GROUPS WITH VANISHING REDUCED 1-COHOMOLOGY

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Abstract. We show that a locally nilpotent group with torsion abelianization has vanishing reduced 1-cohomology. This allows to contradict a conjecture stating that all groups with vanishing reduced 1-cohomology are direct limits of groups with Property (T).

1. Introduction

Our object of study is the 1-cohomology of unitary representations of locally compact groups. Let G be a locally compact group, π a unitary representation in a Hilbert space $\mathscr{H}$. The space $Z^1(G,\pi)$ is defined as the space of continuous functions $b : G \to \mathscr{H}$ satisfying the 1-cocycle condition: for all $g, h \in G$, $b(gh) = \pi(g)b(h) + b(g)$. The subspace of $Z^1(G,\pi)$ of 1-coboundaries, namely, 1-cocycles that can be written as $g \mapsto \xi - \pi(g)\xi$ for some $\xi \in \mathscr{H}$, is denoted by $B^1(G,\pi)$. The *first cohomology group* of π is defined as

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

Locally compact, σ-compact groups G such that $H^1(G,\pi) = \{0\}$ for every unitary representation form a well-understood class since Delorme and Guichardet [4, 5] have proved that it coincides with the class of locally compact groups with Kazhdan's Property (T).

However, the 1-cohomology space $H^1(G,\pi)$ has a bad behaviour in some respects, as Guichardet as pointed out [6]. Given a family (π_i) of representations, it may happen that $H^1(G,\pi_i) = \{0\}$ for every i, but $H^1(G,\bigoplus \pi_i) \neq \{0\}$; this phenomenon arises even when $G = \mathbf{Z}$ and π_i is a well-chosen family of one-dimensional representations.

The *first reduced cohomology group* of π is defined as

$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)},$$

where $Z^1(G,\pi)$ is endowed with the topology of uniform convergence on compact subsets. In contrast to the non-reduced case, Guichardet [6, Chap. III, §2.4] has proved that the reduced cohomology is well-behaved under orthogonal decompositions, and, more generally, direct integral of unitary representations.

In this paper, we focus on the class of groups with vanishing reduced 1-cohomology, i.e. groups G such that $\overline{H^1}(G,\pi) = \{0\}$ for every unitary representation π. The main result in this field is due to Shalom [13]: the class of locally compact, *compactly generated* groups G with vanishing reduced 1-cohomology coincides with the class of locally compact groups with Kazhdan's Property (T). On the other hand, the investigation of non-compactly generated locally compact groups with vanishing 1-cohomology has been launched by F. Martin [9]. He observes that groups that are

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direct limits of groups with Property (T) have vanishing reduced 1-cohomology. For a countable discrete *hypercentral*[1] group G, he proves the equivalence between the following properties:

(i) $\overline{H^1}(G,\pi)=\{0\}$ for every unitary representation π of G;
(ii) G_{ab} is locally finite;
(iii) $H^1(G,\pi)=\{0\}$ for every irreducible unitary representation π of G;
(iv) G is locally finite.

The main goal of this paper is to extend the equivalence between (i) and (ii) to all locally compact, locally nilpotent groups. A locally compact group is locally nilpotent if every compact subset is contained in a closed nilpotent subgroup; discrete hypercentral groups are locally nilpotent[2].

Theorem 1. *Let G be a locally compact locally nilpotent group. The following are equivalent.*

(i) $\overline{H^1}(G,\pi)=\{0\}$ *for every unitary representation π of G;*
(ii) $\mathrm{Hom}(G,\mathbf{R})=\{0\}$.

There appears a new phenomenon: while a discrete hypercentral group G such that $\mathrm{Hom}(G,\mathbf{R})=\{0\}$ is locally finite [11, Lemma 4], there exist nontrivial torsion-free perfect locally nilpotent groups. Such groups are not direct limits of groups with Property (T) and have vanishing reduced 1-cohomology by Theorem 1. This answers negatively a conjecture by F. Martin [9, Conjecture 5.1.7], and seems to indicate that a non-trivial characterization of general locally compact groups with vanishing reduced 1-cohomology is out of reach. By the way, we exhibit a locally finite group not satisfying (iii), namely, the group of permutations with finite support of a countable set.

The proof of Theorem 1 uses, in a crucial way, some new relative notions of vanishing of 1-cohomology, introduced in the preliminaries below, which are variants of relative Property (T). In Section 3, we prove Theorem 1 for discrete groups. The case of non-discrete groups requires some further arguments which are independent of the rest of the paper, and is carried out in Section 4.

2. Preliminaries

Let G be a locally compact group, and H a closed subgroup. Recall that the pair (G,H) has *relative Property (FH)* if, for every isometric action of G on an affine Hilbert space, H has a fixed point. Equivalently, for every unitary representation π of G, the natural morphism in 1-cohomology $H^1(G,\pi)\to H^1(H,\pi)$ is zero. (If G is σ-compact, then it is equivalent to relative Property (T), see [8].)

In analogy, we say that the pair (G,H) has *relative Property* ($\overline{FH}$) if, for every isometric action α of G on an affine Hilbert space, H almost has fixed points, that is, for every compact $K\subset H$ and every $\varepsilon>0$, there exists a (K,ε)-fixed point for the action, i.e. a point v such that $\sup_{g\in K}\|v-\alpha(g)v\|\le\varepsilon$.

[1]A topological group G is hypercentral if $G=N_\alpha(G)$ for sufficiently large α, where $(N_\alpha(G))$ denotes the transfinite ascending central series.

[2]This is not true in the non-discrete case: indeed, any residually nilpotent group (such as a congruence subgroup in $\mathrm{SL}_n(\mathbf{Z})$) embeds in a product of finite nilpotent groups, which is a hypercentral compact group.

This means that, for every unitary representation π of G, the natural morphism $\overline{H^1}(G,\pi) \to \overline{H^1}(H,\pi|_H)$ is zero. If (G,G) has relative Property $(\overline{\mathrm{FH}})$, we say that G has Property $(\overline{\mathrm{FH}})$.

Finally, we say that pair (G,H) has *relative Property (FHI)* [respectively $(\overline{FHI})$] if, for every *irreducible* unitary representation π of G, the natural morphism $H^1(G,\pi) \to H^1(H,\pi|_H)$ [resp. $\overline{H^1}(G,\pi) \to \overline{H^1}(H,\pi|_H)$] is zero.

Proposition 2. *Let G be a σ-compact, locally compact group and H a closed subgroup. The following are equivalent.*

(i) (G,H) *has relative Property* $(\overline{\mathrm{FH}})$;
(ii) (G,H) *has relative Property* $(\overline{\mathrm{FHI}})$.

Proof: The implication (i)⇒(ii) is trivial. Conversely let us suppose (ii). By the Kakutani-Kodaira Theorem [3, Theorem 3.7], there exists a compact normal subgroup K of G such that G/K is second countable. Hence, replacing G by G/K, we can suppose that G is second countable. Let π be any unitary representation of G, and disintegrate it as a direct integral $\pi = \int^{\oplus} \pi_x dx$ of irreducible representations. Let $b \in Z^1(G,\pi)$, and disintegrate it as $b = \int^{\oplus} b_x dx$, where $b_x \in Z^1(G,\pi_x)$. Then, by (ii), $b_x|_H \in \overline{B^1(H,\pi_x)}$ for every x. By [6, Chap. 3, §2], it follows that $b \in \overline{B^1(H,\pi)}$. Thus (G,H) has relative Property $(\overline{\mathrm{FH}})$. ∎

Corollary 3. *Under the hypotheses of Proposition 2, relative Property* (FHI) *for a given pair implies relative Property* $(\overline{\mathrm{FH}})$. ∎

3. Reduced 1-cohomology of locally nilpotent groups

We use the following result of Guichardet [5, Corollaire 5].

Proposition 4 (Guichardet)**.** *Let G be a nilpotent locally compact group, and π a nontrivial irreducible representation. Then $H^1(G,\pi) = \{0\}$.*

As an immediate consequence, we have:

Corollary 5. *Let G be a nilpotent locally compact group. The pair $(G,D(G))$ has relative Property* (FHI).

Proof: Let π be an irreducible unitary representation of G. If π is nontrivial, $H^1(G,\pi) = \{0\}$ by Proposition 4. If $\pi = 1_G$, a 1-cocycle $b \in Z^1(G,\pi)$ is a morphism into the abelian group $\mathbf{C}$, hence vanishes on $D(G)$. ∎

Remark 6. F. Martin [9] generalizes Proposition 4, and thus Corollary 5, to hypercentral locally compact groups.

Let G be a locally compact group and H a closed subgroup. The following three lemmas are immediate.

Lemma 7. *Let $H' \supset H$ be another closed subgroup such that H'/H is compact. If (G,H) has relative Property* (FHI)*, then so does (G,H').*

Proof: Let α be an affine action of G on a Hilbert space with irreducible linear part. Let v be a H-fixed point. Then $H'v$ is a compact orbit for H'. By the centre Lemma [7, §3.b], H' fixes a point. ∎

Lemma 8. *Let G' be another subgroup of G containing H. If the pair (G',H) has relative Property* $(\overline{\mathrm{FH}})$*, then so does (G,H).* ∎

Lemma 9. *(G,H) has relative Property $(\overline{\mathrm{FH}})$ if and only if for every compactly generated, closed subgroup M of H, (G,M) has Property $(\overline{\mathrm{FH}})$.* ■

Let G be a discrete, locally nilpotent group. Denote by $T_a(G)$ the inverse image in G of the torsion subgroup of G_{ab}.

Proposition 10. *Let G be a discrete, locally nilpotent group. Then $(G,T_a(G))$ has relative Property $(\overline{\mathrm{FH}})$.*

Proof: Let Γ be a finitely generated subgroup of $T_a(G)$, with generators γ_i, $i = 1,\dots,n$. For suitable positive integers n_i, γ^{n_i} can be written as a product of commutators. This involves finitely many elements of Γ, so that $\Gamma \subset T_a(H)$ for some finitely generated subgroup $H \subset G$. Therefore, in view of Lemmas 8 and 9, we can suppose that G is finitely generated.

By Corollary 5, $(G,D(G))$ has relative Property (FHI). Since $D(G)$ has finite index in $T_a(G)$, by Lemma 7, the pair $(G,T_a(G))$ has relative Property (FHI). By Corollary 3, $(G,T_a(G))$ also has Property $(\overline{\mathrm{FH}})$. ■

Corollary 11. *Let G be a locally nilpotent, discrete group. The following are equivalent.*

(i) *G has Property $(\overline{\mathrm{FH}})$;*
(ii) *G_{ab} is torsion;*
(iii) $\mathrm{Hom}(G,\mathbf{R}) = \{0\}$.

Proof: (ii)⇒(i) If G_{ab} is torsion, then $G = T_a(G)$, so that, by Proposition 10, G has Property $(\overline{\mathrm{FH}})$.

(i)⇒(iii) If G has Property $(\overline{\mathrm{FH}})$, then $\{0\} = \overline{H^1}(G,1_G) = H^1(G,1_G) = \mathrm{Hom}(G,\mathbf{C}) \simeq \mathrm{Hom}(G,\mathbf{R})^2$.

(iii)⇒(ii) This well-known result immediately follows from the injectivity of $\mathbf{R}$ as a $\mathbf{Z}$-module. ■

Remark 12. F. Martin [9, Corollaire 5.4.8] has obtained Corollary 11 in the case when G is a discrete countable hypercentral group. In this case the conditions are also equivalent to: *G is locally finite* (compare the next remark).

Remark 13. Let G be a σ-compact, locally compact group. Consider the following properties:

(1) Every compact subset of G is contained in an open subgroup with Property (T);
(2) Every compact subset of G is contained in an open subgroup H such that (G,H) has relative Property (T);
(3) G has Property $(\overline{\mathrm{FH}})$.

First note that, if G has the Haagerup Property, i.e. acts properly by isometries on a Hilbert space, then each of (1) and (2) is clearly equivalent to saying that $G = \bigcup G_n$ for some increasing sequence of open compact subgroups G_n. Recall [1] that amenable groups have the Haagerup Property, and in particular locally nilpotent groups do.

In general, (1)⇒(2)⇒(3), and, by a result of Shalom [13], these properties are all equivalent to Property (T) if G is compactly generated. F. Martin [9, Conjecture 5.1.7] and A. Valette (oral communication) have asked whether (3)⇒(1) holds in full generality. We deduce from Corollary 11 that the answer is negative. For example, let Γ be the group of infinite $\mathbf{Q} \times \mathbf{Q}$ matrices with integer entries, which are upper triangular with 1 on the diagonal, and with finitely many nonzero non-diagonal terms. Then Γ, usually referred as a "McLain group", is a perfect, locally nilpotent, torsion-free group [12, §12.1.9]. By the remark above, since Γ is locally nilpotent but not locally finite, it cannot satisfy (2); however it satisfies (3) by Corollary 11.

As regards the implication (2)$\Rightarrow$(1), I have no counterexample. However, I conjecture that there exists a countable group satisfying (2) but not (1). I think that the methods that would lead to such a counterexample might be strictly more interesting that the counterexample itself.

Remark 14. Corollary 11 gives no information about which locally nilpotent groups have Property (FHI). I do not know any example of, say, a countable locally nilpotent group G such that $\mathrm{Hom}(G, \mathbf{R}) = \{0\}$ and G does not have Property (FHI).

On the other hand, there exists a countable locally finite group without Property (FHI). Indeed, let G be the group of permutations with finite support of $\mathbf{N}$. Note that, since G is locally finite, G has Property $(\overline{\mathrm{FH}})$. Let π be its natural representation on $\ell^2(\mathbf{N})$. Then it is easily checked that π is irreducible. On the other hand, $H^1(G, \pi)$ is "large": indeed, to every function $f : \mathbf{N} \to \mathbf{C}$ is associated a formal coboundary $g \mapsto f - \pi(g)f$, which is a coboundary if and only if $f \in \mathbf{C} + \ell^2(\mathbf{N})$.

However, I do not know any group with Property (FHI) not satisfying (1) of Remark 13.

Remark 15. Yehuda Shalom has pointed out to me that the class of amenable groups with Property $(\overline{\mathrm{FH}})$ is stable under quasi-isometries (as defined, without finite generation assumption, in [14]). This follows from Theorems 2.1.7 and 3.2.1 of [14]. The McLain group of Remark 13 shows that this class does not coincide with the class of locally finite groups, which is also stable under quasi-isometries (as an easy consequence of the definition).

Remark 16. In contrast with Property (FH), Properties $(\overline{\mathrm{FH}})$ and (FHI) are not preserved under extensions. Indeed, let F be any nontrivial finite abelian group, and let Γ be an infinite group with Property (T). Then Γ, as a group with Property (T), and $F^{(\Gamma)}$, as a locally finite abelian group, have both Properties $(\overline{\mathrm{FH}})$ and (FHI). On the other hand, by a result independently obtained by P.-A. Cherix, F. Martin, A. Valette [2] and Neuhauser [10], the wreath product $G = F \wr \Gamma = F^{(\Gamma)} \rtimes \Gamma$, which is finitely generated, does not have Kazhdan's Property (T). By Shalom's result stated above, G does not have Properties $(\overline{\mathrm{FH}})$ and (FHI).

4. The locally compact case

Theorem 1 is now proved in the discrete case (Corollary 11); it remains to deal with the general case, which is slightly more involved.

Let G be a locally compact group. We say that $x \in G$ is *elliptic* if the subgroup generated by x is relatively compact in G. We say that x is *ab-elliptic* if the image of x in G_{ab} is elliptic. We denote by $E_{\mathrm{ab}}(G)$ the set of ab-elliptic elements in G.

Lemma 17. *For every locally compact group, $E_{\mathrm{ab}}(G)$ is a closed, normal subgroup of G. Moreover $G/E_{\mathrm{ab}}(G)$ is isomorphic to a direct product $\mathbf{R}^n \times \Gamma$, where $n \in \mathbf{N}$ and Γ is a discrete torsion-free abelian group.*

Proof: We can suppose that G is abelian, so that the first assertion is clear.

This second assertion is an easy consequence of the structure of locally compact abelian groups. For convenience, we recall the proof. Taking the quotient by $E_{\mathrm{ab}}(G)$ if necessary, we can suppose that G is abelian without elliptic elements. Under these hypotheses, G has an open subgroup V that is a connected Lie group, hence isomorphic to $\mathbf{R}^n$ for some n. By injectivity of V as a $\mathbf{Z}$-module, V is a direct factor. Since G has no elliptic element, it follows that G/V is torsion-free. ■

Lemma 18. *Let G be a locally compact, compactly generated group, and (N_i) be an increasing net of closed, normal subgroups. Suppose that, for all i, the quotient G/N_i is abelian and has no nontrivial elliptic element. Then the net (N_i) is stationary: for i, j large enough, $N_i = N_j$.*

Proof: By assumption, and using Lemma 17, for all i, G/N_i is isomorphic to $\mathbf{R}^{n_i} \times \mathbf{Z}^{m_i}$ for suitable integers n_i, m_i, which decrease with i. It follows that there exists (n, m) such that $(n_i, m_i) = (n, m)$ for i large enough. Now $\mathbf{R}^n \times \mathbf{Z}^m$ is Hopfian, in the sense that every continuous surjective endomorphism is an isomorphism. The result follows. ■

Lemma 19. *Let G be a locally compact group. Let $K \subset E_{\mathrm{ab}}(G)$ be a compact subset. Then there exists a closed, compactly generated subgroup H of G such that $K \subset E_{\mathrm{ab}}(H)$.*

Proof: Let G_i be a net of open, compactly generated subgroups containing K and covering G. Set $M_i = E_{\mathrm{ab}}(G_i)$, and $M = \bigcup M_i$.

Fix $j \in I$, and observe that $(G_j \cap M_i)_{i \geq j}$ is a net of normal subgroups of G_j, and, for all $i \geq j$, $G_j/(G_j \cap M_i)$ is abelian without nontrivial elliptic element, since it embeds as an open subgroup in G_i/M_i. By Lemma 18, the net $(G_j \cap M_i)$ is eventually constant, hence equal to $G_j \cap M$.

Since this is true for all j, it follows that M is a closed, normal subgroup of G, and $G_j/(G_j \cap M)$ is abelian without elliptic elements for all j. Hence G/M is abelian without elliptic elements, so that the image of K in G/M is trivial, i.e. $K \subset M$. Now fix j. Since $K \subset G_j$, $K \subset M \cap G_j$, which is equal to $M_i \cap G_j$ for some i. Accordingly, $K \subset E_{\mathrm{ab}}(G_i)$. ■

We can now generalize Proposition 10 to the locally compact case.

Proposition 20. *Let G be a locally nilpotent locally compact group. Then $(G, E_{\mathrm{ab}}(G))$ has relative Property $(\overline{\mathrm{FH}})$.*

Proof: Let Ω be a closed subgroup of $T_a(G)$, generated by a compact subset K. By Lemma 19, there exists an open subgroup H containing K such that $K \subset E_{\mathrm{ab}}(H)$, so that $\Omega \subset E_{\mathrm{ab}}(H)$. Therefore, in view of Lemmas 8 and 9, we can suppose that G is compactly generated. So we can go on as in the proof of Proposition 10. ■

Corollary 21. *Let G be a locally nilpotent locally compact group. The following are equivalent.*

(i) *G has Property $(\overline{\mathrm{FH}})$;*
(ii) *G_{ab} is elliptic;*
(iii) $\mathrm{Hom}(G, \mathbf{R}) = \{0\}$.

Proof: (ii)⇒(i)⇒(iii) are proved as in Corollary 11.

(iii)⇒(ii) We must show that $G/E_{\mathrm{ab}}(G)$ is trivial. By Lemma 17, it is isomorphic to $\mathbf{R}^n \times \Gamma$ for some torsion-free abelian group Γ. Now (iii) implies $n = 0$, and, using the injectivity of $\mathbf{R}$ as a $\mathbf{Z}$-module, (iii) also implies $\Gamma = \{1\}$. ■

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References

[1] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, Alain Valette. *Groups with the Haagerup Property.* Birkhäuser, Progress in Mathematics 197, 2001.

[2] Pierre-Alain Cherix, Florian Martin, Alain Valette. *Spaces with measured walls, the Haagerup Property and Property (T).* Ergod. Th. & Dynam. Sys. **24**, 1895-1908, 2004.

[3] W. Wistar Comfort. *Topological groups.* p.1143-1263 in: "Handbook of Set-Theoretic Topology", edited by K. Kunen and J. E. Vaughan, North Holland, Amsterdam, 1984.

[4] Patrick Delorme. *1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations.* Bull. Soc. Math. France **105**, 281-336, 1977.

[5] Alain Guichardet. *Sur la cohomologie des groupes topologiques, II.* Bull. Sc. Math. (2) **96**, 305-332, 1972.

[6] Alain Guichardet. "Cohomologie des groupes topologiques et des algèbres de Lie". Paris, Cédic-Nathan, 1980.

[7] Pierre de la Harpe, Alain Valette. "La propriété (T) de Kazhdan pour les groupes localement compacts". Astérisque **175**, SMF, 1989.

[8] Paul Jolissaint. *On Property (T) for pairs of groups.* Enseign. Math. (2) **51**, 31-45, 2005.

[9] Florian Martin. "Analyse harmonique et 1-cohomologie réduite des groupes localement compacts". *PhD Thesis.* Université de Neuchâtel, 2003.

[10] Markus Neuhauser. *Relative property (T) and related properties of wreath products.* Math. Z. **251**(1), 167-177, 2005.

[11] Derek J. S. Robinson *On the homology of hypercentral groups.* Arch. Math. (Basel) **32**, no. 3, 223-226, 1979.

[12] Derek J. S. Robinson. *A Course in the Theory of Groups.* Graduate Texts in Mathematics 80, Springer-Verlag, 1982.

[13] Yehuda Shalom. *Rigidity of commensurators and irreducible lattices.* Invent. Math. **141**, no. 1, 1-54, 2000.

[14] Yehuda Shalom. *Harmonic analysis, cohomology, and the large scale geometry of amenable groups.* Acta Mathematica **193**, 119-185, 2004.

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