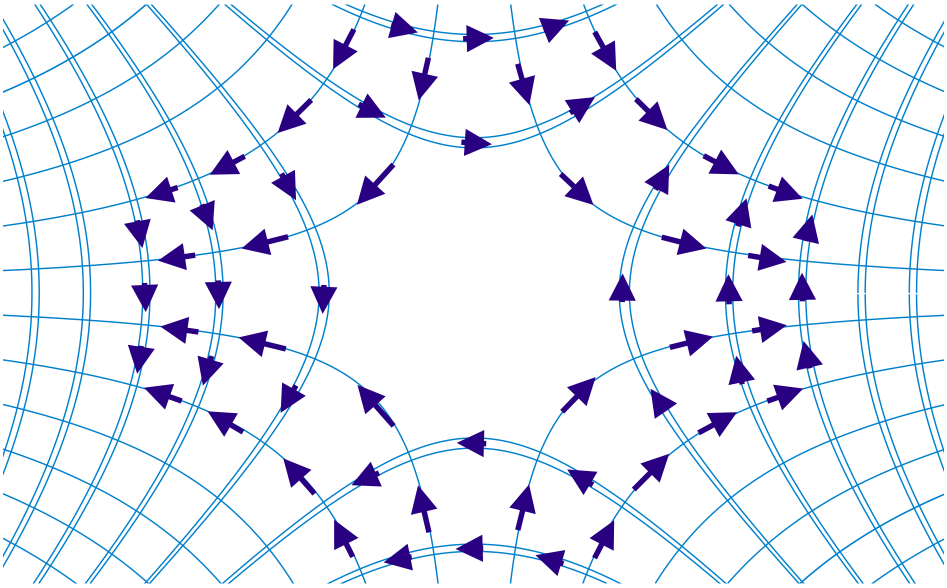
NEAR ACTIONS (LECTURES IN IMPAN, WARSAW, APRIL 2019)

Abstract. Introductory lectures to near actions. The actual lectures (6 hours + 1h of Problem session) were given at the Mathematical Institute of the Polish Academy of Sciences (IMPAN), April 16-18, 2019. Main reference: the monograph [Cor1].



1. Motivation: Schreier graphs, examples of groups defined as infinite permutation groups

Let G be a group and S a generating subset. Let X be a G-set. The Schreier graph of the G-action on X is the graph whose set of vertices is X, and with an edge from x to sx, labeled by s, for every $(s,x) \in S \times X$. If S is finite, this is a locally finite graph. It is connected if and only if the G-action is transitive (or X is empty). Beware that the G-action usually does not preserve the graph structure (however it does when G is abelian). However, the G-action has the property of acting as permutations with bounded displacement.

The possibly best known example is when $X = G$ under left action, in which case it is known as left Cayley graph. In this case, the *right*-action of G preserves the graph structure. In general, the Schreier graph can have few automorphisms, even after forgetting the labeling.

Let us provide various examples of groups that are naturally defined as permutation groups.

(a) We softly start with the Cayley graph of $\mathbf{Z}$:

$$\cdots \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \cdots$$

It can be described as a single infinite cycle. In general, an action of $\mathbf{Z}$ can be described using the cycle decomposition of the generating element.

(b1) Now consider the infinite dihedral group D_∞, with its generating subset consisting of two elements of order 2. Here is the Cayley graph:

$$\cdots \;\text{———}\; \circ = \!\!= \circ \;\text{———}\; \circ = \!\!= \circ \;\text{———}\; \cdots$$

(b2) Here is an infinite Schreier graph (with a self-loop on the left):

$$\subset \circ = \!\!= \circ \;\text{———}\; \circ = \!\!= \circ \;\text{———}\; \cdots$$

(c) The possible earliest use of (non-Cayley) Schreier graphs of infinite groups was the proof in the 20's by O. Schreier himself that the free groups are residually finite. The proof consists in proving that the free product G of 3 copies $\langle a\rangle*\langle b\rangle*\langle c\rangle$ of $\mathbf{Z}/2\mathbf{Z}$ is residually finite (the result for finitely generated free groups follows, since they embed in such a free product). Namely, let w be any nontrivial element in this free product: its reduced form is a nontrivial word (say of length n) in the letters a, b and c, with no two consecutive occurrences of the same letter. Consider a graph Γ_w consisting of $n+1$ vertices written consecutively, each joined to the next one, the n edges being labeled by the corresponding letters of w. Complete this to a Schreier graph of G by adding self-loops of the missing letters at every vertex.

Then w acting (on the left) on the right-hand vertex maps it to the left-hand vertex, and in particular acts non-trivially.

$$\Gamma_w \text{ for } w = abacbc : \qquad \overset{b,c}{\circ} \underset{a}{\text{——}} \overset{c}{\circ} \underset{b}{\text{——}} \overset{c}{\circ} \underset{a}{\text{——}} \overset{b}{\circ} \underset{c}{\text{——}} \overset{a}{\circ} \underset{b}{\text{——}} \overset{a}{\circ} \underset{c}{\text{——}} \overset{a,b}{\circ}$$

(Self-loops are not drawn but indicated as labels on vertices.)

If instead we consider an infinite sequence and choose a labeling of the edges in $\{a, b, c\}$ on which any two consecutive edges have distinct labels, and any finite reduced word occurs somewhere (and again complete the graph using self-loops), then we obtain a faithful transitive action of G for which the Schreier graph is isomorphic to a combinatorial half-line (in particular, it has an invariant mean).

For the next examples, we refer to pictures appended at the end.

(d) Houghton's groups (see (z4) page 21). For $n \geq 2$, consider the set $X_n = \mathbf{N} \times \mathbf{Z}/n\mathbf{Z}$. For $i \in \mathbf{Z}/n\mathbf{Z}$, define a permutation σ_i of X_n by $\sigma_i(n, i) = (n+1, i)$, $\sigma_i(n, i-1) = (n-1, i-1)$, $\sigma_i(0, i-1) = (0, i)$, and identity elsewhere. The group

H_n generated by these permutations is actually an elementary-amenable finitely generated group, and is known to be finitely presented for $n \geq 3$ (but not for $n = 2$).

(e) Wreath products (see page 20). This is a rich source of permutation groups. Let us start with a particular case: given a group G, we consider the set $G \times \{-1, 1\}$, with the action of G on the left $g \cdot (h, e) = (gh, e)$, $e = \pm 1$, and the permutation τ exchanging $(1_G, 1) \leftrightarrow (1_G, -1)$ and fixing all other elements. When G is given as a Cayley graph, the Schreier graph consists of two copies of the Cayley graph of G, joined by a single edge labeled by τ (page 20: (s1) for $G = \mathbf{Z}$ and (s2) for $G = \mathbf{Z}^2$: the first is 4-ended and the second is 2-ended with quadratic growth).

Write $C_2 = \mathbf{Z}/2\mathbf{Z}$. This actually defines faithful actions of the lamplighter group $C_2 \wr G = C_2^{(G)} \rtimes G$. Here $C_2^{(G)}$ is the subgroup of C_2^G consisting of finitely supported elements, and G acts by permuting the factors.

In general, the wreath product is naturally defined for permutation groups. Call permutation group the data of a group G with an action α on a set X. If we refer to it as α, we write $G = G_\alpha$ and $X = X_\alpha$. The wreath product $\beta \wr \alpha$ is the permutation group whose underlying group is known as the (restricted) permutational wreath product

$$G_{\beta\wr\alpha} = G_\beta \wr_{X_\alpha} G_\alpha = G_\beta^{(X_\alpha)} \rtimes G_\alpha,$$

$$X_{\beta\wr\alpha} = X_\beta \times X_\alpha$$

with action $g \cdot (y, x) = (y, gx)$ and

$$f \cdot (y, x) = (f(x)y, x), \quad g \in G_\alpha,\ f \in G_\beta^{(X_\alpha)},\ (y, x) \in X_{\beta\wr\alpha}.$$

If α and β are transitive, then so is $\beta \wr \alpha$. If α and β are faithful (and X_β is nonempty) then so is $\beta \wr \alpha$. See page 20 (s3) for $D_\infty \wr \mathbf{Z}$ and (s4) for $(\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}) \wr \mathbf{Z}$. Notably, they have interesting space of ends: for $D_\infty \wr \mathbf{Z}$ the space of ends consists of a countable space with 2 accumulation points, and in the second case it consists of a "dusty Cantor space" (this is a space, uniquely defined up to homeomorphism by the property of being metrizable, compact, totally disconnected, disjoint union of a Cantor space and a dense set of isolated points).

(f) B.H. Neumann's groups (see page 20, (s5)). Let $(a_n)_{n\geq 0}$ be an increasing sequence of integers ≥ 5. Define $X[a]$ as disjoint union of subsets $X[a]_n$, a copy of $\mathbf{Z}/a_n\mathbf{Z}$. Define two permutations t, s of $X[a]$ by $t(x) = x+1$, and s is transposition exchanging 0 and 1 on each $X[a]_n$. Let Γ_a be the group generated by u and v. It acts faithfully on $X[a]$ with finite orbits (namely the $X[a]_n$ for $n \geq 0$) so it is residually finite.

It is not hard to check that the intersection of Γ_a with the group of finitely supported permutations contains for each n the alternating group on each $X[a]_n$ (acting as the identity outside $X[a]_n$).

(g) All the Schreier graphs of page 21 are given by 2 generators, say a, b. Moreover, they all share the property that the two generators commute outside a finite subset, that is, $[a, b]$ has finite support: this will be interpreted as being a near action of $\mathbf{Z}^2$ (an action would require that $[a, b] = 1$). In all case, the resulting faithful permutation group maps onto $\mathbf{Z}^2$ (or a quotient thereof), with a locally finite kernel (acting by finitely supported permutations). Some of them are known.

- (z1) is just the Cayley graph of $\mathbf{Z}^2$.
- (z2) is obtained after removing one vertex and adding two edges to "repair" the action; I do not know if the resulting group is "known";
- (z3) results in a faithful action of the lamplighter group $(\mathbf{Z}/2\mathbf{Z}) \wr \mathbf{Z}$.
- (z4) is the definition of Houghton's group H_3;
- (z5) is obtained by taking a 2-fold covering of the Cayley graph of $\mathbf{Z}^2$. The resulting permutation group is metabelian, a non-split extension of the abelian group $\mathbf{F}_p[x^{\pm 1}, y^{\pm 1}]$ by $\mathbf{Z}^2$ ((n, m) acting by multiplication by $x^n y^m$);
- (z6) consists in cutting the Cayley graph of $\mathbf{Z}^2$ along the ray $\{0\} \times (-\mathbf{N})$ and gluing after translating one side by $(0, -1)$, and then adding a copy of $\mathbf{N}$ (the vertical generator acting trivially and the horizontal shifting) so as to obtain two permutations.

2. Motivation: commensurating actions

For two subsets M, M' of a set, we say $M \sim M'$ if $M \sim M'$ is finite, where $\triangle$ denotes symmetric difference.

Let a group G act on a set X. A subset $M \subset X$ is G-**commensurated** if $gM \sim M$ is finite for every $g \in G$. We refer to (X, M) as a commensurating action of G.

The most obvious reason to be G-commensurated is when there exists a G-invariant subset M' such that $M \sim M'$, in which case we say that M is G-**transfixed** (and the commensurating action is transfixing).

The simplest example of a non-transfixed commensurated subset for an action is possibly the case of $\mathbf{Z}$ acting on itself by translations, commensurating $\mathbf{N}$.

In general, given a finitely generated group G and G-set X, those G-commensurated subsets are precisely those subsets M of X whose boundary $\partial_S M$ is finite. Here the boundary $\partial_S M$ is taken in the Schreier graph with respect to some finite symmetric generating subset S of G: $\partial_S M = \{x \in M : \exists s \in S : sx \notin M\}$.

For a G-set X, define $\mathrm{Comm}_G(X)$ as the set of G-commensurated subsets of X. It is saturated under the equivalence relation $\sim$.

We say that X is 0-ended if X is finite: this means that $\mathrm{Comm}_G(X)/\sim$ is a singleton; we say that X is 1-ended if $\mathrm{Comm}_G(X)/\sim$ has exactly 2 elements. This means that X is infinite and $\mathrm{Comm}_G(X)$ is reduced to finite subsets of X and their complements. Note that this forces the action to consists of a single infinite orbit and possibly finitely many additional points. For G finitely generated, this means that the Schreier graph of this infinite orbit is 1-ended in the "classical" sense.

Remark 2.1. The set of G-commensurated subsets $\mathrm{Comm}_G(X)$ is a Boolean algebra of subsets of X, and so is the quotient $\mathrm{Comm}_G(X)/\sim$. Stone duality tells us that there exists a canonically defined compact, totally disconnected space $E_G(X)$ such that one has a natural identification of $\mathrm{Comm}_G(X)/\sim$ with the set of clopen subsets of $E_G(X)$. This space $E_G(X)$ is called the space of ends of the G-set X. (We have $E_G(X)$ of cardinal 0 or 1 if and only if X is 0-ended or 1-ended.)

Definition 2.2. A group G has Property FW if every commensurating G-action is transfixing.

Example 2.3. $\mathbf{Z}$ does not have Property FW because of the commensurating action $(\mathbf{Z}, \mathbf{N})$. Hence, every group having $\mathbf{Z}$ as a quotient also fails to have Property FW.

Remark 2.4. One can check that a finitely generated group G has Property FW if and only if all its infinite transitive Schreier graphs are 1-ended.

Example 2.5. Finite groups have Property FW, and in a sense these are the only trivial examples. Groups with Kazhdan's Property T have Property FW. This gives many examples, such as $\mathrm{SL}_d(\mathbf{Z})$ for $d \geq 3$. Examples of groups with Property FW but not T are $\mathrm{SL}_2(\mathbf{Z}[\sqrt{k}])$ when k is a positive non-square.

3. Balanced near actions

Let $\mathfrak{S}(X)$ be the symmetric group on a (typically infinite) set X, consisting of all permutations of X. It has a remarkable normal subgroup, $\mathfrak{S}_{\mathrm{fin}}(X)$ (or "fin" for short, when the context is clear), consisting of those finitely supported permutations, i.e., those permutations that are identity outside a finite subset.

Recall that an action of a group G on a set X is a homomorphism $\alpha : G \to \mathfrak{S}(X)$. In analogy, we define:

Definition 3.1. A **balanced near action** of a group G on a set X is a homomorphism $\alpha : G \to \mathfrak{S}(X)/\mathrm{fin}$. When endowed with α, we call X a balanced near G-set.

The notion of near action will be slightly more general, but many illustrating examples fit into this setting. The purpose is to treat near actions using intuition from usual actions. It will turn out that the framework of balanced near action is inconvenient for this purpose, essentially because we typically need to decompose balanced near actions into disjoint union of not necessarily balanced near actions: we come back to this later.

One main source of balanced near actions is simply actions: every action gives rise to a near action. Near actions occurring in this way are called **realizable**.

More examples appear once we observe that the faithful action of a group can define a near action of a quotient group. This is illustrated on the Schreier graphs page 21, which all factor through a near action of the group $\mathbf{Z}^2$.

A remark is that two actions α, β of the group G on the set X define the same near action if and only if $\alpha(g)^{-1}\beta(g) \in \mathfrak{S}_{\mathrm{fin}}(X)$ for every g. That is, for every g, $\alpha(g)$ and $\beta(g)$ coincide almost everywhere (here "almost everywhere" means "outside a finite subset"). We then say that β and α are **finite perturbation** of each other.

Example 3.2. Let α be the action of $\mathbf{Z}$ on itself by powers of the cycle $n \mapsto n+1$. Let β be the action on itself by powers of the permutation $n \mapsto n+1$ for $n \neq -1, 0$, mapping -1 to 1 and fixing 0: $\ldots \mapsto -2 \mapsto -1 \mapsto 1 \mapsto 2$. Since α is transitive while β has 2 orbits (the singleton $\{0\}$ and its complement), these actions are not conjugate at all. However, they are finite perturbations of each other.

Given an action α, the trivial way of defining finite perturbation is to define another action β by $\beta(g) = \sigma \circ \alpha(g) \circ \sigma^{-1}$ for some $\sigma \in \mathrm{fin}$. It is not the only one as the above example of the infinite cycle shows: but we will see examples of nontrivial actions α all of whose finite perturbations are trivial.

4. Examples of balanced near actions from dynamics

4.A. Piecewise continuous self-transformations of the circle.

Definition 4.1. Let $\widehat{\mathrm{PC}}(\mathbf{S}^1)$ be the group of piecewise continuous permutations of $\mathbf{S}^1$, that is, those permutations of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ with only finitely discontinuity points (this condition, in the setting of the circle, is stable under taking inverses).

Define $\mathrm{PC}(\mathbf{S}^1) = \widehat{\mathrm{PC}}(\mathbf{S}^1)/\mathrm{fin}$ (here $\mathrm{fin} = \mathfrak{S}_{\mathrm{fin}}(\mathrm{S}^1)$).

By definition $\widehat{\mathrm{PC}}(\mathbf{S}^1)$ acts faithfully on the circle $\mathbf{S}^1$. However, the induced near action is not faithful: the kernel of the corresponding composite homomorphism $\widehat{\mathrm{PC}}(\mathbf{S}^1) \to \mathfrak{S}(\mathbf{S}^1)/\mathrm{fin}$ is precisely equal to fin. Hence, we get a balanced near action of the quotient $\mathrm{PC}(\mathbf{S}^1)$ on $\mathbf{S}^1$.

We think of $\mathrm{PC}(\mathbf{S}^1)$ as the group of piecewise continuous self-transformations of the circle. However, it does not really act on the circle, since individual values are not clearly defined.

If we restrict to piecewise orientation-preserving self-transformations, we define, in a similar fashion, subgroups $\widehat{\mathrm{PC}^+}(\mathbf{S}^1) \subset \widehat{\mathrm{PC}}(\mathbf{S}^1)$ and $\mathrm{PC}^+(\mathbf{S}^1) \subset \mathrm{PC}(\mathbf{S}^1)$. The latter by restriction, near acts, and this action is indeed realizable, by mapping any f to its unique left-continuous representative.

Nevertheless, this trick does not work in the non-orientation-preserving setting: the left-continuous representative is not even bijective. And even while individual elements always admit a bijective representative, there is no reason that they should be chosen in a way compatible with multiplication. This is the object of study of the article [Cor2], in which the following result is established:

Theorem 4.2. *The near action of* $\mathrm{PC}(\mathbf{S}^1)$ *on* $\mathbf{S}^1$ *is not realizable.*

4.B. **Near automorphism of trees: Neretin groups.** Let T_d be a regular tree of finite valency $d+1 \geq 3$, identified with its vertex set. Let $\widehat{\mathrm{Ner}}_d$ be the group of permutations of T_d that almost everywhere preserve the graph structure. It contains the group of finitely supported permutations: let Ner_d be the quotient $\widehat{\mathrm{Ner}}_d/\mathrm{fin}$, also called near automorphism group of T_d.

For the same reason as the previous example, the action of $\widehat{\mathrm{Ner}}_d$ on T_d induces a near action of Ner_d on T_d.

5. Definition of the near symmetric group and of near actions

5.A. **Definition.** Let X be a set. Define $\mathcal{I}^{\mathrm{cof}}(X)$ as the set of partial bijections f of X with cofinite (=finite complement) domain D_f and codomain D'_f. In other words, this is the set of subsets of X^2 both of whose projections on X are injective with cofinite image. This is naturally a monoid: composition is defined whenever possible, namely $g \circ f$ is defined on those $x \in D_f$ such that $f(x) \in D_g$, and the assigned value is $g(f(x))$.

For X nonempty, it is not a group, since for every cofinite subset X' of X, the identity of X' is an idempotent.

For $f, g \in \mathcal{I}^{\mathrm{cof}}(X)$, say $f \sim g$ if they coincide on a cofinite subset. Define $\mathfrak{S}^\star(X) = \mathcal{I}^{\mathrm{cof}}(X)/\sim$. Then the monoid structure passes to the quotient (in the language of semigroup $\sim$ is called a "semigroup congruence"), and furthermore this quotient monoid $\mathfrak{S}^\star(X)$ is a group, called **near symmetric group** of X; its elements are called **near permutations** of X.

For $f \in \mathcal{I}^{\mathrm{cof}}(X)$ defined as a bijection $X \smallsetminus F_1 \to X \smallsetminus F_2$, define $\phi(f) = |F_2| - |F_1|$. Then ϕ_X, called index map, is an additive homomorphism, and furthermore factors through a group homomorphism $\mathfrak{S}^\star(X) \to \mathbf{Z}$.

We have $\phi(f) = 0$ if and only if f can be extended to a permutation. This shows that the following sequence is exact:

$$1 \to \mathfrak{S}(X)/\mathrm{fin} \to \mathfrak{S}^\star(X) \xrightarrow{\phi_X} \mathbf{Z};$$

moreover the right-hand map is surjective as soon as X is infinite.

A **near action** of a group G on X is a homomorphism $\alpha : G \to \mathfrak{S}^\star(X)$. Its **index character** is $\phi_X \circ \alpha \in \mathrm{Hom}(G, \mathbf{Z})$. A near action is **balanced** if its index character is zero; this matches the initial definition.

5.B. **Alternative definition: Hilbert's hostel.** Introducing monoids as intermediate step to define groups may sound unusual but it is a natural approach in this context. Nevertheless, it is useful to be able to define $\mathfrak{S}^\star(X)$ while remaining in the realm of groups.

Hilbert's hostel is usually viewed as a way to convey to non-mathematicians the intuition of infinity. Rooms are indexed by positive integers. The hotel is full, but you need a room. Well, each person moves to the next room, and the first room is free![1]

The obstruction, given by the index, to realize a near permutation as a permutation is a finiteness phenomenon, which can be fixed in a similar fashion.

Let X be a set. Define $\mathrm{Hilb}(X)$ as the group of permutations f of the disjoint union $X \sqcup \mathbf{N}$ such that for some integer $m = m_f \in \mathbf{Z}$ and all large enough $n \in \mathbf{N}$, we have $f(n) = n - m_f$ (note that m is clearly unique, whence the notation). Then $\mathrm{Hilb}(X)$ is a subgroup of $\mathfrak{S}(X \sqcup \mathbf{N})$, and $f \mapsto m_f$ is a group homomorphism $\mathrm{Hilb}(X) \to \mathbf{Z}$, surjective as soon as X is infinite. Clearly, $\mathfrak{S}_{\mathrm{fin}}(X \sqcup \mathbf{N}) \subset \mathrm{Hilb}(X)$. Then one can define $\mathfrak{S}^{\star\prime}(X) = \mathrm{Hilb}(X)/\mathrm{fin}$, and factor $f \mapsto m_f$ through a homomorphism $\phi'_X : \mathfrak{S}^{\star\prime}(X) \to \mathbf{Z}$. We leave to the reader to check that there is a canonical isomorphism $\mathfrak{S}^\star(X) \to \mathfrak{S}^{\star\prime}(X)$, which intertwines the index homomorphisms ϕ_X and ϕ'_X.

5.C. **Isomorphisms between near G-sets.** It is important to clarify this, because there is not just one, but there are **two natural non-equivalent ways** of defining isomorphisms between near G-sets.

First, we recall the notion of isomorphism between G-sets given by actions $\alpha : G \to \mathfrak{S}(X)$, $\beta : G \to \mathfrak{S}(Y)$. An isomorphism between two G-sets means a G-equivariant bijection. In other words, this means a bijection $f \in \mathfrak{S}(X, Y)$ (set of bijections from X to Y) such that, denoting by f_* the induced isomorphism $\mathfrak{S}(X) \to \mathfrak{S}(Y)$ (given by $f_*(\sigma) = f \circ f^{-1}$), we have $\beta = f_* \circ \alpha$.

In the "near" setting, define $\mathcal{I}^{\mathrm{cof}}(X, Y)$ as the same of partial bijections between a cofinite subset of X and a cofinite subset of Y, and similarly define the index map $\mathcal{I}^{\mathrm{cof}}(X, Y) \to \mathbf{Z}$, and the cofinite coincidence relation $\sim$ on $\mathcal{I}^{\mathrm{cof}}(X, Y)$, and $\mathfrak{S}^\star(X, Y) = \mathcal{I}^{\mathrm{cof}}(X, Y)/\sim$ (so $\mathfrak{S}^\star(X, X) = \mathfrak{S}^\star(X)$). Then we have a well-defined composition map $\mathfrak{S}^\star(X, Y) \times \mathfrak{S}^\star(Y, Z) \to \mathfrak{S}^\star(X, Z)$, and every $f \in \mathfrak{S}^\star(X, Y)$ defines by conjugation an isomorphism $\mathfrak{S}^\star(X) \to \mathfrak{S}^\star(Y)$.

Given two near actions $\alpha : G \to \mathfrak{S}^\star(X)$ and $\beta : G \to \mathfrak{S}^\star(Y)$, a **near isomorphism** (or **isomorphism of near G-sets**) from the first to the second is $f \in \mathfrak{S}^\star(X, Y)$ such that $\beta = f_* * \alpha$. It is called **balanced** if f has index 0, i.e., can be represented by a bijection $X \to Y$. We say that the near G-sets X and Y are **near isomorphic** if there exists such an isomorphism, and **balanceably near isomorphic** if furthermore this isomorphism can be chosen to be balanced.

Remark 5.1. Near G-sets X, Y are near isomorphic if and only if there exist finite sets F, F' such that $X \sqcup F$ and $Y \sqcup F'$ are balanceably isomorphic. Moreover, one can arrange that either F or F' is empty.

[1]This the cheapest Hilbert hostel, and we stick with this one for budget reasons (and possibly some mathematical motivation, too). In fancier versions we do not consider here, you can ask people to go from room n to room $2n$ and thereby get infinitely many free rooms.

Note that if X is finite, it is a (balanced) near G-set in a unique way (since $\mathfrak{S}^\star(X)$ is then reduced to $\{1\}$). All finite near G-sets are near isomorphic. All finite balanced near G-sets of a given cardinal are balanceably near isomorphic.

Here are two infinite examples:

(a) let $C_2 = \langle a \rangle$ be a cyclic group of order 2. Let X be an infinite free C_2-set. Then X is balanceably isomorphic to X minus $2n$ points, but not to X minus $2n+1$ points.

(b) let D_∞ be the infinite dihedral group, with generators a, b of order 2. Then $D_\infty/\langle a \rangle$ and $D_\infty/\langle b \rangle$ are near isomorphic, but not balanceably (each one is balanceably isomorphic to the other minus a point). The non-balanced isomorphism fact actually follows from (a) by restricting to $\langle a \rangle$.

5.D. **Commensurated subsets of near actions, and completability.** Let $\alpha : G \to \mathfrak{S}^\star(X)$ be a near action. A subset M of X is G-**commensurated** if

$$\forall g \in G,\ \forall^\star x \in M,\ gx \in M.$$

Here, $\forall^\star$ means "for all with finitely many exceptions", and the finite set of exception may depend on g. Moreover gx is not defined. However, for each g, we can lift $\alpha(g)$ to a cofinite partial bijection of X, and thus gx makes sense for all but finitely many x, and two different choices coincide outside a finite subset, and hence the given condition does not depend on choices.

If M is a G-commensurated subset of a near G-set, then we obtain, by restriction, a near action of G on M, and being G-commensurated is precisely the condition for which this works. Hence, G-commensurated subsets can be viewed as the near G-subsets. Note that the analogue for G-sets are just invariant subsets, while commensurated subsets of G-sets implicitly refer to the underlying near action.

As in the setting of actions, we define $\mathrm{Comm}_G(X)$ as the set of G-commensurated subsets of X. It is saturated under $\sim$.

We say that X is 0-ended, 1-ended, finitely-ended if X is finite, resp. if X is infinite and $\mathrm{Comm}_G(X)$ consists of only finite and cofinite subsets, resp. if $\mathrm{Comm}_G(X)/\sim$ is finite. A near G-set is 1-ended if and only if it is infinite and not isomorphic to a disjoint union of two infinite near G-sets.

Being finitely-ended is a strong and convenient finiteness conditions for near G-sets. Indeed, it means that it can be decomposed into a disjoint union of 1-ended near G-sets, and this decomposition is unique up to $\sim$.

It is tempting to ask which near G-sets "come from actions". One interpretation gives rise to the notion of realizability: the near action is induced by an action on the same set. But from actions we obtain more near actions by passing to commensurated subsets:

Definition 5.2. A near G-set is **completable** if it is near isomorphic to a commensurated subset of a G-set. In other words, this means that there exists a near G-set Y such that $X \sqcup Y$ is realizable.

Let us illustrate this to the Houghton near action.

Proposition 5.3. *For every $n \geq 3$, the action of Houghton's group H_n (described in (d) of §1 and depicted as (z4) page 21 for $n = 3$) factors through a near action of $\mathbf{Z}^{n-1}$, which is completable but not realizable.*

Proof. We prove the case $n = 3$, since the general case is similar.

That it factors through a near action of $\mathbf{Z}^2$ is just because the commutator of the two generators has finite support (it is a transposition). That it is completable is seen as follows: we can split the near action as disjoint union of the three branches, and each of these branches can be extended to a 2-sided branch, which is a realizable action.

For a near G-set and $g \in G$, the set $X^{(g)}$ of points fixed by G depends on the choice of some lift of g as a cofinite-partial bijection, but is uniquely defined modulo $\sim$ (finite symmetric difference). In particular, if the near action is given with a realization, one representative is X^g, the set of fixed points for this realization.

If G is abelian, $X^{(g)}$ is G-commensurated and in particular, defines a near action of G that is well-defined up to near isomorphism; in particular its index character is well-defined. In case the near action is realizable, it is realizable as the sub-action X^g, and hence its index character is zero. Hence, if the index character of $X^{(g)}$ is nonzero for some g, then the near action is not realizable.

In the present case the set of fixed points of either of the two generators is the branch on which it "does not act", and the index character is clearly nonzero since the other generator acts with index $+1$ on this branch. $\square$

A similar argument also applies to the near action of $\mathbf{Z}^2$ (or $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$) depicted in (z3) page 21: it is completable and not realizable.

6. Near actions of finitely generated groups

Let G be a group and S a finite generating subset. Let $\alpha : G \to \mathfrak{S}^\star(X)$ be a near action. For each $s \in S$, choose a cofinite-partial bijection $\tilde{s}$ representing $\alpha(s)$, and join each x in its domain to $\tilde{s}x$ by an edge labeled by s. Then, for given S the resulting labeled graph is uniquely defined up to changing finitely many edges. For each choice, we call it a near Schreier graph of the near action.

In addition to this choice, we have the choice of S, which affects more than finitely many edges. Some properties of the resulting graph, for instance being connected, or having a single infinite connected component, are sensitive to the choices. Some others are independent of the choice, and thus reflect properties of the near action. One such property is having finitely many components.

Definition 6.1. A near action of a finitely generated group is **of finite type** if some/every near Schreier graph has finitely many connected components.

This is the main finiteness property for near actions of finitely generated groups. A simple verification, left to the reader, is that finitely-ended implies finite type.

The converse is not true: the left-action of an infinitely-ended group (such as the free group of rank 2) on itself is infinitely-ended, but of finite type.

The following is not hard, but termed theorem because of the role it plays.

Theorem 6.2. *Let G be a finitely generated group and X a near G-set. Suppose that either*

- *G is finitely presented, or*
- *X is a completable near G-set.*

Then there is a decomposition $X = Y \sqcup Z$ into commensurated subsets, such that Y is of finite type and Z is realizable.

Sketch. Start with $G = \mathbf{Z}^2 = \langle a, b \rangle$. Use its standard generating subset (including inverses) and draw a near Schreier graph. Say that a vertex $v \in X$ is bad if, for this choice of near Schreier graph,

- (1) some generator fails to be defined at v, or
- (2) $abv \neq bav$, or one is not defined.

There are finitely many bad vertices, and thus on any component without bad vertex, the near Schreier graph is actually Schreier graph of a $\mathbf{Z}^2$-action. Defining Z as the union of components without bad vertex and Y its complement, Y is of finite type, Z is realizable.

For an arbitrary finitely presented group G, one replaces (2) with a similar condition stating that every relator indeed acts trivially.

For a completable action of a finitely generated group, we embed it into an action and thus define the near Schreier graph by intersecting with the given subset. In this case, all relations between generators are satisfied since they are satisfied in the larger action. $\square$

Example 6.3. The decomposition of Theorem 6.2 fails for arbitrary near actions of arbitrary finitely generated groups. Consider the group $G = \mathfrak{S}_{\text{fin}}(\mathbf{Z}) \rtimes \mathbf{Z}$: it is generated by $t(n) = n + 1$ and s, transposition of 0 and 1, and under these generators has the infinite presentation

$$\langle t, s \mid s^2, (tst^{-1}s)^3, [t^n st^{-n}, s], n \geq 2 \rangle.$$

Consider the B.H. Neumann action depicted on (s5) 20 ,and described in (f) of §1), with generators t (blue) and s of order 2 (red). Then we see that for every $n \geq 2$, the relator $[t^n st^{-n}, s]$ acts trivially on the cycle X_m for all large m (namely as soon as X_m has length $\geq n + 2$), and hence has finite support. Hence, this action factors through a near action of G (that is $\star$-faithful, i.e., faithful as near action).

This near action of G is locally finite, in the sense that some/every near Schreier graph has only finite components. If it had a decomposition $Y \sqcup Z$ with Y finite type and Z realizable, then Y would also be locally finite and hence finite, thus realizable, and so X would be realizable as some action α'. Being locally finite, the realizing action would be locally finite. Since the near action of $\mathfrak{S}_{\text{fin}}(\mathbf{Z}) \rtimes \mathbf{Z}$ is faithful, so is α'. But having a faithful locally finite action implies being residually finite. Since G is not residually finite (since it has the alternating group on $\mathbf{Z}$ as a subgroup, which is infinite simple), this is a contradiction.

Given Theorem 6.2, this also shows that this near action of G is not completable.

7. Rigidity for 1-ended groups

Let G be a group; view it as G-set under left multiplication. If X is a G-set, it is an exercise to check that the G-equivariant maps $G \to X$ are precisely the maps $g \mapsto gx$, for $x \in X$.

Given G-sets X, Y, we say that map from X to Y is near G-equivariant if it satisfies:

$$\forall g \in G, \forall^\star x \in X : f(gx) = gf(x).$$

(Between near G-sets we should assume that f is finite-to-one for this to make sense.)

A group is always endowed with its own left action here. We say that G is 1-ended if it is a 1-ended G-set. This matches the usual definition for finitely generated groups (for arbitrary groups this has been introduced by Specker in 1950 but remains far less known).

Proposition 7.1. *Let G be a 1-ended group. Let X be a G-set. Then for every near G-equivariant map $X \to Y$, there exists $x \in X$ (unique) such that $\forall^\star g \in G$, $f(g) = gx$.*

Remark 7.2. The existence result is actually trivial (and void) when G is finite, but uniqueness fails: uniqueness is clear for G infinite. When G has ≥ 2 ends, it is easy to check that existence fails with $X = G$.

Proof. Define $u(g) = g^{-1}f(g)$. Then the condition $\forall g$, $\forall^\star h$ $f(gh) = gf(h)$ yields $\forall g$, $\forall^\star h$ $u(gh) = u(h)$. It follows that for every subset Y of X, $u^{-1}(Y)$ is G-commensurated.

If u has an infinite image, we can partition this image into two infinite subsets Y_1, Y_2, and then $u^{-1}(Y_1)$ and $u^{-1}(Y_2)$ are disjoint infinite G-commensurated subsets, contradiction with G being 1-ended.

So fibers of u form a finite partition of G into G-commensurated subsets, which are finite or cofinite, so exactly one fiber is cofinite: there exists $x \in X$ such that $\forall^\star g \in G$, $u(g) = x$. This means that $\forall^\star g \in G$, $f(g) = gx$. □

Theorem 7.3. *Let G be a 1-ended group. Then every finite perturbation of the left action of G on itself is conjugate, by a unique finitely supported permutation, to the original left action.*

Proof. The centralizer in $\mathfrak{S}(G)$ of G (acting on the left) is G (acting on the right) so contains no nontrivial finitely supported permutation. This proves the uniqueness part.

Apply Proposition 7.1 to the identity map from (G, λ) to (G, λ'), where λ is the left action and λ' is the perturbed action. So there exists $g_0 \in G$ such that $\forall^\star g \in G$, $g = \lambda'(g)g_0$. Hence the orbit $\lambda'(G)g_0$ is cofinite.

The map $g \mapsto \lambda'(g)g_0$ is G-equivariant from (G, λ) to (G, λ'), hence all its fibers have the same cardinal; since it coincides outside a finite subset with an injective map (the identity map) and G is infinite, we deduce that this common cardinal is 1: $g \mapsto \lambda'(g)g_0$ is injective. Its index is equal to the cardinal of $G \smallsetminus \lambda'(G)g_0$, but is also zero since the index is $\sim$-invariant and the identity map has index zero. Hence $g \mapsto \lambda'(g)g_0$ is a permutation of G.

So there exists a permutation s of G such that $\lambda'(g)(h) = \lambda_s(g)h$, where $\lambda_s(g)h = s^{-1}(gs(h))$ for all $g \in G$ and $h \in H$. For every $g \in G$, we have $\forall^\star h$, $gh = s(gs^{-1}(h))$. That is, $s(gh) = gs(h)$. Thus, by Proposition 7.1 applied to (G, λ) twice now, we see that s coincides with a right translation outside a finite subset. Hence, conjugating with a finitely supported permutation yields λ_s for some right translation s, but this just the same as λ. □

Corollary 7.4. *Let G be a 1-ended group. Then the left near action of G on $G \smallsetminus \{1\}$ is not realizable.*

Proof. To say that it is realizable would be the same as saying that the left action of G on itself can be realized as an action fixing 1. This is in contradiction with Theorem 7.3. □

Another application is a non-completability result.

Proposition 7.5. *Let G be a 1-ended group. Let X be a 1-ended completable near G-set. Let f be a finite-to-one near equivariant map $X \to G$. Then f is a near isomorphism.*

Proof. Write $Z = X \sqcup Y$ with Z a G-set. For $g \in G$, define $F_g = f^{-1}(\{g\}) \subset X \subset Z$. Then $g \mapsto F_g$ is a near equivariant map from G to the set of finite subsets of Z.

By Proposition 7.1, there exists a finite subset F of Z such that $\forall^\star g \in G$ (say $g \in G \smallsetminus T$), we have $f^{-1}(\{g\}) = gF$.

The gF, for $g \in G$, are pairwise disjoint: indeed, otherwise $gF \cap g'F$ is nonempty for some $g \neq g'$, and then we can choose $h \in G$ such that $hg, hg' \notin T$, and then $hgF \cap hg'F = f^{-1}(\{hg\}) \cap f^{-1}(\{hg'\})$ is nonempty, a contradiction.

In particular, G acts freely on $X' = \bigsqcup_{g \in G} gF$, which thus has $|F|$ ends. Also, X' has finite symmetric difference with $\bigsqcup_{g \in G} f^{-1}(\{g\}) = X$, which is 1-ended. Hence F is a singleton, and f is a near isomorphism. □

Corollary 7.6. *The near action of $\mathbf{Z}^2$ depicted in (z5) page 21 is not completable.*

Proof. Indeed, if X is the set of this near action, it is 1-ended and has a 2-to-1 near equivariant map onto $\mathbf{Z}^2$. Assuming it is completable, Proposition 7.5 leads to a contradiction. □

8. Kapoudjian 2-cocycle

The index character of a near action is what is inherited from the 1-cohomology with trivial coefficients of the group $\mathfrak{S}^\star(X)$. We can do something similar in degree 2.

Write $\mathfrak{S}(X)$ as extension with kernel fin and quotient $\mathfrak{S}(X)/\mathrm{fin}$. Modding out by the alternating group (the subgroup fin of even permutations), we deduce a central extension

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathfrak{S}(X)/\mathfrak{A}(X) \to \mathfrak{S}(X)/\mathrm{fin} \to 1.$$

Let $\omega^X \in H^2(\mathfrak{S}(X)/\mathrm{fin}, \mathbf{Z}/2\mathbf{Z})$ be the cohomology class of this extension (it is zero for X finite).

Let $\alpha : G \to \mathfrak{S}(X)/\mathrm{fin}$ be a balanced near action. Then the **Kapoudjian class** of the near action is $\alpha^*\omega^X \in H^2(G, \mathbf{Z}/2\mathbf{Z})$; in particular ω^X is the Kapoudjian class of the near action of $\mathfrak{S}(X)/\mathrm{fin}$ on X.

It is easy to check that central extension $\hat{G}$ of a group G with kernel $\langle c \rangle$ of order 2, is non-split if and only if c is a product of squares in $\tilde{G}$ (note that any commutator is a product of squares).

For X infinite, every element of $\mathfrak{S}(X)$ is a product of squares (Vitali, 1915) and thus $\omega^X \neq 0$.

The group $H^2(\mathbf{Z}^2, \mathbf{Z}/2\mathbf{Z})$ is cyclic of order 2. For a balanced near action of $\mathbf{Z}^2 = \langle a, b \rangle$, lift a, b as permutations $\tilde{a}$, $\tilde{b}$. Then the Kapoudjian class is zero or nonzero according to whether $[\tilde{a}, \tilde{b}]$ is even or odd.

Kapoudjian notably proved that the Kapoudjian class of the near action of the Neretin group Ner_d on T_d is nonzero for every $d \geq 2$.

Sergiescu checked that $H^2(\mathfrak{S}(X)/\mathrm{fin}, \mathbf{Z}/2\mathbf{Z})$ is reduced to $\{0, \omega^X\}$. He actually proved more generally that for X infinite, $H_2(\mathfrak{S}(X)/\mathrm{fin}, \mathbf{Z})$ is cyclic of order 2.

For non-balanced near actions, the Kapoudjian class can still be defined. Namely, we use the definition of $\mathfrak{S}^\star(X)$ using the Hilbert hostel. We get a central extension

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{Hilb}(X)/\mathfrak{A}(X \sqcup \mathbf{N}) \to \mathfrak{S}^\star(X) \to 1;$$

denote by $\omega^X \in H^2(\mathfrak{S}^\star(X), \mathbf{Z}/2\mathbf{Z})$ its cohomology class: then its restriction to $\mathfrak{S}(X)/\mathrm{fin}$ is indeed ω^X. Actually, $H^2(\mathfrak{S}^\star(X), \mathbf{Z}/2\mathbf{Z})$ is reduced to $\{0, \omega^X\}$. Given a near action $\alpha : G \to \mathfrak{S}^\star(X)$, its Kapoudjian class is again defined as $\alpha^*\omega^X$.

9. Almost/near automorphism groups of relational structures

Let I be a set, called index set, with a function $a : I \to \mathbf{N}$ called arity function. An (I, a)-**relational structure** on a set X is the data, for each $i \in I$, of a subset $P_i \subset X^{a(i)}$. The latter is encoded in the subset $P = \bigsqcup P_i \times \{i\} \subset \bigsqcup X^{a(i)} \times \{i\}$.

For instance, for $I = \{i\}$ a singleton and $a(i) = 2$, this encodes an oriented graph structure. Every incidence relation (non-oriented graph structure) can be encoded this way, just by defining P as the (symmetric) set of incident pairs.

For I arbitrary and $a = 2$, this encodes a colored oriented graph structure, in the sense that edges are labeled by elements of I.

For each n, the group $\mathfrak{S}(X)$ naturally acts on X^n (acting on each coordinate). For an (I, a)-relational structure P on A, one defines its **automorphism group**

$$\mathrm{Aut}(X, P) = \{\sigma \in \mathfrak{S}(X) : \sigma(P) = P\} = \{\sigma \in \mathfrak{S}(X) : \forall \in I,\ \sigma(P_i) = P_i\}.$$

An important basic theorem is that every closed subgroup of $\mathfrak{S}(X)$ is automorphism group of some relational structure (exercise: show it for X finite!).

From now on, for convenience let us suppose that P is **locally finite**, in the sense that each $x \in X$ occurs as coordinates of only finitely many elements of P. For instance in the graph setting, it means that each x belongs to only finitely many edges.

The **almost automorphism group** of the (I, a)-relational structure is defined as

$$\mathrm{AAut}(X, P) = \{\sigma \in \mathfrak{S}(X) : \sigma(P) \sim P\} \subset \mathfrak{S}(X).$$

Note that this means that $\sigma(P_i) = P_i$ for all but finitely many i, and $\sigma(P_i) \sim P_i$ for all i (recall that $\sim$ means having finite symmetric difference). We have $\mathfrak{S}_{\mathrm{fin}}(X) \subset \mathrm{AAut}(X, P)$.

If the relational structure encodes a non-oriented regular tree of valency $d+1$, then its almost automorphism group is by definition equal to the Neretin group $\widehat{\mathrm{Ner}}_d$.

For $n \geq 1$, let $T_{1,n}$ be a tree with one vertex of degree n, and all others having degree 2. Then the almost automorphism group of $T_{1,n}$ is known as Houghton's group H_n.

Recall that $\mathfrak{S}(X)$ is naturally endowed with the pointwise convergence topology, which is a group topology (for X countable, this makes it a Polish group).

The group $\mathrm{AAut}(X, P)$ can be endowed as a dense subgroup of $\mathfrak{S}(X)$; however it has a more interesting finer topology. Namely let $[P]$ be the set of those relational structures P' such that $P' \sim P$; by definition $\mathrm{AAut}(X, P)$ preserves $[P]$. We endow it with the topology endowed by its (closed) inclusion into $\mathfrak{S}(X) \times \mathfrak{S}([P])$. In particular, this makes $\mathrm{Aut}(X, P)$ an open subgroup of $\mathrm{AAut}(X, P)$.

Proposition 9.1 (W. Scott, Bergman-Shelah)**.** *Every closed subgroup of* $\mathfrak{S}(X)$ *contained in* $\mathfrak{S}_{\mathrm{fin}}(X)$ *is finite.*

Definition 9.2. We say that the relational structure P on X is **fillable** if $\mathrm{Aut}(X, P) \cap \mathrm{fin}$ is finite, or equivalently if it is closed in $\mathrm{Aut}(X, P)$ (and hence in $\mathrm{AAut}(X, P)$).

This means that there exists a finite subset F_0 such that for every finite subset $F \subset X \smallsetminus F_0$, every element of $\mathrm{Aut}(X, P)$ is determined by its restriction to $X \smallsetminus F$ (whence the terminology). When $\mathrm{Aut}(X, P) \cap \mathrm{fin}$ is trivial, we can choose F_0 to be empty.

For instance, $P = \emptyset$ on X infinite is not fillable. If P encodes a tree without (or with finitely many) vertices of valency 1, then P is fillable.

The group $\mathfrak{S}^\star(X)$ naturally acts on the power set $2^X/\sim$ modulo finite symmetric difference. This also induces an action on the set of locally finite (I, a)-relational structures modulo finite symmetric difference. We thus define

$$\mathrm{NAut}(X, P) = \{\sigma \in \mathfrak{S}^\star(X) : \sigma([P]) = [P]\} \subset \mathfrak{S}^\star(X).$$

The index map $\mathrm{NAut}(X, P) \to \mathbf{Z}$ is sometimes zero and sometimes not, and its image is sometimes a nonzero proper subgroup of $\mathbf{Z}$.

The group $\mathrm{NAut}(X, P)$ can be endowed with a natural topology, which makes $\mathrm{AAut}(X, P)/\mathrm{fin}$ an open subgroup, endowed with the quotient topology (which is thus Hausdorff if and only if P is fillable). When the index map is zero, it is just defined by taking the quotient topology. In general, the trick is to use the "Hilbert hostel": add an extra-element j to the index set I with $a_j = 2$, thus defining $I' = I \sqcup \{j\}$, and define $P'_i = P_i$ for $i \in I$ and $P_j = \{(n, n+1) : n \in \mathbf{N}\}$. This defines an (I', a)-relational structure on $X \sqcup I$, and we can identify $\mathrm{NAut}(X, P)$ to $\mathrm{AAut}(X \sqcup \mathbf{N}, P')/\mathrm{fin}$; the topology is thus the quotient topology.

10. Finitely generated abelian groups

Recall that for a finitely presented group, every near action is disjoint union of a realizable and a finite type one (Theorem 6.2). Therefore, the study essentially reduces to understanding finite type near actions.

We also use the following (from the problem session): every near action of a finite group is realizable.

Every group action has a unique decomposition as disjoint union of isotypic subset: two points are in the same isotypic subset if and only if their stabilizers are conjugate. For a finite group, this is a finite decomposition. Hence, for a near action of a finite group, this decomposition, which depends on the choice of a realization, is unique up to finite symmetric difference.

Hence if we consider a near action of $G = H \times L$, for some finite group H and group L, then the isotypic decomposition under the near action of H is commensurated by G. This allows to reduce to the case of near actions that are isotypic under H.

In the case of $G = H \times \mathbf{Z}$ with H finite abelian, given a near G-set of finite type on which H acts isotypically, say with stabilizer L, it is not hard to decompose it into a finite disjoint union of copies of the 1-ended near G-sets $(H/L) \times \mathbf{N}$ and $(H/L) \times (-\mathbf{N})$. From all this it follows in particular that every near G-set is completable.

This handles the case when G is finitely generated abelian of $\mathbf{Q}$-rank ≤ 1. Next, new phenomena appear, and the most interesting ones appear for $\mathbf{Z}^2$. Let us start with a general result:

Theorem 10.1. *Let G be a finitely generated abelian group. Then every near G-set of finite type is finitely-ended.*

The proof is complicated and involves combinatorial work on the near Schreier graph. Note that the same result for completable near actions would just follow from the fact that transitive Schreier graphs of G are ≤ 2-ended, but it is not clear that the general case follows.

The theorem reduces the classification of near G-sets to 1-ended ones. We call a near action $\alpha : G \to \mathfrak{S}^\star(X)$ is injective.

Theorem 10.2. *Let G be a finitely generated abelian group of $\mathbf{Q}$-rank ≥ 3 (i.e., $\mathbf{Z}^d \times H$ with $d \geq 3$ and H finite abelian). Then every 1-ended, $\star$-faithful near G-set is near isomorphic to the simply transitive action of G on itself.*

This result is the easiest to state, but is based on the case of rank 2, where all most remarkable phenomena happen. We just state it in the case of $\mathbf{Z}^2$. Define $X_{n,v}$, for $n \in \mathbf{N}_{\geq 1}$ and $v \in \mathbf{Z}^2$ as follows.

$X_{n,0}$ is obtained as a connected n-fold covering of the standard Cayley graph. $X_{n,v}$ is obtained similarly, but using a shifting of the gluing by v.

Theorem 10.3. *The 1-ended, $\star$-faithful near $\mathbf{Z}^2$-set are, up to isomorphism, precisely the $X_{n,v}$, for $n \in \mathbf{N}_{\geq 1}$ and $v \in \mathbf{Z}^2$. Among those, only $X_{n,0}$ (the simply connected action) is completable.*

11. Problem session

Near actions: problem session (April 18, 2019) Exercises can be made independently.

1/ **Finite groups.** Show that every near action of a finite group is realizable. Deduce that the same holds for any free product of finite groups.

2/ **Using transpositions.**

a) (Vitali 1915) Let X be infinite. Show that every transposition is a commutator in the infinite symmetric group $\mathfrak{S}(X)$.

b) Deduce that the surjective homomorphism $\mathfrak{S}(X) \to \mathfrak{S}(X)/\mathrm{fin}$ is not split. (Hint: argue by contradiction and mod out by the alternating subgroup.)

Note: whether it splits was asked by W. Scott in 1956 and negatively answered by himself in 1964, using a completely different (and slightly more complicated) approach, of independent interest however.

3/ **Stable realizability.** (Fix a group.) A near action X is stably realizable (resp. finitely stably realizable) if there exists a set (resp. finite set) F with trivial near action such that $X \sqcup F$ is realizable.

(Thus realizable $\Rightarrow$ finitely stably realizable $\Rightarrow$ stably realizable$\Rightarrow$ completable.)

a) Show that X is finitely stably realizable iff it is isomorphic (as near action) to a realizable near action.

b) Suppose that G is finitely generated. Show that stably realizable implies finitely stably realizable.

4/ **Near d-regular trees.** Fix $d \geq 2$. Call near d-tree an infinite simple graph, of finite valency, finitely many components, such that all but finitely many vertices have valency d. Say that two simple graphs of finite valency are near isomorphic if there exists a bijection between cofinite subsets that is a graph isomorphism.

For a near d-tree T, denoting by $\delta(v)$ the valency of a vertex v, define

$$\theta(T) = \sum_{v \in T} (\delta(v) - d) + 2\#\pi_0(T).$$

Show that θ modulo $(d-2)\mathbf{Z}$ is a near isomorphism invariant. Deduce an explicit bijection between the class of near d-trees up to isomorphism, and $\mathbf{Z}/(d-2)\mathbf{Z}$.

(Near automorphism groups of such graphs are known as Higman-Thompson groups.)

5/ **Prüfer group $\mathbf{Z}[1/p]/\mathbf{Z}$ and p-adic invariant; application to Higman-Thompson groups.** Let G be a group. A near G-action is near free if every $g \in G \smallsetminus \{1\}$ has finitely many fixed points.

a) Let F be a finite group. For a near free near action on a set X, choose a realization. Check that the number of non-free points (points with non-trivial stabilizer) is finite and modulo $|F|$, does not depend on the choice of realization.

b) Let G be a locally finite group and X a near free near G-set. For every finite subgroup F of G, let $\nu_F(X)$ be the previous number (belonging to $\mathbf{Z}/|F|\mathbf{Z}$). Define $\nu(X)$ as the resulting element of the projective limit $\varprojlim \mathbf{Z}/|F|\mathbf{Z}$.

Check that $\nu(X)$ is well-defined, and is a balanced isomorphism invariant of X, and is additive under taking disjoint unions of near free near G-sets.

c) For $G = \mathbf{Z}[1/p]/\mathbf{Z}$, $\nu(X)$ belongs to the group of p-adics $\mathbf{Z}_p$. Show that if X is a near free near G-set and is realizable (resp. finitely stably realizable) then $\nu(X)$ belongs to $\mathbf{N} = \{0, 1, \dots\}$ (resp. to $\mathbf{Z}$). Deduce that the near action of G on $G \smallsetminus \{0\}$ is not realizable.

d) Application: show that for $d \geq 2$ the near action of Thompson's group V_d on the regular rooted d-tree (in which every vertex has d successors) is not realizable for $d \geq 2$, and not (finitely) stably realizable for $d \geq 3$.

6/ **Realizability for amalgams of finite groups, $\mathrm{SL}_2(\mathbf{Z})$.**

a) Let $F_1 \supseteq F \subseteq F_2$ be finite groups. Suppose that every action of F on any finite set extends to F_1. Show that every near action of the amalgam $G = F_1 *_F F_2$ is realizable.

Hint: choose realizations α_1, α_2 for each of F_1, F_2 (using the result of Exercise 1). Consider K as the set of x such that $\alpha_1(g)x \neq \alpha_2(g)x$ for some $g \in F$, and define Y as the smallest $\alpha_1(F_1)$-invariant subset.

b) Deduce that every near action of $\mathrm{SL}_2(\mathbf{Z}) \simeq C_6 *_{C_2} C_4$ is realizable.

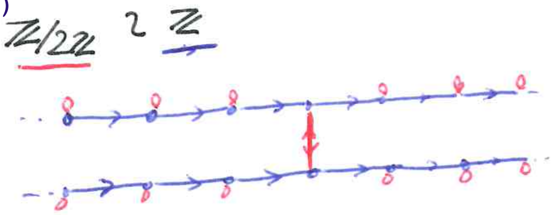
c) We now consider a prime p, $n \geq 1$ divisible by p, and a near action of the amalgam

$$G = C_{np} *_{C_p} C_{p^2} = \langle t, u : t^{pn} = u^{p^2} = 1, t^n = u^p \rangle$$

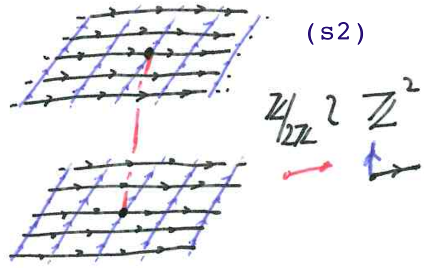
on a set X. Choose realizations as above. For Y a finite $\alpha_1(t^n)$-invariant subset, define c_Y as the number of p-cycles in Y. Show that Y is a large enough $\alpha_2(u)$-invariant subset, then the value modulo p of c_Y does not depend on the choice of Y. Write $s(X) \in \mathbf{Z}/p\mathbf{Z}$ this value. Show that $s(X) \in \mathbf{Z}/p\mathbf{Z}$ is an isomorphism invariant of the near G-set X. Show that X is realizable if and only if $s(X) = 0$, if and only if it is realizable. Check that the disjoint union of p copies of X is always realizable.

d) Check that every element of $\mathbf{Z}/p\mathbf{Z}$ is achieved as $s(X)$ for some near G-set X.

(s1)



(s2)



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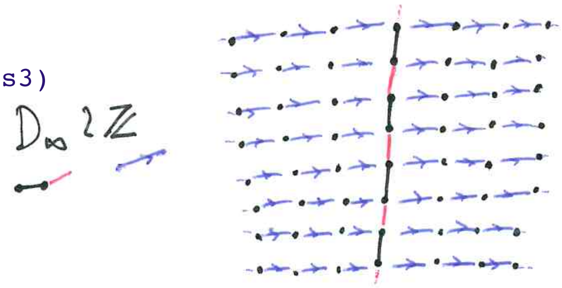

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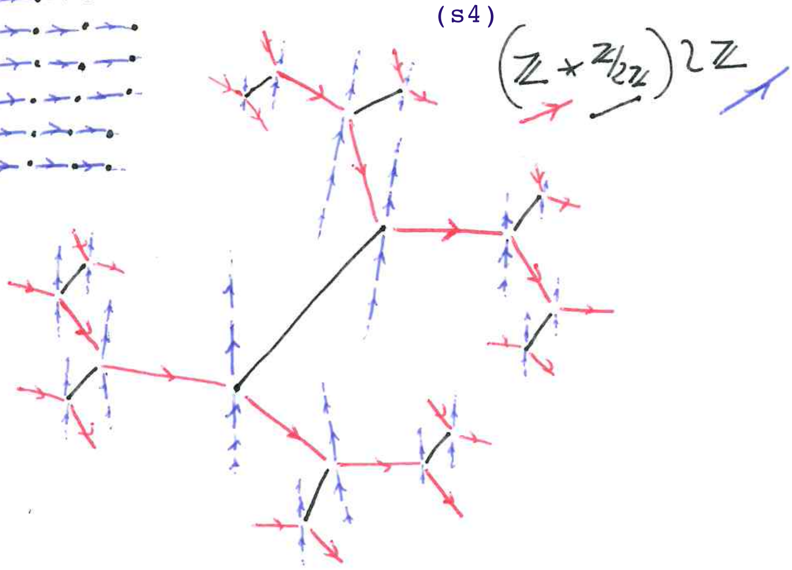
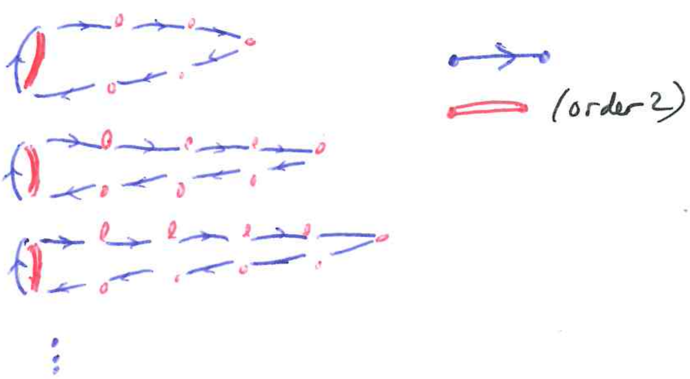


(s5)

B.H. Neumann (1937):



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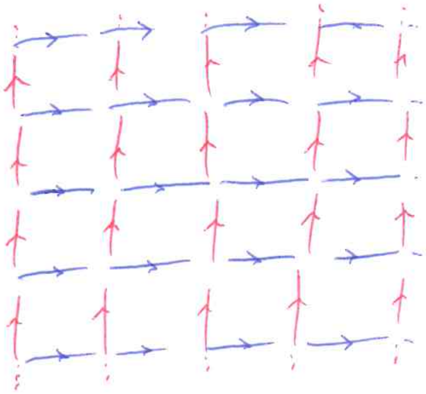
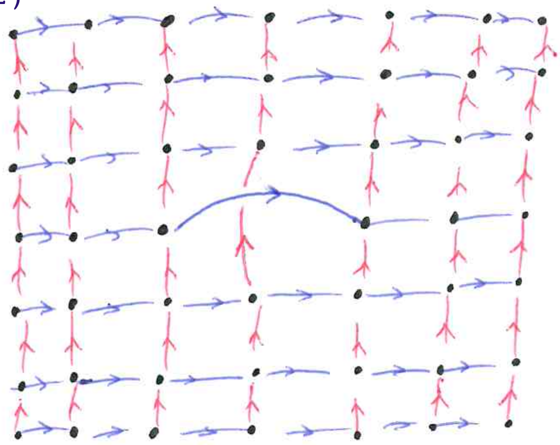
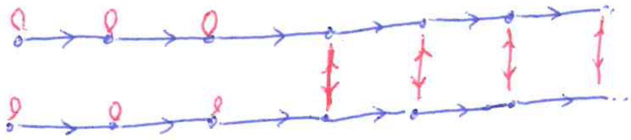


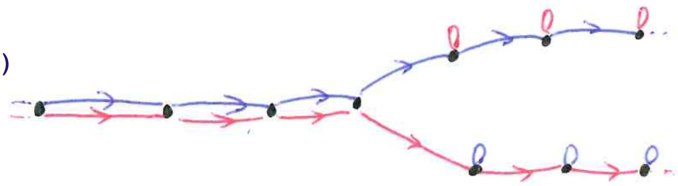
(z2)



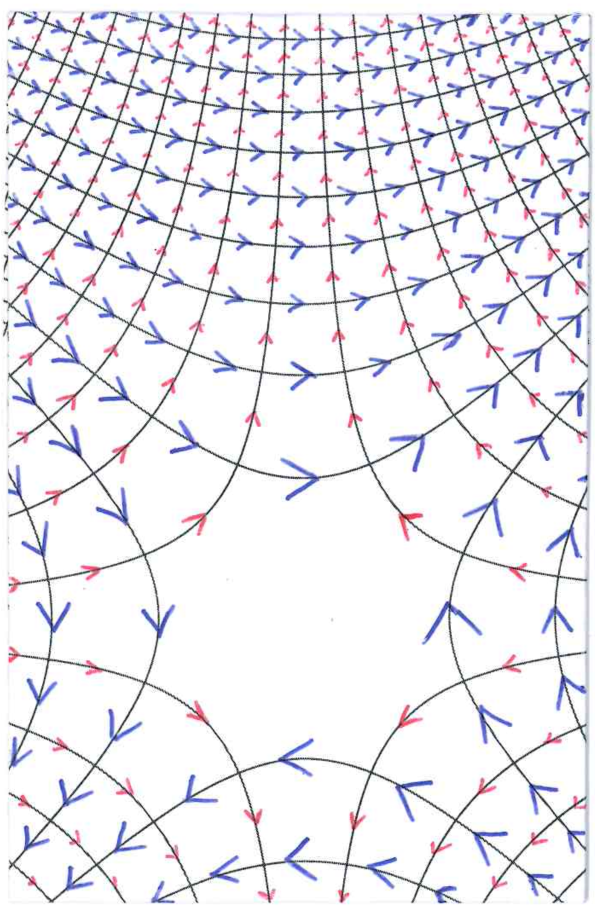
(z3)



(z4)



(z5)



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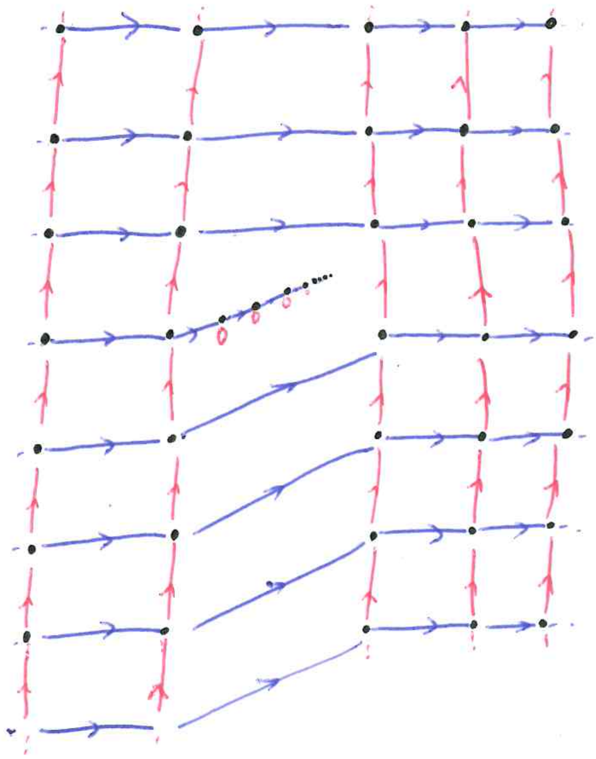

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