Commensurating actions, Properties FW and PW

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Definitions. X set, $A, B \subset X$ are *commensurable* if the symmetric difference $A \triangle B$ is finite.

G group, X G-set. A subset $A \subset X$ is *commensurated* by G if A and gA are commensurable for all $g \in G$. We write $\ell_A(g) = \#(A \triangle gA)$.

Examples. (i) A G-invariant ($\Longleftrightarrow \ell_A = 0$)

(ii) more generally, A commensurable to G-invariant subset B (so ℓ_A is bounded, namely by $2\#(A \triangle B)$. These are called the *trivial* commensurated subsets.

(iii) First non-trivial example: $G = X = \mathbf{Z}$ (action by translation), $A = \mathbf{N} = \{0, 1, 2, \ldots\}$. Here $\ell_A(n) = |n|$. More generally, if Γ is a finitely generated group and $X = \Gamma$ with left translation action, then there is a non-trivial commensurated subset if and only if Γ has at least 2 ends.

Converse to (ii): Proposition: if ℓ_A is bounded then A is commensurable to an invariant subset B [Brailovsky–Pasechnik–Praeger 1995; P.Neumann 1996: B can be chosen with

$$\#(B \triangle A) \leq \max(0, \sup \ell_A - 1)$$

(optimal)]. Other proofs (folklore) use Jung's circumcenter lemma.

Generalization of the first statement in (iii): let G act on a tree T and fix a vertex v. Let X be the set of oriented edges in T (pairs (x, y) such that $\{x, y\}$ is an edge). Let A be the set of oriented edges pointing towards v. Then G commensurates A and

$$\ell_A(g) = d(g, vg).$$

In particular, A is non-trivial if and only G has unbounded orbits.

Definition. G group. A *cardinal definite length* on G is a function $G \to \mathbf{R}$ of the form ℓ_A with X G set, $A \subset$ commensurated.

Proposition. Every cardinal definite is conditionally negative definite.

Proof. For $p \geq 1$ define

$$\ell^p_A(X) = \{\xi : X \to \mathbf{R} : \ \xi - 1_A \in \ell^p(X)\};$$

this is an affine space over $\ell^p(X)$. The G-action on $\mathbf{R}^X$ preserves $\ell^p_A(X)$. We have

$$d_p(1_{gA}, 1_A)^p = \#(A \triangle gA) = \ell_A(g).$$

For $p = 2$ this shows that ℓ_A is conditionally negative definite (with a concrete realization of the associated affine isometric action). $\square$

Remark: this is a particular case of an observation by Robertson and Steger (1998).

Definition G has *Property PW* if there exists on G a proper cardinal definite function and has *Property FW* if every cardinal definite function on G is bounded (or equivalently if actions of G have no nontrivial commensurated subsets).

Clearly (PW and FW)$\Longleftrightarrow$ finite. Also, since Haagerup Property and Property T have the same definition replacing "cardinal definite" by "conditionally negative definite", we have, as corollary of the proposition

$$\text{PW} \Longrightarrow \text{Haagerup}$$
$$\text{Property T} \Longrightarrow \text{FW}$$

Examples of PW groups: free, surface groups. Coxeter groups. Stable under taking subgroups, wreath products.

Haglund (2007): Haagerup $\not\Longrightarrow$ PW.

Haglund's lemma. In a finitely generated group G with Property PW, any infinite cyclic subgroup $\langle x \rangle$ is *undistorted*, i.e. the word length of x^n has linear growth.

Proof. We present a combinatorial proof. (Haglund's original proof uses dynamics of isometries in infinite-dimensional cube complexes.) Let $\mathbf{Z}$ act on X commensurating A. Write $n \cdot x = T^n x$ where T is a permutation of X. Decompose $X = \bigsqcup_{i \in I} X_i$ into $\mathbf{Z}$-orbits. Set $A_i = A \cap X_i$. So $\ell_A = \sum_{i \in I} \ell_{A_i}$. We have $\ell(1) < \infty$ so

$$J = \{i \in I : \; A_i \triangle T(A_i) \neq \emptyset\}$$

is finite. For $i \notin J$ A_i is $\mathbf{Z}$-invariant, so

$$\ell = \sum_{i \in J} \ell_i \quad \text{(finite sum)}$$

For $i \in J$, if X_i is finite then ℓ_{A_i} is bounded.

Suppose the $\mathbf{Z}$-orbit X_i is infinite. So we can identify x_i with $\mathbf{Z}$ with T acting as translation by $+1$. We thus have

$$A_i \triangle (A_i + 1) \text{ finite}$$

and therefore A_i is commensurable to one of: $\emptyset$, $\mathbf{Z}$, $\mathbf{N}$, $-\mathbf{N}$. In the first two cases ℓ_{A_i} is bounded. In the last two cases we get

$$\ell_{A_i}(n) = |n| + \text{(bounded function)}.$$

We conclude

$$\exists k \in \mathbf{N},\ \forall n \in \mathbf{Z},\ \ell_A(n) = kn + \text{(bounded function)}$$

So if $k = 0$, ℓ_A is bounded and otherwise it has linear growth.

Now if X is a G-set and A a commensurated subset such that ℓ_A is proper and x has infinite order, then ℓ_A is unbounded on $\mathbf{Z} = \langle x \rangle$ so grows linearly on $\mathbf{Z}$. Also ℓ_A is sub-additive on G so is bounded by a multiple of the word length, whence the conclusion. $\square$

Proposition [C]. FW $\not\Longrightarrow$ T.

First example: irreducible lattices in $G \times H$, G connected Lie group without Property T and H locally compact group with Property T (for instance arithmetic lattices in SO$(4,1) \times$ SO$(3,2)$).

Second example: $\mathrm{SL}_2(\mathbf{Z}[\sqrt{n}])$, n positive non-square integer. Thus these infinite groups are both Haagerup and FW. **Main lines of proof:** As discrete subgroups (actually lattices) in $\mathrm{SL}_2(\mathbf{R})^2$, these groups are Haagerup. On the other hand, elementary matrices are undistorted (consequence of the existence of infinite order units (Dirichlet's unit theorem)). The proof of Haglund's lemma shows that every cardinal definite function is bounded on these subgroups. By bounded generation by unipotents (Carter-Keller), Property FW follows.

Conjecture. Every irreducible lattice in $\mathrm{SL}_2(\mathbf{R})^2$ has Property FW.

Proposition [C]. Every central extension

$$1 \longrightarrow \mathbf{Z} \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

with $\tilde{\Gamma}$ having Property PW is virtually split, in the sense that there exists a subgroup K with $K \cap \mathbf{Z} = \{e\}$ and $K\mathbf{Z}$ has finite index in Γ.

Example: Γ_g a closed surface group and $\tilde{\Gamma}_g$ its inverse image in $\widetilde{\mathsf{SL}_2}$. Then $\tilde{\Gamma}_g$ does *not* have Property PW (although it is Haagerup and has no distorted cyclic subgroups). Since $\tilde{\Gamma}_g$ is quasi-isometric to $\Gamma_g \times \mathbf{Z}$, this shows that PW is not stable under quasi-isometries (it's unknown whether Haagerup is closed under quasi-isometries).

In short, the proposition says that any central $\mathbf{Z}$ is virtually a direct factor. The result extends (with slightly more complicated proof) to show that every normal $\mathbf{Z}^k$ in an PW-group is virtually a direct factor.

Proof of the proposition. Consider a $\widetilde{\Gamma}$-set X with commensurated subset A. Let ℓ_A be the associated cardinal definite function and assume it unbounded on $\mathbf{Z}$. Decompose X into $\mathbf{Z}$-orbits: $X = \bigsqcup X_i$ and $A_i = A \cap X_i$. Define

$$J = \{i : \ A_i \text{ and } X_i \smallsetminus A_i \text{ are infinite}\}.$$

Then $Y = \bigcup_{i \in J} X_i$ is $\widetilde{\Gamma}$-invariant. On $\mathbf{Z}$, we have

$$\ell_A = \sum_{i \in J} \ell_{A_i} + \sum_{i \notin J} \ell_{A_i}.$$

The proof of Haglund's lemma shows that the second term is bounded on $\mathbf{Z}$ (so that the first term is unbounded) and that J is finite. The group $\widetilde{\Gamma}$ acts by permuting the X_i, $i \in J$. So some finite index subgroup Λ of Γ (containing $\mathbf{Z}$) stabilizes all the X_i ($i \in J$). Since $\mathbf{Z}$ is central, it acts by translation on each X_i. Fix $j \in J$ and let K be the kernel of the action of Λ on X_j. Then $\Lambda = \mathbf{Z} \times K$. $\square$

Definition. V G-set. A *walling* $\mathcal{W}$ on V is the data of a G-set X (index set) and a G-equivariant map $X \to 2^V$, $x \mapsto W_x$ such that for all $u, v \in V$, the set of x such that W_x separates u and v is finite. Its cardinality is denoted by $d_{\mathcal{W}}(u, v)$; $d_{\mathcal{W}}$ is called the *wall pseudo-distance* on V.

This was (up to a minor modification) coined by Haglund and Paulin (1998) although implicit in Robertson-Steger (1998) and some other works.

Lemma. $f : G \to \mathbf{R}$ is cardinal definite if and only if there exists a G-set V, $v \in V$, and a G-walling such that $f(g) = d_{\mathcal{W}}(v, gv)$ for all g. (Actually we can pick $(V, v) = (G, 1)$ with the left translation action.)

Proof: instructive exercise!

A CAT(0) cube complex is a combinatorial complex whose cells are cubes satisfying some further conditions (not written here). **Main example:** let X be a set and A a subset. If B, B' are commensurable to A, with $B \subset B'$, define $c(B, B')$ to be the set of subsets intermediate between B and B'. Consider the cube complex $C(X, A)$ whose k-cubes are the $c(B, B')$ when B, B' range over subsets commensurable to A with $B \subset B'$ and $\#(B' \smallsetminus B) = k$. If cubes are endowed with the ℓ^p-metric, its completion is $\ell^p_A(X)$.

Theorem. A function $\ell : G \to \mathbf{R}$ is cardinal definite if and only if there is CAT(0) a cube complex C with a combinatorial G-action and a vertex v such that in the ℓ^1-metric we have $d(v, gv) = \ell(g)$ for all g. Actually if $\ell = \ell_A$ with $A \subset X$, we can pick C to be $\ell^1_A(X)$ with its action.

The last construction is the Niblo-Roller construction (1998). Conversely, it was proved by Sageev (1995) that CAT(0) cube complexes have a canonical wall structure, proving the "if" part. However, other cubulations are of interest, especially finite-dimensional cubulations. The origin of CAT(0) cube complexes are cubulations of 3-dimensional manifolds (Aitchison-Rubinstein). A particular class of finite-dimensional CAT(0) cube complexes, called *special* was introduced by Haglund and Wise and recently had remarkable applications, notably to the classification of 3-manifold groups (Wise, Agol. . .).

On the other hand, there exist groups with Property PW not having any proper action on a *finite-dimensional* CAT(0) cube complex. Thompson's group F and the lamplighter group over the free group are examples of such groups.