SUBGROUPS APPROXIMATIVELY OF FINITE INDEX AND WREATH PRODUCTS

YVES CORNULIER

Abstract. In a group, we introduce a notion of subgroup approximatively of finite index, which extends the notion of groups approximable by finite groups. We deduce a characterization of wreath products approximable by finite groups.

Contents

1. Introduction

Recall that a group G is approximable by finite groups, abbreviated here as AF if for every finite subset S of G, the marked group $(\langle S\rangle, S)$ is a limit of finite groups in the space of groups marked by S. Another terminology is "LEF", which stands for "locally embeddable into finite groups".

This property, which is a particular instance of a very general notion due to Malcev, was introduced and studied in the group-theoretic setting by Gordon-Vershik [GV97] and Stëpin [St84] is a very natural one and can be characterized in many ways, for instance G is approximable by finite groups if and only if it is isomorphic to a subgroup of an ultraproduct of finite groups, if and only if it is isomorphic to an inductive limit of residually finite groups. A finitely presented group is approximately finite if and only it is residually finite; however for more general groups residual finiteness is a stronger property than approximate finiteness.

Here we introduce a generalization of this property for a pair of a group and a subgroup.

Date: May 20, 2017.

If G is a group, we denote by $\mathcal{N}(G) \subset \mathcal{S}(G)$ the set of its normal subgroups and of its subgroups, endowed with the topology induced by inclusion in 2^G with the product topology. The spaces $\mathcal{N}(G)$ and $\mathcal{S}(G)$ are compact, Hausdorff, and totally disconnected.

Definition 1.1. Let G be a group and H a subgroup. We say that H is approximately of finite index (AFI) in G if for every finitely presented group F and every homomorphism $\rho : F \to G$, the subgroup $\rho^{-1}(H)$ is a limit in $\mathcal{S}(F)$ of a sequence (L_n) of finite index subgroups of F.

A simple verification is that for every group, the subset $\mathcal{S}_{\mathrm{AFI}}(G)$ of $\mathcal{S}(G)$ of those subgroups H that are AFI in G, is closed. In particular, it contains the closure of the set of finite index subgroups. In general it is larger: for instance if G is approximable by finite groups but not residually finite then $\{1\}$ is API in G (this precisely characterizes that G be approximately finite) but is not a limit of finite index subgroups. However, for G finitely presented we have

Proposition 1.2. *If G is finitely presented, then $H \leq G$ is API in G if and only it is a limit in $\mathcal{S}(G)$ of finite index subgroups.*

A consequence of the definition is the following characterization, which is extracted from Proposition 2.10.

Proposition 1.3. *Let G be a group and H a subgroup. Then H is AFI in G if and only if there exists a set I, an ultrafilter ω on I, a family (Γ_i) of groups, a family (Λ_i) of subgroups $\Lambda_i \subset \Gamma_i$ of finite index, and an injective homomorphism j from G to the ultraproduct $\prod_{i\in I}^{\omega} \Gamma_i$, such that $H = j^{-1}\left(\prod_{i\in I}^{\omega} \Lambda_i\right)$.*

Remark 1.4. This proposition shows that if we consider the algebraic structure given by pairs (G, H) with G a group and H a subgroup, then pairs (G, H) such that H is AFI in G are precisely those pairs (G, H) that are "locally embeddable in" the class of pairs (G', H') where G' ranges over groups and H' ranges over finite index subgroups of G', in the sense of [Ma73].

We initially introduced this definition in order to obtain a characterization of permutational wreath products that are approximable by finite groups, namely Proposition 3.1, from which we extract:

Proposition 1.5. *Let B be a nontrivial group, G a group and H a subgroup. Consider the wreath product*

$$W = B \wr_{G/H} G = \left(\bigoplus_{x \in G/H} B \right) \rtimes G.$$

Then W is AF if and only both B, G are AF, and H is AFI in G.

The following simple corollary (for $H = \{1\}$) is due to Gordon and Vershik [GV97].

Corollary 1.6 (Gordon and Vershik)**.** *If B, G are groups, the standard wreath product $B \wr G$ is AF if and only if B and G are both AF.*

This simple statement can appear surprisingly simple by contrast with Gruenberg's theorem, which asserts that $B \wr G$ is residually finite if and only if B is residually finite, and [either G is finite, or B is abelian]. For completeness, we include its version for permutational wreath products (Proposition 3.2), from which we extract:

Proposition 1.7. *Let $W = B \wr_{G/H} G$ be a wreath product, where B is non-trivial. Then W is residually finite if and only if B, G are residually finite, and either*

- *H are profinitely closed in G and B is abelian, or*
- *H has finite index in G.*

Here "profinitely closed" means closed in the profinite topology, which for a subgroup of G just means being an intersection of finite index subgroups of G.

If Γ is a group, a statement stronger that just saying whether Γ is residually finite is to describe the largest residually finite quotient Γ_{rf} of Γ. For wreath products this can be stated as follows, leaving aside the degenerate cases:

Proposition 1.8. *Let $W = B \wr_{G/H} G$ be a wreath product with G/H infinite. Let $\overline{H} \subset G$ be the closure of H in the profinite topology, and let B_{abr} be the residually finite abelianization of B. Then there is a canonical isomorphism*

$$W_{\mathrm{rf}} \simeq B_{\mathrm{abr}} \wr_{G/\overline{H}} G_{\mathrm{rf}}.$$

In particular, if $W = B \wr G$ is a standard wreath product with G infinite, there is a canonical isomorphism

$$W_{\mathrm{rf}} \simeq B_{\mathrm{abr}} \wr G_{\mathrm{rf}}.$$

Here B_{abr} is defined as the largest abelian residually finite quotient of B, namely the quotient of the abelianization B_{ab} by its subgroup of divisible elements. Note that if B is finitely generated then $B_{\mathrm{abr}} = B_{\mathrm{ab}}$.

This simple result can be useful to compute the virtual first Betti number. If Γ is a group, recall that its first virtual Betti number $\mathrm{vb}^1(\Gamma) = \sup \mathrm{b}^1(\Lambda)$, where Λ ranges over finite index subgroups, and the first Betti number $\mathrm{b}^1(\Lambda)$ is defined as the $\mathbf{Q}$-rank of $\mathrm{Hom}(\Lambda, \mathbf{Z})$. (If the numbers are infinite we just write ∞ and ignore a discussion between infinite cardinals here.) Actually there are several non-equivalent alternative definitions of the first Betti number (because $\mathrm{b}^1(\mathbf{Q}) = 0$ with this definition and taking homomorphisms into $\mathbf{Q}$ instead of $\mathbf{Z}$, or considering the $\mathbf{Q}$-rank of the abelianization could yield to a different number), but all of them coincide for finitely generated groups.

Corollary 1.9. *Let $W = B \wr_{G/H} G$ be a wreath product with G/H infinite. Then its first virtual Betti number is given by*

$$\mathrm{vb}^1(W) = \mathrm{b}^1(B)[G : \overline{H}] + \mathrm{vb}^1(G)$$

(where $0 \cdot \infty = 0$.) In particular, if G an infinite residually finite group and B is any group, then the first virtual Betti number of the standard wreath product $B \wr G$ is equal to:

- ∞ *if* $\mathrm{Hom}(B, \mathbf{Z}) \neq 0$;
- $\mathrm{vb}^1(G)$ *if* $\mathrm{Hom}(B, \mathbf{Z}) = 0$.

Example 1.10. If D_∞ is the infinite dihedral group, we have $\mathrm{vb}^1(\mathbf{Z} \wr \mathbf{Z}) = \infty$ and $\mathrm{vb}^1(D_\infty \wr \mathbf{Z}) = 1$. This example is considered by Shalom [Sha04] to indicate an example of two quasi-isometric amenable groups that have distinct first virtual Betti number (although this contradicts the claim in [Sha04, p. 121] that $\mathrm{vb}^1(D_\infty \wr \mathbf{Z}) = 2$).

Recall that a group is called large if some finite index subgroup admits the free group of rank 2 as a quotient. Note that a large group Γ satisfies $\mathrm{vb}^1(\Gamma) = \infty$. The following corollary shows that wreath products do not provide interesting examples of large groups.

Corollary 1.11. *Let $W = B \wr_{G/H} G$ be a wreath product with G/H infinite. Then W is large if and only if G is large.*

Finite presentability of wreath products was considered in [Cor06]; unlike in the standard case due to G. Baumslag [Ba61], there are many interesting instances in the permutational case.

Corollary 1.12. *Let $W = B \wr_{G/H} G$ be a finitely presented wreath product. Then* $\mathrm{vb}^1(W) = \infty$ *if and only if* $\mathrm{vb}^1(G) = \infty$, *or* $\mathrm{b}^1(B) \neq 0$ *and $\overline{H}$ has infinite index in G. In particular, if W is finitely presented,* $\mathrm{vb}^1(W) = \infty$ *if and only if* $\mathrm{vb}^1(G) = \infty$.

Remark 1.13. It is unknown whether there exists a finitely presented group Γ that is not large but satisfies $\mathrm{vb}^1(\Gamma) = \infty$. (In the finitely generated setting, wreath products provide plenty of examples, the simplest of which is the well-known $\mathbf{Z} \wr \mathbf{Z}$.)

Remark 1.14. A characterization of permutational wreath products that are linear over a field of given characteristic was obtained by Wehrfritz [We97]. The case of standard wreath products was done much before in several papers by Vapne and Wehrfritz (see the references in [We97]); for instance a standard wreath product $B \wr G$ with G infinite is linear in characteristic zero if and only if B is torsion-free abelian and G is virtually torsion-free abelian. In particular, if G is an arbitrary infinite residually finite group, then $\mathbf{Z} \wr G$ is always residually finite, but is linear in characteristic zero only if G is virtually abelian (actually it is never linear over a field of positive characteristic).

2. AFI subgroups

Let G be a group and H a subgroup. The following definition is useful to deal with the definition of being AFI.

Definition 2.1. Let F be a finitely presented group with a homomorphism ρ into G.

The pair (G, H) is AFI with respect to (F, ρ) the subgroup $\rho^{-1}(H)$ is a limit in $\mathcal{S}(F)$ of a sequence (L_n) of finite index subgroups of F.

The pair (G, H) is KAFI with respect to (F, ρ) if moreover it satisfies the "kernel condition": the above sequence (L_n) can be chosen to satisfy: for every finite subset U of $\mathrm{Ker}(\rho)$, eventually L_n contains the normal closure of U.

Thus, by definition, H is AFI in G if and only if for every finitely presented group F and homomorphism $\rho : F \to G$, the pair (G, H) is AFI with respect to (F, ρ).

Lemma 2.2. *Let G be a group. Then the set of subgroups $H \leq G$ such that H is AFI in G is closed in $\mathcal{S}(G)$.*

Proof. If (F, ρ) is given and $H_i \to H$, then $\rho^{-1}(H_i) \to \rho^{-1}(H)$. Therefore if each $\rho^{-1}(H_i)$ is a limit of finite index subgroups, then so is $\rho^{-1}(H)$. $\square$

Lemma 2.3. *If $f : G' \to G$ is a homomorphism and H is AFI in G, then $f^{-1}(H)$ is AFI in G'. More generally, if $\rho : F \to G'$ is a homomorphism with F finitely presented and (G, H) is AFI (resp. KAFI) with respect to $(F, f \circ \rho)$ then $(G', f^{-1}(H))$ is AFI (resp. KAFI) with respect to (F, ρ).*

Also, if (G, H) is AFI (resp. KAFI) with respect to (F, ρ) and $u : F' \to F$ is a homomorphism between finitely presented groups, then (G, H) is AFI (resp. KAFI) with respect to $(F', \rho \circ u)$.

Proof. This is immediate. $\square$

Lemma 2.4. *Assume that G is finitely presented and let $H \leq G$ be a subgroup. The following conditions are equivalent:*

(i) H is AFI in G;

(ii) H is a limit in $\mathcal{S}(G)$ of finite index subgroups.

Actually the implication (ii)⇒(i) holds without assuming G finitely presented.

Proof. (i)⇒(ii) is trivial (picking $F = G$, since G is finitely presented). Assume (ii) (G being arbitrary), which means that (G, H) is AFI with respect to (G, id). Let F be a finitely presented group with a homomorphism into G. Then by the last statement in Lemma 2.3, (G, H) is AFI with respect to (F, ρ). $\square$

Remark 2.5. When G is a group and H is a finitely generated subgroup, the condition (ii) means that H is profinitely closed in G, i.e., is an intersection of finite index subgroups of G.

Example 2.6. Recall that a group G is called LERF ("locally extended residually finite"), or "subgroup separable", if every finitely generated subgroup is profinitely closed in G. In such a group, every subgroup is AFI: indeed, the set of AFI subgroups is closed (Lemma 2.2) and contains all finitely generated subgroups by Lemma 2.4. Since the set of finitely generated subgroups is always dense, it follows that the set of AFI subgroups is all of $\mathcal{S}(G)$.

In particular, if F is a free group, then every subgroup is AFI in F. Indeed the LERF property for F was established by M. Hall [Hal49]. Actually, it follows from Lemma 2.4 and Remark 2.5 that if G is a finitely presented group, then G is LERF if and only every subgroup of G is AFI in G.

On the other hand, Lemma 2.4 provides examples of non-AFI subgroups. Indeed, if G is a finitely presented group with no nontrivial finite quotients (e.g. infinite and simple), then its only AFI subgroup is G itself.

Lemma 2.7. *Assume that G is finitely generated. The following assertions are equivalent:*

(i) H is AFI in G;
(ii) (G, H) is AFI with respect to (F, ρ) for every finitely presented group F and every surjective homomorphism $\rho : F \to G$;
(iii) (G, H) is KAFI with respect to (F, ρ) for every finitely presented group F and every surjective homomorphism $\rho : F \to G$;
(iv) (G, H) is KAFI with respect to (F, ρ) for some finitely presented group F and some surjective homomorphism $\rho : F \to G$.

Proof. (i)⇒(iii) Given (F, ρ), fix a finite generating subset in F so that the n-ball makes sense. Also write $\mathrm{Ker}(\rho)$ as an ascending union of normal subgroup finitely generated qua normal subgroups N_n. For every n, in F/N_n, there exists a finite index subgroup coinciding with H on the n-ball. Let L_n be its inverse image in F. Then $\lim L_n = \rho^{-1}(H)$ and (L_n) satisfies the kernel condition.

(ii)⇒(i) Let F be a finitely presented group and $\rho : F \to G$ be a homomorphism. There exists a free group of finite rank F' so that we can extend ρ to a surjective homomorphism $\rho' : F * F' \to G$. Write $(\rho')^{-1}(H) = \lim L_n$. Then $\rho^{-1}(H) = (\rho')^{-1}(H) \cap F$ and is the limit of the sequence $(F \cap L_n)$ in $\mathcal{S}(F)$.

(iii)⇒(ii) and (iii)⇒(iv) are trivial.

(iv)⇒(iii) Suppose that $\rho_0 : F_0 \to G$ satisfies the condition defining AFI. Let $\rho : F \to G$ be a surjective homomorphism. By a simple verification [CGP07, Lemma 1.3(3)] there exists a finitely presented group F' with surjective homomorphisms $\rho' : F' \to F$, $\rho'_0 : F' \to F_0$ such that $u = \rho_0 \circ \rho'_0 = \rho \circ \rho'$.

Write $\rho_0^{-1}(H) = \lim L_n$ with L_n of finite index and define $M_n = (\rho'_0)^{-1}(L_n)$. So $(\rho_0 \circ \rho'_0)^{-1}(H) = \lim M_n$, which have finite index in F', and satisfy the kernel condition (for the homomorphism u). Since the kernel of ρ' is finitely generated as a normal subgroup, the kernel condition implies that it is contained in $(\rho'_0)^{-1}(L_n)$ for n large enough. It follows that $\rho'(M_n)$, which has finite index in F, satisfies $\lim \rho'(M_n) = \lim \rho'((\rho_0 \circ \rho'_0)^{-1}(H)) = \rho^{-1}(H)$ and satisfies the kernel condition. □

Remark 2.8. On the other hand, the condition "(G, H) is AFI with respect to (F, ρ) for some finitely presented group F and some surjective homomorphism

$\rho : F \to G$", akin to Lemma 2.7, is not interesting as it is satisfied by every pair (G, H) with G finitely generated. Indeed, just pick a finitely generated free group with a surjective homomorphism onto G and use Example 2.6.

Lemma 2.9. *If P is a subgroup of G and H is AFI in G then $H \cap P$ is AFI in $G \cap P$. Conversely, if $H \cap P$ is AFI in $G \cap P$ for every finitely generated subgroup P of G then H is AFI in G.*

Proof. This is trivial. □

Proposition 2.10. *The following statements are equivalent:*

(i) the subgroup H is AFI in G;
(ii) there exists a set I, an ultrafilter ω on I, a family of groups Γ_i and a family of finite index subgroups $\Lambda_i \subset \Gamma_i$, and a homomorphism j of G into the ultraproduct $\prod^\omega \Gamma_i$ such that $j^{-1}\left(\prod^\omega \Lambda_i\right) = H$.
(iii) Same as (ii), with j injective.

In (ii) and (iii), the Γ_i can be chosen to be finitely presented; if G is countable the set I can be chosen to be countable. Also if H is normal in G then the Λ_i can be chosen to be normal, and the statements are then also equivalent to: G/H is isomorphic to a subgroup of some ultraproduct of finite groups.

Proof. (iii)⇒(ii) is trivial.

(i)⇒(iii) Suppose that H is AFI in G. If X is a set denote by $\mathbf{F}_X$ the free group over X, whose basis we denote by $(e_x)_{x\in X}$; if $X \subset G$ let p_X be the unique homomorphism $\mathbf{F}_X \to G$ defined by the assignment $e_x \to x$.

Let I be the set of triples (X, N, Λ) where X ranges over finite subsets of G, where N ranges over normal subgroups of $\mathbf{F}_X$ that are finitely generated qua normal subgroup and contained in $\mathrm{Ker}(p_X)$, and Λ ranges over finite index subgroups of $\mathbf{F}_X/N$. Denote by π_N the projection $\mathbf{F}_X \to \mathbf{F}_X/N$ (note that p_X factors through π_N). If

- Y is a finite subset of G,
- Z a finite subset of $\mathrm{Ker}(p_Y)$,
- U is a finite subset of $p_Y^{-1}(H)$,
- V is a finite subset of $p_Y^{-1}(G \smallsetminus H)$,

let $I_{Y,Z,U,V}$ be the set of (X, N, Λ) in I such that

- $X \supset Y$,
- $N \supset Z$,
- $\pi_N(U) \subset \Lambda$,
- $\pi_N(V) \cap \Lambda = \emptyset$.

Then $I_{Y,Z,U,V}$ is non-empty, as a consequence of H being AFI in G. Besides, for all (Y, Z, U, V) and (Y', Z', U', V'), we have

$$I_{Y,Z,U,V} \cap I_{Y',Z',U',V'} \supset I_{Y\cup Y',Z\cup Z',U\cup U',V\cup V'}.$$

It follows that there exists some ultrafilter ω on I containing $I_{Y,Z,U,V}$ for all Y,Z,U,V.

If $(X,N,\Lambda)\in I$, consider the (set) map $j_{X,N,\Lambda}:G\to\mathbf{F}_X/N$ mapping $x\in X$ to its canonical image, and $y\notin X$ to the trivial element. This induces a map j from G to $\prod^{\omega}_{(X,N,\Lambda)\in I}\mathbf{F}_X/N$. We claim that the latter is an injective homomorphism.

If $g,h\in G$, define $Y=\{g,h,gh\}$ and $Z=\{e_ge_he_{gh}^{-1}\}$. Then for every $(X,N)\in I_{Y,Z,\emptyset,\emptyset}$ we have $j_{X,N}(gh)=j_{X,N}(g)j_{X,N}(h)$, and hence $j(gh)=j(g)j(h)$. Similarly, $j(g^{-1})=j(g)^{-1}$. Finally, if $g\neq 1$ and $Y=\{g\}$, then for every $(X,N,\Lambda)\in I_{Y,\emptyset,\emptyset,\emptyset}$ we have $j_{X,N,\Lambda}(g)\neq 1$. It follows that $\mathrm{Ker}(j)=\{1\}$.

If $h\in H$, and $Y=\{h\}$ and $U=\{e_h\}$, the element $j_{X,N,\Lambda}(h)$ is equal to $\pi_N(e_h)$, which by assumption belongs to Λ, so $j_{X,N,\Lambda}(h)\in\Lambda$ for all $(X,N,\Lambda)\in I_{\{h\},\emptyset,\{h\},\emptyset}$. Similarly, if $g\notin H$, we obtain $j_{X,N,\Lambda}(h)\notin\Lambda$ for all $(X,N,\Lambda)\in I_{\{h\},\emptyset,\emptyset,\{h\}}$. It follows that $j^{-1}(\prod^{\omega}_{(X,N,\Lambda)}\Lambda)=H$.

(ii)⇒(i) Suppose there is such a homomorphism $j:G\to\prod^{\omega}_{i\in I}\Gamma_i$ such that $H=j^{-1}(\prod^{\omega}\Lambda_i)$. Let F be a finitely presented group and ρ a homomorphism $F\to G$. Since F is finitely presented, there exists a homomorphism $\prod_i f_i:F\to\prod_{i\in I}\Gamma_i$ lifting $j\circ\rho$. Then $L_i=f_i^{-1}(\Lambda_i)$ has finite index in F. Let us check that $\lim_\omega L_i=\rho^{-1}(H)$ in $\mathcal{S}(F)$.

Suppose $x\in F$. Then $\rho(x)=(f_i(x))^{\omega}$. Then

$$x\in\rho^{-1}(H)\Leftrightarrow j\circ\rho(x)\in\prod^{\omega}\Lambda_i\Leftrightarrow\forall_\omega i,f_i(x)\in\Lambda_i\Leftrightarrow\forall_\omega i,x\in L_i;$$

this exactly means that $\lim_\omega L_i=\rho^{-1}(H)$.

All assertions in the last statement, except at first sight the one when H is normal, hold by construction (in the proof of (i)⇒(iii)). Actually, if N is a normal subgroup in a group A, then H is a limit of finite index subgroups in $\mathcal{S}(A)$ if and only if it is a limit of finite index normal subgroups (taking the normal closure). It follows that assuming H normal, if in the proof of (i)⇒(iii)) we define I by only considering those (X,N,Λ) with Λ normal, then the proof works with no change. The last (well-known) characterization holds since if we have $j:G\to\prod^{\omega}\Gamma_i$ with $H=j^{-1}(\prod^{\omega}\Lambda_i)$ and Λ_i is normal in Γ_i, then G/H embeds as a subgroup of $\prod^{\omega}\Gamma_i/\Lambda_i$. □

3. Wreath products

3.1. Approximate finiteness.

Proposition 3.1. *Let $W=B\wr_X G$ be a wreath product, where B is non-trivial and X is a nonempty union of orbits G/H_i. Then W is AF if and only if B,G are AF and all H_i are AFI in G.*

Proof. Suppose that the condition is satisfied. We can suppose B,G are finitely generated and X has finitely many G-orbits. Since W embeds in the direct product of all $B\wr_{G/H_i}G$, we can reduce to the transitive case $X=G/H$. Writing, in a suitable space of marked groups, $B = \lim B_n$ with B_n finite, we have $W = \lim B_n \wr_X G$. So we can suppose B finite. Write $G = \lim G_n$ where G is a quotient of the finitely presented group G_n; let H_n be the inverse image of H, so H_n is AFI in G. Then $W = \lim B \wr_{G_n/H_n} G_n$. So we are in addition reduced to G finitely presented. Now, using Lemma 2.4, we can write $H = \lim H_n$ with H_n of finite index in G. Then $W = \lim B \wr_{G/H_n} G$. So we are reduced to the case when G/H is finite. Such a group is abstractly commensurable with a direct product $B^k \times G$ and thus is AF.

Conversely assume that W is AF. Then (because $X \neq \emptyset$) it follows that B and G are AF. Let us show H_i is AFI in G, writing $H = H_i$. We can suppose that G is finitely generated. If b is a nontrivial element and d the element of $B^{(G/H)}$ supported by the base-point with value b, then the centralizer of d in G is exactly H. Fix a finitely presented group F with a surjective homomorphism p onto $U = \langle G, d \rangle$, with kernel N. Since U is AF, there is a sequence of normal subgroups of finite index $M_n \subset F$ tending to N. Write $C_n = \{x \in F \mid [x,d] \subset M_n\}$. Then (C_n) tends to

$$C = \{x \in F \mid [x,d] \subset N\} = p^{-1}(H).$$

Note that $M_n \subset C_n$. It follows that for every finite subset S of N, eventually C_n contains the normal closure of S (because it contains M_n which is normal and eventually contains S). So (U, H) is KAFI with respect to (F, p). By Lemma 2.7, it follows that H is AFI in U. By Lemma 2.9, it follows that H is AFI in G.

By definition of AFI, it follows that C/N (the centralizer of b) is AFI in $\langle b, G \rangle$, and restricting to G it follows that H is AFI in G. □

3.2. **Residual finiteness.**

Proof of Proposition 1.8. We first need to show that if B is abelian and residually finite, if H is closed in the profinite topology and G is residually finite then the wreath product $B \wr_{G/H} G$ is residually finite. If $\gamma = (b_x)_{x \in G/H} g$ is a nontrivial element, let us find a finite quotient where it survives; if $g \neq 1$ this is clear; assume $g = 1$. Then there exists a finite index subgroup L containing H such that the support of (b_x), namely the set of x such that $b_x \neq 1$, is mapped injectively into G/L by the canonical projection $\pi : G/H \to G/L$. Since B is abelian, the canonical homomorphism $B \wr_{G/H} G \to B \wr_{G/L} G$ mapping $(b'_x)_{x \in G/H} g'$ to $(b''_y)_{y \in G/L} g'$, where $b''_y = \sum_{\{x \mid \pi(x) = y\}} b'_x$, is well-defined. It maps γ to a nontrivial element.

So it is enough to check that $W' = B \wr_{G/L} G$ is residually finite. Since L has finite index, and hence contains a normal subgroup N of finite index, W' admits a finite index subgroup isomorphic to $B^{[G:L]} \times N$, which is residually finite; so W' is residually finite as well.

Therefore, in the setting of Proposition 1.8, there is a natural surjective homomorphism $W_{\mathrm{rf}} \to B_{\mathrm{abr}} \wr_{G/\overline{H}} G_{\mathrm{rf}}$. Let us show it is an isomorphism. This amounts to showing that

(1) elements of $\mathrm{Ker}(G \to G_{\mathrm{rf}})$ have a trivial image in W_{rf};
(2) elements of $[B, \overline{H}]$ have a trivial image in W_{rf};
(3) elements of $\mathrm{Ker}(B \to B_{\mathrm{rf}})$ have a trivial image in W_{rf}.

(1) is trivial. If Γ is a group, let $\mathfrak{pf}(\Gamma)$ be the profinite topology on Γ. If X is a subset of G or B, let $\hat{X}$ be its image in W_{rf}.

For (2), observe that the centralizer of $\hat{B}$ in $\hat{G}$ is closed in $(\hat{G}, \mathfrak{pf}(\hat{G}))$ and contains H, hence contains its profinite closure.

For (3), observe that the set $\{g \in \hat{G} \mid [g\hat{B}g^{-1}, \hat{B}] = 1\}$ is closed and contains $\widehat{G \smallsetminus H}$; since H has infinite index in G, it has an empty interior in $(G, \mathfrak{pf}(G))$. So $\widehat{G \smallsetminus H}$ is dense in $(\hat{G}, \mathfrak{pf}(\hat{G}))$, and hence, using that $(\hat{G}, \mathfrak{pf}(\hat{G}))$ is Hausdorff, we deduce that $[g\hat{B}g^{-1}, \hat{B}] = 1$ for all $g \in G$; in particular for $g = 1$ we deduce that $\hat{B}$ is abelian. It follows that elements of $[B, B]$ have a trivial image in W_{rf}; since W_{rf} is residually finite, (3) follows. □

Proposition 3.2. *Let $W = B \wr_X G$ be a wreath product, where B is a non-trivial group, and X is a non-empty disjoint union of G-orbits G/H_i. Then W is residually finite if and only if B, G are residually finite, and either*

- *B is residually finite abelian and all H_i are profinitely closed in G, or*
- *all H_i have finite index in G.*

Proof. Suppose that the condition are satisfied. Then there is a canonical homomorphic embedding of W into $\prod_i B \wr_{G/H_i} G$; each factor is residually finite by Proposition 1.8 and hence so is W.

Conversely, suppose that W is residually finite. Then G is residually finite. Then for each i, $B \wr_{G/H_i} G$ is residually finite, and it follows from Proposition 1.8 that H_i is closed in the profinite topology for every i. If for some i, H_i has infinite index, it follows from Proposition 1.8 that B is residually finite abelian. □

3.3. Virtual first Betti number.

Proposition 3.3. *If G is a group, then $\mathrm{vb}^1(G) = \sup \mathrm{vb}^1(G/N)$, where G/N ranges over finitely generated virtually abelian quotients of G with no nontrivial finite normal subgroup. In particular, $\mathrm{vb}^1(G) = \mathrm{vb}^1(G_{\mathrm{rf}})$.*

Note that if Γ is finitely generated and virtually abelian, say virtually $\mathbf{Z}^k$, then $\mathrm{vb}^1(\Gamma) = k$.

Proof. The inequality $\geq$ is trivial. To show the inequality $\leq$, suppose that $\mathrm{vb}^1(G) \geq k$ and let us find a virtually abelian quotient G/N such that $\mathrm{vb}^1(G/N) \geq k$.

Let M be a finite index normal subgroup of G (say of index ℓ) with $\mathrm{b}^1(M) \geq k$. This means that M has k $\mathbf{Q}$-linearly independent homomorphisms into $\mathbf{Z}$. Hence M has a surjective homomorphism into $\mathbf{Z}^k$, say with kernel P. Note that P is normalized by M. Define $N = \bigcap_{g \in G/M} gPg^{-1}$. Then M/N naturally embeds as a subgroup of $\mathbf{Z}^{k\ell}$, hence is free abelian of finite rank, actually at least equal to k since it admits $M/P \simeq \mathbf{Z}^k$ as a quotient. So G/N is a finitely generated, virtually abelian quotient of G, and $\mathrm{vb}^1(G/N) \geq k$. Actually, we can mod out by its largest finite normal subgroup (this clearly does not change vb^1) to ensure the latter is trivial.

For the last statement, just apply the result to G_{rf} and observe that all finitely generated virtually abelian quotients of G are actually quotients of G_{rf}. □

Lemma 3.4. *Let A be an abelian group and B a subgroup of finite index. If* $\mathrm{Hom}(A, \mathbf{Z}) = 0$ *then* $\mathrm{Hom}(B, \mathbf{Z}) = 0$.

Proof. By contraposition, consider a nonzero homomorphism $B \to \mathbf{Z}$. By injectivity of the $\mathbf{Z}$-module $\mathbf{Q}$, it extends to a nonzero homomorphism $A \to \mathbf{Q}$, whose image is virtually infinite cyclic and abelian, hence cyclic, proving the lemma. □

Proof of Corollary 1.9. By Propositions 3.3 and 1.8, we have

$$\mathrm{vb}^1(W) = \mathrm{vb}^1(W_{\mathrm{rf}}) = \mathrm{vb}^1\left(B_{\mathrm{abr}} \wr_{G/\overline{H}} G_{\mathrm{rf}}\right).$$

Let us now discuss. If $[G : \overline{H}]$ is finite, then W_{rf} is abstractly commensurable to $B_{\mathrm{abr}}^{G/\overline{H}} \times G_{\mathrm{rf}}$, and hence its virtual first Betti number (which is additive under direct products) is given by $\mathrm{b}^1(B)[G : \overline{H}] + \mathrm{vb}^1(G)$ as desired.

If $\mathrm{b}^1(B) = 0$, then for every homomorphism ϕ from W onto a finitely generated virtually abelian group Λ, $\phi(B^{(G/H)}$ is finite (and normal). Indeed if infinite, then we would obtain a nonzero homomorphism from some finite index subgroup of $B^{(G/H)}$ into $\mathbf{Z}$, and thus contradict Lemma 3.4. So assuming Λ has no nontrivial finite normal subgroup, we deduce that $B^{(G/H)}$ is in the kernel. We deduce (in view of Proposition 3.3) that $\mathrm{vb}^1(B \wr_{G/H} G) = \mathrm{vb}^1(G)$.

The remaining case is when $\mathrm{b}^1(B) \neq 0$ and $[G : \overline{H}]$ is infinite, in which case we have to show that $\mathrm{vb}^1(W) = \infty$. Indeed, we can find a finite index subgroup containing H with $[G : L]$ arbitrary large. Then we have surjective homomorphisms

$$B \wr_{G/H} G \twoheadrightarrow \mathbf{Z} \wr_{G/H} G \twoheadrightarrow \mathbf{Z} \wr_{G/L} G.$$

By the case of finite index above, we have $\mathrm{vb}^1(\mathbf{Z} \wr_{G/L} G) \geq [G : L]$, and hence $\mathrm{vb}^1(B \wr_{G/H} G) \geq [G : L]$. Since the latter number can be chosen arbitrary large, it follows that $\mathrm{vb}^1(B \wr_{G/H} G) = \infty$. □

To deal with largeness, we need to replace finitely generated virtually abelian groups by the following notion.

Definition 3.5. An extra-free group is a group Γ which is isomorphic a nontrivial subdirect product of non-abelian free groups, i.e. such that there exists $k \geq 1$ and non-abelian free groups $F_{(1)}, \dots, F_{(k)}$ such that Γ is isomorphic to a subgroup of the direct product $\prod_i F_{(i)}$ all of whose projections on factors are surjective. It is called of finite rank if all $F_{(i)}$ can be chosen of finite rank.

This definition allows to define largeness in terms of quotients (rather than virtual quotients).

Lemma 3.6. *A group* Γ *is large if and only if it has quotient that is virtually extra-free of finite rank. In particular,* Γ *is large if and only if* Γ_{rf} *is large.*

Proof. The "if" condition is trivial. Conversely suppose that Γ is large. Let M be a normal subgroup of finite index with a surjective homomorphism into $F_{(2)}$, with kernel P. Define $N = \bigcap_{g\in\Gamma/M} gPg^{-1}$. Then N is normal in G and is also the kernel of the map defined coordinate-wise as the projection from M to $\prod_{g\in\Gamma/M} M/gPg^{-1}$. Thus this maps induces an embedding of M/N as a subdirect product of the copies M/gPg^{-1} of F_2. Thus M/N is extra-free of finite rank and hence G/N is virtually extra-free of finite rank.

The last statement immediately follows, since virtually extra-free implies residually finite. □

Lemma 3.7. *Let* Γ *be a group. If* Γ *is not large and* f *is a homomorphism from* Γ *to a virtually extra-free group (or more generally virtually residually free), then its image* $f(\Gamma)$ *is virtually abelian.*

Proof. Let Λ admit a residually free subgroup of finite index Λ', and consider a homomorphism $f : \Gamma \to \Lambda$. Define $\Gamma_1 = f^{-1}(\Lambda_1)$. Consider an embedding $i = (i_j)_j$ of Λ_1 into a (possibly infinite) unrestricted product $\prod_j F_{(j)}$ of free groups. Since Γ is not large, for each j, the composite map $i_j \circ f$ (defined on Γ_1) has an abelian image. Hence $i \circ f$ (defined on Γ_1) has an abelian image. Since i is injective, it follows that $f(\Gamma_1)$ is abelian. Hence $f(\Gamma)$ is virtually abelian. □

Proof of Corollary 1.11. Let us assume that G is not large and let us show that W is not large. By Lemma 3.6, it is enough to show that if f is a homomorphism from W to an extra-free group, then f is not surjective.

First observe that since every extra-free group is residually finite, f factors through W_{rf}, which is naturally isomorphic to $B_{\mathrm{abr}} \wr_{G/\overline{H}} G_{\mathrm{wr}}$. Hence the image of $B^{(G/H)}$ by f is an abelian normal subgroup in $f(W)$. Also, by Lemma 3.7, $f(G)$ is virtually abelian. Hence $f(W) = f(B^{(G/H)}f(G)$ is virtually solvable, so f cannot be surjective. □

Proof of Corollary 1.12. The first statement is a trivial consequence of Corollary 1.9; the second follows from the fact that if $B \neq 1$ and $\overline{H}$ has infinite index, then $B \wr_{G/H} G$ is not finitely presented [Cor06]. □

References

[Ba61] Gilbert Baumslag. Wreath products of finitely presented groups. Math. Z. **75** , 22-28, 1961.

[CGP07] Y. Cornulier, L. Guyot, W. Pitsch. On the isolated points in the space of groups. J. Algebra 307(1) (2007), 254-277.

[Cor06] Y. Cornulier. Finitely presented wreath products and double coset decompositions. Geom. Dedicata 122 (2006), 89–108.

[GV97] E. I. Gordon and A. M. Vershik. Groups that are locally embeddable in the class of finite groups. St. Petersburg Math. J. 9 (1998) no. 1, 49–67. Translation from Algebra Anal. 9, No.1, 71-97 (1997)

[Gru57] K. W. Gruenberg. Residual properties of infinite soluble groups. Proc. London Math. Soc. (3) **7**, 1957.

[Hal49] M. Hall Jr. Coset representations in free groups. Trans. Amer. Math. Soc. 67, 421-432, 1949.

[Ma73] A. Malcev. I. Algebraic systems. Posthumous edition, edited by D. Smirnov and M. Taclin. Translated from the Russian by B. D. Seckler and A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 192. Springer-Verlag, New York-Heidelberg, 1973.

[Sha04] Y. Shalom. Harmonic analysis, cohomology, and the large scale geometry of amenable groups. Acta Mathematica 193, 119-185, 2004.

[St84] A. Stëpin. A remark on the approximability of groups. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1984, no. 4, 85–87. English translation: Moscow Univ. Math. Bull. 39 (1984), 90–93.

[We97] B. Wehrfritz. The linearity of wreath products. Mathematika, 44 (1997), 357–367

Laboratoire de Mathématiques, Bâtiment 425, Université Paris-Sud 11, 91405 Orsay, FRANCE

E-mail address: yves.cornulier@math.u-psud.fr