits form. One can look at the representation modulo the kernel of this form; we give a new proof of the following result of Vinberg: if W is non-affine, then this representation remains faithful. Our proof uses relative Kazhdan Property (T).

The following corollary was only known to hold when the Tits form is non-degenerate: the reduced C^* -algebra of W is simple with a unique normalized trace. Some other corollaries are pointed out.

0. FOREWORD

I wrote the present paper in the beginning of 2005. After Luis Paris informed me that its only purportedly original contribution was an old result of Vinberg, I left the paper on my web page as an unpublished note.

1. INTRODUCTION

We recall that a (discrete) group Γ is amenable if it has a left invariant finitely additive probability on its power set. All we need to know about the class of amenable group is that it is stable under taking subgroups, quotient, and direct limits. It immediately follows that every group Γ has a unique biggest amenable normal subgroup, which we call its *amenable radical* and denote it $R(\Gamma)$.

Let S be a set, and $M = (m_{st})_{(s,t) \in S^2}$ a $S \times S$ symmetric matrix, with 1's on the diagonal and coefficients in $\{2, 3, \dots, \infty\}$ outside the diagonal. The group W with presentation

$$\langle (\sigma_s)_{s \in S} \mid ((\sigma_s \sigma_t)^{m_{st}})_{(s,t) \in S^2} \rangle,$$

where (we set $x^\infty = 1$ for all x) is called the Coxeter group associated to the Coxeter matrix M .

The Coxeter matrix M defines a non-oriented labelled graph \mathcal{G} with S as set of vertices and an edge between s and t if and only if $m_{s,t} \geq 3$; this edge being labelled by m_{st} . The decomposition of the graph \mathcal{G} in connected components corresponds to a decomposition of W into a direct sum. Thus, most problems about the group structure of W , such as the determination of $R(W)$, are reduced to the case when M is irreducible, which means by definition that \mathcal{G} is connected and non-empty. By abuse of language, we sometimes say that W is irreducible.

Irreducible Coxeter groups fall into three classes:

- Coxeter groups of spherical type. They are locally finite; they are entirely classified: the corresponding diagrams are called A_n , ($n \geq 1$) B_n ($n \geq 2$), D_n ($n \geq 3$), E_n ($6 \leq n \leq 8$), F_4 , H_3 , H_4 , $I_2(n)$ ($n \geq 3$) for the finitely generated (hence finite) ones, and A_∞ , A'_∞ , B_∞ , D_∞ for the infinitely generated ones.
- Coxeter groups of affine type. They are finitely generated, infinite, virtually abelian. The corresponding diagrams are called \widetilde{A}_n , ($n \geq 1$) \widetilde{B}_n ($n \geq 3$), \widetilde{C}_n ($n \geq 2$) \widetilde{D}_n ($n \geq 4$), \widetilde{E}_n ($6 \leq n \leq 8$), \widetilde{F}_4 , \widetilde{G}_2 .
- Non-affine Coxeter groups: these are the remaining Coxeter groups. They contains a non-abelian free group [Ha1]. In the finitely generated case, they are even *large*, i.e. have a finite index subgroup mapping onto a free non-abelian free group [Gon, MaVi]. This is the class we mainly study.

Date: July 19, 2005 (Minor revisions: Nov. 23, 2012).

2000 *Mathematics Subject Classification.* Primary 20F55; Secondary 22E40, 46L05, 20B15.

The purpose of the present paper is to exhibit a discrete Zariski dense embedding of every non-affine Coxeter group in a simple orthogonal group $O(p, q)$ for some $p \geq 2$, $q \geq 1$. This is actually a combination of a result of Benoist and de la Harpe, and a result of Vinberg, which we provide a new proof based on Kazhdan's relative Property (T). Before this, let us begin by the corollaries.

Corollary 1. *Let W be the Coxeter group associated to a non-affine irreducible Coxeter matrix. Then its amenable radical is trivial: $R(W) = \{1\}$.*

We immediately deduce the amenable radical of an arbitrary Coxeter group.

Corollary 2. *Let W be a Coxeter group. Let $W = \bigoplus W_i \oplus \bigoplus W_j$ be its decomposition as a direct sum of irreducible Coxeter groups, with W_i non-affine and W_j affine or spherical. Then $R(W) = \bigoplus W_j$.*

Using a Theorem of Y. Benoist and P. de la Harpe [BenH], which also plays an essential role in this paper, we obtain several other consequences.

A group Γ is called *primitive* if it has a proper maximal subgroup Λ containing no nontrivial normal subgroup of Γ , i.e. such that the action of Γ on Γ/Λ is faithful. Using a characterization of primitive linear groups due to T. Gelander and Y. Glasner, we obtain:

Corollary 3. *Let W be an infinite, finitely generated Coxeter group. Then W is primitive if and only if it is irreducible and non-affine.*

We also obtain a characterization of Coxeter groups with simple reduced C^* -algebra. If Γ is any discrete group and $g \in \Gamma$, its regular representation of Γ on $\ell^2(\Gamma)$ is defined by $\lambda(g)(f)(h) = f(g^{-1}h)$ for $f \in \ell^2(\Gamma)$. The norm closure of the linear span of the operators $\lambda(g)$ for $g \in \Gamma$ is called the reduced C^* -algebra of Γ and denoted $C_r^*\Gamma$.

Corollary 4. *Let W be a Coxeter group. Then its reduced C^* -algebra C_r^*W is simple if and only if W has only non-affine factors. Moreover, if these conditions are satisfied, then it has a unique normalized trace.*

The existence of a nontrivial amenable normal subgroup is a well-known obstruction for C_r^*W to be simple, and is conjectured to be the only one for linear groups [BekCH]. Actually Corollary 4 reduces to show that any non-affine irreducible Coxeter group has simple reduced C^* -algebra. Actually this had been proved in [Fe] and [BenH, §2, vii] under the additional assumption that the Tits form (introduced below) is non-degenerate; in which case Corollary 1 is an immediate consequence of results of [BenH].

Remark. An extensive discussion on simplicity of group C^* -algebras can be found in [Ha2], where a part of Corollary 4 (in the finitely generated irreducible case) appears as [Ha2, Cor. 18].

Let us now recall the definition of the Tits form. On the vector space $\mathbf{R}^{(S)}$ with basis $(e_s)_{s \in S}$, define a symmetric bilinear form, called the *Tits form*, by $B(e_s, e_t) = -\cos(\pi/m_{st})$. Suppose now that S is finite, and let (p, q, r) be the signature of B , meaning that B is equivalent to the form $\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0I_r \end{pmatrix}$.

By abuse of language, we call (p, q, r) the signature of W . It is known [Bou, Chap. V, §4.8] that the Coxeter system is spherical if and only if $q = r = 0$, i.e. B is a scalar product. If the Coxeter system is irreducible, then $q = 0$ implies $r \leq 1$, and the Coxeter system is affine exactly when $(p, q, r) = (p, 0, 1)$ [Bou, Chap. V, §4.9].

For $s \in S$, set $r_s(v) = v - 2B(e_s, v)e_s$. The mapping $\sigma_s \rightarrow r_s$ extends to a well-defined group morphism α , called the *Tits representation* of the Coxeter group W . By a well-known theorem of Tits, this representation is faithful and has discrete image [Bou, Chap. V, §4.4].

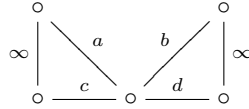
Denote by $O_f(B)$ the subgroup of $GL(\mathbf{R}^{(S)})$ consisting of those linear maps preserving the form B and fixing pointwise the kernel $\text{Ker}(B)$. It is known [Bou, Chap. V, §4.7] that $\alpha(W)$ is contained in $O_f(B)$.

Theorem 5 (Benoist, de la Harpe [BenH]). *Suppose that S is finite, W is irreducible and non-affine. Then the image $\alpha(W)$ of the Tits representation is Zariski-dense in $O_f(B)$.*

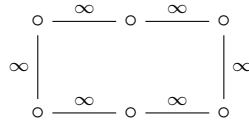
The group $O_f(B)$ is easily seen to be isomorphic to $O(p, q) \times (\mathbf{R}^{p+q})^r$. Accordingly, when $r = 0$, it is isomorphic to $O(p, q)$, whose amenable radical is its centre $\{\pm 1\}$. Since infinite irreducible Coxeter groups have trivial centre [Bou, Chap. V, §4, Exercice 3], one thus obtains that if $r = 0$, then the Coxeter group W has a trivial amenable radical.

However, even if, in a certain sense, “most” Coxeter groups have a non-degenerate Tits form, those with degenerate Tits form are numerous and a classification of those seems out of reach. Here are some examples:

For all $a, b, c, d \in \{2, 3, 4, \dots, \infty\}$ with $c, d \geq 3$, the signature of the Coxeter diagram



is $(3, 1, 1)$. The signature of the Coxeter diagram



is $(3, 1, 2)$. In [BenH, §5], irreducible Coxeter groups with signature $(p, 1, (p + 1)(p - 2)/2)$ are exhibited for all $p \geq 4$.

The bilinear form B naturally factors through a non-degenerate bilinear map \bar{B} on the quotient $\mathbf{R}^{(S)}/\text{Ker}(B)$. Denote by T_f the kernel of the natural map from $O_f(B)$ to $O(\bar{B})$, so that T_f is isomorphic to $(\mathbf{R}^{p+q})^r$. We have the following lemma (recall that the Coxeter system is supposed finite, non-affine, and irreducible):

Lemma 6 (Vinberg [Vin, Proposition 13]; see also [Kra, Proposition 6.1.3]).

$$\alpha(W) \cap T_f = \{1\}.$$

A sketch of our proof of Lemma 6 goes as follows: set $N = \alpha^{-1}(\alpha(W) \cap T_f)$. We must prove that $N = \{1\}$. We prove that (W, N) has relative Property (T), i.e. that every isometric action of W on a Hilbert space has a N -fixed point. On the other hand, all finitely generated Coxeter groups are known [BoJS] to act properly by isometries on some Hilbert space (this is called the Haagerup Property); the combination of these two facts implies that N is finite, hence trivial.

Lemma 6 has the following consequence. Let π be the composite natural projection $O_f(B) \rightarrow O(\bar{B}) \rightarrow \text{PO}(\bar{B})$.

Theorem 7 (Reduced Tits representation). *If W is a non-affine irreducible finitely generated Coxeter group, then the homomorphism $\pi \circ \alpha$ embeds W as a discrete, Zariski dense subgroup of $\text{PO}(\bar{B})$.*

Corollary 8. *If W is a non-affine irreducible finitely generated Coxeter group, then W embeds as a discrete, \mathbf{C} -Zariski dense subgroup of a complex simple Lie group with trivial centre, namely a projective orthogonal group.*

In Section 2, we show how Lemma 6 implies all results above, and in Section 3 we prove Lemma 6. The reader may find superfluous to use Kazhdan’s Property (T) since a geometric proof already exists; but we have included it here to illustrate a surprising application of relative Property (T).

Remerciements. Je remercie Yves Benoist et Pierre de la Harpe pour les discussions à ce sujet, et particulièrement Luis Paris pour m’avoir signalé le résultat de Vinberg. Je remercie Goulnara Arzhantseva pour l’intérêt porté à ce papier.

2. PROOF OF ALL RESULTS FROM LEMMA 6

We use the easy lemma in the proof of Corollary 1:

Lemma 9. *If S has at least 3 elements, and the signature of the corresponding Coxeter group is (p, q, r) , then $p \geq 2$.*

Proof: If S has an edge st with finite label m_{st} (possibly $m_{st} = 2$), then the restriction of B to the plane $\mathbf{R}e_s \oplus \mathbf{R}e_t$ is positive definite, so that $p \geq 2$. Otherwise S is the complete graph with all labels infinite, and, when S has 3 elements, a direct computation shows that the signature is $(2, 1, 0)$. ■

Remark 10. By a less trivial result by Luis Paris [Pa], if S has at least 4 elements and is connected, then $p \geq 3$. On the other hand we have pointed out above that an hexagon with infinite labels has signature $(3, 1, 2)$; we do not know if there exist irreducible Coxeter groups with $|S| \geq 7$ and $p = 3$.

Proof of Theorem 7. The injectivity of the map into $O(\bar{B})$ is the contents of Lemma 6. Since infinite irreducible Coxeter groups have trivial centre [Bou, Chap. V, §4, Exercice 3], the composite map into $PO(\bar{B})$ is still injective.

The Zariski density of its image follows from Theorem 5. It remains to prove that the image $\Gamma = p \circ \alpha(W)$ is discrete. By a theorem of Auslander [Rag, Theorem 8.24], the connected component $\bar{\Gamma}^0$ (in the ordinary topology) is solvable. Since Γ is Zariski dense in the simple group $PO(\bar{B})$, it follows that its normal subgroup $\bar{\Gamma}^0$ must be discrete, hence trivial, i.e. Γ is discrete. ■

Proof of Corollary 8. By Theorem 7, W embeds as a Zariski dense subgroup in the real group $PO(\bar{B}) \simeq PO(p, q)$ with $p \geq 2$ and $q \geq 1$. Taking the complexification, we obtain, unless $(p, q) = (2, 2)$ or $(3, 1)$, a \mathbf{C} -Zariski dense embedding of W in $PO_{p+q}(\mathbf{C})$ and since $4 \neq p + q \geq 3$, we are done. If $(p, q) = (3, 1)$, the group $PO(p, q)$ itself has a structure of a simple complex Lie group so the argument works without complexification. Finally, we cannot have $(p, q) = (2, 2)$. Indeed, by a result of Paris (see Remark 10), if $|S| \geq 4$ then $p \geq 3$. ■

Proof of Corollary 1. We can suppose W irreducible. If W is finitely generated, then the result immediately follows from Corollary 8. If W is infinite, then the Coxeter graph is a direct limit of finite connected non-affine Coxeter subgraphs, and thus W is a direct limit of finitely generated non-affine irreducible Coxeter groups. Since the property of having trivial amenable radical is stable under passing to direct limits, we are done. ■

Proof of Corollary 3. It is a easy fact that if a primitive group Γ decomposes as a nontrivial direct product $\Gamma_1 \times \Gamma_2$, then Γ_1 and Γ_2 are simple non-abelian and isomorphic. It immediately follows that any primitive Coxeter group W must be irreducible.

If W is an affine Coxeter group, then it cannot be primitive: indeed, let M be a maximal subgroup. Then W is virtually abelian, hence is subgroup separable, i.e. every finitely generated subgroup is the intersection of subgroups of finite index containing it. Moreover every subgroup is finitely generated. It immediately follows that every maximal subgroup in W must have finite index. Therefore, the action of the infinite group W on the finite set W/M cannot be faithful.

Let us now suppose that W is irreducible and non-affine. Gelander and Glasner [GeG] prove that an infinite finitely generated linear group Γ is primitive if and only if there exists an algebraically closed field K , a linear algebraic K -group G , and a morphism $\Gamma \rightarrow G(K)$ with Zariski dense image, such that G^0 is semisimple with trivial centre, and the action of Γ by conjugation on $G^0(K)$ is faithful and is transitive on simple factors of G^0 . It follows from Corollary 8 that this criterion is satisfied. ■

Remark 11. The primitive finite Coxeter groups are those of type A_n ($n \geq 1$), D_{2n+1} ($n \geq 1$), E_6 , and $I_2(p)$ for p odd prime (note the redundancies $I_2(3) \simeq A_2$, $A_3 \simeq D_3$). Indeed, Coxeter groups of type B_n ($n \geq 2$), D_{2n} ($n \geq 2$), E_7 , E_8 , F_4 , H_3 , H_4 , $I_2(2n)$ ($n \geq 2$) have centre cyclic of order 2. The only remaining cases are those of type $I_2(n)$ for odd non-prime n , for which the verification is straightforward.

Conversely, the group of type A_n , the symmetric group S_{n+1} , acts primitively and faithfully on $n + 1$ elements. The group of type D_{2n+1} , isomorphic to $S_{n+1} \times (\mathbf{Z}/2\mathbf{Z})_0^n$ acts affinely and

primitively on

$$(\mathbf{Z}/2\mathbf{Z})_0^n = \{(a_1, \dots, a_{n+1}) \in (\mathbf{Z}/2\mathbf{Z})^{n+1}, a_1 + \dots + a_{n+1} = 0\}.$$

Finally, the group W of type E_6 has only one normal subgroup other than trivial ones, namely W^+ , and it follows that if Λ is any maximal subgroup other than W^+ , then the action of W on W/Λ is faithful.

Proof of Corollary 4. If C_r^*W is simple, then W has no nontrivial amenable normal subgroup, so that W has no affine or spherical factor.

Conversely, suppose that W has no non-affine factor. Since the property of having simple reduced C^* -algebra with a unique normalized trace is inherited by direct limits [BekH, Lemma 5.1], we are reduced to the case when W is finitely generated.

The argument is the same as that given in [BenH, §2, vii], except that, thanks to Lemma 6, we can avoid assuming that B is non-degenerate.

We use the following results:

- [BekH, Theorem 1] If a discrete group Γ embeds as a Zariski dense subgroup in a connected real semisimple Lie group without compact factors, then $C_r^*\Gamma$ is simple and has a unique normalized trace.
- [BekCH] If Γ_0 has finite index in Γ , if Γ is i.c.c. (all its nontrivial conjugacy classes are infinite), and if $C_r^*\Gamma_0$ is simple and has a unique normalized trace, then $C_r^*\Gamma$ is also simple and has a unique normalized trace.

First suppose that W is irreducible. By Corollary 1, W is i.c.c.; moreover W has a subgroup of index at most 2 embedding as a Zariski dense subgroup in $\mathrm{PO}_0(p, q)$. So the two criteria above apply.

In general, decompose W as $W = W_1 \times \dots \times W_n$ with each W_i irreducible non-affine. Then W is i.c.c., and has a subgroup of index $\leq 2^n$ that embeds as a Zariski dense subgroup in a connected semisimple Lie group with n noncompact simple factors. It follows that C_r^*W is also simple and has a unique normalized trace. ■

Remark 12. It is not difficult to see that, conversely, Lemma 6 follows from any one among Corollaries 1, 3, or 4.

3. PROOF OF LEMMA 6

By [BoJS], if W is any Coxeter group and l its length function, then there exists an isometric action u of W on a Hilbert space \mathcal{H} , and $v \in \mathcal{H}$ such that $l(g) = \|u(g)v - v\|^2$ for all $g \in W$.

On the other hand, recall that, given a group Γ and a subgroup Λ , the pair (Γ, Λ) has *relative Property (T)* if for every isometric action u of Γ on a Hilbert space \mathcal{H} , and every $v \in \mathcal{H}$, the restriction to Λ of the function $g \mapsto \|u(g)v - v\|$ is bounded.

It follows that if W is a finitely generated Coxeter group, and if $\Lambda \subset W$ is a subgroup such that (W, Λ) has relative Property (T), then Λ is finite. In particular, if Λ is torsion-free, this implies $\Lambda = \{1\}$. Thus Lemma 6 follows from the following lemma:

Lemma 13. *Set $N = \alpha^{-1}(\alpha(W) \cap T_f)$. The pair (W, N) has relative Property (T).*

Proof: It is clear from the definition that relative Property (T) is inherited by images. Consider the semidirect product $W \rtimes N$, with group law $(w_1, n_1) \cdot (w_2, n_2) = (w_1 w_2, w_2^{-1} n_1 w_2 n_2)$. There is an obvious morphism from $W \rtimes N$ to W sending (w, n) to w . Thus the lemma reduces to proving that $(W \rtimes N, N)$ has relative Property (T).

Set $V = N \otimes_{\mathbf{Z}} \mathbf{R}$. Then $W \rtimes N$ naturally embeds as a cocompact subgroup of cofinite volume into $W \rtimes V$. We are going to show that $(W \rtimes V, V)$ has relative Property (T). It trivially implies that $(W \rtimes V, N)$ has relative Property (T), and then, since Property (T) relative to a given normal subgroup is inherited by subgroups of cofinite volume [Jol, Corollary 4.1(2)], this implies that $(W \rtimes V, N)$ has relative Property (T).

So it remains to prove that $(W \rtimes V, V)$ has relative Property (T). By a classical result by Burger, this reduces to proving that the action by conjugation of $W \rtimes V$ on V does not preserve any Borel probability measure on the projective space $P(V^*)$, where V denotes the dual space of V . This

action factors through W , so that we have to show that W does not preserve any probability on the projective space $P(V^*)$.

Let V_1 denote the vector subspace of T_f generated by $T_f \cap \alpha(W)$. We thus have to show that the action by conjugation of $\alpha(W)$ on $P(T_f)$ does not preserve any probability. Otherwise, by results of Furstenberg [Zim, §3.2], some finite index subgroup of $\alpha(W)$ preserve a nonzero subspace of V_1^* on which it acts through the action of a compact group. But this in contradiction with the fact that $\alpha(W)$ is Zariski dense in $O_f(B)$, and that the action by conjugation of the connected semisimple group without compact factors $O_0(\bar{B})$ on T_f has no invariant vectors. ■

REFERENCES

- [BekCH] Bachir BEKKA, Michael COWLING, Pierre DE LA HARPE. *Some groups whose reduced C^* -algebra is simple*. Publ. Math. Inst. Hautes Études Sci. **80**, 117-134, 1994.
- [BekH] Bachir BEKKA, Pierre DE LA HARPE. *Groups with simple reduced C^* -algebras*. Expo. Math. **18**, 215-230, 2000.
- [BenH] Yves BENOIST, Pierre DE LA HARPE. *Adhérence de Zariski des groupes de Coxeter*. Compositio Math. **140**, 1357-1366, 2004.
- [BoJS] M. BOZEJKO, T. JANUSZKIEWICZ, R. SPATZIER. *Infinite Coxeter groups do not have Kazhdan's property*. J. Operator Theory **19**, 63-68, 1988.
- [Bou] N. BOURBAKI. "Groupes et algèbres de Lie. Chapitres 4,5 et 6". Masson, 1981.
- [CCJJV] Pierre-Alain CHERIX, Michael COWLING, Paul JOLISSAINT, Pierre JULG, Alain VALETTE. "Groups with the Haagerup Property". Birkhäuser, Progress in Mathematics 197, 2001.
- [Fe] Gero FENDLER. *Simplicity of the reduced C^* -algebras of certain Coxeter groups*. Ill. J. Math. **47**(3), 883-897, 2003.
- [GeG] Tsachik GELANDER, Yair GLASNER. *Countable primitive groups*. Geom. Functional Anal. **17**(5), 1479-1523, 2008.
- [Gon] C. GONCIULEA. *Virtual epimorphisms of Coxeter groups onto free groups*. PhD thesis, Ohio State University, 2000.
- [Ha1] Pierre DE LA HARPE. *Groupes de Coxeter infinis non affines*. Expo. Math. **5**(1) 91-96, 1987.
- [Ha2] Pierre DE LA HARPE. *On simplicity of reduced C^* -algebras of groups*, Bull. London Math. Soc. **39**(1), 1-26, 2007.
- [HV] Pierre DE LA HARPE, Alain VALETTE. "La propriété (T) de Kazhdan pour les groupes localement compacts". Astérisque **175**, SMF, 1989.
- [Jol] Paul JOLISSAINT. *On Property (T) for pairs of topological groups*. Enseign. Math. (2) **51**, 31-45, 2005.
- [Kra] Daan KRAMMER. *The conjugacy problem for Coxeter groups* Groups Geom. Dyn. **3**(1), 71-171, 2009. Originally: "The conjugacy problem for Coxeter groups". PhD thesis, Universiteit Utrecht, 1994.
- [MaVi] Gregory A. MARGULIS, E. B. VINBERG. *Some linear groups virtually having a free quotient*. J. Lie Theory **10**, 171-180, 2000.
- [Pa] Luis PARIS. "Signature des graphes de Coxeter". Thèse, Université de Genève, 1989.
- [Rag] M. S. RAGHUNATHAN. "Discrete subgroups of Lie groups". Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag 1972.
- [Vin] E. B. VINBERG. *Discrete linear groups generated by reflections*. Math USSR Izvestija **5**, 1083-1119, 1971.
- [Zim] Robert J. ZIMMER. "Ergodic theory and semisimple groups". Boston, Birkhäuser, 1984.

Yves de Cornulier

E-mail: yves.cornulier@math.u-psud.fr