COMMENSURATING ACTIONS OF BIRATIONAL GROUPS AND GROUPS OF PSEUDO-AUTOMORPHISMS

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ABSTRACT. Pseudo-automorphisms are birational transformations acting as regular automorphisms in codimension 1. We import ideas from geometric group theory to study groups of birational transformations, and prove that a group of birational transformations that satisfies a fixed point property on CAT(0) cubical complexes is birationally conjugate to a group acting by pseudo-automorphisms on a non-empty Zariski-open subset. We apply this argument to classify groups of birational transformations of surfaces with this fixed point property up to birational conjugacy.

1. INTRODUCTION

1.1. **Birational transformations and pseudo-automorphisms.** Let X be a quasi-projective variety, over an algebraically closed field $\mathbf{k}$. Denote by $\mathsf{Bir}(X)$ the group of birational transformations of X and by $\mathsf{Aut}(X)$ the subgroup of (regular) automorphisms of X. For the affine space of dimension n, automorphisms are invertible transformations $f\colon \mathbb{A}^n_{\mathbf{k}} \to \mathbb{A}^n_{\mathbf{k}}$ such that both f and f^{-1} are defined by polynomial formulas in affine coordinates:

$$f(x_1,\ldots,x_n) = (f_1,\ldots,f_n),\;\; f^{-1}(x_1,\ldots,x_n) = (g_1,\ldots,g_n)$$

with $f_i, g_i \in \mathbf{k}[x_1,\ldots,x_n]$. Similarly, birational transformations of $\mathbb{A}^n_{\mathbf{k}}$ are given by rational formulas, i.e. $f_i, g_i \in \mathbf{k}(x_1,\ldots,x_n)$.

Birational transformations may contract hypersurfaces. Roughly speaking, **pseudo-automorphisms** are birational transformations that act as automorphisms in codimension 1. Precisely, a birational transformation $f\colon X \dashrightarrow X$ is a pseudo-automorphism if there exist Zariski-open subsets $\mathcal{U}$ and $\mathcal{V}$ in X such that $X \smallsetminus \mathcal{U}$ and $X \smallsetminus \mathcal{V}$ have codimension ≥ 2 and f induces an isomorphism from $\mathcal{U}$ to $\mathcal{V}$. The pseudo-automorphisms of X form a group, which we denote by $\mathsf{Psaut}(X)$. For instance, all birational transformations of Calabi-Yau manifolds are pseudo-automorphisms; and there are examples of such manifolds for which $\mathsf{Psaut}(X)$ is

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infinite while $\mathsf{Aut}(X)$ is trivial (see [8]). Pseudo-automorphisms are studied in Section 2.

Definition 1.1. *Let $\Gamma \subset \mathsf{Bir}(X)$ be group of birational transformations of an irreducible projective variety X. We say that Γ is* **pseudo-regularizable** *if there exists a triple $(Y, \mathcal{U}, \varphi)$ where*

(1) *Y is a projective variety and $\varphi \colon Y \dashrightarrow X$ is a birational map;*
(2) *$\mathcal{U}$ is a non-empty, Zariski open subset of Y;*
(3) *$\varphi^{-1} \circ \Gamma \circ \varphi$ yields an action of Γ by pseudo-automorphisms on $\mathcal{U}$.*

More generally if $\alpha : \Gamma \to \mathsf{Bir}(X)$ is a birational action, we say that it is pseudo-regularizable if $\alpha(\Gamma)$ is pseudo-regularizable.

One goal of this article is to exhibit a class of groups Γ for which every action of Γ by birational transformations on a projective variety M is pseudo-regularizable.

1.2. **Property (FW).** The class of groups we shall be interested in is characterized by a fixed point property; this property appears in several related situations, for instance for actions on CAT(0) cubical complexes. Here, we adopt the viewpoint of commensurated subsets. Let Γ be a group, and $\Gamma \times S \to S$ an action of Γ on a set S. Let A be a subset of S. One says that Γ **commensurates** A if the symmetric difference

$$\gamma(A) \,\Delta A = (\gamma(A) \smallsetminus A) \cup A\,(\smallsetminus \gamma(A))$$

is finite for every element γ of Γ. One says that Γ **transfixes** A if there is a subset B of S such that $A \,\Delta B$ is finite and B is Γ-invariant: $\gamma(B) = B$, for every γ in Γ.

A group Γ has **Property (FW)** if, given any action of Γ on a set S, all commensurated subsets of S are automatically transfixed. For instance, $\mathsf{SL}_2(\mathbf{Z}[\sqrt{5}])$ and $\mathsf{SL}_3(\mathbf{Z})$ do have Property (FW), but free groups do not share this property. Property (FW) is discussed in Section 3.

Let us mention that among various characterizations of Property (FW) (see [9]), one is: every combinatorial action of Γ on a CAT(0) cube complex fixes some cube. Another (for Γ finitely generated) is that all its infinite Schreier graphs are one-ended.

1.3. **Pseudo-regularizations.** Let X be a projective variety. The group $\mathsf{Bir}(X)$ does not really act on X, because there are indeterminacy points; it does not act on the set of hypersurfaces either, because some of them may be contracted. As we shall explain, one can introduce the set $\widetilde{\mathsf{Hyp}}(X)$ of all irreducible and reduced hypersurfaces in all birational models of X (up to a natural identification), and then $\mathsf{Bir}(X)$

acts on this set by strict transforms; moreover, this action commensurates the subset $\mathsf{Hyp}(X)$ of hypersurfaces of X. This construction leads to the following result.

Theorem A. *Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Let Γ be a subgroup of $\mathsf{Bir}(X)$. If Γ has Property (FW), then Γ is pseudo-regularizable.*

There is also a relative version of Property (FW) for pairs of groups $\Lambda \leq \Gamma$, which leads to a similar pseudo-regularization theorem for the subgroup Λ: this is discussed in Section 4.5, with applications to distorted birational transformations.

Remark 1.2. There are two extreme cases for the pair $(Y, \mathcal{U})$ in Theorem A, corresponding to the size of the boundary $Y \smallsetminus \mathcal{U}$. If this boundary is empty, Γ acts by pseudo-automorphisms on a projective variety Y. If it is ample, its complement $\mathcal{U}$ is an affine variety, and then Γ acts in fact by regular automorphisms on $\mathcal{U}$ (see Section 2.4). Thus, in the study of groups of birational transformations, *pseudo-automorphisms of projective varieties and regular automorphisms of affine varieties deserve specific attention.*

1.4. **Classification in dimension 2.** In dimension 2, pseudo-automorphisms do not differ much from automorphisms; for instance, $\mathsf{Psaut}(X)$ coincides with $\mathsf{Aut}(X)$ if X is a smooth projective surface. Thus, for groups with Property (FW), Theorem A can be used to reduce the study of birational transformations to the study of automorphisms of quasi-projective surfaces, a subject which has been intensively studied.

Theorem B. *Let X be a smooth, projective, and irreducible surface, over an algebraically closed field. Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$. If Γ has Property (FW), there is a birational map $\varphi \colon Y \dashrightarrow X$ such that*

(1) *Y is the projective plane $\mathbb{P}^2_{\mathbf{k}}$, a Hirzebruch surface $\mathbb{F}_m$ with $m \geq 1$, or the product of a curve C by the projective line $\mathbb{P}^1_{\mathbf{k}}$;*
(2) *$\varphi^{-1} \circ \Gamma \circ \varphi$ is contained in $\mathsf{Aut}(Y)$.*

If the characteristic of the field is positive, Y is the projective plane $\mathbb{P}^2_{\mathbf{k}}$.

For those surfaces Y, the group $\mathsf{Aut}(Y)$ has finitely many connected components. Thus, changing Γ into a finite index subgroup Γ_0, one gets a subgroup of $\mathsf{Aut}(Y)^0$. Here $\mathsf{Aut}(Y)^0$ denotes the connected component of the identity of $\mathsf{Aut}(Y)$. This is an algebraic group, acting algebraically on Y.

Example 1.3. Groups with Kazhdan Property (T) satisfy Property (FW). Thus, Theorem B extends Theorem A of [6]. Moreover, the present article offers a new proof of this result.

Theorem B may also be applied to the group $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$, where $d \geq 2$ is a non-square positive integer. Thus, every action of this group on a projective surface by birational transformations is conjugate to an action by regular automorphisms on $\mathbb{P}^2_{\mathbf{k}}$, the product of a curve C by the projective line $\mathbb{P}^1_{\mathbf{k}}$, or a Hirzebruch surface. In this case, we can make use of Margulis' superrigidity to get a more precise result, see §8.

In general, for a variety X one can ask whether $\mathsf{Bir}(X)$ transfixes $\mathsf{Hyp}(X)$, or equivalently is pseudo-regularizable. For a surface X, this holds precisely when X is not birationally equivalent to the product of the projective line with a curve. See §5 for more precise results.

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2. Pseudo-automorphisms

2.1. **Birational transformations, indeterminacies, strict and total transforms.** Let X and Y be two (irreducible, reduced) algebraic varieties over an algebraically closed field $\mathbf{k}$. If $f\colon X \dashrightarrow Y$ is a birational map, we denote by $\mathrm{Ind}(f)$ its indeterminacy set. When X and Y are projective and normal, $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ have codimension ≥ 2. The transformation of the affine plane $(x,y) \mapsto (x, y/x)$ is birational, and its indeterminacy locus is the line $\{x = 0\}$: this set of co-dimension 1 is mapped "to infinity" (if the affine plane is compactified by the projective plane, the transformation becomes $[x:y:z] \mapsto [x^2 : yz : xz]$, with two indeterminacy points). The graph $\mathsf{Gr}(f)$ of f is, by definition, the Zariski closure of $\{(x, f(x)) : x \in X \smallsetminus \mathrm{Ind}(f)\}$.

The **jacobian determinant** $\mathsf{Jac}(f)(x)$ is defined in local coordinates as the determinant of the differential df_x; $\mathsf{Jac}(f)$ depends on the coordinates, but its zero locus does not. The **exceptional set** of f is the subset of X along which f is not a local isomorphism onto its image ; it coincides with the union of $\mathrm{Ind}(f)$ and the zero locus of $\mathsf{Jac}(f)$.

The **total** transform of a subset $Z \subset X$ is denoted by $f_*(Z)$. If Z is not contained in $\mathrm{Ind}(f)$, we denote by $f_\circ(Z)$ its **strict** transform, defined as the Zariski closure of $f(Z \smallsetminus \mathrm{Ind}(f))$. We say that a hypersurface $W \subset Z$ is contracted if the codimension of its strict transform is larger than 1.

2.2. **Pseudo-isomorphisms.** A birational map $f\colon X \dashrightarrow Y$ is a **pseudo-isomorphism** if one can find Zariski open subsets $\mathcal{U} \subset X$ and $\mathcal{V} \subset Y$ such that

(i) f realizes a regular isomorphism from $\mathcal{U}$ to $\mathcal{V}$ and
(ii) $X \setminus \mathcal{U}$ and $Y \setminus \mathcal{V}$ have codimension ≥ 2.

Pseudo-isomorphisms from X to X are called **pseudo-automorphisms** (see § 1.2). For an example, start with the standard birational involution $\sigma_n\colon \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$ which is defined in homogeneous coordinates by

$$\sigma_n[x_0 : \ldots : x_n] = [x_0^{-1} : \ldots : x_n^{-1}].$$

Blow-up the $(n+1)$ vertices of the simplex $\Delta_n = \{[x_0 : \ldots : x_n];\ \prod x_i = 0\}$; this provides a smooth rational variety X_n together with a birational morphism $\pi\colon X_n \to \mathbb{P}^n_{\mathbf{k}}$. Then, $\pi^{-1} \circ \sigma_n \circ \pi$ is a pseudo-automorphism of X_n, and is an automorphism if $n \leq 2$.

Proposition 2.1. *Let $f\colon X \dashrightarrow Y$ be a birational map between two (irreducible, reduced) algebraic varieties. Assume that the codimension of the indeterminacy sets of f and f^{-1} is at least 2. Then, the following properties are equivalent:*

(1) *f and f^{-1} do not contract any hypersurface.*
(2) *The jacobian determinant of f (resp. of f^{-1}) does not vanish on $X \setminus \mathrm{Ind}(f)$ (resp. on $Y \setminus \mathrm{Ind}(f^{-1})$).*
(3) *For every $q \in X \setminus \mathrm{Ind}(f)$ (resp. $q \in Y \setminus \mathrm{Ind}(f^{-1})$), f (resp. f^{-1}) is a local isomorphism from a neighborhood of q to a neighborhood of $f(q)$ (resp. $f^{-1}(q)$).*
(4) *f is a pseudo-isomorphism from X to Y.*

The proof of this proposition is straightforward. As a corollary, the set of pseudo-automorphisms of X is a subgroup $\mathsf{Psaut}(X)$ of $\mathsf{Bir}(X)$.

Example 2.2. Let X be a smooth projective variety with trivial canonical bundle K_X. Let Ω be a non-vanishing section of K_X, and let f be a birational transformation of X. Then, $f^*\Omega$ extends from $X \setminus \mathrm{Ind}(f)$ to X and determines a new section of K_X; this section does not vanish identically because f is dominant, hence it does not vanish at all because K_X is trivial. As a consequence, $\mathsf{Jac}(f)$ does not vanish, f is a pseudo-automorphism of X, and $\mathsf{Bir}(X) = \mathsf{Psaut}(X)$. We refer to [8] for families of Calabi-Yau varieties with an infinite group of pseudo-automorphisms.

2.3. **Projective varieties.**

Proposition 2.3 (see [4])**.** *Let $f\colon X \dashrightarrow Y$ be a pseudo-isomorphism between two projective varieties. Then*

(1) *the total transform of* $\mathrm{Ind}(f)$ *by* f *is equal to* $\mathrm{Ind}(f^{-1})$*;*
(2) f *has no isolated indeterminacy point;*
(3) *if* $\dim(X) = 2$*, then* f *is a regular isomorphism.*

Proof. Let $p \in X$ be an indeterminacy point of the pseudo-isomorphism $f\colon X \dashrightarrow Y$. Then f^{-1} contracts a subset $C \subset Y$ of positive dimension on p. Since f and f^{-1} are local isomorphisms on the complement of their indeterminacy sets, C is contained in $\mathrm{Ind}(f^{-1})$. The total transform of a point $q \in C$ by f^{-1} is a connected subset of X that contains p and has dimension ≥ 1. This set D is contained in $\mathrm{Ind}(f)$ because f is a local isomorphism on the complement of $\mathrm{Ind}(f)$; since $p \in D \subset \mathrm{Ind}(f)$, p is not an isolated indeterminacy point. This proves Assertions (1) and (2). The third assertion follows from the second one because indeterminacy sets of birational transformations of projective surfaces are finite sets. □

Let W be a hypersurface of X, and let $f\colon X \to Y$ be a pseudo-isomorphism. The divisorial part of the total transform $f_*(W)$ coincides with the strict transform $f_\circ(W)$. Indeed, $f_*(W)$ and $f_\circ(W)$ coincide on the open subset of Y on which f^{-1} is a local isomorphism, and this open subset has codimension ≥ 2.

Theorem 2.4. *The action of pseudo-isomorphisms on Néron-Severi groups is functorial:* $(g \circ f)_* = g_* \circ f_*$ *for all pairs of pseudo-isomorphisms* $f\colon X \dashrightarrow Y$ *and* $g\colon Y \dashrightarrow Z$*. If* X *is a smooth projective variety, the group* $\mathsf{Psaut}(X)$ *acts linearly on the Néron-Severi group* $\mathrm{NS}(X)$*; this provides a morphism*

$$\mathsf{Psaut}(X) \to \mathsf{GL}\left(\mathrm{NS}(X)\right).$$

The kernel of this morphism is contained in $\mathsf{Aut}(X)$ *and contains* $\mathsf{Aut}(X)^0$ *as a finite index subgroup.*

As a consequence, if X is projective the group $\mathsf{Psaut}(X)$ is an extension of a discrete linear subgroup of $\mathsf{GL}\left(\mathrm{NS}(X)\right)$ by an algebraic group.

Proof. The first statement follows from the equality $f_* = f_\circ$ on divisors. The second follows from the first. To study the kernel K of the linear representation $\mathsf{Psaut}(X) \to \mathsf{GL}\left(\mathrm{NS}(X)\right)$, fix an embedding $\varphi\colon X \to \mathbb{P}^m_{\mathbf{k}}$ and denote by H the polarization given by hyperplane sections in $\mathbb{P}^m_{\mathbf{k}}$. For every f in K, $f_*(H)$ is an ample divisor, because its class in $\mathrm{NS}(X)$ coincides with the class of H. From Matsusaka-Mumford theorem, we deduce that f is an automorphism of X (see [24], and [18] exercise 5.6). To conclude, note that $\mathsf{Aut}(X)^0$ has finite index in the kernel of the action of $\mathsf{Aut}(X)$ on $\mathrm{NS}(X)$ (see [20]). □

2.4. **Affine varieties.** The group $\mathsf{Psaut}(\mathbb{A}^n_{\mathbf{k}})$ coincides with the group $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ of polynomial automorphisms of the affine space $\mathbb{A}^n_{\mathbf{k}}$: this is a special case of the following proposition.

Proposition 2.5. *If Z is a smooth affine variety, the group* $\mathsf{Psaut}(Z)$ *coincides with the group* $\mathsf{Aut}(Z)$.

Proof. Fix an embedding $Z \to \mathbb{A}^m_{\mathbf{k}}$. Rational functions on Z are restrictions of rational functions on $\mathbb{A}^m_{\mathbf{k}}$. Thus, every birational transformation $f\colon Z \to Z$ is given by rational formulas $f(x_1,\ldots,x_m) = (f_1,\ldots,f_m)$ where each f_i is a rational function

$$f_i = \frac{p_i}{q_i} \in \mathbf{k}(x_1,\ldots,x_m);$$

here, p_i and q_i are relatively prime polynomial functions. Since the local rings $O_{Z,x}$ are unique factorization domains, we may assume that the hypersurfaces $W_Z(p_i) = \{x \in Z;\ p_i(z) = 0\}$ and $W_Z(q_i) = \{x \in Z;\ q_i(z) = 0\}$ have no common components. Then, the generic point of $W_Z(q_i)$ is mapped to infinity by f. Since f is a pseudo-isomorphism, $W_Z(q_i)$ is in fact empty; but if q_i does not vanish on Z, f is a regular map. □

3. Groups with Property (FW)

3.1. **Commensurated subsets and cardinal definite length functions (see** [9]**).** Let G be a group, and $G \times S \to S$ an action of G on a set S. Let A be a subset of S. One says that G **commensurates** A if the symmetric difference $g(A)\,\Delta\, A$ is finite for every element g of G. One says that G **transfixes** A if there is a subset B of S such that $A\,\Delta\, B$ is finite and B is G-invariant: $g(B) = B$, for every g in G. If A is transfixed, then it is commensurated. Actually, A is transfixed if and only if the function $g \mapsto \#(A\,\Delta\, gA)$ is bounded on G.

A group G has **Property (FW)** if, given any action of G on a set S, all commensurated subsets of S are automatically transfixed. More generally, if H is a subgroup of G, then (G,H) has **relative Property (FW)** if every commensurating action of G is transfixing in restriction to H. This means that, if G acts on a set S and commensurates a subset A, then H transfixes automatically A. The case $H = G$ is Property (FW) for G.

We refer to [9] for a detailed study of Property (FW). For instance, if G_0 is a finite index subgroup of G, then G has Property (FW) if, and only if G_0 has it ([9], Prop. 5.B.1). The two main known sources of Property (FW) and its relative version are, on the one hand, Kazhdan's Property (T), and, on the other hand, distorted elements, as explained in the next paragraphs.

Remark 3.1. Property (FW) should be thought of as a rigidity property. For instance, suppose that K is a group with and action on a set S that commensurates a subset $A \subset S$ without transfixing it; then, if G has Property (FW), there are restrictions on the homomorphisms $G \to K$. To illustrate this idea, one shall say that a group K has Property (PW) if it admits a commensurating action on a set S, with a commensurating subset C such that the function $g \mapsto \#(C \Delta gC)$ has finite fibers. For such a group, the only subgroups H such that (K,H) has relative Property (FW) are the finite subgroups.

3.2. **Property (FW) and Property (T).** One can rephrase Property (FW) as follows: G has Property (FW) if and only if every isometric action on an "integral Hilbert space" $\ell^2(X,\mathbf{Z})$ (X any discrete set) has bounded orbits. A group has Property (FH) if all its isometric actions on Hilbert spaces have fixed points. More generally, if G is a group and H a subgroup, the pair (G,H) has relative Property (FH) if every isometric G-action on a Hilbert space has an H-fixed point. Thus, the relative Property (FH) implies the relative Property (FW).

By a theorem of Delorme and Guichardet, Property (FH) is equivalent to Kazhdan's Property (T) for countable groups (see [11]). Thus, (T) implies (FW).

Kazhdan's Property (T) is satisfied by lattices in semisimple Lie groups all of whose simple factors have Property (T), for instance if all simple factors have real rank ≥ 2. For example, $\mathsf{SL}_3(\mathbf{Z})$ satisfies Property (T).

Property (FW) is actually conjectured to hold for all irreducible lattices in semisimple Lie groups of real rank ≥ 2, such as $\mathsf{SL}_2(\mathbf{R})^k$ for $k \geq 2$. (Recall that irreducible means that the projection of the lattice *modulo* every simple factor is dense.) This is known in the case of a semisimple Lie group admitting at least one noncompact simple factor with Kazhdan's Property (T), for instance in $\mathsf{SO}(2,3) \times \mathsf{SO}(1,4)$, which admits irreducible lattices (see [10]).

3.3. **Distortion.** Let G be a group. An element g of G is **distorted** in G if there exists a finite subset Σ of G generating a subgroup $\langle \Sigma \rangle$ containing g, such that $\lim_{n\to\infty} \frac{1}{n}|g^n|_\Sigma = 0$; here, $|g|_\Sigma$ is the length of g with respect to the set Σ. If G is finitely generated, this condition holds for some Σ if and only if it holds for every finite generating subset of G. For example, every finite order element is distorted.

Example 3.2. Let K be a field. The distorted elements of $\mathsf{SL}_n(K)$ are exactly the virtually unipotent elements, that is, those elements whose eigenvalues are all roots of unity (in positive characteristic, these are elements of finite order). By results of Lubotzky, Mozes, and Raghunathan (see [22, 21]), the same characterization holds in the group $\mathsf{SL}_n(\mathbf{Z})$, as soon as $n \geq 3$; it also holds in $\mathsf{SL}_n(\mathbf{Z}[\sqrt{d}])$ for $n \geq 2$ and non-square $d \geq 2$. However, in $\mathsf{SL}_2(\mathbf{Z})$, every element of infinite order is undistorted.

Lemma 3.3 (see [9])**.** *Let G be a group, and H a finitely generated abelian subgroup of G consisting of distorted elements. Then, the pair (G,H) has relative Property (FW).*

This fact can be used to get many examples. For instance, if G is any finitely generated nilpotent group and G' is its derived subgroup, then (G,G') has relative Property (FH); this result is due to Houghton, in a more general formulation encompassing polycyclic groups (see [9]). Bounded generation by distorted unipotent elements can also be used to obtain nontrivial examples of groups with Property (FW), including the above examples $\mathsf{SL}_n(\mathbf{Z})$ for $n \geq 3$, and $\mathsf{SL}_n(\mathbf{Z}[\sqrt{d}])$. The case of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ is particularly interesting because it does not have Property (T).

3.4. **Subgroups of $\mathsf{PGL}_2(\mathbf{k})$ with Property (FW).** If a group G acts on a tree T by graph automorphisms, then G acts on the set E of directed edges of T (T is non-oriented, so each edge gives rise to a pair of opposite directed edges). Let E_v be the set of directed edges pointing towards a vertex v. Then $E_v \,\Delta\, E_w$ is the set of directed edges lying in the segment between v and w; in particular it is finite of cardinality $2d(v,w)$, where d is the graph distance. Thus G commensurates the subset E_v for every v, and $\#(E_v \,\Delta\, gE_v) = 2d(v,gv)$. As a consequence, if G has Property (FW), then it has Property (FA) in the sense that every action of G on a tree has bounded orbits. A similar statement holds for relative properties.

Lemma 3.4 (See [9])**.** *Let Γ be a group with Property (FW), then all finite index subgroups of Γ have Property (FW), and hence have Property (FA).*

In what follows, we denote by $\overline{\mathbf{Z}} \subset \overline{\mathbf{Q}}$ the ring of algebraic integers (in an algebraic closure of $\mathbf{Q}$). If $\mathbf{k}$ is an algebraically closed field of positive characteristic, algebraic integers are roots of unity.

Theorem 3.5. *Let $\mathbf{k}$ be an algebraically closed field. Let Γ be a countable subgroup of $\mathsf{GL}_2(\mathbf{k})$. Assume that all finite index subgroups of Γ have Property (FA). Then, Γ is conjugate, by a matrix $B \in \mathsf{GL}_2(\mathbf{k})$, to a subgroup of $\mathsf{GL}_2(A)$, where A is the ring of algebraic integers:*

- *if the characteristic of $\mathbf{k}$ is positive, Γ is conjugate to a subgroup of $\mathsf{GL}_2(\mathbf{k}')$, for some finite subfield $\mathbf{k}'$ of $\mathbf{k}$;*
- *if the characteristic of $\mathbf{k}$ is 0, Γ is conjugate to a subgroup of $\mathsf{GL}_2(\overline{\mathbf{Z}})$.*

The same result applies if we replace $\mathsf{GL}_2(\mathbf{k})$ by $\mathsf{PGL}_2(\mathbf{k})$ or $\mathsf{SL}_2(\mathbf{k})$, and we shall refer to it several times for $\mathsf{PGL}_2(\mathbf{k})$.

Lemma 3.6. *Let Λ be a subgroup of $\mathsf{GL}_2(\mathbf{k})$. Assume that all finite index subgroups of Λ have Property (FA). If the action of Λ on the projective line preserves a non-empty, finite set, then Λ is finite.*

Proof of the Lemma. A finite index subgroup Λ_0 of Λ fixes each point of the finite invariant subset of $\mathbb{P}^1(\mathbf{k})$. By a projective change of coordinates, we may assume that the point $\infty = [1:0]$ is fixed. This means that Λ_0 is contained in the group of upper triangular matrices. The diagonal coefficients determine a morphism from Λ_0 to $\mathbf{k}^* \times \mathbf{k}^*$. Since Λ_0 has Property (FA), a finite index subgroup Λ_1 of Λ_0 is made of upper triangular matrices with diagonal coefficients equal to 1. By assumption, Λ_1 inherits Property (FA). Being abelian, Λ_1 is finite, and Λ is finite too. □

Proof of Theorem 3.5. This directly follows from the GL_2-subgroup theorem of Hyman Bass (see [2, 1]). First, note that Property (FA) is equivalent to the conjunction of the following three properties: Γ is finitely generated, Γ is not a non-trivial amalgam, and the abelianization $\Gamma/[\Gamma,\Gamma]$ is finite. This is proved in [26], Theorem 15, Section 6. Then, apply Corollary 2 of [2], as well as Remark (2) on the same page. Either Γ is contained in a conjugate of $\mathsf{GL}_2(A)$, with A the ring of algebraic integers, or in a conjugate of the group of matrices

$$\left(\begin{array}{cc} a & b \\ 0 & d \end{array}\right)$$

with a and d roots of unity. In this last case, Lemma 3.6 shows that Γ is finite; as such, it is conjugate to a subgroup of $\mathsf{GL}_2(A)$. □

Remark 3.7. Let $K \subset \mathbf{k}$ be a field extension, with $\mathbf{k}$ algebraically closed and of characteristic 0. In Theorem 3.5, assume that the group Γ is contained in $\mathsf{GL}_2(K)$ (or $\mathsf{PGL}_2(K)$), and that Γ is infinite. If K contains $\overline{\mathbf{Q}}$, then one can choose the conjugacy matrix B in $\mathsf{GL}_2(K)$.

To prove this result, we first note that Γ is absolutely irreducible. Otherwise, Γ would be finite, by Lemma 3.6. Thus, Γ generates the vector space of 2×2 matrices $\mathsf{Mat}_2(K)$. Write

$$B = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

with coefficients a, b, c, and d in $\mathbf{k}$, and multiply B by a non-zero element of $\mathbf{k}$ to assume that $a = 1$ or $a = 0$ and $b = 1$. Then, consider the conjugacy relation

$B\Gamma B^{-1} \subset \mathsf{GL}_2(\overline{\mathbf{Z}})$ and form a linear combination, with coefficients in K, to obtain

$$B\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B^{-1} \in \mathsf{GL}_2(K);$$

this is possible because Γ generates $\mathsf{Mat}_2(K)$ and K contains $\overline{\mathbf{Z}}$. A direct computation implies that b/d is in K (if $d \neq 0$). Performing similar computations with the other elementary matrices, one obtains that all coefficients of B are in K.

Corollary 3.8. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* C *be a projective curve over* $\mathbf{k}$*, and let* $\mathbf{k}(C)$ *be the field of rational functions on the curve* C*. Let* Γ *be an infinite subgroup of* $\mathsf{PGL}_2(\mathbf{k}(C))$*. If* Γ *has Property (FW), then*

(1) *the field* $\mathbf{k}$ *has characteristic* 0*;*
(2) *there is an element of* $\mathsf{PGL}_2(\mathbf{k}(C))$ *that conjugates* Γ *to a subgroup of* $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \subset \mathsf{PGL}_2(\mathbf{k}(C))$.

The proof is a direct consequence of Theorem 3.5 and the previous remark.

4. From birational transformations to pseudo-automorphisms

4.1. **An example.** Consider the birational transformation $f(x,y) = (x+1, xy)$ of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. The vertical curves $C_i = \{x = -i\}$, $i \in \mathbf{Z}$, are exceptional curves for the cyclic group $\Gamma = \langle f \rangle$: each of these curves is contracted by an element of Γ onto a point, namely $f_\circ^{i+1}(C_i) = (1,0)$.

Let $\varphi\colon Y \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ be a birational map, and let $\mathcal{U}$ be a non-empty open subset of Y. Consider the subgroup $\Gamma_Y := \varphi^{-1} \circ \Gamma \circ \varphi$ of $\mathsf{Bir}(Y)$. If i is large enough, $\varphi_\circ^{-1}(C_i)$ is an irreducible curve $C_i' \subset Y$, and these curves C_i' are pairwise distinct, so that most of them intersect $\mathcal{U}$. For positive integers m, f^{i+m} maps C_i onto $(m,0)$, and $(m,0)$ is not an indeterminacy point of φ^{-1} if m is large. Thus, $\varphi^{-1} \circ f^m \circ \varphi$ contracts C_i', and $\varphi^{-1} \circ f^m \circ \varphi$ is not a pseudo-automorphism of $\mathcal{U}$. This argument proves the following lemma.

Lemma 4.1. *Let* X *be the surface* $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$*. Let* $f\colon X \dashrightarrow X$ *be defined by* $f(x,y) = (x+1, xy)$*, and let* Γ *be the subgroup generated by* f^ℓ*, for some* $\ell \geq 1$*. Then the cyclic group* Γ *is not pseudo-regularizable.*

This shows that Theorem A requires an assumption on the groups Γ. More generally, assume that $\Gamma \subset \mathsf{Bir}(X)$

(a) contracts a family of hypersurfaces $W_i \subset X$ whose union is Zariski dense
(b) the union of the family of strict transforms $f_\circ(W_i)$, for $f \in \Gamma$ contracting W_i, form a subset of X whose Zariski closure has codimension at most 1.

Then, Γ cannot be pseudo-regularized.

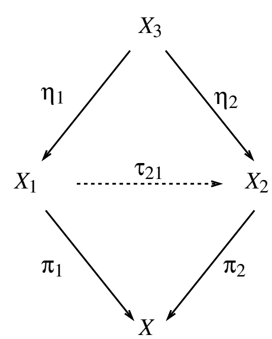
4.2. **The space of hypersurfaces** $\tilde{\mathsf{Hyp}}(X)$**.** Let X be a smooth projective variety, over an algebraically closed field $\mathbf{k}$; we assume that the characteristic of $\mathbf{k}$ is 0 (we make use of the resolution of singularities), but in dimension 2 such an assumption is not necessary. Let $\mathsf{Hyp}(X)$ be the set of irreducible and reduced hypersurfaces of X.

Let $\pi_1 \colon X_1 \to X$ be a birational morphism between smooth projective varieties. The strict transform $\pi_1^\circ(W)$ of every irreducible hypersurface $W \in \mathsf{Hyp}(X)$ is a an irreducible hypersurface of X_1; this determines an injective map

$$\pi_1^\circ \colon \mathsf{Hyp}(X) \to \mathsf{Hyp}(X_1)$$

and $\mathsf{Hyp}(X_1) = \pi_1^\circ(\mathsf{Hyp}(X)) \cup \{E_1, \ldots, E_\ell\}$ where the E_i are the irreducible hypersurfaces that are contracted by π_1 (i.e. the components of the zero locus of $\mathsf{Jac}(\pi_1)$).

Let $\pi_2 \colon X_2 \to X$ be another birational morphism and denote by $\tau_{21} \colon X_1 \dashrightarrow X_2$ the birational map $\pi_2^{-1} \circ \pi_1$. Let $\mathsf{Gr}(\tau_{21}) \subset X_1 \times X_2$ be the graph of τ_{21} and $\eta \colon X_3 \to \mathsf{Gr}(\tau_{21})$ be a desingularization; the projective variety X_3 is smooth and the maps $\eta_i \colon X_3 \to X_i$, $i = 1, 2$, $\eta_i = \mathrm{proj}_i \circ \eta$ are two birational morphisms such that the following diagram commutes:



For $W \in \mathsf{Hyp}(X)$, we have $\eta_1^\circ \pi_1^\circ W = \eta_2^\circ \pi_2^\circ W$; this shows that the embeddings of $\mathsf{Hyp}(X)$ in $\mathsf{Hyp}(X_3)$ via π_1° and η_1° and via π_2° and η_2° coincide. Thus, one can define the set $\tilde{\mathsf{Hyp}}(X)$ of all irreducible hypersurfaces in all birational models of X as the injective limit

$$\tilde{\mathsf{Hyp}}(X) = \lim_{\pi \colon Y \to X} \tilde{\mathsf{Hyp}}(Y)$$

where $\pi \colon Y \to X$ runs over the set of birational morphisms from a smooth projective variety Y to X. In [23], Manin proposes a similar construction for surfaces.

4.3. **Action of** $\mathsf{Bir}(X)$ **on** $\tilde{\mathsf{Hyp}}(X)$**.** Let $g\colon X' \dashrightarrow X$ be a birational map. Let $\mathsf{Gr}(g) \subset X' \times X$ be its graph, and let $Y \to \mathsf{Gr}(g)$ be a desingularization. This provides two morphisms $\varepsilon\colon Y \to X'$ and $\pi\colon Y \to X$ such that $g = \pi \circ \varepsilon^{-1}$.

Since ε is a birational morphism, $\mathsf{Hyp}(Y)$ is naturally identified to a subset of $\tilde{\mathsf{Hyp}}(X')$ containing $\mathsf{Hyp}(X')$. Since π is a birational morphism, π° determines an embedding of $\mathsf{Hyp}(X)$ into $\mathsf{Hyp}(Y)$, hence an embedding into $\tilde{\mathsf{Hyp}}(X')$ via the inclusion $\mathsf{Hyp}(Y) \subset \tilde{\mathsf{Hyp}}(X')$. We define $g^\bullet$ on $\mathsf{Hyp}(X)$ by

$$g^\bullet(W) = \pi^\circ(W) \in \tilde{\mathsf{Hyp}}(X').$$

This provides an injection $g^\bullet\colon \mathsf{Hyp}(X) \to \tilde{\mathsf{Hyp}}(X')$.

Given another resolution of the indeterminacies of g by two birational morphisms $\varepsilon\colon Y' \to X'$ and $\pi'\colon Y' \to X$, one can find a smooth variety Z and two birational morphisms $\tau\colon Z \to Y$ and $\tau'\colon Z \to Y'$ such that

(a) Z is a desingularization of the graph of the birational transformation
$$\mu = \varepsilon'^{-1} \circ \varepsilon = \pi'^{-1} \circ \pi = \tau' \circ \tau^{-1}.$$

(b) $(\pi \circ \tau)^\circ(W) = (\pi' \circ \tau')^\circ(W)$ for every $W \in \mathsf{Hyp}(X)$ because
$$(\pi \circ \tau)^{-1} \circ (\pi' \circ \tau') = \tau^{-1} \circ \pi^{-1} \circ \pi' \circ \tau' = \tau^{-1} \circ \mu^{-1} \circ \tau' = \mathsf{id}_{X'}.$$

From property (b) we deduce that the strict transforms $\pi^\circ(W) \in \mathsf{Hyp}(Y)$ and $(\pi')^\circ(W) \in \mathsf{Hyp}(Y')$ coincide in $\tilde{\mathsf{Hyp}}(X')$; this shows that $g^\bullet(W) = \pi^\circ(W) = (\pi')^\circ(W)$ does not depend on the resolution of its indeterminacies.

Next, we want to define $g^\bullet$ as an embedding of $\tilde{\mathsf{Hyp}}(X)$ (instead of just $\mathsf{Hyp}(X)$) into $\tilde{\mathsf{Hyp}}(X')$. Consider a birational morphism $\pi_1 : X_1 \to X$ and lift g to a birational map $g_1 : X' \dashrightarrow X_1$ with $\pi \circ g_1 = g$. Then, $g_1^\bullet\colon \mathsf{Hyp}(X_1) \to \tilde{\mathsf{Hyp}}(X')$ coincides with $g^\bullet\colon \mathsf{Hyp}(X) \to \tilde{\mathsf{Hyp}}(X')$ on $\mathsf{Hyp}(X)$ via the identification $\pi_1^\circ\colon \mathsf{Hyp}(X) \hookrightarrow \mathsf{Hyp}(X_1)$ because $\pi_1 \circ g_1 = g$. Thus, one can extend $g^\bullet$ to $\mathsf{Hyp}(X_1)$ by $g_1^\bullet$. These extensions are compatible and determine an injective map

$$g^\bullet\colon \tilde{\mathsf{Hyp}}(X) \to \tilde{\mathsf{Hyp}}(X').$$

Since g is a birational transformation, one checks that $g^\bullet$ is a bijection whose inverse map is $g_\bullet := (g^{-1})^\bullet$. Similarly, if $f\colon X'' \dashrightarrow X'$ is another birational map, then $(g \circ f)^\bullet = f^\bullet \circ g^\bullet$.

Proposition 4.2. *The group* $\mathsf{Bir}(X)$ *acts faithfully by permutations on the set* $\tilde{\mathsf{Hyp}}(X)$ *via the homomorphism*

$$\begin{array}{rcl} \mathsf{Bir}(X) & \to & \mathsf{Perm}(\tilde{\mathsf{Hyp}}(\mathsf{X})) \\ g & \mapsto & g_\bullet \end{array}$$

Proof. We already proved that this action is well-defined. To show it is faithful, remark that $g_\bullet = \mathsf{id}_X$ implies $g_\circ(W) = W$ for every irreducible hypersurface $W \subset X$; this implies that $g = \mathsf{id}_X$. □

In the following statement, we denote by $\mathsf{exc}_X(g)$ the number of hypersurfaces $W \in \mathsf{Hyp}(X)$ that are contracted by the birational transformation g.

Lemma 4.3. *The subset* $\mathsf{Hyp}(X)$ *of* $\tilde{\mathsf{Hyp}}(X)$ *is commensurated by the action of* $\mathsf{Bir}(X)$ *on* $\tilde{\mathsf{Hyp}}(X)$*: For every* g *in* $\mathsf{Bir}(X)$*,*

$$|g_\bullet(\mathsf{Hyp}(X))\,\Delta\,\mathsf{Hyp}(X)| \leq \mathsf{exc}_X(g) + \mathsf{exc}_X(g^{-1}).$$

Proof. Let $W \subset X$ be an irreducible hypersurface. If there exists a point $x \in W$ around which g is a local isomorphism then $g_\bullet(W) = g_\circ(W)$ is an irreducible hypersurface in X. If not, W is contracted by g and $g_\bullet(W)$ corresponds to a hypersurface in $\tilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$ coming from a resolution $\pi_1 : X_1 \to X$ of the indeterminacies of g^{-1}. This shows that

$$|g_\bullet(\mathsf{Hyp}(X)) \smallsetminus \mathsf{Hyp}(X)| \leq \mathsf{exc}_X(g).$$

Similarly, $|\mathsf{Hyp}(X) \smallsetminus g_\bullet(\mathsf{Hyp}(X))| = |(g^{-1})_\bullet(\mathsf{Hyp}(X)) \smallsetminus \mathsf{Hyp}(X)|$ is bounded by the number $\mathsf{exc}_X(g^{-1})$. □

Example 4.4. Let g be a birational transformation of $\mathbb{P}^n_{\mathbf{k}}$ of degree d, meaning that $g^*(H) \simeq dH$ where H denotes a hyperplane of $\mathbb{P}^n_{\mathbf{k}}$, or equivalently that g is defined by $n+1$ homogeneous polynomials of the same degree d without common factor of positive degree. The exceptional set of g has degree $(n+1)(d-1)$; as a consequence, $\mathsf{exc}_{\mathbb{P}^n_{\mathbf{k}}}(g) \leq (n+1)(d-1)$. More generally, if H is a polarization of X, then $\mathsf{exc}_X(g)$ is bounded from above by a function that depends only on the degree $\deg_H(g) := (g^*H)\cdot H^{\dim(X)-1}$.

4.4. **Pseudo-regularization.** Let X be a smooth projective variety. Let Γ be a subgroup of $\mathsf{Bir}(X)$. Assume that the action of Γ on $\tilde{\mathsf{Hyp}}(X)$ fixes (globally) a subset $A \subset \tilde{\mathsf{Hyp}}(X)$ such that

$$|A\,\Delta\,\mathsf{Hyp}(X)| < +\infty.$$

In other words, A is obtained from $\mathsf{Hyp}(X)$ by removing finitely many hypersurfaces $W_i \in \mathsf{Hyp}(X)$ and adding finitely many hypersurfaces $W'_j \in \tilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$. Each W'_j comes from an irreducible hypersurface in some model $\pi_j : X_j \to X$, and there is a model $\pi : Y \to X$ that covers all of them (i.e. $\pi \circ \pi_j^{-1}$ is a morphism from Y to X_j for every j). Then, $\pi^\circ(A)$ is a subset of $\mathsf{Hyp}(Y)$. Changing X into Y and Γ into $\pi^{-1} \circ \Gamma \circ \pi$, we may assume that

(1) $A = \mathsf{Hyp}(X) \smallsetminus \{E_1, \ldots, E_\ell\}$ where the E_i are ℓ distinct irreducible hypersurfaces of X,
(2) the action of Γ on $\tilde{\mathsf{Hyp}}(X)$ fixes the set A.

In what follows, we denote by $\mathcal{U}$ the non-empty Zariski open subset $X \smallsetminus \cup_i E_i$ and by ∂X the boundary $X \smallsetminus \mathcal{U} = E_1 \cup \cdots \cup E_\ell$; ∂X is considered as the boundary of the compactification X of $\mathcal{U}$.

Lemma 4.5. *The group Γ acts by pseudo-automorphisms on the open set $\mathcal{U}$. If there is an ample divisor D whose support coincides with ∂X, then Γ acts by automorphisms on $\mathcal{U}$.*

In this statement, we say that the support of a divisor D coincides with ∂X if $D = \sum_i a_i E_i$ with $a_i > 0$ for every $1 \leq i \leq \ell$.

Proof. Let g be an element of Γ. Let W be a hypersurface of X that intersects $\mathcal{U}$. Since $g_\bullet(W)$ is an element of $A \subset \mathsf{Hyp}(X)$, g does not contract W; since A is g-invariant, $g_\bullet(W)$ is not one of the E_i and it must intersect $\mathcal{U}$. In particular, no hypersurface of $\mathcal{U}$ is mapped to the boundary ∂X and the set

$$E(g) = \{x \in U;\, x \in \mathrm{Ind}(g) \text{ or } \mathsf{Jac}(g)(x) = 0\}$$

has codimension ≥ 2. Moreover, if x is in $\mathcal{U} \smallsetminus E(g)$, then g is a local isomorphism from a neighborhood $\mathcal{V}$ of x in $\mathcal{U}$ to a neighborhood $\mathcal{V}'$ of $g(x)$ in X; if $g(x)$ is in ∂X and E_i contains $g(x)$, then g^{-1} maps E_i to a hypersurface Z_i that contains x, in contradiction with the first argument because Z_i intersects $\mathcal{U}$ and is mapped to E_i by g. We have shown that g is a local isomorphism from $\mathcal{U} \smallsetminus E(g)$ to $\mathcal{U}$. The same argument applies to g^{-1}, and Proposition 2.1 implies that g determines a pseudo-automorphism of $\mathcal{U}$.

If D is an ample divisor, some positive multiple mD is very ample, and the complete linear system $|mD|$ provides an embedding of X in a projective space. The divisor mD corresponds to a hyperplane section of X in this embedding, and the open set $\mathcal{U}$ is an affine variety if the support of D is equal to ∂X. Proposition 2.5 concludes the proof of the lemma. □

Every subgroup of $\mathsf{Bir}(X)$ acts on $\tilde{\mathsf{Hyp}}(X)$ and commensurates $\mathsf{Hyp}(X)$. If Γ transfixes $\mathsf{Hyp}(X)$, there is an invariant subset A of $\tilde{\mathsf{Hyp}}(X)$ for which $A \,\Delta\, \mathsf{Hyp}(X)$ is finite. Thus, one gets the following characterization (the converse being immediate).

Theorem 4.6. *Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Let Γ be a subgroup of $\mathsf{Bir}(X)$. Then Γ transfixes the subset*

$\mathsf{Hyp}(X)$ *of* $\tilde{\mathsf{Hyp}}(X)$ *if and only if* Γ *is pseudo-regularizable (in the sense of Definition 1.1).*

Remark 4.7. As explained in Remark 1.2, there are two extreme cases, corresponding to an empty or an ample boundary $B = \cup_i E_i$.

- If $\mathcal{U} = Y$, Γ acts by pseudo-automorphisms on the projective variety Y. As explained in Theorem 2.4, Γ is an extension of a subgroup of $\mathsf{GL}(\mathsf{NS}(Y))$ by an algebraic group (which is almost contained in $\mathsf{Aut}(Y)^0$).
- If $\mathcal{U}$ is affine, Γ acts by automorphisms on $\mathcal{U}$. The group $\mathsf{Aut}(\mathcal{U})$ may be huge (for instance if $\mathcal{U}$ is the affine space), but there are techniques to study groups of automorphisms that are not available for birational transformations. For instance Γ is residually finite and virtually torsion free if Γ is a group of automorphisms generated by finitely many elements (see [3]).

4.5. **Distorted elements.** Theorem 4.6 may be applied when Γ has Property (FW), or for pairs (Λ, Γ) with relative Property (FW) such that Λ acts on X by birational transformations. In particular, we obtain the following corollary.

Corollary 4.8. *Let X be a smooth projective variety over an algebraically closed field of characteristic* 0*. Let Γ be a distorted cyclic subgroup of* $\mathsf{Bir}(X)$*. Then Γ is pseudo-regularizable.*

5. BIRATIONAL GROUPS OF SURFACES

Theorem 5.1. *Let X be an irreducible projective surface over an algebraically closed field. The following are equivalent:*

1. $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
2. *the Kodaira dimension of X is $-\infty$;*
3. *X is birationally equivalent to the product of the projective line with a curve.*

Proof. The equivalence between (2) and (3) is classical.

Suppose that X is a product $C \times \mathsf{P}^1$. Let H_t be the hypersurface $\{t\} \times \mathsf{P}^1 \subset X$. For each $\tau \in \mathsf{P}^1$, choose a meromorphic function f_τ on C having a zero at t. Then the birational map $(t,x) \mapsto (t, f_\tau(t)x)$ contracts $H_\tau \in \mathsf{Hyp}(X)$, hence maps it to an element $H'_\tau \in \tilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$. The H'_τ, for $\tau \in \mathsf{P}^1$ are pairwise distinct. If by contradiction $\mathsf{Hyp}(X)$ were transfixed, by definition there is an invariant subset $A \subset \tilde{\mathsf{Hyp}}(X)$ with $\mathsf{Hyp}(X) \triangle A$ finite. So for all but finitely many $t \in \mathsf{P}^1$ we have $H_t \in A$ and $H'_t \notin A$. Since for every t we can find f as above, this is a contradiction.

Conversely, if X has nonnegative Kodaira dimension, then $\mathsf{Bir}(X)$ acts by automorphisms on its minimal model Y. Thus it stabilizes $\mathsf{Hyp}(Y)$, and hence transfixes $\mathsf{Hyp}(X)$. ☐

Theorem 5.2. *Let X be an irreducible projective surface over an algebraically closed field. The following are equivalent:*

(1) *some finitely generated subgroup of* $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
(2) *some cyclic subgroup of* $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
(3) *X is birationally equivalent to the product of the projective line with a curve of genus 0 or 1.*

Proof. Trivially (2) implies (1).

Suppose that (3) holds and let us prove (2). The case of $\mathsf{P}^1 \times \mathsf{P}^1$ is already covered by Lemma 4.1. The case $X = C \times \mathsf{P}^1$, where C is an elliptic curve, is similar. Namely, let s be a translation of infinite order of C. Fix $t_0 \in C$. Let f be a meromorphic function on C vanishing at t_0 and with no poles or zero at any other point of the orbit $\{s^n(t_0) : n \in \mathbf{Z}\}$. Define a birational self-transformation of X by $u(t,x) = (s(t), f(t)x)$. Let H be the hypersurface $\{t_0\} \times C$. Then for $n \in \mathbf{Z}$, we have $(u_\bullet)^n H \in \mathsf{Hyp}(X)$ if and only if $n \leq 0$. Hence the action of the cyclic group $\langle u \rangle$ does not transfix $\mathsf{Hyp}(X)$.

Suppose now that (1) holds and let us prove (3). Applying Theorem 5.1, and changing X to a birationally equivalent surface if necessary, we have $X = C \times \mathbb{P}^1$ for some (smooth irreducible) curve C. Assuming that C has genus ≥ 2, we have to show that every finitely generated group Γ of self-transformations of X transfixes $\mathsf{Hyp}(X)$. Since the genus of C is at least 1, the birational action of $\mathsf{Bir}(X)$ on X preserves the fibration $X \to C$, and thus induces a (surjective) homomorphism $\mathsf{Bir}(X) \to \mathsf{Aut}(C)$. Since moreover the genus of C is at least 2, $\mathsf{Aut}(C)$ is finite. It follows that replacing Γ by a finite index subgroup (which does not affect its transfixing property) we can assume that every $\gamma \in \Gamma$ has the form $(t,x) \mapsto (t, \phi_t(x))$, where $t \mapsto \phi_t$ is a morphism $C \to \mathsf{PGL}_2$ defined on an open Zariski-dense subset U_γ of C. Then if Γ is generated by a finite subset T, we see that Γ acts by automorphisms on the open Zariski-dense subset $\bigcap_{\gamma \in T} U_\gamma \times \mathsf{P}^1$. Hence Γ transfixes $\mathsf{Hyp}(X)$. ☐

6. Birational transformations of surfaces I

From now on, we work in dimension 2: X, Y, and Z will be smooth projective surfaces over the algebraically closed field $\mathbf{k}$.

6.1. **Regularization.** In this section, we refine Theorem 4.6, in order to apply results of Danilov and Gizatullin. Recall that a curve C in a smooth surface Y has **normal crossings** if each of its singularities is a simple node with two transverse tangents. In the complex case, this means that is locally analytically equivalent to $\{xy = 0\}$ (two branches intersecting transversally) in a neighborhood of each of its singularities.

Theorem 6.1. *Let X be a smooth projective surface, defined over an algebraically closed field $\mathbf{k}$. Let Γ be a subgroup of $\mathsf{Bir}(X)$ that transfixes the subset $\mathsf{Hyp}(X)$ of $\tilde{\mathsf{Hyp}}(X)$. There exists a smooth projective surface Z, a birational map $\varphi\colon Z \dashrightarrow X$ and an open subset $\mathcal{U} \subset Z$ such that, writing the boundary $\partial Z := Z \smallsetminus \mathcal{U}$ as a finite union of irreducible components $E_i \subset Z$, $1 \leq i \leq \ell$:*

(1) *The boundary ∂Z is a curve with normal crossings.*
(2) *The group $\Gamma_Z := \varphi^{-1} \circ \Gamma \circ \varphi$ acts by automorphisms on $\mathcal{U}$.*
(3) *$\forall 1 \leq i \leq \ell$, $\forall g \in \Gamma_Z$, the strict transform of E_i under the action of g on Z is contained in ∂Z: either $g_\circ(E_i)$ is a point of ∂Z or $g_\circ(E_i)$ is an irreducible component E_j of ∂Z.*
(4) *$\forall 1 \leq i \leq \ell$, there exists an element $g \in \Gamma_Z$ that contracts E_i onto a point $g_\circ(E_i) \in \partial Z$. In particular, E_i is rational.*
(5) *The pair $(Z, \mathcal{U})$ is minimal for the previous properties: if one contracts a smooth curve of self-intersection -1 in ∂Z, then the boundary stops to be a normal crossing divisor.*

Proof. We apply Theorem 4.6 (which works in positive characteristic because X is a surface), and get a birational morphism $\varphi_0\colon Y_0 \to X$ and an open subset $\mathcal{U}_0$ of Y_0 that satisfy properties (1) and (3), except that we only know that the action of $\Gamma_0 := \varphi_0^{-1} \circ \Gamma \circ \varphi_0$ on $\mathcal{U}_0$ is by pseudo-automorphisms (not yet by automorphisms). We shall progressively modify the triple $(Y_0, \mathcal{U}_0, \varphi_0)$ to obtain a surface Z with properties (1) to (5).

Step 1.– First, we blow-up the singularities of the curve $\partial Y_0 = Y_0 \smallsetminus \mathcal{U}_0$ to get a boundary that is a normal crossing divisor. This replaces the surface Y_0 by a new one, still denoted Y_0. This modification adds new components to the boundary ∂Y_0 but does not change the fact that Γ_0 acts by pseudo-automorphisms on $\mathcal{U}_0$.

Step 2.– Consider a point q in $\mathcal{U}_0$, and assume that there is a curve E_i of ∂Y_0 that is contracted to q by an element $g \in \Gamma_0$; fix such a g, and denote by D the union of the curves E_j such that $g_\circ(E_j) = q$. By construction, g is a pseudo-automorphism of $\mathcal{U}_0$. The curve D does not intersect the indeterminacy set of g, since otherwise there would be a curve C containing q that is contracted by g^{-1}. And D is a connected component of ∂Y_0, because otherwise g maps one of the E_j to a curve that intersects $\mathcal{U}_0$. Thus, there are small neighborhoods $\mathcal{W}$ of D and $\mathcal{W}'$ of q such that $\mathcal{W} \cap \partial Y_0 = D$ and g realizes an isomorphism from $\mathcal{W} \smallsetminus D$ to $\mathcal{W}' \smallsetminus \{q\}$, contracting D onto the smooth point $q \in Y_0$. As a consequence, there is a birational morphism $\pi_1 \colon Y_0 \to Y_1$ such that

(1) Y_1 is smooth
(2) π_1 contracts D onto a point $q_1 \in Y_1$
(3) π_1 is an isomorphism from $Y_0 \smallsetminus D$ to $Y_1 \smallsetminus \{q_1\}$.

In particular, $\pi_1(\mathcal{U}_0)$ is an open subset of Y_1 and $\mathcal{U}_1 = \pi_1(\mathcal{U}_0) \cup \{q_1\}$ is an open neighborhood of q_1 in Y_1.

Then, $\Gamma_1 := \pi_1 \circ \Gamma_0 \circ \pi_1^{-1}$ acts birationally on Y_1, and by pseudo-automorphisms on $\mathcal{U}_1$. The boundary $\partial Y_1 = Y_1 \smallsetminus \mathcal{U}_1$ contains ℓ_1 irreducible components, with $\ell_1 < \ell$ (the difference is the number of components of D), and is a normal crossing divisor because D is a connected component of ∂Y_0.

Repeating this process, we construct a sequence of surfaces $\pi_k \colon Y_{k-1} \to Y_k$ and open subsets $\pi_k(\mathcal{U}_{k-1}) \subset \mathcal{U}_k \subset Y_k$ such that the number of irreducible components of $\partial Y_k = Y_k \smallsetminus \mathcal{U}_k$ decreases. After a finite number of steps (at most ℓ), we may assume that $\Gamma_k \subset \mathsf{Bir}(Y_k)$ does not contract any boundary curve to a point of the open set $\mathcal{U}_k$. On such a model, Γ_k acts by automorphisms on $\mathcal{U}_k$.

In what follows, we fix such a model, which we denote by the letters Y, $\mathcal{U}$, ∂Y, φ . The birational map $\varphi \colon Y \dashrightarrow X$ is the composition of φ_0 with the inverse of the morphism $Y_0 \to Y_k$. On such a model, properties (1), (1) and (2) are satisfied. Moreover, (3) follows from (2). We now modify Y further to get property (4).

Step 3.– Assume that the curve E_i is not contracted by Γ. Let F be the orbit of E_i: $F = \cup_{g \in \Gamma} g_\circ(E_i)$; this curve is contained in the boundary ∂Y of the open subset $\mathcal{U}$. Changing $\mathcal{U}$ into

$$\mathcal{U}' = \mathcal{U} \cup (F \smallsetminus \overline{\partial Y \smallsetminus F}),$$

the group Γ also acts by pseudo-automorphisms on $\mathcal{U}'$. This operation decreases the number ℓ of irreducible components of the boundary. Thus, combining steps 2 and 3 finitely many times, we reach a model that satisfies Properties (1) to (4).

Step 4.– If the boundary ∂Y contains a smooth (rational) curve E_i of self-intersection -1, it can be blown down to a smooth point q by a birational morphism $\pi \colon Y \to Y'$; the open set $\mathcal{U}$ is not affected, but the boundary $\partial Y'$ has one component less. If E_i was a connected component of ∂Y, then $\mathcal{U}' = \pi(\mathcal{U}) \cup \{q\}$ is a neighborhood of q and one replaces $\mathcal{U}$ by $\mathcal{U}'$, as in step 2. Now, two cases may happen. If the boundary $\partial Y'$ ceases to be a normal crossing divisor, we come back to Y and do not apply this surgery. If $\partial Y'$ has normal crossings, we replace Y by this new model. In a finite number of steps, looking successively at all (-1)-curves and iterating the process, we reach a new surface Z on which all six properties are satisfied. □

Remark 6.2. One may also remove property (5) and replace property (1) by

(2') The E_i are rational curves, and none of them is a smooth rational curve with self-intersection -1.

But doing so, we may lose the normal crossing property. To get property (2'), apply the theorem and argue as in step 4.

6.2. **Constraints on the boundary.** We now work on the new surface Z given by Theorem 6.1. Thus, Z is the surface, Γ the subgroup of $\mathsf{Bir}(Z)$, $\mathcal{U}$ the open subset on which Γ acts by automorphisms, and ∂Z the boundary of $\mathcal{U}$.

Proposition 6.3 (Gizatullin, [13] § 4)**.** *There are four possibilities for the geometry of the boundary* $\partial Z = Z \smallsetminus \mathcal{U}$*.*

1. ∂Z *is empty.*
2. ∂Z *is a cycle of rational curves.*
3. ∂Z *is a chain of rational curves.*
4. ∂Z *is not connected; it is the disjoint union of finitely many smooth rational curves of self-intersection* 0*.*

Moreover, in cases (2) and (3), the open set $\mathcal{U}$ *is the blow-up of an affine surface.*

Thus, there are four possibilities for ∂Z, which we study successively. We shall start with (1) and (4) in sections 6.3 and 6.4. Then case (3) is dealt with in Section 6.5. Case (2) is slightly more involved: it is treated in Section 7.

Before that, let us explain how Proposition 6.3 follows from Section 5 of [13]. First, we describe the precise meaning of the statement, and then we explain how the original results of [13] apply to our situation.

The boundary and its dual graph .– Consider the dual graph $\mathcal{G}_Z$ of the boundary ∂Z. The vertices of $\mathcal{G}_Z$ are in one to one correspondence with the irreducible components E_i of ∂Z. The edges correspond to singularities of ∂Z: each singular point q gives rise to an edge connecting the components E_i that determine the two local branches of ∂Z at q. When the two branches correspond to the same irreducible component, one gets a loop of the graph $\mathcal{G}_Z$.

We say that ∂Z is a **chain** of rational curves if the dual graph is of type A_ℓ: ℓ is the number of components, and the graph is linear, with ℓ vertices. Chains are also called **zigzags** by Gizatullin and Danilov.

We say that ∂Z is a **cycle** if the dual graph is isomorphic to a regular polygon with ℓ vertices. There are two special cases: when ∂Z is reduced to one component, this curve is a rational curve with one singular point and the dual graph is a loop (one vertex, one edge); when ∂Z is made of two components, these components intersect in two distinct points, and the dual graph is made of two vertices with two edges between them. For $\ell = 3, 4, \ldots$, the graph is a triangle, a square, ...

Gizatullin's original statement.– To describe Gizatullin's article, let us introduce some useful vocabulary. Let S be a projective surface, and $C \subset S$ be a curve; C is a union of irreducible components, which may have singularities. Assume that S is smooth in a neighborhood of C. Let S_0 be the complement of C in S, and let $\iota\colon S_0 \to S$ be the natural embedding of S_0 in S. Then, S is a **completion** of S_0: this completion is marked by the embedding $\iota\colon S_0 \to S$, and its boundary is the curve C. Following [13] and [14, 15], we only consider completions of S_0 by curves (i.e. $S \setminus \iota(S_0)$ is of pure dimension 1), and we always assume S to be smooth in a neighborhood of the boundary. Such a completion is

(i) **simple** if the boundary C has normal crossings;
(ii) **minimal** if it is simple and minimal for this property: if $C_i \subset C$ is an exceptional divisor of the first kind then, contracting C_i, the image of C is not a simple normal crossing divisor anymore. Equivalently, C_i intersects at least three other components of C. Equivalently, if $\iota'\colon S_0 \to S'$ is another simple completion, and $\pi\colon S \to S'$ is a birational morphism such that $\pi \circ \iota = \iota'$, then π is an isomorphism.

If S is a completion of S_0, one can blow-up boundary points to obtain a simple completion, and then blow-down some of the boundary components C_i to reach a minimal completion.

Now, consider the group of automorphisms of the open surface S_0. This group $\mathsf{Aut}(S_0)$ acts by birational transformations on S. An irreducible component E_i of the boundary C is **contracted** if there is an element g of $\mathsf{Aut}(S_0)$ that contracts E_i: $g_\circ(E_i)$ is a point of C. Let E be the union of the contracted components. In [13], Gizatullin proves that E satisfies one of the four properties stated in Proposition 6.3; moreover, in cases (2) and (3), E contains an irreducible component E_i with $E_i^2 > 0$ (see Corollary 4, Section 5 of [13]).

Thus, Proposition 6.3 follows from the properties of the pair $(Z, \mathcal{U}, \Gamma)$: the open set $\mathcal{U}$ plays the role of S_0, and Z is the completion S; the boundary ∂Z is the curve C: it is a normal crossing divisor, and it is minimal by construction. Since every component of ∂Z is contracted by at least one element of $\Gamma \subset \mathsf{Aut}(\mathcal{U})$, ∂Z coincides with Gizatullin's curve E. The only thing we have to prove is the last sentence of the Proposition concerning the structure of the open set $\mathcal{U}$.

First, let us show that $E = \partial Z$ supports an effective divisor D such that $D^2 > 0$ and $D \cdot F \geq 0$ for every irreducible curve. To do so, fix an irreducible component E_0 of ∂Z with positive self-intersection. Assume that ∂Z is a cycle, and list cyclically the other irreducible components: E_1, E_2, ..., up to E_m, with E_1 and E_m intersecting E_0. Then, one defines $a_1 = 1$, then one chooses $a_2 > 0$ such that $a_1E_1 + a_2E_2$ intersects positively E_1, then $a_3 > 0$ such that $a_1E_1 + a_2E_2 + a_3E_3$ intersects positively E_1 and E_2, ..., up to $\sum_{i=1}^{m} a_iE_i$ that intersects all components E_i, $1 \leq i \leq m-1$ positively. Since $E_0^2 > 0$ and E_0 intersects E_m, one can find a coefficient a_0 for which the divisor

$$D = \sum_{i=0}^{m} a_iE_i$$

satisfies $D^2 > 0$ and $D \cdot E_i > 0$ for all E_i, $0 \leq i \leq m$. This implies that D intersects every irreducible curve F non-negatively. Thus, D is big and nef (see [19], Section 2.2). A similar proof applies when ∂Z is a zigzag.

Let W be the subspace of $\mathrm{NS}(X)$ spanned by classes of curves F with $D \cdot F = 0$. Since $D^2 > 0$, Hodge index theorem implies that the intersection form is negative definite on W. Thus, Mumford-Grauert contraction theorem provides a birational morphism $\tau\colon Z \to Z'$ that contracts simultaneously all curves F with $[F] \in W$ and is an isomorphism on $Z \setminus F$; in particular, τ is an isomorphism from a neighborhood $\mathcal{V}$ of ∂Z onto its image $\tau(\mathcal{V}) \subset Z'$. In other words, the modification τ may contract curves that are contained in $\mathcal{U}$, and may create singularities for the new open set $\mathcal{U}' = \tau(\mathcal{U})$, but does not modify Z near the boundary ∂Z. Now, on Z', the divisor $D' = \tau_*(D)$ intersects every effecitve curve positively and satisfies $(D')^2 > 0$. Nakai-Moishezon criterion shows that D' is ample (see [19], Section 1.2.B); consequently, there is an embedding of Z' into a projective space and a hyperplane section H of Z' for which $Z' \setminus H$ coincides with $\mathcal{U}'$. This proves that $\mathcal{U}$ is a blow-up of the affine (singular) surface $\mathcal{U}'$.

6.3. **Projective surfaces and automorphisms.** In this section, we (almost always) assume that Γ acts by regular automorphisms on a projective surface X. This corresponds to case (1) in Proposition 6.3. Our goal is the special case of Theorem B which is stated below as Theorem 6.8.

6.3.1. *Action on the Néron-Severi group.* The intersection form is a non-degenerate quadratic form q_X on the Néron-Severi group $\mathrm{NS}(X)$, and Hodge index theorem asserts that its signature is $(1, \rho(X)-1)$, where $\rho(X)$ denotes the Picard number, i.e. the rank of the lattice $\mathrm{NS}(X) \simeq \mathbf{Z}^\rho$.

The action of $\mathsf{Aut}(X)$ on the Néron-Severi group $\mathrm{NS}(X)$ provides a linear representation preserving the intersection form q_X. This gives a morphism

$$\mathsf{Aut}(X) \to \mathsf{O}(\mathrm{NS}(X); q_X).$$

Fix an ample class a in $\mathrm{NS}(X)$ and consider the hyperboloid

$$\mathbb{H}_X = \{u \in \mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{R};\ q_X(u,u) = 1 \text{ and } q_X(u,a) > 0\}.$$

This set is one of the two connected components of $\{u; q_X(u,u) = 1\}$. With the riemannian metric induced by $(-q_X)$, it is a copy of the hyperbolic space of dimension $\rho(X)-1$; the group $\mathsf{Aut}(X)$ acts by isometries on this space (see [7]).

Proposition 6.4. *Let X be a smooth projective surface. Let Γ be a subgroup of* $\mathsf{Aut}(X)$. *If Γ has Property (FW), its action on* $\mathrm{NS}(X)$ *fixes a very ample class, the image of Γ in* $\mathsf{O}(\mathrm{NS}(X); q_X)$ *is finite, and a finite index subgroup of Γ is contained in* $\mathsf{Aut}(X)^0$.

Proof. The image Γ^* of Γ is contained in the arithmetic group $\mathsf{O}(\mathrm{NS}(X); q_X)$. The Néron-Severi group $\mathrm{NS}(X)$ is a lattice $\mathbf{Z}^\rho$ and q_X is defined over $\mathbf{Z}$. Thus, $\mathsf{O}(\mathrm{NS}(X); q_X)$ is a standard arithmetic group in the sense of [5], § 1.1. The main results of [5] imply that the action of Γ^* on the hyperbolic space $\mathbb{H}_X$ has a fixed point. Let u be such a fixed point. Since q_X is negative definite on the orthogonal complement $u^\perp$ of u in $\mathrm{NS}(X)$, and Γ^* is a discrete group acting by isometries on it, we deduce that Γ^* is finite. If a is a very ample class, the sum $\sum_{\gamma \in \Gamma^*} \gamma^*(a)$ is an invariant, very ample class.

The kernel $K \subset \mathsf{Aut}(X)$ of the action on $\mathrm{NS}(X)$ contains $\mathsf{Aut}(X)^0$ as a finite index subgroup. Thus, if Γ has Property (FW), it contains a finite index subgroup that is contained in $\mathsf{Aut}(X)^0$. □

6.3.2. *Non-rational surfaces.* In this paragraph, we assume that the surface X is not rational. The following proposition classifies subgroups of $\mathsf{Bir}(X)$ with Property (FW); in particular, such a group is finite if the Kodaira dimension of X is non-negative (resp. if the characteristic of $\mathbf{k}$ is positive).

Proposition 6.5. *Let X be a smooth, projective, and non-rational surface, over the algebraically closed field $\mathbf{k}$. Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$ with Property (FW). Then $\mathbf{k}$ has characteristic 0, and there is a birational map $\varphi\colon X \dashrightarrow C\times\mathbb{P}^1_{\mathbf{k}}$ that conjugates Γ to a subgroup of $\mathsf{Aut}(C\times\mathbb{P}^1_{\mathbf{k}})$. Moreover, there is a finite index subgroup Γ_0 of Γ such that $\varphi\circ\Gamma_0\circ\varphi^{-1}$, is a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$, acting on $C\times\mathbb{P}^1_{\mathbf{k}}$ by linear projective transformations on the second factor.*

Proof. Assume, first, that the Kodaira dimension of X is non-negative. Let $\pi\colon X\to X_0$ be the projection of X on its (unique) minimal model (see [16], Thm. V.5.8). The group $\mathsf{Bir}(X_0)$ coincides with $\mathsf{Aut}(X_0)$; thus, after conjugacy by π, Γ becomes a subgroup of $\mathsf{Aut}(X_0)$, and Proposition 6.4 provides a finite index subgroup $\Gamma_0\leq\Gamma$ that is contained in $\mathsf{Aut}(X_0)^0$. Note that Γ_0 inherits Property (FW) from Γ.

If the Kodaira dimension of X is equal to 2, the group $\mathsf{Aut}(X_0)^0$ is trivial; hence $\Gamma_0=\{\mathrm{Id}_{X_0}\}$ and Γ is finite. If the Kodaira dimension is equal to 1, $\mathsf{Aut}(X_0)^0$ is either trivial, or isomorphic to an elliptic curve, acting by translations on the fibers of the Kodaira-Iitaka fibration of X_0 (this occurs, for instance, when X_0 is the product of an elliptic curve with a curve of higher genus). If the Kodaira dimension is 0, then $\mathsf{Aut}(X_0)^0$ is also an abelian group (either trivial, or isomorphic to an abelian surface). Since abelian groups with Property (FW) are finite, the group Γ_0 is finite, and so is Γ.

We may now assume that the Kodaira dimension $\mathsf{kod}(X)$ is negative. Since X is not rational, then X is birationally equivalent to a product $S=C\times\mathbb{P}^1_{\mathbf{k}}$, where C is a curve of genus $g(C)\geq 1$. Denote by $\mathbf{k}(C)$ the field of rational functions on the curve C. We fix a local coordinate x on C and denote the elements of $\mathbf{k}(C)$ as functions $a(x)$ of x. The semi-direct product $\mathsf{Aut}(C)\ltimes\mathsf{PGL}_2(\mathbf{k}(C))$ acts on S by birational transformations of the form

$$(x,y)\in C\times\mathbb{P}^1_{\mathbf{k}}\mapsto\left(f(x),\frac{a(x)y+b(x)}{c(x)y+d(x)}\right),$$

and $\mathsf{Bir}(S)$ coincides with this group $\mathsf{Aut}(C)\ltimes\mathsf{PGL}_2(\mathbf{k}(C))$; indeed, the first projection $\pi\colon S\to C$ is equivariant under the action of $\mathsf{Bir}(S)$ because every rational map $\mathbb{P}^1_{\mathbf{k}}\to C$ is constant.

Since $\mathsf{Aut}(C)$ is virtually abelian and Γ has Property (FW), there is a finite index, normal subgroup $\Gamma_0\leq\Gamma$ that is contained in $\mathsf{PGL}_2(\mathbf{k}(C))$. By Corollary 3.8, every subgroup of $\mathsf{PGL}_2(\mathbf{k}(C))$ with Property (FW) is conjugate to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$ or a finite group if the characteristic of the field $\mathbf{k}$ is positive.

We may assume now that the characteristic of $\mathbf{k}$ is 0 and that $\Gamma_0 \subset \mathsf{PGL}_2(\overline{\mathbf{Z}})$ is infinite. Consider an element g of Γ; it acts as a birational transformation on the surface $S = C \times \mathbb{P}^1_{\mathbf{k}}$, and it normalizes Γ_0:

$$g \circ \Gamma_0 = \Gamma_0 \circ g.$$

Since Γ_0 acts by automorphisms on S, the finite set $\mathrm{Ind}(g)$ is Γ_0-invariant. But a subgroup of $\mathsf{PGL}_2(\mathbf{k})$ with Property (FW) preserving a non-empty, finite subset of $\mathbb{P}^1(\mathbf{k})$ is a finite group. Thus, $\mathrm{Ind}(g)$ must be empty. This shows that Γ is contained in $\mathsf{Aut}(S)$. □

6.3.3. *Rational surfaces.* We now assume that X is rational, that $\Gamma \leq \mathsf{Bir}(X)$ is an infinite subgroup with Property (FW), and that Γ contains a finite index, normal subgroup Γ_0 that is contained in $\mathsf{Aut}(X)^0$. Every exceptional curve of the first kind $E \subset X$ is determined by its class in $\mathrm{NS}(X)$ and is therefore invariant under the action of $\mathsf{Aut}(X)^0$. Contracting (-1)-curves one by one, we obtain the following lemma.

Lemma 6.6. *There is a birational morphism $\pi\colon X \to Y$ onto a minimal rational surface Y that is equivariant under the action of Γ_0; Y does not contain any exceptional curve of the first kind and Γ_0 becomes a subgroup of* $\mathsf{Aut}(Y)^0$.

Let us recall the classification of minimal rational surfaces and describe their groups of automorphisms. First, we have the projective plane $\mathbb{P}^2_{\mathbf{k}}$, with $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{PGL}_3(\mathbf{k})$ acting by linear projective transformations. Then comes the quadric $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, with

$$\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})^0 = \mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$$

acting by linear projective transformations on each factor; the group of automorphisms is the semi-direct product of $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$ with the group of order 2 generated by the permutation of the two factors, $\eta(x,y) = (y,x)$. Then, for each integer $m \geq 1$, we have the Hirzebruch surface $\mathbb{F}_m$; this is the projectivization of the rank 2 bundle $O \oplus O(m)$ over $\mathbb{P}^1_{\mathbf{k}}$; it may be characterized as the unique ruled surface $Z \to \mathbb{P}^1_{\mathbf{k}}$ with a section C of self-intersection $-m$. Its group of automorphisms is connected and preserves the ruling. This provides a homomorphism $\mathsf{Aut}(\mathbb{F}_m) \to \mathsf{PGL}_2(\mathbf{k})$ that describes the action on the base of the ruling, and it turns out that this homomorphism is surjective. If we choose coordinates for which the section C intersects each fiber at infinity, the kernel J_m of this homomorphism acts by transformations of type

$$(x,y) \mapsto (x, \alpha y + \beta(x))$$

where $\beta(x)$ is a polynomial function of degree $\leq m$. In particular, J_m is solvable. In other words, $\mathsf{Aut}(\mathbb{F}_m)$ is isomorphic to the group

$$(\mathsf{GL}_2(\mathbf{k})/\mu_m) \ltimes W_m$$

where W_m is the linear representation of $\mathsf{GL}_2(\mathbf{k})$ on homogeneous polynomials of degree m in two variables, and μ_m is the kernel of this representation: it is the subgroup of $\mathsf{GL}_2(\mathbf{k})$ given by scalar multiplications by roots of unity of order dividing m.

Lemma 6.7. *The group Γ is also contained, after conjugacy by $\pi\colon X \to Y$, in the group of automorphisms of the minimal, rational surface Y.*

Proof. Assume that the surface Y is the quadric $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. Then, according to Theorem 3.5, Γ_0 is conjugate to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$. If g is an element of Γ, its indeterminacy locus is a finite subset $\mathrm{Ind}(g)$ of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that is invariant under the action of Γ_0, because g normalizes Γ_0. Since Γ_0 is infinite and has Property (FW), this set $\mathrm{Ind}(g)$ is empty (Lemma 3.6). Thus, Γ is contained in $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$.

The same argument applies for Hirzebruch surfaces. Indeed, Γ_0 is an infinite subgroup of $\mathsf{Aut}(\mathbb{F}_m)$ with Property (FW). Thus, up to conjugacy, its projection in $\mathsf{PGL}_2(\mathbf{k})$ is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}})$. If it is finite, a finite index subgroup of Γ_0 is contained in the solvable group J_m, and must therefore be finite too by Property (FW); this contradicts $|\Gamma_0| = \infty$. Thus, the projection of Γ_0 in $\mathsf{PGL}(\overline{\mathbf{Z}})$ is infinite. If g is an element of Γ, $\mathrm{Ind}(g)$ is a finite, Γ_0-invariant subset, and by looking at the projection of this set in $\mathbb{P}^1_{\mathbf{k}}$ one deduces that it is empty (Lemma 3.6). This proves that Γ is contained in $\mathsf{Aut}(\mathbb{F}_m)$.

Let us now assume that Y is the projective plane. Fix an element g of Γ, and assume that g is not an automorphism; the indeterminacy and exceptional sets of g are Γ_0 invariant. Consider an irreducible curve C in the exceptional set of g, together with an indeterminacy point q of g on C. Changing Γ_0 in a finite index subgroup, we may assume that Γ_0 fixes C and q; in particular, Γ_0 fixes q, and permutes the tangent lines of C through q. But the algebraic subgroup of $\mathsf{PGL}_3(\mathbf{k})$ preserving a point q and a line through q does not contain any infinite group with Property (FW) (Lemma 3.6). Thus, again, Γ is contained in $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$. □

6.3.4. *Conclusion, in case (1).* Putting everything together, we obtain the following particular case of Theorem B.

Theorem 6.8. *Let X be a smooth projective surface over an algebraically closed field* $\mathbf{k}$. *Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$ with Property (FW). If a finite index subgroup of Γ is contained in $\mathsf{Aut}(X)$, there is a birational morphism $\varphi\colon X \to Y$ that conjugates Γ to a subgroup Γ_Y of $\mathsf{Aut}(Y)$, with Y in the following list:*

1. *Y is the product of a curve C by $\mathbb{P}^1_{\mathbf{k}}$, the field $\mathbf{k}$ has characteristic 0, and a finite index subgroup Γ'_Y of Γ_Y is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}})$, acting by linear projective transformations on the second factor;*
2. *Y is $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, the field $\mathbf{k}$ has characteristic 0, and Γ_Y is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$;*
3. *Y is a Hirzebruch surface $\mathbb{F}_m$ and $\mathbf{k}$ has characteristic 0;*
4. *Y is the projective plane $\mathbb{P}^2_{\mathbf{k}}$.*

In particular, $Y = \mathbb{P}^2_{\mathbf{k}}$ if the characteristic of $\mathbf{k}$ is positive.

Remark 6.9. Denote by $\varphi\colon X \to Y$ the birational map given by the theorem. Changing Γ in a finite index subgroup, we may assume that it acts by automorphisms on both X and Y.

If $Y = C \times \mathbb{P}^1$, then φ is in fact an isomorphism. To prove this fact, denote by ψ the inverse of φ. The indeterminacy set $\mathrm{Ind}(\psi)$ is Γ_Y invariant because both Γ and Γ_Y act by automorphisms. From Lemma 3.6, applied to $\Gamma'_Y \subset \mathsf{PGL}_2(\mathbf{k})$, we deduce that $\mathrm{Ind}(\psi)$ is empty and ψ is an isomorphism. The same argument implies that the conjugacy is an isomorphism if $Y = \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ or a Hirzebruch surface $\mathbb{F}_m$, $m \geq 1$.

Now, if Y is $\mathbb{P}^2_{\mathbf{k}}$, φ is not always an isomorphism. For instance, $\mathsf{SL}_2(\mathbf{C})$ acts on $\mathbb{P}^2_{\mathbf{k}}$ with a fixed point, and one may blow up this point to get a new surface with an action of groups with Property (FW). But this is the only possible example, *i.e.* X is either $\mathbb{P}^2_{\mathbf{k}}$, or a single blow-up of $\mathbb{P}^2_{\mathbf{k}}$ (because $\Gamma \subset \mathsf{PGL}_3(\mathbf{C})$ can not preserve more than one base point for φ^{-1} without loosing Property (FW)).

6.4. **Invariant fibrations.** We now assume that Γ acts by automorphisms on $\mathcal{U} \subset X$, and that the boundary $\partial X = X \smallsetminus \mathcal{U}$ is the union of $\ell \geq 2$ pairwise disjoint rational curves E_i; each of them has self-intersection $E_i^2 = 0$ and is contracted by at least one element of Γ. This corresponds to the fourth possibility in Gizatullin's result. Since $E_i \cdot E_j = 0$, the Hodge index theorem implies that the classes $e_i = [E_i]$ span a line in $\mathrm{NS}(X)$.

From Section 6.3.2, we may, and do assume that X is a rational surface. In particular, the Euler characteristic of the structural sheaf is equal to 1: $\chi(O_X) = 1$, and Riemann-Roch formula gives

$$h^0(X, E_1) - h^1(X, E_1) + h^2(X, E_1) = \frac{E_1^2 - K_X \cdot E_1}{2} + 1.$$

The genus formula implies $K_X \cdot E_1 = -2$, and Serre duality shows that $h^2(X,E_1) = h^0(X,K_X - E_1) = 0$ because otherwise $-2 = (K_X - E_1)\cdot E_1$ would be non-negative. From this, we obtain

$$h^0(X,E_1) = h^1(X,E_1) + 2 \geq 2.$$

Since $E_1^2 = 0$, we conclude that the space $H^0(X,E_1)$ has dimension 2 and determines a fibration $\pi\colon X \to \mathbb{P}^1_{\mathbf{k}}$; the curve E_1, as well as the E_i for $i \geq 2$, are fibers of π.

If f is an automorphism of $\mathcal{U}$ and $F \subset \mathcal{U}$ is a fiber of π, then $f(F)$ is a (complete) rational curve. Its projection $\pi(f(F))$ is contained in $\mathbb{P}^1_{\mathbf{k}} \smallsetminus \cup_i \pi(E_i)$ and must therefore be reduced to a point. Thus, $f(F)$ is a fiber of π and f preserves the fibration. This proves the following lemma.

Lemma 6.10. *There is a fibration* $\pi\colon X \to \mathbb{P}^1_{\mathbf{k}}$ *such that*

(1) *every component E_i of ∂X is a fiber of π: $\mathcal{U} = \pi^{-1}(\mathcal{V})$ for an open subset $\mathcal{V} \subset \mathbb{P}^1_{\mathbf{k}}$;*
(2) *the generic component of π is a smooth rational curve;*
(3) *Γ permutes the fibers of π: there is a morphism $\rho\colon \Gamma \to \mathsf{PGL}_2(\mathbf{k})$ such that $\pi \circ f = \rho(f) \circ \pi$ for every $f \in \Gamma$.*

The open subset $\mathcal{V} \subsetneq \mathbb{P}^1_{\mathbf{k}}$ is invariant under the action of $\rho(\Gamma)$; hence $\rho(\Gamma)$ $\rho(\Gamma)$ is finite by Lemma 3.6. Let Γ_0 be the kernel of this morphism. Then, denote by $\varphi\colon X \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ a birational map that conjugates the fibration π to the first projection $\tau\colon \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$. The group Γ_0 is conjugate to a subgroup of $\mathsf{PGL}_2(\mathbf{k}(x))$ acting on $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ by linear projective transformations of the fibers of τ. From Corollary 3.8, a new conjugacy by an element of $\mathsf{PGL}_2(\mathbf{k}(x))$ changes Γ_0 in an infinite subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$. Then, as in Sections 6.3.2 and 6.3.3 we conclude that Γ becomes a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$, with a finite projection on the first factor.

Proposition 6.11. *Let Γ be an infinite group with Property (FW), with $\Gamma \subset \mathsf{Aut}(\mathcal{U})$, and $\mathcal{U} \subset Z$ as in case (4) of Proposition 6.3. There exists a birational map $\psi\colon Z \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that conjugates Γ to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$, with a finite projection on the first factor.*

6.5. **Completions by zigzags.** Two cases remain to be studied: ∂Z can be a chain of rational curves (a zigzag in Gizatullin's terminology) or a cycle of rational curves (a loop in Gizatullin's terminology). Cycles are considered in Section 7. In this section, we rely on difficult results of Danilov and Gizatullin to treat the case of chains of rational curves (case (3) in Proposition 6.3). Thus, in this section

(i) ∂X is a chain of smooth rational curves E_i
(ii) $\mathcal{U} = X \smallsetminus \partial X$ is an affine surface (singularities are allowed)
(iii) every irreducible component E_i is contracted to a point of ∂X by at least one element of $\Gamma \subset \mathsf{Aut}(\mathcal{U}) \subset \mathsf{Bir}(X)$.

In [14, 15], Danilov and Gizatullin introduce a set of "standard completions" of the affine surface $\mathcal{U}$. As in Section 6.2, a completion (or more precisely a "marked completion") is an embedding $\iota\colon \mathcal{U} \to Y$ into a complete surface such that $\partial Y = Y \smallsetminus \iota(\mathcal{U})$ is a curve (this boundary curve may be reducible). Danilov and Gizatullin only consider completions for which ∂Y is a chain of smooth rational curves and Y is smooth in a neighborhood of ∂Y; the surface X provides such a completion. Two completions $\iota\colon \mathcal{U} \to Y$ and $\iota'\colon \mathcal{U} \to Y'$ are isomorphic if the birational map $\iota' \circ \iota^{-1}\colon Y \to Y'$ is an isomorphism; in particular, the boundary curves are identified by this isomorphism. The group $\mathsf{Aut}(\mathcal{U})$ acts by pre-composition on the set of isomorphism classes of (marked) completions.

Among all possible completions, Danilov and Gizatullin distinguish a class of "standard (marked) completions". We refer to [14] for a definition. There are elementary links (corresponding to certain birational mappings $Y \dashrightarrow Y'$) between standard completions, and one can construct a graph $\Delta_{\mathcal{U}}$ whose vertices are standard completions and there is an edge between two completions if one can pass from one to the other by an elementary link.

Example 6.12. A completion is m-standard, for some $m \in \mathbf{Z}$, if the boundary curve ∂Y is a chain of $n+1$ consecutive rational curves $E_0, E_1, \ldots, E_n$ ($n \geq 1$) such that

$$E_0^2 = 0, \;\; E_1^2 = -m, \;\; \text{and} \;\; E_i^2 = -2 \;\; \text{if} \;\; i \geq 2.$$

Blowing-up the intersection point $q = E_0 \cap E_1$, one creates a new chain starting by E_0' with $(E_0')^2 = -1$; blowing down E_0', one creates a new $(m+1)$-standard completion. This is one of the elementary links.

Standard completions are defined by constraints on the self-intersections of the components E_i. Thus, the action of $\mathsf{Aut}(\mathcal{U})$ on completions permutes the standard completions; this action determines a morphism from $\mathsf{Aut}(\mathcal{U})$ to the group of isometries (or automorphisms) of the graph $\Delta_{\mathcal{U}}$ (see [14]):

$$\mathsf{Aut}(\mathcal{U}) \to \mathsf{Iso}\,(\Delta_{\mathcal{U}}).$$

Theorem 6.13 (Danilov and Gizatullin, [14, 15])**.** *The graph $\Delta_{\mathcal{U}}$ of all isomorphism classes of standard completions of $\mathcal{U}$ is a tree. The group $\mathsf{Aut}(\mathcal{U})$ acts by isometries* *of this tree. The stabilizer of a vertex* $\iota\colon \mathcal{U} \to Y$ *is the subgroup* $G(\iota)$ *of automorphisms of the complete surface* Y *that fix the curve* ∂Y*. This group is an algebraic subgroup of* $\mathsf{Aut}(Y)$.

The last property means that $G(\iota)$ is an algebraic group that acts algebraically on Y. It coincides with the subgroup of $\mathsf{Aut}(Y)$ fixing the boundary ∂Y; the fact that it is algebraic follows from the existence of a $G(\iota)$-invariant, big and nef divisor which is supported on ∂Y (see the last sentence of Proposition 6.3).

The crucial assertion in this theorem is that $\Delta_{\mathcal{U}}$ is a simplicial tree (typically, infinitely many edges emanate from each vertex). There are sufficiently many links to assure connectedness, but not too many in order to prevent the existence of cycles in the graph $\Delta_{\mathcal{U}}$.

Corollary 6.14. *If* Γ *is a subgroup of* $\mathsf{Aut}(\mathcal{U})$ *that has the fixed point property on trees, then* Γ *is contained in* $G(\iota) \subset \mathsf{Aut}(Y)$ *for some completion* $\iota\colon \mathcal{U} \to Y$.

If Γ has Property (FW), it has Property (FA) (see Section 3.4). Thus, if it acts by automorphisms on $\mathcal{U}$, Γ is conjugate to the subgroup $G(\iota)$ of $\mathsf{Aut}(Y)$, for some zigzag-completion Y of $\mathcal{U}$. Theorem 6.8 of Section 6.3.3 implies that the action of Γ on the initial surface X is conjugate to a regular action on $\mathbb{P}^2_{\mathbf{k}}$, $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ or $\mathbb{F}_m$, for some Hirzebruch surface $\mathbb{F}_m$. This action preserves a curve, namely the image of the zigzag into the surface Y.

Example 6.15. Consider the projective plane $\mathbb{P}^2_{\mathbf{k}}$, together with a subgroup $\Gamma \subset \mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ that preserves a curve C and has Property (FW). Then, C must be a smooth rational curve: either a line, or a smooth conic. If C is the line "at infinity", then Γ acts by affine transformations on the affine plane $\mathbb{P}^2_{\mathbf{k}} \setminus C$. If the curve is the conic $x^2 + y^2 + z^2 = 0$, Γ becomes a subgroup of $\mathsf{PO}_3(\mathbf{k})$.

Example 6.16. When Γ is a subgroup of $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$ that preserves a curve C and has Property (FW), then C must be a smooth curve because Γ has no finite orbit. Similarly, the two projections $C \to \mathbb{P}^1_{\mathbf{k}}$ being equivariant with respect to the morphisms $\Gamma \to \mathsf{PGL}_2(\mathbf{k})$, they have no ramification points. Thus, C is a smooth rational curve, and its projections onto each factor are isomorphisms. Thus, the action of Γ on C and on each factor are conjugate. From these conjugacies, one deduces that the action of Γ on $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is conjugate to a diagonal embedding

$$\gamma \in \Gamma \ \mapsto \ (\rho(\gamma), \rho(\gamma)) \in \mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$$

preserving the diagonal.

Example 6.17. Similarly, the group $\mathsf{SL}_2(\mathbf{C})$ acts on the Hirzebruch surface $\mathbb{F}_m$, preserving the zero section of the fibration $\pi\colon \mathbb{F}_m \to \mathbb{P}^1_{\mathbf{k}}$. This gives examples of groups with Property (FW) acting on $\mathbb{F}_m$ and preserving a big and nef curve C.

Starting with one of the above examples, one can blow-up points on the invariant curve C, and then contract C, to get examples of zigzag completions Y on which Γ acts and contracts the boundary ∂Y.

7. Birational transformations of surfaces II

In this section, $\mathcal{U}$ is a (singular) affine surface with a completion X by a cycle of ℓ rational curves. Every irreducible component E_i of the boundary $\partial X = X \setminus \mathcal{U}$ is contracted by at least one automorphism of $\mathcal{U}$. Our goal is to classify subgroups Γ of $\mathsf{Aut}(\mathcal{U}) \subset \mathsf{Bir}(X)$ that are infinite and have Property (FW): in fact, we shall show that no such group exists.

Example 7.1. Let $(\mathbb{A}^1_{\mathbf{k}})^*$ denote the complement of the origin in the affine line $\mathbb{A}^1_{\mathbf{k}}$; it is isomorphic to the multiplicative group $\mathbb{G}_m$ over $\mathbf{k}$. The surface $(\mathbb{A}^1_{\mathbf{k}})^* \times (\mathbb{A}^1_{\mathbf{k}})^*$ is an open subset in $\mathbb{P}^2_{\mathbf{k}}$ whose boundary is the triangle of coordinate lines $\{[x : y : z];\, xyz = 0\}$. Thus, the boundary is a cycle (of length $\ell = 3$). The group of automorphisms of $(\mathbb{A}^1_{\mathbf{k}})^* \times (\mathbb{A}^1_{\mathbf{k}})^*$ is the semi-direct product

$$\mathsf{GL}_2(\mathbf{Z}) \ltimes (\mathbb{G}_m(\mathbf{k}) \times \mathbb{G}_m(\mathbf{k}));$$

it does not contain any infinite group with Property (FW).

7.1. **Resolution of indeterminacies.** Let us order cyclically the irreducible components E_i of ∂X, so that $E_i \cap E_j \neq \emptyset$ if and only if $i - j = \pm 1 (\mathrm{mod}\, \ell)$. Blowing up finitely many singularities of ∂X, we may assume that $\ell = 2^m$ for some integer $m \geq 1$; in particular, every curve E_i is smooth. (with such a modification, one may a priori create irreducible components of ∂X that are not contracted by the group Γ)

Lemma 7.2. *Let f be an automorphism of $\mathcal{U}$ and let f_X be the birational extension of f to the surface X. Then*

1. *Every indeterminacy point of f_X is a singular point of ∂X, i.e. one of the intersection points $E_i \cap E_{i+1}$.*
2. *Indeterminacies of f_X are resolved by inserting chains of rational curves.*

Property (2) means that there exists a resolution of the indeterminacies of f_X, given by two birational morphisms $\varepsilon\colon Y \to X$ and $\pi\colon Y \to X$ with $f \circ \varepsilon = \pi$, such that $\pi^{-1}(\partial X) = \varepsilon^{-1}(X)$ is a cycle of rational curves. Some of the singularities of ∂X have been blown-up into chains of rational curves to construct Y.

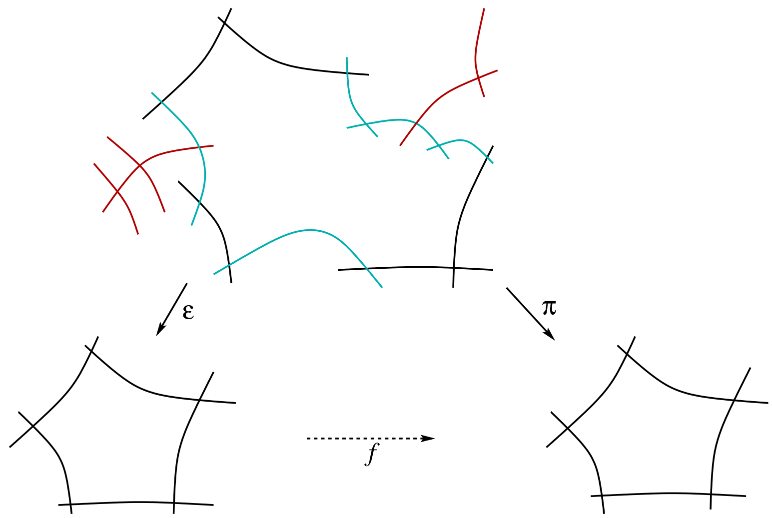


FIGURE 1. A blow-up sequence creating two (red) branches. No branch of this type appears for minimal resolution.

Proof. Consider a minimal resolution of the indeterminacies of f_X. It is given by a finite sequence of blow-ups of the base points of f_X, producing a surface Y and two birational morphisms $\varepsilon\colon Y \to X$ and $\pi\colon Y \to X$ such that $f_X = \pi \circ \varepsilon^{-1}$. Since the indeterminacy points of f_X are contained in ∂X, all necessary blow-ups are centered on ∂X.

The total transform $F = \varepsilon^*(\partial X)$ is a union of rational curves: it is made of a cycle, together with branches emanating from it. One of the assertions (1) and (2) fails if and only if F is not a cycle; in that case, there is at least one branch.

Each branch is a tree of smooth rational curves, which may be blown-down onto a smooth point; indeed, these branches come from smooth points of the main cycle that have been blown-up finitely many times. Thus, there is a birational morphism $\eta\colon Y \to Y_0$ onto a smooth surface Y_0 that contracts the branches (and nothing more).

The morphism π maps F onto the cycle ∂X, so that all branches of F are contracted by π. Thus, both ε and π induce (regular) birational morphisms $\varepsilon_0\colon Y_0 \to X$ and $\pi_0\colon Y_0 \to X$. This contradicts the minimality of the resolution. □

Let us introduce a family of surfaces

$$\pi_k\colon X_k \to X.$$

First, $X_1 = X$ and π_1 is just the identity map. Then, X_2 is obtained by blowing-up the ℓ singularities of ∂X_1; X_2 is a compactification of $\mathcal{U}$ by a cycle ∂X_2 of $2\ell = 2^{m+1}$

smooth rational curves. Then, X_3 is obtained by blowing up the singularities of ∂X_2, and so on. In particular, ∂X_k is a cycle of $2^{k-1}\ell$ curves.

Denote by $\mathcal{D}_k$ the dual graph of ∂X_k: vertices of $\mathcal{D}_k$ correspond to irreducible components E_i of ∂X_k and edges to intersection points $E_i \cap E_j$. A simple blow-up modifies both ∂X_k and $\mathcal{D}_k$ locally as follows

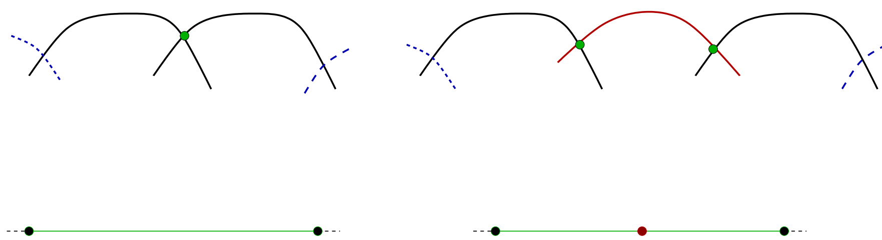


FIGURE 2. Blowing-up one point.

The group $\mathsf{Aut}(\mathcal{U})$ acts on $\tilde{\mathsf{Hyp}}(X)$ and Lemma 7.2 shows that its action stabilizes the following subset $\mathcal{B}$ of $\tilde{\mathsf{Hyp}}(X)$:

$$\mathcal{B} = \left\{ C \in \tilde{\mathsf{Hyp}}(X);\ \exists k \geq 1, C \text{ is an irreducible component of } \partial X_k \right\}$$

7.2. **Farey and dyadic parametrizations.** Consider an edge of the graph $\mathcal{D}_1$, and identify this edge with the unit interval $[0,1]$. Its endpoints correspond to two adjacent components E_i and E_{i+1} of ∂X_1, and the edge corresponds to their intersection q. Blowing-up q creates a new vertex (see Figure 2). The edge is replaced by two adjacent edges of $\mathcal{D}_2$ with a common vertex corresponding to the exceptional divisor and the other vertices corresponding to (the strict transforms of) E_i and E_{i+1}; we may identify this part of $\mathcal{D}_2$ with the segment $[0,1]$, the three vertices with $\{0,1/2,1\}$, and the two edges with $[0,1/2]$ and $[1/2,1]$.

Subsequent blow-ups may be organized in two different ways by using either a dyadic or a Farey algorithm (see Figure 3).

In the dyadic algorithm, the vertices are labelled by dyadic numbers $m/2^k$. The vertices of $\mathcal{D}_{k+1}$ coming from an initial edge $[0,1]$ of $\mathcal{D}_1$ are the points $\{n/2^k;\ 0 \leq n \leq 2^k\}$ of the segment $[0,1]$. We denote by $\mathrm{Dyad}(k)$ the set of dyadic numbers $n/2^k \in [0,1]$; thus, $\mathrm{Dyad}(k) \subset \mathrm{Dyad}(k+1)$. We shall say that an interval $[a,b]$ is a **standard dyadic** interval if a and b are two consecutive numbers in $\mathrm{Dyad}(k)$ for some k.

In the Farey algorithm, the vertices correspond to rational numbers p/q. Adjacent vertices of $\mathcal{D}_k$ coming from the initial segment $[0,1]$ correspond to pairs of rational numbers $(p/q, r/s)$ with $ps - qr = \pm 1$; two adjacent vertices of $\mathcal{D}_k$ give birth to a new, middle vertex in $\mathcal{D}_{k+1}$: this middle vertex is $(p+r)/(q+s)$ (in the dyadic algorithm, the middle vertex is the "usual" euclidean middle). We shall say that an interval $[a,b]$ is a **standard Farey** interval if $a=p/q$ and $b=r/s$ with $ps-qr=-1$. We denote by $\mathrm{Far}(k)$ the finite set of rational numbers $p/q\in[0,1]$ that is given by the k-th step of Farey algorithm; thus, $\mathrm{Far}(1)=\{0,1\}$ and $\mathrm{Far}(k)$ is a set of 2^{k+1} rational numbers p/q with $0\leq p\leq q$. (One can check that $1\leq q\leq \mathrm{Fib}(k)$, with $\mathrm{Fib}(k)$ the k-th Fibonacci number.)

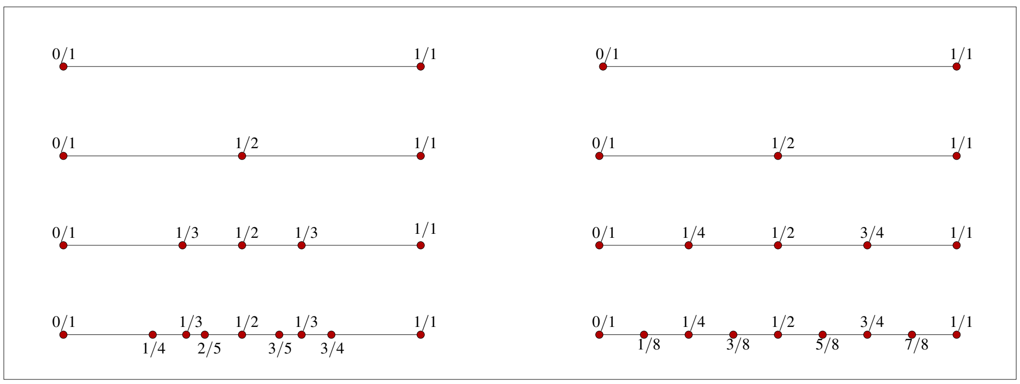


FIGURE 3. On the left, the Farey algorithm. On the right, the dyadic one.

By construction, the graph $\mathcal{D}_1$ has $\ell=2^m$ edges. Recall that the edges of $\mathcal{D}_1$ are in one to one correspondance with the singularities q_j of ∂X_1. Each edge determines a subset $\mathcal{B}_j$ of $\mathcal{B}$; the elements of $\mathcal{B}_j$ are the curves $C\subset\partial X_k$ ($k\geq 1$) such that $\pi_k(C)$ contains the singularity q_j determined by the edge. Using the dyadic algorithm (resp. Farey algorithm), the elements of $\mathcal{B}_j$ are in one-to-one correspondence with dyadic (resp. rational) numbers in $[0,1]$. Gluing these segments cyclically together one gets a circle $\mathbb{S}^1$, together with a nested sequence of subdivisions in ℓ, 2ℓ, ..., $2^{k-1}\ell$, ... intervals; each interval is a standard dyadic interval (resp. standard Farey interval) of one of the initial edges .

Since there are $\ell=2^m$ initial edges, we may identify the graph $\mathcal{D}_1$ with the circle $\mathbb{S}^1=\mathbf{R}/\mathbf{Z}=[0,1]/_{0\simeq 1}$ and the initial vertices with the dyadic numbers in $\mathrm{Dyad}(m)$ modulo 1 (resp. with the elements of $\mathrm{Far}(m)$ modulo 1). Doing this, the vertices of $\mathcal{D}_k$ are in one to one correspondence with the dyadic numbers in $\mathrm{Dyad}(k+m-1)$ (resp. in $\mathrm{Far}(k+m-1)$).

Remark 7.3. (a).– By construction, the interval $[p/q,r/s]\subset[0,1]$ is a standard Farey interval if and only if $ps-qr=-1$, iff it is delimited by two adjacent elements of $\mathrm{Far}(m)$ for some m.

(b).– If $h\colon [x,y] \to [x',y']$ is a homeomorphism between two standard Farey intervals mapping rational numbers to rational numbers and standard Farey intervals to standard Farey intervals, then h is the restriction to $[x,y]$ of a unique linear projective transformation with integer coefficients:

$$h(t) = \frac{at+b}{ct+d}, \text{ for an element } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ of } \mathsf{PGL}_2(\mathbf{Z}).$$

(c).– Similarly, if h is a homeomorphism mapping standard dyadic intervals to intervals of the same type, then h is the restriction of an affine dyadic map

$$h(t) = 2^m t + \frac{u}{2^n}, \text{ with } m,n \in \mathbf{Z}.$$

In what follows, we denote by G_{Far} the group of self-homeomorphisms of $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ that are piecewise $\mathsf{PGL}_2(\mathbf{Z})$ mapping with respect to a finite decomposition of the circle in standard Farey intervals $[p/q, r/s]$. In other words, if f is an element of G_{Far}, there are two partitions of the circle into consecutive intervals I_i and J_i such that the I_i are intervals with rational endpoints, h maps I_i to J_i, and the restriction $f\colon I_i \to J_i$ is the restriction of an element of $\mathsf{PGL}_2(\mathbf{Z})$.

Theorem 7.4. *Let $\mathcal{U}$ be an affine surface with a compactification $\mathcal{U} \subset X$ such that $\partial X := X \setminus \mathcal{U}$ is a cycle of smooth rational curves. In the Farey parametrization of the set $\mathcal{B} \subset \widetilde{\mathsf{Hyp}}(X)$ of boundary curves, the group $\mathsf{Aut}(\mathcal{U})$ acts on $\mathcal{B}$ as a subgroup of G_{Far}.*

Remark 7.5. There is a unique orientation preserving self-homeomorphism of the circle that maps $\mathrm{Dyad}(k)$ to $\mathrm{Far}(k)$ for every k. This self-homeomorphism conjugates G_{Far} to the group G_{Dya} of self-homeomorphisms of the circle that are piecewise affine with respect to a dyadic decomposition of the circle, with slopes in $\pm 2^{\mathbf{Z}}$, and with translation parts in $\mathbf{Z}[1/2]$. Using the parametrization of $\mathcal{B}$ by dyadic numbers, the image of $\mathsf{Aut}(\mathcal{U})$ becomes a subgroup of G_{Dya}.

Remark 7.6. The reason why we keep in parallel the dyadic and Farey viewpoints is the following: the Farey viewpoint is more natural for algebraic geometers (this is related to toric –i.e. monomial– maps and appears clearly in [17]), while the dyadic viewpoint is more natural to geometric group theorists, because this is the classical setting used in the study of Thompson groups.

Proof. Lemma 7.2 is the main ingredient. Consider the action of the group $\mathsf{Aut}(\mathcal{U})$ on the set $\mathcal{B}$. Let f be an element of $\mathsf{Aut}(\mathcal{U}) \subset \mathsf{Bir}(X)$. Consider an irreducible curve $E \in \mathcal{B}$, and denote by F its image: $F = f_\bullet(E)$ is an element of $\mathcal{B}$ by Lemma 7.2.

There are integers k and l such that $E \subset \partial X_k$ and $F \subset \partial X_l$. Replacing X_k by a higher blow-up $X_m \to X$, we may assume that $f_{lm} := \pi_l^{-1} \circ f \circ \pi_m$ is regular on a neighborhood of the curve E. Let q_k be one of the two singularities of ∂X_m that are contained in E, and let E' be the second irreducible component of ∂X_m containing q. If E' is blown down by f_{lm}, its image is one of the two singularities of ∂X_l contained in F (by Lemma 7.2). Consider the smallest integer $n \geq l$ such that ∂X_n contains the strict transform $F' = f_\bullet(E')$; in X_n, the curve F' is adjacent to the strict transform of F (still denoted F), and f is a local isomorphism from a neighborhood of q in X_m to a neighborhood of $q' := F \cap F'$ in X_n.

Now, if one blows-up q, the exceptional divisor D is mapped by $f_\bullet$ to the exceptional divisor D' obtained by a simple blow-up of q: f lifts to a local isomorphism from a neighborhood of D to a neighborhood of D', the action from D to D' being given by the differential df_q. The curve D contains two singularities of ∂X_{m+1}, which can be blown-up too: again, f lifts to a local isomorphism if one blow-ups the singularities of $\partial X_{n+1} \cap D'$. We can repeat this process indefinitely. Let us now phrase this remark differently. The point q determines an edge of $\mathcal{D}_m$, hence a standard Farey interval $I(q)$. The point q' determines an edge of $\mathcal{D}_n$, hence another standard Farey interval $I(q')$. Then, the points of $\mathcal{B}$ that are parametrized by rational numbers in $I(q)$ are mapped by $f_\bullet$ to rational numbers in $I(q')$ and this map respects the Farey order: if we identify $I(q)$ and $I(q')$ to $[0,1]$, $f_\bullet$ is the restriction of a monotone map that sends $\mathrm{Far}(k)$ to $\mathrm{Far}(k)$ for every k. Thus, on $I(q)$, $f_\bullet$ is the restriction of a linear projective transformation with integer coefficients (see Remark 7.3-(b)). This shows that $f_\bullet$ is an element of G_{Far}. □

7.3. **Conclusion.** The following result is proved in Section 9.

Theorem 7.7 (Farley, Navas [12, 25])**.** *Every subgroup of G_{Far} with Property (FW) is a finite cyclic group.*

Thus, if Γ is a subgroup of $\mathsf{Aut}(\mathcal{U})$ with Property (FW), it contains a finite index subgroup Γ_0 that acts trivially on the set $\mathcal{B} \subset \tilde{\mathsf{Hyp}}(X)$. This means that Γ_0 extends as a group of automorphisms of X fixing the boundary ∂X. Since ∂X supports a big and nef divisor, Γ_0 contains a finite index subgroup Γ_1 that is contained in $\mathsf{Aut}(X)^0$.

Note that Γ_1 has Property (FW) because it is a finite index subgroup of Γ. It preserves every irreducible component of the boundary curve ∂X, as well as its singularities. As such, it must act trivially on ∂X. When we apply Theorem 6.8 to Γ_1, the conjugacy $\varphi \colon X \to Y$ cab not contract ∂X, because the boundary supports an ample divisor. Thus, Γ_1 is conjugate to a subgroup of $\mathsf{Aut}(Y)$ that fixes a curve pointwise. This is not possible if Γ_1 is infinite (Lemma 3.6).

We conclude that Γ is finite in case (2) of Proposition 6.3.

8. Birational actions of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$

We develop here Example 1.3. Let σ_1, σ_2 be the distinct embeddings of $\mathbf{Q}(\sqrt{d})$ into $\mathbf{k}$. Let j_1, j_2 the resulting embeddings of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ into $\mathsf{SL}_2(\mathbf{k})$, and $j = j_1 \times j_2$ the resulting embedding into

$$\mathsf{G} = \mathsf{SL}_2(\mathbf{k}) \times \mathsf{SL}_2(\mathbf{k}).$$

Theorem 8.1. *Let Γ be a finite index subgroup of of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$. Let X be an irreducible projective surface over an algebraically closed field $\mathbf{k}$. Let $\alpha : \Gamma \to \mathsf{Bir}(X)$ be a homomorphism with infinite image. Then $\mathbf{k}$ has characteristic zero, and there exists a birational map $\varphi : Y \dashrightarrow X$ such that*

(1) *Y is the projective plane $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_m$, or $C \times \mathbb{P}^1$ for some curve C;*
(2) $\varphi^{-1}\alpha(\Gamma)\varphi \subset \mathsf{Aut}(Y)$*;*
(3) *There is a unique algebraic homomorphism $\beta : \mathsf{G} \to \mathsf{Aut}(Y)$ such that, for some finite index subgroup Γ' of Γ, we have $\varphi^{-1}\alpha(\gamma)\varphi = \beta(j(\gamma))$ for every $\gamma \in \Gamma'$.*

Using Theorem B ensures (1) and (2). If Y is $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_m$, then $\mathsf{Aut}(Y)$ is a linear algebraic group. If Y is a product $C \times \mathbb{P}^1$, a finite index subgroup of Γ preserves the projection onto $\mathbb{P}^1$, so that it acts via an embedding into the linear algebraic group $\mathsf{Aut}(\mathbb{P}^1) = \mathsf{PGL}_2(\mathbf{k})$.

When $\mathbf{k}$ has positive characteristic, Y is the projective plane, and the Γ-action is given by a homomorphism $\Gamma \to \mathsf{PGL}_3(\mathbf{k})$. Then we use the fact that for any n, every homomorphism $f : \Gamma \to \mathsf{GL}_n(\mathbf{k})$ has finite image. Indeed, it is well-known that $\mathsf{GL}_n(\mathbf{k})$ has no infinite order distorted elements: elements of infinite order have some transcendental eigenvalue and the conclusion easily follows. Since Γ has an exponentially distorted cyclic subgroup, f has infinite kernel, and infinite normal subgroups of Γ have finite index.

On the other hand, in characteristic zero we conclude the proof of Theorem 8.1 with the following lemma.

Lemma 8.2. *Let $\mathbf{k}$ be any field extension of $\mathbf{Q}(\sqrt{2})$. Consider the embedding j of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ into $G = \mathsf{SL}_2(\mathbf{k}) \times \mathsf{SL}_2(\mathbf{k})$ given by the standard embedding into the* *left-hand* SL_2 *and its Galois conjugate in the right-hand* SL_2. *Then for every linear algebraic group* H *and homomorphism* $f : \mathsf{SL}_2(\mathbf{Z}[\sqrt{d}]) \to H(\mathbf{k})$, *there exists a unique homomorphism* $\bar{f} : G \to H$ *of* $\mathbf{k}$*-algebraic groups such that the homomorphisms* f *and* $\tilde{f} \circ j$ *coincide on some finite index subgroup of* Γ.

Proof. The uniqueness is a consequence of Zariski density of the image of j. Let us prove the existence. Zariski density allows to reduce to the case when $H = \mathsf{SL}_n$. First, the case $\mathbf{k} = \mathbf{R}$ is given by Margulis' superrigidity, along with the fact that every continuous real representation of $\mathsf{SL}_n(\mathbf{R})$ is algebraic. The case of fields containing $\mathbf{R}$ immediately follows, and in turn it follows for subfields of overfields of $\mathbf{R}$ (as soon as they contain $\mathbf{Q}(\sqrt{d})$). □

9. Appendix: subgroups of G_{Far} and G_{Dya}

Consider the group G^*_{Dya} of self-homeomorphisms of the circle $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ that are piecewise affine with respect to a finite partition of $\mathbf{R}/\mathbf{Z}$ into dyadic intervals $[x_i, x_{i+1}[$ with x_i in $\mathbf{Z}[1/2]/\mathbf{Z}$ for every i, and satisfy

$$h(t) = 2^{m_i} t + a_i$$

with $m_i \in \mathbf{Z}$ and $a_i \in \mathbf{Z}[1/2]$ for every i. This group is known as the Thompson group of the circle, and is isomorphic to the group G^*_{Far} of orientation preserving self-homeomorphisms in G_{Far} (defined in §7.2). We want to prove Theorem 7.7, which we state here as follows: *Every subgroup of* G^*_{Dya} *with Property (FW) is a finite cyclic group.*

In [12, 25], Farley and Navas study subgroups of G^*_{Dya} with Kazhdan Property (T), but their proof applies to groups with Property (FW). We follow [25] to provide a proof of Theorem 7.7.[1]

9.1. The space of standard dyadic intervals. Again, refer to §7.2 for the notion of standard dyadic interval. Consider the subgroup H of G^*_{Dya} defined by

$$h \in H \;\text{ iff }\; h = \text{ identity on } [0, 1/2].$$

Let Q be the quotient space G^*_{Dya}/H; this is a countable set on which G^*_{Dya} acts by left translations. Define $A \subset Q$ by

$$A = \{g \cdot H;\; g_{[0,1/2]} \text{ is affine and } g[0,1/2] \text{ is a standard dyadic interval }\}.$$

This means that $g(t) = 2^m t + a$ for some $m \in \mathbf{Z}$ and $a \in \mathbf{Z}[1/2]$, and that $g[0,1/2] = [b/2^k, (b+1)/2^k]$ for some $k \geq 1$ and $0 \leq b \leq 2^k - 1$. For instance, $g(t) = t/2 + 1/8$ is affine but $g[0,1/2] = [1/8, 3/8]$ is not standard.

Remark 9.1. Let h be an element of G^*_{Dya}.

[1] There is a little gap in Farley's argument, namely in Prop. 2.3 and Thm. 2.4 of [12], but besides that, the proof is exactly the one of Farley.

(a).– Assume that h is affine on the interval J, with $h_{|J}(t) = t/2^N + a/2^M$, $1 \leq a \leq 2^M - 1$. Let $I \subset J$ be a standard dyadic interval of length 2^{-L}: $I = [b/2^L, (b+1)/2^L]$. Then $h(I)$ is the interval

$$\left[\frac{b}{2^{N+L}} + \frac{a}{2^M}, \frac{b+1}{2^{N+L}} + \frac{a}{2^M}\right].$$

If $M \leq N+L$, $h(I)$ is a standard interval. Thus, if the length of I is less than 2^{-M+N}, $h(I)$ is a standard interval.

(b).– There exists only finitely many standard intervals J such that h_J is not affine, or is affine but $h(J)$ is not standard.

(c).– If $g \in G^*_{\mathrm{Dya}}$ is affine on $[0,1/2]$, with $g(t) = t/2^L + a/2^M$ on this interval, then $L \geq 0$ because g is a self-homeomorphism of the circle.

(d).– If $I = [b/2^L, (b+1)/2^L]$ is standard, then $h(t) = t/2^{L-1} + b/2^L$ maps the interval $[0,1/2]$ onto I, and h is the unique affine map of type $2^m t + c/2^n$ mapping $[0,1/2]$ onto I. Thus the elements $g \cdot H$ of A are in one to one correspondance with standard dyadic intervals $g[0,1/2]$.

Lemma 9.2. *The action of G^*_{Dya} on Q by left translations commensurates the subset $A \subset Q$: $\forall h \in G^*_{\mathrm{Dya}}$,*

$$|h(A) \,\Delta\, A| \leq M(h) + M(h^{-1})$$

where $M(h)$ is the number of standard dyadic intervals $I \subset \mathbb{S}^1$ such that h is not affine on I, or h is affine on I and $h(I)$ is not standard.

The proof follows directly from the definition of A and the previous remarks: the map $g \cdot H \in A \to hg \cdot H \in Q$ corresponds to the action of h on standard dyadic intervals by Remark 9.1 (d), and $M(h)$ is finite by Remarks 9.1 (a) and (b).

9.2. **Conclusion.** Now, consider a subgroup Γ of G^*_{Dya} with Property (FW). Since Γ commensurates $A \subset Q$. There exists a subset $B \subset Q$ such that

(i) $|B \,\Delta\, A| < +\infty$
(ii) $\forall \gamma \in \Gamma$, $\gamma(B) = B$.

From (i), we know that there exists a finite set $\{g_1 \cdot H, \ldots, g_m \cdot H\} \subset Q \smallsetminus A$ and a finite set $\{h_1 \cdot H, \ldots, h_n \cdot H\} \subset A$ such that

$$B = A \cup \{g_1 \cdot H, \ldots, g_m \cdot H\} \smallsetminus \{h_1 \cdot H, \ldots, h_n \cdot H\}.$$

The set A corresponds to standard dyadic intervals, and the invariance of B means that there are only finitely many standard intervals I on which at least one of the elements of Γ is not affine, or is affine but does not map I to a standard interval. Let 2^{-N+1} be the minimum of the lengths of these intervals. By construction, every element of Γ is affine on every interval J of length $\leq 2^{-N}$; in particular, the slopes are bounded from above by 2^N and every $\gamma \in \Gamma$ is of type $t \mapsto 2^m t + a$ with $-N \leq m \leq N$ on such intervals J. Then, the translation part $a \in \mathbf{Z}[1/2]$ satisfies $|a| \geq 2^{-2N}$ because otherwise $\gamma(J)$ would not be standard (see Remark 9.1 (a) above). Since there are only finitely many choices for the pieces J of length 2^{-N}, the slopes 2^m, and the translation parts a, the group Γ is finite. Since Γ acts faithfully on the circle by orientation preserving self-homeomorphisms, Γ is cyclic.

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