ZOOMING IN ON THE LARGE-SCALE GEOMETRY OF LOCALLY COMPACT GROUPS

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1. INTRODUCTION

The purpose of this survey is to describe how locally compact groups can be studied as geometric objects. We will emphasize the main ideas and skip or just sketch most proofs, often referring the reader to to our much more detailed book [CH–14].

It is now classical to view any finitely generated group G as a geometric object. If S is a finite generating set, the Cayley graph $\Gamma(G, S)$ is the graph whose vertex set is G, and whose edges are the pairs (g, gs) where (g, s) ranges over $G \times S$. Note that this can be viewed as an oriented and labeled graph, possibly with self-loops (if $1 \in S$), but this does not play any role in the sequel. Since S generates G, this graph is connected, and therefore there is a well-defined metric on the vertex set G, for which the distance $d_S(g, h)$ between $g, h \in G$ is the number $\ell_S(g^{-1}h)$ defined as the smallest k such that $g^{-1}h$ can be written as a product of k elements in $S^{\pm 1}$. This metric, called the **word metric** (with respect to S), enjoys the following properties:

(1) it is left-invariant, i.e. the left action of G on itself is by isometries;
(2) it is proper, in the sense that bounded subsets are finite;
(3) it satisfies the following geodesicity property: for all integers n and $g, h \in G$ such that $d_S(g, h) = n$, there exist $g_0, g_1, \ldots, g_n \in G$ such that $d_S(g_{i-1}, g_i) = 1$ for all $i = 1, \ldots, n$ and $(g_0, g_n) = (g, h)$.

The main drawback of this metric is that it depends on the choice of a finite generating set S; in particular, a metric property of this metric need not be intrinsic to G. Nevertheless, if S' is another finite generating set, an easy induction shows that, for some constants $c, c' > 0$, we have

$$cd_{S'} \leq d_S \leq c'd_{S'}.$$

In other words, the identity $(G, d_S) \to (G, d_{S'})$ is a **bilipschitz map**.

Word metrics on finitely generated groups have proved useful on several occasions, for example in [Dehn–11], [Švar–55, Miln–68], and [Grom–81, Grom–84, Grom–93]. It is natural to wonder how this concept generalizes to a broader setting.

Let us first discuss a generalization to discrete groups, beyond finitely generated ones. If we consider the word metric with respect to a generating subset S, then obviously S is bounded; if we require the properness of the metric, S is necessarily finite, so, in a sense, the word metric is only suitable for finitely generated groups.

Beyond the case of word metrics, it is natural to consider left-invariant proper metrics. For instance, for a group G with a finite generating set S and a subgroup $H \subset G$, the restriction of d_S to H is a left-invariant proper metric on H. This is not, in general, a word metric on H. Actually, by a theorem of Higman-Neumann-Neumann [HiNN–49], every countable group is isomorphic to a subgroup of a finitely generated group, and thus admits a left-invariant proper metric by the above construction. Conversely, it is clear that the existence of a proper metric implies the countability of the group. The uniqueness up to bilipschitz maps fails for infinite groups, since when d is a proper left-invariant metric on G then so is $\sqrt{d}$. However, if G is a discrete group with two proper left-invariant metrics d, d', there exist non-decreasing functions Φ_-, Φ_+ from the set of nonnegative numbers to itself, tending to $+\infty$ at $+\infty$, such that $\Phi_- \circ d' \leq d \leq \Phi_+ \circ d'$. This is interpreted by saying that the identity $(G, d) \to (G, d')$ is a **coarse equivalence**.

A further generalization is to consider topological groups, especially locally compact groups. Given a topological group G, we consider metrics (or pseudo-metrics); we *do not* consider the topology defined by these metrics and only refer to the given topology on G. It is natural to require that compact subsets are bounded (noting that this is automatic when the metric or pseudo-metric is continuous). The properness assumption is that bounded subsets have a compact closure. A convenient setting is to assume the topological group to be locally compact (and in particular, Hausdorff, by definition); note that this includes discrete groups as an important particular case.

To avoid local topological issues, we allow pseudo-metrics.

In §2, we introduce some general metric notions, including coarsely Lipschitz maps and coarse equivalences. In §3, we define the coarse language in the context of locally compact groups, and we characterize σ-compact locally compact groups in a metric way. In §4, we introduce the coarse and large-scale geodesic notions, which allow to characterize compactly generated locally compact groups in a metric way. In §5, we introduce coarsely ultrametric spaces, which provide a coarse characterization of locally elliptic locally compact groups, which generalize locally finite groups from the discrete setting. In §6, we introduce the notion of coarse properness for metric spaces, which allows to define in a coarse setting the notion of growth and of amenability. In §7, we introduce coarsely simply connected metric spaces and use them to characterize metrically compactly presented groups, which generalize finitely presented groups in the setting of locally compact groups. In the last two sections, we illustrate compact presentability: in §8, we describe the Bieri-Strebel Theorem, which provides constraints for surjective homomorphisms of compactly presented locally groups onto $\mathbf{Z}$, and §9 provides further examples.

2. Metric categories

We denote by $\mathbf{R}_+$ the set of nonnegative real numbers. The standard metric d is defined on $\mathbf{R}_+$ by $d(x, y) = |y - x|$.

The objects we will consider are pseudo-metric spaces, that is, pairs (X, d) where d is a symmetric function $X \times X \to \mathbf{R}_+$, satisfying the triangle inequality. By a common abuse of notation, a pair (X, d) will often be identified with the underlying set X.

Definition 2.1. A map $f : X \to Y$ between pseudo-metric spaces is

- **coarsely Lipschitz** if there exists a nondecreasing map $\Phi_+ : \mathbf{R}_+ \to \mathbf{R}_+$ such that $d(f(x), f(x')) \leq \Phi_+(d(x, x'))$ for all $x, x' \in X$; we say that f is Φ_+-coarse;
- **large-scale Lipschitz** if it is Φ_+-coarse for some affine function Φ_+.

Two maps $f, f' : X \to Y$ are **close**, written $f \sim f'$, if $\sup_{x \in X} d(f(x), f'(x)) < \infty$. Equivalence classes of this equivalence relation are called **closeness classes**.

For instance, the map $f_a : \mathbf{R}_+ \to \mathbf{R}_+$ mapping x to x^a $(a > 0)$ is coarse if and only if it is large-scale Lipschitz, if and only if $a \leq 1$. If (X, d) is an arbitrary unbounded metric space, then the identity map $(X, d) \to (X, \sqrt{d})$ is large-scale Lipschitz (hence coarse), while its inverse is coarse but not large-scale Lipschitz.

It is clear that, if f is coarse (respectively large-scale Lipschitz) and $f \sim f'$, then f' satisfies the same property.

Definition 2.2. The **metric coarse category** (resp. **large-scale category**) is the category whose objects are pseudo-metric spaces and arrows are closeness classes of coarsely Lipschitz maps (resp. of large-scale Lipschitz maps).

Definition 2.3. Let $f : X \to Y$ be a map between pseudo-metric spaces.
- The map f is **essentially surjective** if $\sup_{y \in Y} d(y, f(X)) < \infty$.
- The map f is **coarsely expansive** if there exists a non-decreasing function[1] $\Phi_- : \mathbf{R}_+ \to \mathbf{R}_+$ tending to infinity at infinity, such that $d(f(x), f(x')) \geq \Phi_-(d(x, x'))$ for all $x, x' \in X$; we say that f is Φ_--coarsely expansive.
- The map f is **large-scale expansive** if it is Φ-coarsely expansive for some affine function Φ.
- The map f is a **coarse equivalence** if it is coarse, coarsely expansive and essentially surjective.
- The map f is a **quasi-isometry** if it is large-scale Lipschitz, large-scale expansive and essentially surjective.
- Two metric spaces X, Y are **coarsely equivalent** (resp. **quasi-isometric**) if there exists a coarse equivalence (resp. quasi-isometry) $X \to Y$.

Proposition 2.4. *Let $f : X \to Y$ be a map between pseudo-metric spaces.*
  (1) f induces an isomorphism in the metric coarse category if and only if f is a coarse equivalence;
  (2) f induces an isomorphism in the large-scale category if and only if f is a quasi-isometry.

In particular, to be coarsely equivalent (resp. quasi-isometric) is an equivalence relation between pseudo-metric spaces.

Example 2.5.   (1) Let X be a pseudo-metric space and let $\widehat{X}$ be its Hausdorffization, namely the metric space obtained from X by identifying points at distance 0. Then the canonical projection $X \to \widehat{X}$ is a quasi-isometry (and hence a coarse equivalence). Thus, in the metric coarse category and in the large-scale category, the full subcategories where objects are metric spaces are essential.
  (2) Let X be a metric space and $\overline{X}$ its completion. Then the canonical injection $X \to \overline{X}$ is a quasi-isometry (it is indeed an isometry onto a dense subset).

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[1]One could equally consider functions $\Phi_- : \mathbf{R}_+ \to \mathbf{R}_+ \cup \{\infty\}$. This would not change the theory.

(3) Let X be a pseudo-metric space and let $Y \subset X$ be a subset maximal for the property that any two points in Y have distance ≥ 1. Then the isometric injection $Y \subset X$ is essentially surjective and thus is a quasi-isometry. Thus every metric space is quasi-isometric to a discrete one.

(4) To be bounded is invariant under coarse equivalence, and all non empty bounded pseudo-metric spaces are quasi-isometric.

3. Coarse category of locally compact groups

3.1. The abstract coarse category.

Definition 3.1. Let $f : G \to H$ be a map between locally compact groups (not necessarily a homomorphism or continuous). We say that f is a **coarse map** if, for every compact subset $K \subset G$, there exists a compact subset $L \subset H$ such that, for all $g, g' \in G$, we have $g^{-1}g' \in K \Rightarrow f(g)^{-1}f(g') \in L$.

Let $f' : G \to H$ be another map. We say that f and f' are **close** if the set $\{h \in H : h = f(g)^{-1}f'(g) \text{ for some } g \in G\}$ has a compact closure. Equivalence classes of this equivalence relation are called **closeness classes**.

For instance, any continuous homomorphism between locally compact groups is a coarse map. If we have a semidirect product of locally compact groups $G = H \rtimes K$ with K compact, then the mapping $hk \mapsto h$, for $(h, k) \in H \times K$, is close to the identity of G (but is in general not a homomorphism).

Definition 3.2. The **coarse category of locally compact groups** is the category in which objects are locally compact groups and arrows are closeness classes of coarse maps.

Definition 3.3. If G, H are locally compact groups, a map $f : G \to H$ is **essentially surjective** if there exists a compact subset $L \subset H$ such that $H = f(G)L := \{f(g)l : g \in G, l \in L\}$.

The map $f : G \to H$ is **coarsely expansive** if for every compact subset $L \subset H$ there exists a compact subset $K \subset G$ such that for all $g, g' \in G$ we have $g^{-1}g' \notin K \Rightarrow f(g)^{-1}f(g') \notin L$.

When locally compact groups are treated as metric objects, this terminology could be in conflict with the metric notions of §2; nevertheless Proposition 3.9 will show that these are equivalent notions.

Proposition 3.4. *A map $f : G \to H$ between locally compact groups induces an isomorphism in the coarse category if and only if f is a coarse map, is coarsely expansive, and is essentially surjective.*

An important example is the case of continuous homomorphisms.

Proposition 3.5. *Let $f : G \to H$ be a continuous homomorphism of locally compact groups. Then f is a coarse map. It is coarsely expansive if and only if it is proper. It is essentially surjective if and only if $H/\overline{f(G)}$ is compact. In particular, it induces an isomorphism in the coarse category if and only it is proper and has a cocompact image.*

Recall that a continuous map between Hausdorff topological spaces is by definition proper if the inverse image of every compact subset is compact. If a continuous homomorphism $G \to H$ is proper, then it has a compact kernel and a closed image. The converse is true if G is σ-compact.

3.2. **Locally compact groups as pseudo-metric spaces.** The abstract coarse theory can be expressed using the language of pseudo-metric spaces, under a minor assumption on the locally compact groups, namely when they are σ-**compact**, that is, are countable unions of compact subsets.

Definition 3.6. Let G be a locally compact group. An **adapted pseudo-metric** on G is a pseudo-metric which is

- left-invariant,
- locally bounded (compact subsets are bounded);
- proper (bounded subsets have a compact closure).

Theorem 3.7. *A locally compact group admits an adapted pseudo-metric if and only if it is σ-compact (i.e. is a countable union of compact subsets).*

Proof (sketch). One direction is clear. Conversely, suppose that G is σ-compact and write $G = \bigcup K_n$ with K_n a compact subset, contained in the interior of K_{n+1}. Consider the metric graph with G as set of vertices and an edge (g, gs) of length n for all n and every $(g, s) \in G \times K_n$. Then this graph is connected (since there is at least one edge between any two vertices), the graph metric on the set G of vertices is left-invariant, and each compact subset of G is bounded, being contained in some K_n. Moreover, bounded subsets have a compact closure: indeed, for every $n \geq 1$ the n-ball around 1 is contained in the union of the $K_{n_1} \ldots K_{n_k}$, where $k \geq 1$ and $(n_1, \ldots, n_k)$ ranges over the k-tuples of positive integers with sum n. Thus G admits an adapted pseudo-metric (indeed a metric, since any two distinct points are at distance ≥ 1). $\square$

Proposition 3.8. *Let G be a locally compact group. For any two adapted pseudo-metrics d, d' on G, the identity map of pseudo-metric spaces $(G, d) \to (G, d')$ is a coarse equivalence.*

If G is a σ-compact locally compact group, it admits an adapted pseudo-metric d by Theorem 3.7, and this allows to view (G, d) as a well-defined object in the metric coarse category.

The following proposition shows that, for σ-compact locally compact groups, on which the definitions of §2 and §3.1 both make sense, the definitions are consistent.

Proposition 3.9. *If (G, d) and (G', d') are locally compact groups with adapted pseudo-metrics, a map $f : G \to G'$ is a coarse map of locally compact groups (in the sense of §3.1) if and only if is a coarsely Lipschitz map of pseudo-metric spaces (in the sense of §2). The same holds for coarsely expansive maps, essentially surjective maps, coarse equivalences, and closeness.*

Also, being σ-compact is a coarse invariant among locally compact groups.

Proposition 3.10. *If G and H are coarsely equivalent locally compact groups and G is σ-compact then so is H.*

4. Geodesic metric notions and compactly generated groups

4.1. **Coarse connectedness and geodesic notions.** We turn back to the metric setting, with in the scope the goal of characterizing, in the coarse setting, compactly generated locally compact groups among σ-compact locally compact groups.

Definition 4.1. A pseudo-metric space X is **coarsely connected** if there exists $c > 0$ such that the equivalence relation generated by "being at distance $\leq c$" identifies all points in X. That is, for any two points $x, y \in X$, there exist n and $x = x_0, x_1, \ldots, x_n = y$ in X with $\sup_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq c$.

The pseudo-metric space X is **coarsely geodesic** if there exists a nondecreasing function $\Phi : \mathbf{R}_+ \to \mathbf{R}$ and $c > 0$ such that, for any two points $x, y \in X$, there exist $n \leq \Phi(d(x, y))$ and $x = x_0, x_1, \ldots, x_n = y$ in X with $\sup_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq c$. It is **large-scale geodesic** if the above (Φ, c) can be chosen with Φ an affine function.

A basic observation is that being coarsely connected or coarsely geodesic are coarse invariants; for instance, if there is a coarse equivalence between a coarsely geodesic pseudo-metric space and another pseudo-metric space, then the latter is coarsely geodesic as well.

Similarly, being large-scale geodesic is a quasi-isometry invariant. However, it is not a coarse invariant: if (X, d) is an unbounded large-scale geodesic metric space, then it is coarsely equivalent to $(X, \sqrt{d})$, but the latter is not large-scale geodesic. It can actually be checked that a pseudo-metric space is coarsely geodesic if and only if it is coarsely equivalent to a large-scale geodesic metric space.

We saw in §2 examples of coarsely Lipschitz maps that are not large-scale Lipschitz. Nonetheless, we have the following useful proposition.

Theorem 4.2. *Let $f : X \to Y$ be a map between pseudo-metric spaces. Assume that X is large-scale geodesic and that f is a coarsely Lipschitz map. Then f is large-scale Lipschitz.*

Corollary 4.3. *Let $f : X \to Y$ be a coarse equivalence between large-scale geodesic pseudo-metric spaces. Then f is a quasi-isometry.*

4.2. **Compactly generated locally compact groups.** By definition, a locally compact group is compactly generated if it is generated, as a group, by a compact subset. For instance, for a discrete group it means being finitely generated.

Theorem 4.4. *Let G be a σ-compact locally compact group and d an adapted pseudo-metric on G. Then G is compactly generated if and only if (G, d) is coarsely geodesic, if and only if it is coarsely connected.*

Moreover, when this holds, there exists and adapted pseudo-metric d' on G such that (G, d') is large-scale geodesic.

Sketch of proof. If (G, d) is coarsely connected and c is the constant given in Definition 4.1, then a simple verification shows that the c-ball centred at 1 in G has a compact closure, and generates G.

Conversely, if G is compactly generated, then the word metric d' with respect to a given compactly generated subset is adapted and (G, d') is coarsely geodesic. Since it is coarsely equivalent to (G, d), by Proposition 3.8, and since being coarsely geodesic is a coarse invariant, we deduce that (G, d) is coarsely geodesic as well. $\square$

Combining this with Proposition 3.5, we obtain a geometric proof of the following corollary.

Corollary 4.5. *Let $f : G \to H$ be a continuous proper homomorphism with cocompact image between locally compact groups. Then G is compactly generated if and only if H is compactly generated.*

Also, with Corollary 4.3 we obtain

Corollary 4.6. *Between compactly generated locally compact groups, every coarse equivalence is a quasi-isometry.*

In particular, the classification of σ-compact locally compact groups up to coarse equivalence extends the classification of compactly generated locally compact groups up to quasi-isometry.

Definition 4.7. A pseudo-metric d on a compactly generated locally compact group is **geodesically adapted** if it is equivalent to the word length d' with respect to some/any compact generating subset, in the sense that the identity map $(G, d) \to (G, d')$ is a quasi-isometry.

Analogously with Proposition 3.8, we have:

Proposition 4.8. *Let G be a compactly generated locally compact group. For any two geodesically adapted pseudo-metrics d, d' on G, the identity map of pseudo-metric spaces $(G, d) \to (G, d')$ is a quasi-isometry.*

Example 4.9. If G is a connected Lie group, we have two natural families of geodesically adapted pseudo-metrics:

- the metrics associated to left-invariant Riemannian metrics on G;
- the words metric associated to compact generating subsets of G (observe that by compactness, any compact subset with non empty interior generates G).

Then the identity map of G for any two of these metrics is a quasi-isometry, by Proposition 4.8.

5. Coarsely ultrametric spaces and locally elliptic locally compact groups

Definition 5.1. A pseudo-metric space is **coarsely ultrametric** if, for every $r \geq 0$, the equivalence relation generated by the relation "being at distance $\leq r$" has orbits of bounded diameter.

This is a coarse invariant. Indeed, a simple verification shows the following:

Proposition 5.2. *A pseudo-metric space is coarsely ultrametric if and only if it is coarsely equivalent to an ultrametric space.*

Note that an immediate consequence of the definition is that, if a pseudo-metric space is both coarsely ultrametric and coarsely geodesic, then it is bounded. More generally, every coarsely Lipschitz map from a coarsely geodesic pseudo-metric space to a coarsely ultrametric pseudo-metric space has a bounded image.

Definition 5.3. A locally compact group is **locally elliptic** if every compact subset is contained in a compact subgroup.

Note that such a locally compact group has a compact identity component. Discrete locally elliptic locally compact groups are better known as *locally finite groups*.

Proposition 5.4. *If G is a σ-compact locally compact group and d an adapted pseudo-metric, then G is locally elliptic if and only if (G, d) is coarsely ultrametric.*

Among locally compact groups, the class of locally elliptic groups is closed under coarse equivalence.

The verifications of the first claim is straightforward. In the σ-cmpact case, the second claim folllows from the first one.

6. Coarse properness, growth, and amenability

6.1. The metric notions.

Definition 6.1. The **uniform growth function** of a quasi-metric space (X, d) is the function mapping $r \geq 0$ to the supremum $b_X(r)$ of the cardinalities of all subsets of diameter $\leq r$. A quasi-metric space is **uniformly locally finite (ULF)** if the function $b_X(\cdot)$ takes finite values.

Among non-decreasing functions $\mathbf{R}_+ \to \mathbf{R}_+$, write $f \preceq g$ if there exist constants $c, c', c'' > 0$ such that $f(r) \leq cg(c'r) + c''$ for all $r > 0$. Say that f and g are **asymptotically equivalent**, written $f \simeq g$, if $f \preceq g \preceq f$.

Lemma 6.2. *If two ULF metric spaces are quasi-isometric, they have asymptotically equivalent growth functions.*

This allows to extend the notion of growth (up to asymptotic equivalence) to a broader setting.

Definition 6.3. A pseudo-metric space X is **uniformly coarsely proper**[2] if there exists a nondecreasing function $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ and $r_0 > 0$ such that, for every $r \geq r_0$, every subset of X of diameter $\leq r$ is covered by at most $\Psi(r)$ subsets of diameter $\leq r_0$.

Note that being uniformly coarsely proper is a coarse invariant of metric spaces. More generally, if $X \to Y$ is a coarse embedding and if Y is uniformly coarsely proper, then so is X; in the case of an isometric embedding, the function Ψ of Definition 6.3 can be chosen to be the same for X as for Y.

Proposition 6.4. *A pseudo-metric space is uniformly coarsely proper if and only if it is quasi-isometric to a ULF metric space.*

Sketch of proof. Let us only comment the forward implication. Assume that X is uniformly coarsely proper, with (Ψ, r_0) as in the definition. Using Zorn's lemma, there exists a maximal subset Y in which any two distinct points have distance $\geq 2r_0$. The isometric inclusion $Y \subset X$ is a quasi-isometry; indeed any point in X is at distance $\leq 2r_0$ of at least one point in Y. Then in Y, for every $r \geq r_0$, any subset of diameter $\leq r$ is covered by $\leq \Psi(r)$ subsets of diameter $\leq r_0$, and these are singletons. $\square$

Definition 6.5. If X is a uniformly coarsely proper metric space, the asymptotic equivalence class of the growth of a ULF metric space Y quasi-isometric to X is called the **growth class** of X (it does not depend on Y, by Lemma 6.2).

In a metric space X, for $Y \subset X$ and $r \geq 0$, define $B_X(Y, r)$ as the set of points at distance $\leq r$ to Y.

Definition 6.6. A ULF metric space is called **amenable** if for any $\varepsilon > 0$ and $r > 0$, there exists a nonempty finite subset $F \subset X$ such that $\#(B_X(F, r))/\#(F) \leq 1 + \varepsilon$.

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[2]Uniformly coarsely proper is often called "of bounded geometry".

Proposition 6.7. *Let X, Y be coarsely equivalent ULF metric spaces. Then X is amenable if and only if Y is amenable.*

Proof. It is enough to show that, if Y is amenable, then so is X. We can assume that X and Y are non empty.

Let $f : X \to Y$ and $g : Y \to X$ be coarsely Lipschitz maps and $c > 0$ a constant be such that $\sup_{x \in X} d_X(g(f(x)), x) \leq c$ and $\sup_{y \in X} d_Y(f(g(y)), y) \leq c$. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $d_Y(f(x), f(x')) \leq \Phi(d_X(x, x'))$ for all $x, x' \in X$. There exist $k, \ell > 0$ such that $\#(f^{-1}(y)) \leq k$ for all $y \in Y$ and $\#(g^{-1}(x)) \leq \ell$ for all $x \in X$.

Fix $r, \varepsilon > 0$; we can suppose $r \geq c$. Let $F \subset Y$ be a non empty finite subset such that

$$\#(B_Y(F, \Phi(r + c)))/\#(F) \leq 1 + \frac{\varepsilon}{k\ell}.$$

Define $F' = \{x \in X \mid d_Y(f(x), F) \leq c\}$. Then F' contains $g(F)$, so that $\#(F') \geq \frac{1}{\ell}\#(F)$.

Let $x \in X$ be such that $0 < d_X(x, F') \leq r$. Then $c < d_Y(f(x), F) \leq \Phi(r) + c$, that is $f(x) \in B_Y(F, \Phi(r) + c) \smallsetminus F$. Since the cardinal of $B_Y(F, \Phi(r) + c) \smallsetminus F$ is at most $\varepsilon\#(F)/k\ell$, the cardinal of $\{x \in X \mid 0 < d_X(x, F') \leq r\}$ is at most $\varepsilon\#(F)/\ell$, and a fortiori at most $\varepsilon\#(F')$. It follows that the cardinal of $B_X(F', r)$ is at most $(1 + \varepsilon)\#(F')$. $\square$

In view of Proposition 6.7, the following definition is valid.

Definition 6.8. A uniformly coarsely proper metric space X is called **amenable** if it is quasi-isometric to an amenable ULF metric space, or equivalently if every ULF metric space coarsely equivalent to X is amenable.

Example 6.9. If X is a nonempty ULF metric space and $\liminf_{n \to \infty} b_X(n+1)/b_X(n) = 1$, then X is amenable.

If X is a non empty coarsely uniformly proper metric space of subexponential growth, then X is amenable.

6.2. The case of locally compact groups.

Theorem 6.10. *Let G be a σ-compact locally compact group and d an adapted pseudo-metric (Definition 3.6). Then (G, d) is uniformly coarsely proper.*

In particular, the notion of metric amenability makes sense for any σ-compact locally compact group. The notion of growth (up to asymptotic equivalence) makes sense for any compactly generated locally compact group, by considering the growth of (G, d) for a geodesically adapted pseudo-metric d on G, in the sense of Definition 4.7.

It can also be shown that this notion of growth is equivalent to that involving the Haar measures of balls in the group.

Definition 6.11. A σ-compact locally compact group G is **metrically amenable** if (G, d) is amenable.

This is closely related, but not equivalent, to the notion of amenability. Recall that a locally compact group G, endowed with a *left* Haar measure λ, is **amenable** if, for every compact subset S and every $\varepsilon > 0$, there exists a measurable subset F of finite nonzero measure such that $\lambda(SF) \leq (1 + \varepsilon)\lambda(F)$.

Besides, recall that a locally compact group G is **unimodular** if the action of G on itself by conjugation preserves some (and hence every) left Haar measure.

Proposition 6.12. *A σ-compact, locally compact group is metrically amenable if and only if it is amenable and unimodular.*

On the proof. Let us say that a locally compact group is **right-amenable** if, for every compact subset S and every $\varepsilon > 0$, there exists a measurable subset F of finite nonzero measure such that $\lambda(FS) \leq (1 + \varepsilon)\lambda(F)$. Note that, in comparison with amenability, SF has been replaced by FS, while we still have a *left* Haar measure. The subset FS can be thought of as a metric thickening of F, and a routine verification shows that a σ-compact locally compact group is metrically amenable if and only if it is right-amenable. Now on the one hand, for a *unimodular* group, it is clear that amenability and right-amenability are equivalent properties. On the other hand, if a locally compact G is not unimodular, if s is an element with $\Delta(s) > 1$, so that $\lambda(Fs) = \Delta(s)\lambda(F)$, the condition of right-amenability fails for $S = \{s\}$. $\qquad\Box$

Corollary 6.13. *To be amenable and unimodular is a coarse invariant among locally compact groups. In particular, to be amenable is a coarse invariant among discrete groups.*

Note that this is not true when unimodularity is dropped. Indeed, there are many cocompact closed inclusions of groups $H \subset G$ with H amenable (necessarily non-unimodular), and G non-amenable: fix $n \geq 2$ and let $\mathbf{K}$ be a nondiscrete locally compact field, e.g. $\mathbf{K} = \mathbf{R}$:

- $G = \mathrm{GL}_n(\mathbf{K})$, $H = \mathrm{T}_n(\mathbf{K})$, the subgroup of upper triangular matrices;
- $G = \mathrm{GL}_n(\mathbf{K}) \ltimes \mathbf{K}^n$ (the group of affine transformations), $H = \mathrm{T}_n(\mathbf{K}) \ltimes \mathbf{K}^n$ (here G is also non-unimodular).

7. COMPACTLY PRESENTED GROUPS

7.1. **Coarsely simply connected metric spaces.** Let X be a pseudo-metric space, c a positive real number, and k is a positive integer. The **Rips complex** $\mathrm{Rips}_c^k(X)$ is the simplicial complex whose set of vertices is X, and a subset $Y \subset X$ forms a simplex if its cardinal is $\leq k + 1$ and its points are pairwise at distance $\leq c$. The k-simplices are endowed with the metric induced by the standard ℓ^∞-norm on $\mathbf{R}^{k+1}$.

For instance, the pseudo-metric space X is coarsely connected (Definition 4.1) if and only if $\mathrm{Rips}_c^1(X)$ is connected for some c (then $\mathrm{Rips}_{c'}^k(X)$ is connected for all $c' \geq c$ and $k \geq 1$).

Definition 7.1. The pseudo-metric space X is **coarsely simply connected** if there exist $c \geq 0$ and $c' \geq c$ such that $\mathrm{Rips}_c^1(X)$ is connected and every loop in $\mathrm{Rips}_c^1(X)$ is homotopically trivial in $\mathrm{Rips}_{c'}^2(X)$.

It is possible to interpret the latter condition by a certain discrete connectedness property, along with the requirement that every discrete path in X has a discrete homotopy to the trivial loop. The precise statement is technical and we refer to [CH–14].

Proposition 7.2. *To be coarsely simply connected is a coarse invariant of pseudo-metric spaces.*

7.2. Compactly presented groups.

Definition 7.3. A **bounded presentation** is a presentation of the form $\langle S \mid R \rangle$, where S is an arbitrary set and $R \subset F_S$ (where F_S is the free group over S) is a set of words of bounded length with respect to S.

A locally compact group G is **compactly presented** if there exists an isomorphism of a boundedly presented group $\langle S \mid R \rangle$ onto G such that the image of S is a compact generating subset of G.

In other words, G has a presentation by a compact subset of generators and relators of bounded length.

Example 7.4. A discrete group is compactly presented if and only if it is finitely presented.

Recall that, if S is a generating subset of a group, the Cayley graph $\mathcal{G}(G,S)$ is the graph whose set of vertices is G and for which (g,h) is an edge whenever $g^{-1}h \in S$. Observe that $\mathcal{G}(G,S)$ is $\mathrm{Rips}_1^1(X)$ when $(X,d) = (G, d_S)$.

Standard homotopy arguments show the following:

Proposition 7.5. *Let G be a compactly generated locally compact group; consider its Cayley graph $\mathcal{G}(G,S)$ with respect to some compact generating subset S.*

Then G is compactly presented if and only if $\pi_1(\mathcal{G}(G,S))$ is generated by loops of bounded size, in other words if and only if $\mathcal{G}(G,S)$ can be filled in a G-invariant way by gons of bounded size, so that the resulting 2-complex is simply connected.

Here, "loops of bounded size" more precisely means loops of the form $\gamma c \gamma^{-1}$, for paths γ starting from 1, and loops c of bounded diameter, based at the end of γ.

Proposition 7.6. *Let G be a σ-compact locally compact group and d an adapted pseudo-metric on G. Then G is compactly presented if and only if (G,d) is coarsely simply connected. In particular, to be compactly presented is invariant under coarse equivalence among locally compact groups.*

Standard facts about finitely presented groups carry over to compactly presented groups.

Proposition 7.7. *Let G be a locally compact group, N a closed normal subgroup and $Q = G/N$.*

(1) If G is compactly presented and N is compactly generated qua normal subgroup, then Q is compactly presented;

(2) if N and Q are compactly presented then so is G;

(3) if G is compactly generated and Q is compactly presented, then N is compactly generated qua normal subgroup.

There are no free groups in the context of locally compact groups. Nevertheless, we have the following:

Proposition 7.8. *Every compactly generated locally compact group Q is isomorphic to the quotient of some compactly presented locally compact group G by a discrete normal subgroup N.*

Note that, by Proposition 7.7, G is compactly presented if and only if N is finitely generated as a normal subgroup.

In the case of totally disconnected groups, this can be refined.

Definition 7.9. A **tree-like** locally compact group is a locally compact group admitting a proper cocompact action on some tree of bounded valency.

If the group is assumed to be compactly generated, it can be shown that "cocompact" can be removed from the definition. Note that a finitely generated group is tree-like if and only if it is virtually free.

For the next theorem, we refer to [CH–14, Theorem 8.A.20].

Theorem 7.10. *Every compactly generated, totally disconnected (or more generally, which a compact unit component) locally compact group is isomorphic to the quotient of some tree-like locally group by some discrete normal subgroup.*

8. The Bieri-Strebel Theorem

Theorem 8.1. *Let $G = G_\infty$ be a compactly generated locally compact group with a continuous homomorphism $\pi = \pi_\infty$ of G onto $\mathbf{Z}$. Then there exists*

- *a sequence $(G_n)_{n \geq 0}$ of locally compact groups, with surjective continuous homomorphisms $\pi_n : G_n \to \mathbf{Z}$,*
- *surjective continuous homomorphisms $\varphi_{n,m} : G_m \to G_n$ with discrete kernel, for $n \leq m \leq \infty$,*

such that

- *the $\varphi_{n,m}$ are compatible with each other ($\varphi_{n,m} \circ \varphi_{m,\ell}$ for all $\ell \leq m \leq n \leq \infty$) and compatible with the projections ($\pi_m = \pi_n \circ \varphi_{n,m}$ for all $m \leq n \leq \infty$),*
- *G_n splits as an HNN-extension over some compactly generated open subgroup of $\mathrm{Ker}(\pi_n)$, for all $n < \infty$.*

This theorem is an approximation theorem. Note that, when $\mathrm{Ker}(\pi)$ is compactly generated, it is an empty statement, since we can choose $G_n = G$ for all n. However, it provides useful information when this is not part of the assumptions; when G is compactly presented, then $\varphi_{\infty,n}$ has to be an isomorphism for some $n < \infty$, which provides:

Corollary 8.2. *Let G be a compactly presented locally compact group along with a homomorphism π of G onto $\mathbf{Z}$. Then G splits as an HNN-extension over some compactly generated open subgroup of $\mathrm{Ker}(\pi)$.*

An HNN-extension involves one base group H and two subgroups H_1, H_2 of H along with an isomorphism between H_1 and H_2. When it is non-ascending (that is, $H_1 \neq H \neq H_2$), it contains a discrete non-abelian free subgroup.

Corollary 8.3. *Let $G = N \rtimes \mathbf{Z}$ be a compactly presented locally compact group, with $\mathbf{Z}$ acting through powers of some topological group automorphism α of N. Assume that G has no non-abelian discrete free subgroup. Then there exists $\beta \in \{\alpha, \alpha^{-1}\}$ and a compactly generated open subgroup H of N such that $\beta(H) \subset H$ and $N = \bigcup_{n \geq 0} \beta^{-n}(H)$.*

9. Examples

Let us give various sources of examples.

Proposition 9.1. *If G is a locally compact group and G/G° is compact, then G is compactly presented.*

In the situation of Proposition 9.1, G can be shown to admit a proper transitive continuous action by isometries on some Riemannian manifold homeomorphic to some Euclidean space.

Proposition 9.2. *Let H be a compactly presented locally compact group and f a continuous endomorphism of H with open image. Then the ascending HNN-extension associated to (H, f) is compactly presented.*

A generalization of the previous proposition is the following:

Proposition 9.3. *Consider a locally compact group with a topological semidirect product decomposition $G = N \rtimes \mathbf{Z}^k$, such that some element α of $\mathbf{Z}^k$ and some compactly presented open subgroup H of N satisfy $\alpha(H) \subset H$ and $\bigcup_{n \geq 0} \alpha^{-n}(H) = N$. Then G is compactly presented.*

Example 9.4. Let N_1, N_2 be totally disconnected non-compact locally compact groups with contracting topological group automorphisms α_1, α_2 (contracting means that $\lim_{n \to +\infty} \alpha_i^n(g) = 1$ for every $g \in N_i$). For instance, we can choose $N_i = \mathbf{Q}_{p_i}$ for primes p_1, p_2 and α_i the multiplication by $p_i^{n_i}$ for some positive integer n_i.

Consider the automorphism of $N_1 \times N_2$ given by $\alpha = (\alpha_1, \alpha_2^{-1})$. Then the semidirect product $(N_1 \times N_2) \rtimes_\alpha \mathbf{Z}$ is not compactly presented.

Example 9.5. Let $\mathbf{K}_1, \mathbf{K}_2$ be ultrametric nondiscrete locally compact fields. Fix $\lambda_1 \in \mathbf{K}_1^*$ and $\lambda_2 \in \mathbf{K}_2^*$. Consider the semidirect product $G = (\mathbf{K}_1 \times \mathbf{K}_2)_{(\lambda_1, \lambda_2)} \rtimes \mathbf{Z}$. Then

- If either λ_1 or λ_2 is equal to 1, then G is not compactly generated.
- if $|\lambda_1| < 1 < |\lambda_2|$ or $|\lambda_2| < 1 < |\lambda_1|$, then G is compactly generated but not compactly presented (as a particular case of Example 9.4)
- if $|\lambda_1|$ and $|\lambda_2|$ are both < 1 or > 1, then G is compactly presented (by Proposition 8.3).

Example 9.6. Consider distinct primes p, q and, for $i = 1, 2$, the semidirect product $\Gamma_i = \mathbf{Z}[1/pq] \rtimes_{n_i} \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by n_i, with $n_1 = pq$ and $n_2 = p/q$. Then both Γ_1 and Γ_2 are finitely generated. The group Γ_i is a cocompact lattice in the locally compact group $G_i = (\mathbf{R} \times \mathbf{Q}_p \times \mathbf{Q}_q) \rtimes_{n_i} \mathbf{Z}$.

It follows from Example 9.5 that $G_1/\mathbf{R}$ is compactly presented and $G_2/\mathbf{R}$ is not compactly presented. Since $\mathbf{R}$ itself is compactly presented, it follows from Theorem 7.7 that G_1 is compactly presented and G_2 is not compactly presented. By Proposition 3.5, the inclusion of Γ_i into G_i is a coarse equivalence. By Proposition 7.6, we deduce that Γ_1 is finitely presented while Γ_2 is not.

Example 9.7. If $\mathbf{G}$ is a reductive $\mathbf{K}$-group, for some non-discrete locally compact field $\mathbf{K}$, then $G = \mathbf{G}(\mathbf{K})$ is compactly presented. Indeed, when $\mathbf{K}$ is Archimedean then G has finitely many components and Proposition 9.1 applies; otherwise G admits some closed cocompact (solvable) subgroup satisfying the hypotheses of Proposition 9.3.

Example 9.8. Every compactly generated nilpotent locally compact group is compactly presented. Indeed, it can be shown that such a group always admits a proper homomorphism with cocompact image into a Lie group with finitely many connected components.

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