FINITELY PRESENTABLE, NON-HOPFIAN GROUPS WITH KAZHDAN'S PROPERTY (T) AND INFINITE OUTER AUTOMORPHISM GROUP

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Abstract. We give simple examples of Kazhdan groups with infinite outer automorphism groups. This answers a question of Paulin, independently answered by Ollivier and Wise by completely different methods. As arithmetic lattices in (non-semisimple) Lie groups, our examples are in addition finitely presented.

We also use results of Abels about compact presentability of p-adic groups to exhibit a finitely presented non-Hopfian Kazhdan group. This answers a question of Ollivier and Wise.

1. Introduction

Recall that a locally compact group is said to have Property (T) if every weakly continuous unitary representation with almost invariant vectors[1] has nonzero invariant vectors.

It was asked by Paulin in [HV, p.134] (1989) whether there exists a group with Kazhdan's Property (T) and with infinite outer automorphism group. This question remained unanswered until 2004; in particular, it is Question 18 in [Wo].

This question was motivated by the two following special cases. The first is the case of lattices in *semisimple* groups over local fields, which have long been considered as prototypical examples of groups with Property (T). If Γ is such a lattice, Mostow's rigidity Theorem and the fact that semisimple groups have finite outer automorphism group imply that $\mathrm{Out}(\Gamma)$ is finite. Secondly, a new source of groups with Property (T) appeared when Zuk [Zu] proved that certain models of random groups have Property (T). But they are also hyperbolic, and Paulin proved [Pa] that a hyperbolic group with Property (T) has a finite outer automorphism group.

However, it turns out that various arithmetic lattices in appropriate *non-semisimple* groups provide examples. For instance, consider the additive group $\mathrm{Mat}_{m\times n}(\mathbf{Z})$ of $m \times n$ matrices over $\mathbf{Z}$, endowed with the action of $\mathrm{GL}_n(\mathbf{Z})$ by left multiplication.

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[1]A representation $\pi : G \to \mathcal{U}(\mathcal{H})$ almost has invariant vectors if for every $\varepsilon > 0$ and every finite subset $F \subseteq G$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(g)\xi - \xi\| < \varepsilon$ for every $g \in F$.

Proposition 1.1. *For every $n \geq 3$, $m \geq 1$, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z})$ is a finitely presented linear group, has Property (T), is non-coHopfian*[2]*, and its outer automorphism group contains a copy of $\mathrm{PGL}_m(\mathbf{Z})$, hence is infinite if $m \geq 2$.*

We later learned that Ollivier and Wise [OW] had independently found examples of a very different nature. They embed any countable group G in $\mathrm{Out}(\Gamma)$, where Γ has Property (T), is a subgroup of a torsion-free hyperbolic group, satisfying a certain "graphical" small cancelation condition (see also [BS]). In contrast to our examples, theirs are not, a priori, finitely presented; on the other hand, our examples are certainly not subgroups of hyperbolic groups since they all contain a copy of $\mathbf{Z}^2$.

They also construct in [OW] a non-coHopfian group with Property (T) that embeds in a hyperbolic group. Proposition 1.1 actually answers two questions in their paper: namely, whether there exists a finitely presented group with Property (T) and without the coHopfian Property (resp. with infinite outer automorphism group).

Remark 1.2. Another example of a non-coHopfian group with Property (T) is $\mathrm{PGL}_n(\mathbf{F}_p[X])$ when $n \geq 3$. This group is finitely presentable if $n \geq 4$ [RS] (but not for $n = 3$ [Be]). In contrast with the previous examples, the Frobenius morphism Fr induces an isomorphism onto a subgroup of *infinite* index, and the intersection $\bigcap_{k\geq 0} \mathrm{Im}(\mathrm{Fr}^k)$ is reduced to $\{1\}$.

Ollivier and Wise also constructed in [OW] the first examples of non-Hopfian groups with Property (T). They asked whether a finitely presented example exists. Although linear finitely generated groups are residually finite, hence Hopfian, we use them to positively answer their question.

Theorem 1.3. *There exists a S-arithmetic lattice Γ, and a central subgroup $Z \subset \Gamma$, such that Γ and Γ/Z are finitely presented, have Property (T), and Γ/Z is non-Hopfian.*

The group Γ has a simple description as a matrix group from which Property (T) and the non-Hopfian property for Γ/Z are easily checked (Proposition 2.7). Section 3 is devoted to prove finite presentability of Γ. We use here a general criterion for finite presentability of S-arithmetic groups, due to Abels [A2]. It involves the computation of the first and second cohomology group of a suitable Lie algebra.

2. Proofs of all results except finite presentability of Γ

We need some facts about Property (T).

Lemma 2.1 (see [HV, Chap. 3, Théorème 4])**.** *Let G be a locally compact group, and Γ a lattice in G. Then G has Property (T) if and only if Γ has Property (T).* □

The next lemma is an immediate consequence of the classification of semisimple algebraic groups over local fields with Property (T) (see [Ma, Chap. III, Theorem 5.6]) and S. P. Wang's results on the non-semisimple case [Wa, Theorem 2.10].

Lemma 2.2. *Let $\mathbf{K}$ be a local field, G a connected linear algebraic group defined over $\mathbf{K}$. Suppose that G is perfect, and, for every simple quotient S of G, either*

[2]A group is coHopfian (resp. Hopfian) if it is isomorphic to no proper subgroup (resp. quotient) of itself.

S has $\mathbf{K}$-rank ≥ 2, or $\mathbf{K} = \mathbf{R}$ and S is isogeneous to either $\mathrm{Sp}(n,1)$ *($n \geq 2$) or* $\mathrm{F}_{4(-20)}$. *If* $\mathrm{char}(\mathbf{K}) > 0$, *suppose in addition that G has a Levi decomposition defined over $\mathbf{K}$. Then $G(\mathbf{K})$ has Property (T).* □

Proof of Proposition 1.1. The group $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z})$ is linear in dimension $n+m$. As a semidirect product of two finitely presented groups, it is finitely presented. For every $k \geq 2$, it is isomorphic to its proper subgroup $\mathrm{SL}_n(\mathbf{Z}) \ltimes k\mathrm{Mat}_{m\times n}(\mathbf{Z})$ of finite index k^{mn}.

The group $\mathrm{GL}_m(\mathbf{Z})$ acts on $\mathrm{Mat}_{m\times n}(\mathbf{Z})$ by right multiplication. Since this action commutes with the left multiplication of $\mathrm{SL}_n(\mathbf{Z})$, $\mathrm{GL}_m(\mathbf{Z})$ acts on the semidirect product $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z})$ by automorphisms, and, by an immediate verification, this gives an embedding of $\mathrm{GL}_m(\mathbf{Z})$ if n is odd or $\mathrm{PGL}_m(\mathbf{Z})$ if n is even into $\mathrm{Out}(\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z}))$ (it can be shown that this is an isomorphism if n is odd; if n is even, the image has index two). In particular, if $m \geq 2$, then $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z})$ has infinite outer automorphism group.

On the other hand, in view of Lemma 2.1, it has Property (T) (actually for all $m \geq 0$): indeed, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{Z})$ is a lattice in $\mathrm{SL}_n(\mathbf{R}) \ltimes \mathrm{Mat}_{m\times n}(\mathbf{R})$, which has Property (T) by Lemma 2.2 as $n \geq 3$. □

We now turn to the proof of Theorem 1.3. The following lemma is immediate, and already used in [Ha, Th. 4(iii)] and [A1].

Lemma 2.3. *Let Γ be a group, Z a central subgroup. Let α be an automorphism of Γ such that $\alpha(Z)$ is a proper subgroup of Z. Then α induces a surjective, non-injective endomorphism of Γ/Z, whose kernel is $\alpha^{-1}(Z)/Z$.* □

Definition 2.4. Fix $n_1, n_2, n_3, n_4 \in \mathbf{N} - \{0\}$ with $n_2, n_3 \geq 3$. We set $\Gamma = G(\mathbf{Z}[1/p])$, where p is any prime, and G is algebraic the group defined as matrices by blocks of size n_1, n_2, n_3, n_4:

$$\begin{pmatrix} I_{n_1} & (*)_{12} & (*)_{13} & (*)_{14} \\ 0 & (**)_{22} & (*)_{23} & (*)_{24} \\ 0 & 0 & (**)_{33} & (*)_{34} \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix},$$

where $(*)$ denote any matrices and $(**)_{ii}$ denote matrices in SL_{n_i}, $i = 2, 3$.

The centre of G consists of matrices of the form $\begin{pmatrix} I_{n_1} & 0 & 0 & (*)_{14} \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix}$. Define Z as the centre of $G(\mathbf{Z})$.

Remark 2.5. This group is related to an example of Abels: in [A1] he considers the same group, but with blocks 1×1, and GL_1 instead of SL_1 in the diagonal. Taking the points over $\mathbf{Z}[1/p]$, and taking the quotient by a cyclic subgroup if the centre, this provided the first example of a finitely presentable non-Hopfian solvable group.

Remark 2.6. If we do not care about finite presentability, we can take $n_3 = 0$ (i.e. 3 blocks suffice), as in P. Hall's original solvable example [Ha, Th. 4(iii)].

We begin by easy observations. Identify GL_{n_1} to the upper left diagonal block. It acts by *conjugation* on G as follows:

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & A_{12} & A_{13} & A_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & uA_{12} & uA_{13} & uA_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix}.$$

This gives an action of GL_{n_1} on G, and also on its centre, and this latter action is faithful. In particular, for every commutative ring R, $\mathrm{GL}_{n_1}(R)$ embeds in $\mathrm{Out}(G(R))$.

From now on, we suppose that $R = \mathbf{Z}[1/p]$, and $u = pI_{n_1}$. The automorphism of $\Gamma = G(\mathbf{Z}[1/p])$ induced by u maps Z to its proper subgroup Z^p. In view of Lemma 2.3, this implies that Γ/Z is non-Hopfian.

Proposition 2.7. *The groups Γ and Γ/Z are finitely generated, have Property (T), and Γ/Z is non-Hopfian.*

Proof. We have just proved that Γ/Z is non-Hopfian. By the Borel-Harish-Chandra Theorem [BHC], Γ is a lattice in $G(\mathbf{R}) \times G(\mathbf{Q}_p)$. This group has Property (T) as a consequence of Lemma 2.2. By Lemma 2.1, Γ also has Property (T). Finite generation is a consequence of Property (T) [HV, Lemme 10]. Since Property (T) is (trivially) inherited by quotients, Γ/Z also has Property (T). □

Remark 2.8. This group has a surjective endomorphism with nontrivial finite kernel. We have no analogous example with infinite kernel. Such examples might be constructed if we could prove that some groups over rings of dimension ≥ 2 such as $\mathrm{SL}_n(\mathbf{Z}[X])$ or $\mathrm{SL}_n(\mathbf{F}_p[X,Y])$ have Property (T), but this is an open problem [Sh]. The non-Hopfian Kazhdan group of Ollivier and Wise [OW] is torsion-free, so the kernel is infinite in their case.

Remark 2.9. It is easy to check that $\mathrm{GL}_{n_1}(\mathbf{Z}) \times \mathrm{GL}_{n_4}(\mathbf{Z})$ embeds in $\mathrm{Out}(\Gamma)$ and $\mathrm{Out}(\Gamma/Z)$. In particular, if $\max(n_1, n_4) \geq 2$, then these outer automorphism groups are infinite.

We finish this section by observing that Z is a finitely generated subgroup of the centre of Γ, so that finite presentability of Γ/Z immediately follows from that of Γ.

3. Finite presentability of Γ

We recall that a Hausdorff topological group H is *compactly presented* if there exists a compact generating subset C of H such that the abstract group H is the quotient of the group freely generated by C by relations of bounded length. See Abels [A2, §1.1] for more about compact presentability.

Kneser [Kn] has proved that for every linear algebraic $\mathbf{Q}_p$-group, the S-arithmetic lattice $G(\mathbf{Z}[1/p])$ is finitely presented if and only if $G(\mathbf{Q}_p)$ is compactly presented. A characterization of the linear algebraic $\mathbf{Q}_p$-groups G such that $G(\mathbf{Q}_p)$ is compactly presented is given in [A2]. This criterion requires the study of a solvable cocompact subgroup of $G(\mathbf{Q}_p)$, which seems tedious to carry out in our specific example.

Let us describe another sufficient criterion for compact presentability, also given in [A2], which is applicable to our example. Let U be the unipotent radical in G, and let S denote a Levi factor defined over $\mathbf{Q}_p$, so that $G = S \ltimes U$. Let $\mathfrak{u}$ be the

Lie algebra of U, and D be a maximal $\mathbf{Q}_p$-split torus in S. We recall that the first homology group of $\mathfrak{u}$ is defined as the abelianization

$$H_1(\mathfrak{u}) = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}],$$

and the second homology group of $\mathfrak{u}$ is defined as $\mathrm{Ker}(d_2)/\mathrm{Im}(d_3)$, where the maps

$$\mathfrak{u} \wedge \mathfrak{u} \wedge \mathfrak{u} \xrightarrow{d_3} \mathfrak{u} \wedge \mathfrak{u} \xrightarrow{d_2} \mathfrak{u}$$

are defined by:

$$d_2(x_1 \wedge x_2) = -[x_1, x_2] \quad \text{and} \quad d_3(x_1 \wedge x_2 \wedge x_3) = x_3 \wedge [x_1, x_2] + x_2 \wedge [x_3, x_1] + x_1 \wedge [x_2, x_3].$$

We can now state the result by Abels that we use (see [A2, Theorem 6.4.3 and Remark 6.4.5]).

Theorem 3.1 (Abels)**.** [3]

Let G be a connected linear algebraic group over $\mathbf{Q}_p$. Suppose that G is unipotent-by-semisimple, i.e. $G = US$, where U is the unipotent radical and S is a semisimple Levi factor. We furthermore assume that S is split semisimple without any simple factors of rank one. Then $G(\mathbf{Q}_p)$ is compactly presented if and only if the following two conditions are satisfied

(i) $H_1(\mathfrak{u})^S = \{0\}$*;*
(ii) $H_2(\mathfrak{u})^S = \{0\}$.

On the proof. This relies on [A2, Theorem 6.4.3 and Remark 6.4.5]. A few comments are necessary:

- Condition (1a) of [A2, Remark 6.4.5] involves the orthogonal $\Phi^\perp$ of the subspace generated by roots. It states that $\Phi^\perp$ does not contain dominant weights ω_1, ω_2 of the S-module $H_1(\mathfrak{u})$ with $0 \in [\omega_1, \omega_2]$. Since S is semisimple, $\Phi^\perp = \{0\}$ and (1a) just means that 0 is not a dominant weight, which is exactly Condition (i) above. (Note that (i) is actually a necessary and sufficient condition for $G(\mathbf{Q}_p)$ to be compactly *generated*, see [A2, Theorem 6.4.4].)
- As noticed in [A2, p. 132], Condition (1b) of [A2, Remark 6.4.5] is superfluous when S has no factors of rank ≤ 1.
- (ii) is a restatement of Condition 2 of [A2, Theorem 6.4.3]. □

We now return to our particular example G from Definition 2.4; it is unipotent-by-semisimple and the semisimple Levi factor $S = \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$ is split with no factor of rank one, so it fulfills the assumptions of Theorem 3.1. So its compact presentability is equivalent to Conditions (i) and (ii) of this theorem. Keep the previous notation S, D, U, $\mathfrak{u}$, so that S (resp. D) denoting in our case the diagonal by blocks (resp. diagonal) matrices in G, and U denotes the matrices in G all of whose diagonal blocks are the identity. The set of indices of the matrix is partitioned

[3] In the published version, the corresponding theorem is a misquotation of Abels' theorem. This version is corrected. The numbering of the lemmas is not affected. The reference to the erratum is published as Proc. Amer. Math. Soc. 139 (2011), 383–384. It essentially includes the correct statement of Theorem 3.1 (as given here) and the subsequent lines "on the proof". The verifications in the published version were enough to be applicable to the correct version of Theorem 3.1, so no change in the computations has been done.

as $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, with $|I_j| = n_j$ as in Definition 2.4. It follows that, for every field K,

$$\mathfrak{u}(K) = \left\{ T \in \mathrm{End}(K^I),\ \forall j,\ T(K^{I_j}) \subset \bigoplus_{i<j} K^{I_i} \right\}.$$

Throughout, we use the following notation: a letter such as i_k (or j_k, etc.) implicitly means $i_k \in I_k$. Define, in an obvious way, subgroups U_{ij}, $i < j$, of U, and their Lie algebras $\mathfrak{u}_{ij}$.

We begin by checking Condition (i) of Theorem 3.1. This follows from the following lemma.

Lemma 3.2. [4] *For any two weights of the action of D on $H_1(\mathfrak{u})$, 0 is not on the segment joining them. In particular, 0 is not a weight of the action of D on $H_1(\mathfrak{u})$.*

Proof. Recall that $H_1(\mathfrak{u}) = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$. So it suffices to look at the action on the supplement D-subspace $\mathfrak{u}_{12} \oplus \mathfrak{u}_{23} \oplus \mathfrak{u}_{34}$ of $[\mathfrak{u},\mathfrak{u}]$. Identifying S with $\mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, we denote (A, B) an element of $D \subset S$. We also denote by e_{pq} the matrix whose coefficient (p, q) equals one and all others are zero.

$$(A,B) \cdot e_{i_1 j_2} = a_{j_2}^{-1} e_{i_1 j_2}, \quad (A,B) \cdot e_{j_2 k_3} = a_{j_2} b_{k_3}^{-1} e_{j_2 k_3}, \quad (A,B) \cdot e_{k_3 \ell_4} = b_{k_3} e_{k_3 \ell_4}.$$

Since $S = \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, the weights for the adjoint action on $\mathfrak{u}_{12} \oplus \mathfrak{u}_{23} \oplus \mathfrak{u}_{34}$ live in M/P, where M is the free $\mathbf{Z}$-module of rank $n_2 + n_3$ with basis $(u_1, \dots, u_{n_2}, v_1, \dots, v_{n_3})$, and P is the plane generated by $\sum_{j_2} u_{j_2}$ and $\sum_{k_3} v_{k_3}$. Thus, the weights are (modulo P) $-u_{j_2}$, $u_{j_2} - v_{k_3}$, v_{k_3} $(1 \le j_2 \le n_2,\ 1 \le k_3 \le n_3)$.

Using that $n_2, n_3 \ge 3$, it is clear that no nontrivial positive combination of two weights (viewed as elements of $\mathbf{Z}^{n_2+n_3}$) lies in P. □

We must now check Condition (ii) of Theorem 3.1, and therefore compute $H_2(\mathfrak{u})$ as a D-module.

Lemma 3.3. $\mathrm{Ker}(d_2) \subset \mathfrak{u} \wedge \mathfrak{u}$ *is linearly spanned by*

(1) $\mathfrak{u}_{12} \wedge \mathfrak{u}_{12}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{34} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{34}$.
(2) $\mathfrak{u}_{14} \wedge \mathfrak{u}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
(3) $e_{i_1 j_2} \wedge e_{k_2 \ell_3}$ *$(j_2 \neq k_2)$*, $e_{i_2 j_3} \wedge e_{k_3 \ell_4}$ *$(j_3 \neq \ell_3)$*.
(4) $e_{i_1 j_2} \wedge e_{k_2 \ell_4}$ *$(j_2 \neq k_2)$*, $e_{i_1 j_3} \wedge e_{k_3 \ell_4}$ *$(j_3 \neq k_3)$*.
(5) *Elements of the form* $\sum_{j_2} \alpha_{j_2}(e_{i_1 j_2} \wedge e_{j_2 k_3})$ *if* $\sum_{j_2} \alpha_{j_2} = 0$, *and* $\sum_{j_3} \alpha_{j_3}(e_{i_2 j_3} \wedge e_{j_3 k_4})$ *if* $\sum_{j_3} \alpha_{j_3} = 0$.
(6) *Elements of the form* $\sum_{j_2} \alpha_{j_2}(e_{i_1 j_2} \wedge e_{j_2 k_4}) + \sum_{j_3} \beta_{j_3}(e_{i_1 j_3} \wedge e_{j_3 k_4})$ *if* $\sum_{j_2} \alpha_{j_2} + \sum_{j_3} \beta_{j_3} = 0$.

Proof. First observe that $\mathrm{Ker}(d_2)$ contains $\mathfrak{u}_{ij} \wedge \mathfrak{u}_{kl}$ when $[\mathfrak{u}_{ij}, \mathfrak{u}_{kl}] = 0$. This corresponds to (1) and (2). The remaining cases are $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{34}$.

[4] Compared to the published version, only the second sentence in the statement of Lemma 3.2 has been added, and the proof has not been modified, although the statement is stronger than what is actually needed for the corrected version given here of Theorem 3.1.

On the one hand, $\mathrm{Ker}(d_2)$ also contains $e_{i_1j_2} \wedge e_{k_2\ell_3}$ if $j_2 \neq k_2$, etc.; this corresponds to elements in (3), (4). On the other hand, $d_2(e_{i_1j_2} \wedge e_{j_2k_3}) = -e_{i_1k_3}$, $d_2(e_{i_2j_3} \wedge e_{j_3k_4}) = -e_{i_2k_4}$, $d_2(e_{i_1j_2} \wedge e_{j_2k_4}) = -e_{i_1k_4}$, $d_2(e_{i_1j_3} \wedge e_{j_3k_4}) = -e_{i_1k_4}$. The lemma follows. □

Definition 3.4. Denote by $\mathfrak{b}$ (resp. $\mathfrak{h}$) the subspace spanned by elements in (2), (4), and (6) (resp. in (1), (3), and (5)) of Lemma 3.3.

Proposition 3.5. $\mathrm{Im}(d_3) = \mathfrak{b}$, *and* $\mathrm{Ker}(d_2) = \mathfrak{b} \oplus \mathfrak{h}$ *as* D*-module. In particular,* $H_2(\mathfrak{u})$ *is isomorphic to* $\mathfrak{h}$ *as a* D*-module.*

Proof. We first prove, in a series of facts, that $\mathrm{Im}(d_3) \supset \mathfrak{b}$.

Fact. $\mathfrak{u}_{14} \wedge \mathfrak{u}$ is contained in $\mathrm{Im}(d_3)$.

Proof. If $z \in \mathfrak{u}_{14}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. This already shows that $\mathfrak{u}_{14} \wedge (\mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14})$ is contained in $\mathrm{Im}(d_3)$, since $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14}$.

Now, if $(x, y, z) \in \mathfrak{u}_{24} \times \mathfrak{u}_{12} \times \mathfrak{u}_{34}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{24}, \mathfrak{u}_{12}] = \mathfrak{u}_{14}$, this implies that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{34} \subset \mathrm{Im}(d_3)$. Similarly, $\mathfrak{u}_{14} \wedge \mathfrak{u}_{12} \subset \mathrm{Im}(d_3)$.

Finally we must prove that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} \subset \mathrm{Im}(d_3)$. This follows from the formula $e_{i_1j_4} \wedge e_{k_2\ell_3} = d_3(e_{i_1m_2} \wedge e_{k_2\ell_3} \wedge e_{m_2j_4})$, where $m_2 \neq k_2$ (so that we use that $|I_2| \geq 2$). □

Fact. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$ and, similarly, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, are contained in $\mathrm{Im}(d_3)$.

Proof. If $(x, y, z) \in \mathfrak{u}_{12} \times \mathfrak{u}_{23} \times \mathfrak{u}_{13}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{12}, \mathfrak{u}_{23}] = \mathfrak{u}_{13}$, this implies that $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13} \subset \mathrm{Im}(d_3)$. □

Fact. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$ is contained in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1k_2} \wedge e_{k_2\ell_3} \wedge e_{k_2j_4}) = e_{k_2j_4} \wedge e_{i_1\ell_3} + e_{i_1j_4} \wedge e_{k_2\ell_3}$. Since we already know that $e_{i_1j_4} \wedge e_{k_2\ell_3} \in \mathrm{Im}(d_3)$, this implies $e_{k_2j_4} \wedge e_{i_1\ell_3} \in \mathrm{Im}(d_3)$. □

Fact. The elements in (4) are in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{\ell_3m_4}) = -e_{i_1k_3} \wedge e_{\ell_3m_4}$ if $k_3 \neq \ell_3$. The other case is similar. □

Fact. The elements in (6) are in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{k_3\ell_4}) = -e_{i_1k_3} \wedge e_{k_3\ell_4} + e_{i_1j_2} \wedge e_{j_2\ell_4}$. Such elements linearly span all elements as in (6). □

Conversely, we must check $\mathrm{Im}(d_3) \subset \mathfrak{b}$. By straightforward verifications:

- $d_3(\mathfrak{u}_{14} \wedge \mathfrak{u} \wedge \mathfrak{u}) \subset \mathfrak{u}_{14} \wedge \mathfrak{u}$.
- $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24}) = 0$
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{24})$, $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$ are all contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{23}) \subset \mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, and similarly $d_3(\mathfrak{u}_{23} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34}) \subset \mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24})$ and similarly $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34})$ are contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} + \mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
- The only remaining case is that of $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$: $d_3(e_{i_1j_2} \wedge e_{j_2'k_3} \wedge e_{k_3'\ell_4}) = \delta_{k_3k_3'} e_{i_1j_2} \wedge e_{j_2'\ell_4} - \delta_{j_2j_2'} e_{i_1k_3} \wedge e_{k_3'\ell_4}$, which lies in (4) or in (6).

Finally $\mathrm{Im}(d_3) = \mathfrak{b}$.

It follows from Lemma 3.3 that $\mathrm{Ker}(d_2) = \mathfrak{h} \oplus \mathfrak{b}$. Since $\mathfrak{b} = \mathrm{Im}(d_3)$, this is a D-submodule. Let us check that $\mathfrak{h}$ is also a D-submodule; the computation will be used in the sequel.

The action of S on $\mathfrak{u}$ by *conjugation* is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ 0 & 0 & X_{23} & X_{24} \\ 0 & 0 & 0 & X_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_{12}A^{-1} & X_{13}B^{-1} & X_{14} \\ 0 & 0 & AX_{23}B^{-1} & AX_{24} \\ 0 & 0 & 0 & BX_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We must look at the action of D on the elements in (1), (3), and (5). We fix $(A,B) \in D \subset S \simeq \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, and we write $A = \sum_{j_2} a_{j_2} e_{j_2 j_2}$ and $B = \sum_{k_3} b_{k_3} e_{k_3 k_3}$.

- (1):

$$(A,B) \cdot e_{i_1 j_2} \wedge e_{k_1 \ell_2} = e_{i_1 j_2} A^{-1} \wedge e_{k_1 \ell_2} A^{-1} = a_{j_2}^{-1} a_{\ell_2}^{-1} e_{i_1 j_2} \wedge e_{k_1 \ell_2}. \tag{3.1}$$

  The action on other elements in (1) has a similar form.
- (3) ($j_2 \neq k_2$):

$$(A,B) \cdot e_{i_1 j_2} \wedge e_{k_2 \ell_3} = e_{i_1 j_2} A^{-1} \wedge A e_{k_2 \ell_4} B^{-1} = a_{j_2}^{-1} a_{k_2} b_{\ell_3}^{-1} e_{i_1 j_2} \wedge e_{k_2 \ell_3}. \tag{3.2}$$

  The action on the other elements in (3) has a similar form.
- (5) ($\sum_{j_2} \alpha_{j_2} = 0$)

$$(A,B) \cdot \sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_3}) \;=\; \sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} A^{-1} \wedge A e_{j_2 k_3} B^{-1})$$

$$= \sum_{j_2} \alpha_{j_2} a_{j_2}^{-1} (e_{i_1 j_2} \wedge a_{j_2} b_{k_3}^{-1} e_{j_2 k_3}) \;=\; b_{k_3}^{-1} \left(\sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_3}) \right). \tag{3.3}$$

  The other case in (5) has a similar form. □

Lemma 3.6. *0 is not a weight for the action of D on $H_2(\mathfrak{u})$.*

Proof. As described in the proof of Lemma 3.2, we think of weights as elements of M/P. Hence, we describe weights as elements of $M = \mathbf{Z}^{n_2+n_3}$ rather than M/P, and must check that no weight lies in P.

(1) In (3.1), the weight is $-u_{j_2} - u_{\ell_2}$, hence does not belong to P since $n_2 \geq 3$. The other verifications are similar.
(3) In (3.2), the weight is $-u_{j_2} + u_{k_2} - v_{\ell_3}$, hence does not belong to P. The other verification for (3) is similar.
(5) In (3.3), the weight is $-v_{k_3}$, hence does dot belong to P. The other verification is similar. □

Finally, Lemmas 3.2 and 3.6 imply that the conditions (i) and (ii) of Theorem 3.1 are satisfied, so that Γ is finitely presented. □

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