CROSS-WIRED LAMPLIGHTER GROUPS

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ABSTRACT. We give a necessary and sufficient condition for a locally compact group to be isomorphic to a closed cocompact subgroup in the isometry group of a Diestel-Leader graph. As a consequence of this condition, we see that every cocompact lattice in the isometry group of a Diestel-Leader graph admits a transitive, proper action on some other Diestel-Leader graph. We also give some examples of lattices that are not virtually lamplighters. This implies the class of discrete groups commensurable to lamplighter groups is not closed under quasi-isometries and, combined with work of Eskin, Fisher and Whyte, gives a characterization of their quasi-isometry class.

1. INTRODUCTION

A lamplighter group is a group of the form $F \wr \mathbf{Z}$ where F is some non-trivial finite group. For one particular choice of generators, the Cayley graphs of these groups are examples of Diestel-Leader graphs $\mathrm{DL}(n, n)$ [2, 12, 13], which can be defined as follows (see Section 2 for details): let T_n be the $(n + 1)$-regular tree and b a Busemann function on T_n. The Diestel-Leader graph of degree n is defined as

$$\mathrm{DL}(n, n) = \{(x, y) \in T_n \times T_n : b(x) + b(y) = 0\}.$$

This is a connected graph, where $\{(x, y), (x', y')\}$ is an edge whenever both $\{x, x'\}$ and $\{y, y'\}$ are edges. Eskin, Fisher, and Whyte have shown ([4, 5, 6]) that a finitely generated group is quasi-isometric to a lamplighter group $F \wr \mathbf{Z}$ if and only if it acts properly and cocompactly by isometries on a Diestel-Leader graph $\mathrm{DL}(n, n)$, where n and $|F|$ have a common power. In this paper, we address the question of whether or not the lamplighter groups are quasi-isometrically rigid by studying the cocompact lattices in the isometry group $\mathrm{Isom}(\mathrm{DL}(n, n))$ of the Diestel-Leader graphs. While we show that these lattices are not necessarily lamplighter groups, we do give an algebraic characterization of them.

We give cocompact lattices in $\mathrm{Isom}(\mathrm{DL}(n, n))$ the name of *cross-wired lamplighters*, a terminology we explain briefly at the end of the paper.

An isometry of $\mathrm{DL}(n, n)$ is called *positive* if it is the restriction to $\mathrm{DL}(n, n)$ of some product isometry $(\alpha, \beta) \in \mathrm{Aut}(T_n \times T_n)$ of $T_n \times T_n$. It can be shown that a non-positive isometry of $\mathrm{DL}(n, n)$ is the composition of a positive isometry and the flip $(x, y) \mapsto (y, x)$, so that the group $\mathrm{Isom}^+(\mathrm{DL}(n, n))$ of positive isometries has index two in $\mathrm{Isom}(\mathrm{DL}(n, n))$ (see Section 2). Rather than just proving a theorem about lattices in $\mathrm{Isom}(\mathrm{DL}(n, n))$, we will prove a more general result about closed, cocompact subgroups of $\mathrm{Isom}(\mathrm{DL}(m, n))$. In the case when $m \neq n$, $\mathrm{Isom}(\mathrm{DL}(m, n))$ is not unimodular, so there are no lattices, but understanding closed, cocompact subgroups is also of interest. When $m \neq n$, all isometries are positive.

Theorem 1.1. *Suppose $m, n \geq 2$.*

(a) *Let Γ be a closed, cocompact subgroup of $\mathrm{Isom}^+(\mathrm{DL}(m, n))$. Then Γ has a unique open normal subgroup H such that Γ/H is infinite cyclic. Moreover, if t is any element of Γ mapping to a generator of Γ/H, then H has two open subgroups L, L' such that*
  - *tLt^{-1} and $t^{-1}L't$ are subgroups of index m and n in L and L' respectively;*
  - *$\bigcup_{k \in \mathbf{Z}} t^{-k}Lt^k = \bigcup_{k \in \mathbf{Z}} t^kL't^{-k} = H$ (these are increasing unions);*
  - *$L \cap L'$ is a vertex stabilizer and thus is compact;*
  - *$LL' = H$ (that is, the double coset space $L\backslash H/L'$ is reduced to a point).*

  Moreover, Γ has no non-trivial compact normal subgroup and if Γ is discrete, then $m = n$. If the action of Γ is simply transitive (i.e. $\mathrm{DL}(m, n)$ is a Cayley graph of Γ with its Γ-action), then $L \cap L' = \{1\}$ and $LL' = H$.

(b) *Conversely, let Γ be a locally compact group, with a semidirect product decomposition $\Gamma = H \rtimes \langle t \rangle$, with H non-compact. Assume that H has open subgroups L, L' such that*
  - *tLt^{-1} and $t^{-1}L't$ are finite index subgroups, of index m and n, in L and L' respectively;*
  - *$\bigcup_{k \in \mathbf{Z}} t^{-k}Lt^k = \bigcup_{k \in \mathbf{Z}} t^kL't^{-k} = H$ (these are increasing unions);*
  - *$L \cap L'$ is compact;*
  - *the double coset space $L\backslash H/L'$ is finite of cardinality d.*

  Then H is locally elliptic (i.e. every compact subset of H is contained in a compact subgroup), and Γ has a proper, transitive

action on $\mathrm{DL}(m, n)$, *for which* $L \cap L'$ *is a vertex stabilizer, and whose kernel is* $\bigcap_{k \in \mathbf{Z}} t^{-k}(L \cap L')t^k$. *Moreover if* Γ *is discrete, then it is finitely generated and* $m = n$.

The combination of the two main points of the theorem yields a result we find surprising: namely that any lattice $\Gamma < \mathrm{Isom}(\mathrm{DL}(n, n))$, or more generally any closed cocompact subgroup $\Gamma \leq \mathrm{Isom}(\mathrm{DL}(m, n))$ has an index two subgroup that admits a proper transitive action on another Diestel-Leader graph. We will also give a short geometric proof of this fact that avoids the need to pass to a subgroup of index two, see Proposition 2.4 below. Of course, any cocompact lattice D in the isometry group of any vertex-transitive locally finite graph admits a proper transitive action on some graph, namely its own Cayley graph. What is surprising here is that the transitive action is on another Diestel-Leader graph. One other class of graphs for which a similar statement is known are trees. As Diestel-Leader graphs are clearly closely related to trees, it would be interesting to further study the analogies between cross-wired lamplighters and tree lattices.

We next provide examples of cross-wired lamplighters.

Example 1.2 (lamplighters)**.** A standard wreath product $F \wr \mathbf{Z}$, where F is any non-trivial finite group, is a cross-wired lamplighter (these examples were used by Erschler (Dyubina) [3] to observe that being virtually solvable is not a quasi-isometry invariant). Here we can choose $H = F^{(\mathbf{Z})}$, $L = F^{(\mathbf{Z}_{\geq 0})}$ and $L' = F^{(\mathbf{Z}_{<0})}$. Note that such a group is residually finite if and only if F is abelian [9].

Example 1.3. Let q be a prime power, and consider the ring $\mathbf{A} = \mathbf{F}_q[t^{\pm 1}]$. Let $\mathbb{H}$ be the Heisenberg group, consisting of upper-triangular square matrices of size three with 1's on the diagonal. Consider the action of $\mathbf{Z}$ on $\mathbb{H}(\mathbf{A})$ defined by the automorphism

$$\Phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & tx & t^2 z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the group $\mathbb{H}(\mathbf{A}) \rtimes_\Phi \mathbf{Z}$ is a cross-wired lamplighter, and actually admits $\mathrm{DL}(q^4, q^4)$ as a Cayley graph (see §3.1).

In the two previous examples, we have a natural embedding of Γ as a cocompact lattice in a group of the form

$$(G_1 \times G_2) \rtimes_{(\alpha_1, \alpha_2^{-1})} \mathbf{Z},$$

where (G_i, α_i) is a contraction group, i.e. α_i is a contracting automorphism of G_i in the sense that $\alpha_i^k(g) \to 1$ when $k \to +\infty$ (pointwise, but

this automatically implies the convergence to be uniform on compact subsets). Results of Glockner and Willis on the structure of contraction groups [7] might thus prove useful to prove general structural results about cross-wired lamplighters. However, it is not clear in general if contraction groups are enough. In general, if Γ is a cross-wired lamplighter, then the closure of its projection on $\mathrm{Aut}(T)$, where T is one of the two trees, is of the form

$$G \rtimes_\Phi \mathbf{Z},$$

where Φ contracts G modulo a compact subgroup, i.e. there exists a compact subgroup K of G such that for every g, we have $\Phi^k(g) \to 1$ in G/K, when $k \to +\infty$. Typically, if $G \rtimes_\alpha \mathbf{Z}$ is the full stabilizer of an end in T_n, then α contracts G modulo a (non-trivial) compact subgroup. However, we do not know if a cross-wired lamplighter can project densely on this group.

In Section 3, we develop these examples and ask a few additional questions.

Here are some more questions about cross-wired lamplighter we leave open:

Question 1.4.

(i) Is there a cross-wired lamplighter that is not "virtually symmetric"? Here we define a cross-wired lamplighter to be symmetric if it admits an automorphism α such that if π is the projection to $\mathbf{Z}$ (which is unique up to sign) then $\alpha \circ \pi \circ \alpha^{-1} = -\pi$, and virtually symmetric if it admits a symmetric subgroup of finite index.

(ii) Given a cross-wired lamplighter, is the pair $\{L, L'\}$ unique up to commensurability and automorphisms? Precisely, suppose that an abstract group Γ is given two embeddings as a cross-wired lamplighter, giving rise to pairs (L_1, L'_1) and (L_2, L'_2). Is there (up to swapping L_2 and L'_2) an automorphism β of Γ such that $\beta(L_1)$ is commensurable with L'_1 and $\beta(L'_2)$ with L'_2?

(iii) Let Γ be a closed cocompact subgroup of $\mathrm{Isom}(\mathrm{DL}(n, n))$ *not* contained in $\mathrm{Isom}^+(\mathrm{DL}(n, n))$; it is easy to deduce from Theorem 1.1 that there is a unique open normal subgroup H such that Γ/H is isomorphic to the infinite dihedral group D_∞. Is the extension $1 \to H \to \Gamma \to D_\infty \to 1$ necessarily split?

In Section 2, we recall basic definitions concerning Diestel-Leader graphs and prove Theorem 1.1. In Section 3, we describe various examples of cross-wired lamplighters and we explain the choice of title for this paper and name for these groups.

2. Isometries of Diestel-Leader graphs

2.1. **Preliminary remarks.** We begin by describing Diestel-Leader graphs. More detailed introductions to Diestel-Leader graphs, lamplighter groups, and related concepts are given by Woess [12] and Wortman [13].

Given an integer $n > 0$ we denote by T_n the regular tree of degree $n + 1$. We fix a preferred end $-\infty$ of T_n. Given two vertices $x, y \in V(T_n)$, the rays from x and y to $-\infty$ intersect to give a ray from some other vertex to $-\infty$. We shall denote this vertex by $\widehat{xy}$ and call it the confluent of x and y. By making a choice of preferred vertex o, we can define a *Busemann* (or *height*) *function* $b : V(T_n) \to \mathbf{Z}$ on the vertices of T_n by

$$b(x) := d(x, \widehat{xo}) - d(o, \widehat{xo}).$$

This partitions the vertices of T_n into level sets of b which are called horocycles and denoted $\mathcal{F}_k$. A vertex in the horocycle $\mathcal{F}_k$ has one neighbour, its parent, in $\mathcal{F}_{k-1}$ and n neighbours, its children, in $\mathcal{F}_{k+1}$.

Given two regular trees T_m and T_n and some choice of height functions b and b' respectively, the Diestel-Leader graph $\mathrm{DL}(m, n)$ is given by

$$V(\mathrm{DL}(m, n)) := \{(x, y) \in T_m \times T_n : b(x) + b'(y) = 0\},$$

where two vertices (x, y) and (x', y') are joined be an edge if $\{x, x'\}$ is an edge in T_m and $\{y, y'\}$ is an edge in T_n. Note that this description gives an embedding of $\mathrm{DL}(m, n)$ into $T_m \times T_n$. Any other choice of Busemann function on the trees gives rise to another such embedding (because any two Busemann functions on a regular tree are conjugate under some automorphism of the tree). In particular, each level set of the function $b(x) + b'(y)$ on $T_m \times T_n$ with edges defined analogously will be an embedded $\mathrm{DL}(m, n)$.

Note that we can define a Busemann function on $\mathrm{DL}(n, m)$ just by taking $h(x, y) = b(x) = -b'(y)$. We call the level sets of this Busemann function *horospheres* and denote them by $\mathcal{H}_k$. A geodesic is called *vertical* if it intersects each $\mathcal{H}_k$ exactly once. Note that a horosphere is a product of horocycles, one in T_m one in T_n.

As observed in [1], the Diestel-Leader graphs only have unimodular isometry groups if $n = m$ and so only contain lattices in this case.

Let U_0 denote the isometries of T_n which fix $-\infty$ and preserve the level sets of h. Let U_0' be the corresponding subgroup of isometries of T_m. It is clear that $U = U_0' \times U_0$ is contained in $\mathrm{Isom}(\mathrm{DL}(m, n))$. The

groups U_0, U_0' and U are elliptic, i.e. every compact subset is contained in an open compact subgroup.

Recall that each T_n (or T_m) has a preferred end which we labeled $-\infty$. Let ϕ_n be a hyperbolic isometry of T_n, of translation length 1, and having $-\infty$ as repelling end and define ϕ_m similarly for T_m. Define the isometry $(x, y) \mapsto (\phi_m(x), \phi_n^{-1}(y))$ of $T_m \times T_n$. It restricts to an isometry of $\mathrm{DL}(m, n)$. When $m = n$, there is also an isometry of $T_n \times T_n$ given by interchanging the trees. This isometry restricts to an isometry of $\mathrm{DL}(n, n)$ which we call ψ. We sometimes refer to ψ (or its conjugates) as a *flip*.

Thus $\mathrm{Isom}(\mathrm{DL}(n, n))$ contains the subgroup generated U, ψ and ϕ. The following result of Bartholdi, Neuhauser, and Woess [1] shows that there are no further isometries.

Proposition 2.1. *For $m > n \geq 2$,*

$$\mathrm{Isom}(\mathrm{DL}(m, n)) = (U \rtimes_\phi \mathbf{Z})$$

For any $n \geq 2$,

$$\mathrm{Isom}(\mathrm{DL}(n, n)) = (U \rtimes_\phi \mathbf{Z}) \rtimes_\psi \mathbf{Z}/2\mathbf{Z}.$$

Using the proposition, we fix some notation. We let $G = \mathrm{Isom}^+(\mathrm{DL}(n, n))$ be the index two subgroup $U \rtimes_\phi \mathbf{Z}$ when $m = n$ and $G = \mathrm{Isom}(\mathrm{DL}(m, n))$ when $m \neq n$. We let $\pi : G \to \mathbf{Z}$ be the natural projection so $U = \ker \pi$.

We also deduce some corollaries from the proposition. The first is obvious.

Corollary 2.2. *The full group $\mathrm{Isom}(\mathrm{DL}(m, n))$ acts isometrically on the product $T_m \times T_n$ permuting the level sets of the function $b(x) + b'(y)$.*

In particular, since every level set $b(x) + b'(y) = k$ is another copy of the Diestel-Leader graph $\mathrm{DL}(m, n)$, which we will label for clarity $\mathrm{DL}(m, n)_k$, any group $\Gamma < \mathrm{Isom}(\mathrm{DL}(m, n))$ has infinitely many actions on $\mathrm{DL}(m, n)$. It will be important to us that these actions are usually actually different. Let $\Gamma' = G \cap \Gamma$.

Corollary 2.3. *If $\Gamma \leq \mathrm{Isom}(\mathrm{DL}(m, n))$ is a closed, cocompact subgroup, and define $\Gamma' = \mathrm{Ker}(\pi) \cap \Gamma$. Then there is a natural number m and an element t in Γ such that $\Gamma' = (\Gamma' \cap U) \rtimes \langle t \rangle$.*

Proof. It is clear that the image of π restricted to Γ' must be infinite for Γ to be cocompact. Let b be the (positive) generator of $\pi(\Gamma')$ and choose a lift t of b to Γ. It is straightforward to check that t has the desired properties. $\square$

As promised in the introduction, we now give a direct proof that any group acting properly and cocompactly on $\mathrm{DL}(n,n)$ admits a proper, transitive action on some $\mathrm{DL}(n',n')$.

Proposition 2.4. *Let the locally compact group Γ act properly and cocompactly on $\mathrm{DL}(m,n)$. Then there is a positive integer d such that Γ acts properly and transitively on $\mathrm{DL}(m^d,n^d)$.*

Proof. Given $\Gamma \leq \mathrm{Isom}(\mathrm{DL}(m,n))$ there is an element $t \in \Gamma$ as in Corollary 2.3. Since every element of Γ that does not fix $\mathcal{H}_0$ translates it to some $\mathcal{H}_{kd}$ with k an integer, it follows that the action of Γ on $\mathrm{DL}(m,n)$ can be replaced by an action of Γ on a "collapsed" Diestel-Leader graph $\mathrm{DL}(m^d,n^d)$. We obtain the collapsed Diestel-Leader graph from the first one by simply ignoring all vertices that occur in level sets of h that are not multiples of d and considering the edges to be vertical geodesic segment between vertices in $\mathcal{H}_{dk}$ and vertices in $\mathcal{H}_{(d+1)k}$.

It is now clear that Γ acts transitively on the set of horospheres of $\mathrm{DL}(m^d,n^d)$. To see that the action is transitive, we need to see that the action along some horosphere is transitive. Let the two trees be T^1 and T^2. We let $\Gamma'_0 = \Gamma' \cap U$ the subgroup of elements stabilizing each horocycle.

We consider the horocycle in $\mathcal{H}_0$ at height 0 and note that Γ'_0 is cocompact on it. Choose orbit representatives $(x_1, y_1), ..., (x_l, y_l)$ for this action. It is clear that there is some height k^1 (resp. k^2) such that all the x_j (resp. y_j) are on vertical geodesics going down from a single vertex $\bar{x}$ at height k^1 in T^1 (resp $\bar{y}$ at height k^2 in T^2).

Letting $k = |k^1 + k^2|$ this implies that the Γ'_0 action on the horocycle containing $(\bar{x}, \bar{y})$ in $\mathrm{DL}(m^d,n^d)_k$ is transitive. If it is not, there is some other orbit, with representative (x', y'). Let x'' (resp. y'') be any vertex at height zero below x' (resp. y') in T^1 (resp. T^2). Since Γ'_0 cannot move (x', y') to $(\bar{x}, \bar{y})$, it cannot move (x'', y'') to any of our orbit representatives for the Γ'_0 action on $\mathcal{H}_0$, a contradiction. This shows that the Γ action on the subgraph $\mathrm{DL}(m^d,n^d)_k$ (which is isomorphic to $\mathrm{DL}(m^d,n^d)$) is transitive. $\square$

2.2. **Proof of the theorem.**

Proof. Let us prove (a). Let Γ be a closed, cocompact subgroup of G. By Corollary 2.3 there is $t \in \Gamma$ mapped to a generator of the cyclic group $\pi(\Gamma)$, such that $\Gamma = (U \cap \Gamma) \rtimes \langle t \rangle$. Set $H = U \cap \Gamma$.

For convenience we denote the two trees as T and T'. Note that t acts as a hyperbolic element on both T and T'. Fix vertices $v \in T$ and $v' \in T'$ in the translation axes of t in T and T'. Let $L \subset \Gamma$ be the stabilizer of v and L' the stabilizer of v' (for the action on T and

T' respectively). Then L and L' are open in Γ and contained in H. Since v belongs to the axis of t, we have $tLt^{-1} \subset L$ and $t^{-1}L't \subset L'$. The open subgroup $L \cap L'$ is the stabilizer of (v, v') and therefore is compact. If $h \in \Gamma_0 = \Gamma \cap \mathrm{Ker}(\pi)$, then $t^n h t^{-n} \in L$ for n large enough. Indeed, write $h = (g, g')$. By assumption, $\pi(h) = 0$, so g has a fixed point w in T. So g, hence h also fixes the projection w_1 of w on the axis of v (by definition, w_1 is the unique point of the axis of v minimizing the distance $d(w_1, w)$), because w_1 is in the geodesic ray joining w to the end at $-\infty$. So h is contained in the conjugate of L by some power of t, that is, $\bigcup_{n \in \mathbf{Z}} t^n L t^{-n} = \Gamma_0$. Similarly $\bigcup_{n \in \mathbf{Z}} t^{-n} L' t^n = \Gamma_0$.

Let S be the horosphere $\{x \in T : b(x) = 0\}$ and similarly define S' in T'. We see that two elements of $S \times S'$ are contained in the same orbit under Γ if and only if they are contained in the same orbit under Γ_0. In particular, Γ_0 has finitely many orbits for its action on $\Gamma_0 v \times \Gamma_0 v'$, or equivalently for its action on $\Gamma_0/L \times \Gamma_0/L'$. This exactly means that $L \backslash \Gamma_0 / L'$ is finite.

Therefore there exists a finite subset F of Γ_0 such that $\Gamma_0 = LFL'$. For some $k \leq 0$ we have $F \subset t^k L t^{-k}$.

If we had chosen $t^k v$ instead of v, the open subgroup L would be replaced by $t^k L t^{-k}$, and we would get $\Gamma_0 = LL'$ without altering the other conclusions.

Let us prove (b). Given the data of (b), and denoting by d the cardinality of $L \backslash H / L'$: we first give a construction giving the following:

(*) Γ has a proper, cocompact action on $\mathrm{DL}(m, n)$ with exactly d orbits.

We see that the group Γ is naturally identified with the ascending HNN extension $\mathrm{HNN}(H, L, t)$. So it has a natural action on its Bass-Serre tree T. Recall the definition of this tree. Write $L_n = t^n L t^{-n}$. Its vertex set can be written as the disjoint union $\bigsqcup_{n \in \mathbf{Z}} H/L_n$, and the edges are of the form $\{aL_n, aL_{n+1}\}$; in particular, this tree is regular of degree $1 + [L : tLt^{-1}]$; the action of H on each coset H/L_n is the natural one, and the action of t is given by $t \cdot (aL_n) = (tat^{-1})L_{n+1}$. Since $tL_n t^{-1} = L_{n+1}$, this is a well-defined continuous action. The stabilizer of a vertex is L. If we map T to $\mathbf{Z}$ by sending each vertex in H/L_n to n, we obtain a Busemann function b.

We can carry out the same construction with L' and get a natural action on a tree T', regular of degree $1 + [L' : t^{-1}L't]$, with a Busemann function. The diagonal action on $T \times T'$ has stabilizer $L \cap L'$, which is compact, and since this tree is locally finite, this implies that this diagonal action is proper. In particular, H is locally elliptic. It preserves the Diestel-Leader graph D of equation $b(x) + b'(y) = 0$.

We want to check that this action has exactly d orbits. Using the action of t, every orbit intersects the subset of equation $b(x) = b(y) = 0$. So we are reduced to proving that the action of H on $H/L \times H/L'$ has d orbits, but these orbits exactly correspond to elements of the double coset space $L \backslash H / L'$.

Thus we obtain (*). Now if we carry out the same construction replacing L by its conjugate L_1 by a suitable power of t such that $L_1 L' = H$, the values m, n remain the same and d is replaced by 1, yielding a proper transitive action of Γ on $\mathrm{DL}(m, n)$.

In both (a) and (b) the fact that Γ is discrete implies that it is finitely generated (because it acts properly cocompactly on a locally finite graph) and that $m = n$ because otherwise $\mathrm{Isom}(\mathrm{DL}(m, n))$ is not unimodular and hence does not admit lattices. $\qquad\square$

3. Examples and further comments

3.1. **Examples.** We now provide some more examples of cross-wired lamplighters which are not lamplighters, i.e. which are not of the form $F \wr \mathbf{Z}$ for some finite group F. We provide a specific example of a general construction and indicate the general construction.

Fix a prime power q. We let $\mathbb{H}(R)$ be the Heisenberg group over any commutative ring R. We have two natural ring embeddings of $\mathbf{F}_q[t, t^{-1}]$ into $\mathbf{F}_q((t))$, one taking t to t and the other taking t to t^{-1}. Combining these two embeddings, we obtain a discrete group embedding

$$i: \ H = \mathbb{H}(\mathbf{F}_q[t, t^{-1}]) \ \longrightarrow \ \mathbb{H}(\mathbf{F}_q((t))) \times \mathbb{H}(\mathbf{F}_q((t))).$$

It is standard to note that $\mathbb{H}(\mathbf{F}_q((t)))$ has a contracting automorphism α, this acts as multiplication by t on the x and y coordinates and by multiplication by t^2 on the z coordinate. We define the group $\mathbb{H}(\mathbf{F}_q[t, t^{-1}]) \rtimes \mathbf{Z}$ where the $\mathbf{Z}$-action is defined by restricting $\alpha \times \alpha$ on $\mathbb{H}(\mathbf{F}_q((t))) \times \mathbb{H}(\mathbf{F}_q((t)))$ to $i(\mathbb{H}(\mathbf{F}_q[t, t^{-1}]))$. To verify the statement of Theorem 1.1, we let $L = \mathbb{H}(\mathbf{F}_q[t])$ and $L' = \mathbb{H}(\mathbf{F}_q[t^{-1}])$. It is straightforward to verify that the conditions of the theorem are satisfied.

Actually, we can see $\mathrm{DL}(q^4, q^4)$ as the Cayley graph of $H \rtimes \mathbf{Z}$. By Theorem 1.1, we need to ensure $H = LL'$ and $L \cap L' = \{1\}$. To ensure the latter, we again set $L' = \mathbb{H}(\mathbf{F}_q[t^{-1}])$, but define $L = \mathbb{H}(t\mathbf{F}_q[t])$, i.e., those elements in $\mathbb{H}(\mathbf{F}_q[t])$ all of whose entries above the diagonal are in $t\mathbf{F}_q[t]$. That $LL' = H$ and $L \cap L' = \{1\}$ are immediate.

One can easily adapt this construction to other nilpotent groups over local fields of positive characteristic admitting contracting automorphisms. Note that in any construction of this kind, we are constructing a linear group, which is therefore residually finite. Note that when $\Gamma = F \wr \mathbf{Z}$ then Γ is residually finite if and only if F is abelian [9].

It would be interesting to construct examples of cross-wired lamp-lighters which are not commensurable to either wreath products or linear groups. Determining whether or not such examples exist is necessary to answer the questions asked at the end of the introduction. The reader should note that groups commensurable to wreath products are not necessarily wreath products themselves.

Remark 3.1. Another generalization of lamplighter groups $F \wr \mathbf{Z}$ (F finite abelian) are Cayley machines associated to arbitrary finite groups [11]. It is suggested to us by L. Bartholdi that these might provide other examples of cross-wired lamplighters.

Remark 3.2. If G is a locally compact, compactly generated group and admits a cocompact, continuous, faithful action on a tree with no degree one vertex, then G has no nontrivial compact normal subgroup. Indeed, every cocompact action on such a tree has to be minimal. It follows that every closed cocompact subgroup in $\mathrm{Isom}(\mathrm{DL}(m,n))$ has no nontrivial compact normal subgroup. Thus a locally compact group Γ admitting a proper cocompact action on $\mathrm{DL}(m,n)$ $(m,n \geq 2)$ has a unique maximal compact normal subgroup $W(\Gamma)$, and Γ is actually isomorphic to a closed cocompact subgroup of $\mathrm{DL}(m,n)$ if and only if $W(\Gamma) = \{1\}$.

Here is an example of a discrete group Γ with a proper cocompact action on some $\mathrm{DL}(n,n)$ such that $W(\Gamma_1)$ is nontrivial for every finite index subgroup Γ_1 of Γ. Namely, let F be a finite group and Z a nontrivial central subgroup of F contained in the derived subgroup $[F,F]$. Start from the wreath product $F \wr \mathbf{Z} = F^{(\mathbf{Z})} \rtimes \mathbf{Z}$ and mod out by the subgroup Z_0 of $Z^{(\mathbf{Z})}$ of families $(z_n)_{n \in \mathbf{Z}}$ such that $\sum_{n \in \mathbf{Z}} z_n = 0$, and $\Gamma = (F \wr \mathbf{Z})/Z_0$ (Z_0 is normal in $F \wr \mathbf{Z}$ because Z is central in F). The image of $Z^{(\mathbf{Z})}$ in Γ is a central subgroup isomorphic to Z, let us call it Z. By Gruenberg [9], every finite index subgroup of $F \wr \mathbf{Z}$ contains $[F,F]^{(\mathbf{Z})}$. Thus every finite index subgroup Γ_1 of Γ contains the finite normal subgroup Z (and thus it easily follows that $W(\Gamma_1) = Z$).

3.2. **The name "Cross-wired lamplighters".** A lamplighter group $\sum F \rtimes \mathbf{Z}$ is usually interpreted as follows. The generator of $\mathbf{Z}$ describes the walk of a lamplighter along an infinite row of lamps with $|F|$ states. The generators of F represent the lamplighters ability to change the state of the lamp at his current position. (The description is most intuitive where $F = \mathbf{Z}/2\mathbf{Z}$ and lamps are simply on or off.)

All of our examples of cross-wired lamplighters have a similar but more complex structure. The generator of $\mathbf{Z}$ can again be viewed as letting the lamplighter walk along an infinite row of lamps. And in

each example there is a finite group F which can again be viewed as allowing the lamplighter to change the state of some lamp at his current position. However, in this setting, the wires of the lamps are "crossed" and changing the setting of the lamp at the current position may change the setting of other lamps at other positions. In all of our examples these "cross-wires" arise from a kind of non-commutativity based on a nilpotent structure. It would be interesting to know if all "cross-wired lamplighters" exhibit similar structure.

References

[1] L. Bartholdi, M. Neuhauser, W. Woess. *Horocyclic products of trees.* J. European Math. Society 10 (2008) 771–816.

[2] R. Diestel, I. Leader. *A conjecture concerning a limit of non-Cayley graphs.* J. Algebraic Combin. 14 (2001), no. 1, 17–25.

[3] A. Dyubina. *Instability the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups.* Int. Math. Res. Not. **21** (2000), 1097–1101.

[4] A. Eskin, D. Fisher, K. Whyte. *Quasi-isometries and rigidity of Solvable Groups.* Pure and Applied Mathematics Quarterly 3 (2007) 927–947.

[5] A. Eskin, D. Fisher, K. Whyte. *Coarse Differentiation of Quasi-isometries I: spaces not quasi-isometric to Cayley graphs.* Annals of Math. 176 (2012) 221–260.

[6] A. Eskin, D. Fisher, K. Whyte. *Coarse Differentiation of Quasi-isometries II: rigidity for Sol and Lamplighter groups.* ArXiv 0706.0940v1.

[7] H. Glockner, G. Willis. *Classification of the simple factors appearing in the composition series of totally disconnected contraction groups,* J. für Reine Angew. Math. (Crelles Journal) 643 (2010) 141–169.

[8] O. Golovin. *Nilpotent products of groups.* Amer. Math. Soc. Transl. (2) 2 (1956), 89–115.

[9] K. W. Gruenberg. Residual properties of infinite soluble groups. Proc. London Math. Soc. (3) 7, 1957, 29–62.

[10] T. MacHenry. *The tensor product and the 2nd nilpotent product of groups.* Math. Z. 73 (1960) 134–145.

[11] P. V. Silva and B. Steinberg. *On a class of automata groups generalizing lamplighter groups.* Internat. J. Algebra Comput. 15 (2005), 1213–1235.

[12] W. Woess. *Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions.* Combinatorics, Probability & Computing 14 (2005) 415–433.

[13] K. Wortman. *A finitely presented solvable group with a small quasi-isometry group.* Michigan Math. J. 55 (2007), 3–24.

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