Unirationality and R-equivalence for conic bundles over quasi-finite fields

Elyes Boughattas

Abstract

Yanchevskiĭ had asked whether conic bundle surfaces over $\mathbf{P}^1_k$ are unirational when k is a finite field. We give a partial answer to his question by showing that for quasi-finite fields k (e.g. finite fields) a regular conic bundle X over $\mathbf{P}^1_k$ is unirational if all non-split fibres lie over rational points. For large finite fields k, this beats a previous result of Mestre. Under the same assumption, we also prove that all rational points of X are R-equivalent.

1 Introduction

Let k be a field. A k-variety is said to be *rational* (resp. *unirational*) if there exist $n \in \mathbf{Z}_{\geq 0}$ and a rational map $\mathbf{P}^n_k \dashrightarrow X$ that is birational (resp. dominant). In the case of smooth geometrically rational projective surfaces, the classification of their minimal models by seminal works of Manin [Man66][Man67] and Iskovskikh [Isk79] splits the study of unirationality into two families. The first family consists in del Pezzo surfaces X, that is, smooth projective surfaces with ample anticanonical bundle K_X. To X is associated its degree $d := (K_X)^2 \in \{1, \ldots, 9\}$ and works of Segre, Manin and Kollár showed that X is unirational whenever $X(k) \neq \emptyset$ and $d \geq 3$, over any field k. When $d = 2$, partial results are given by Salgado, Testa and Várilly-Alvarado [STVA14] and the case of finite fields has been fully tackled by Festi and van Luijk [FvL16]. The second family consists in conic bundles over a smooth curve. To define them, we recall that a *conic* over k is the vanishing locus of a quadric in $\mathbf{P}^2_k$.

Definition 1.1. For any scheme S, a *conic bundle over S* is a proper and flat morphism $f : X \to S$ where X is integral, with a smooth generic fibre and such that all geometric fibres of f are isomorphic to a conic. The conic bundle f is called *regular* if X is regular.

By Lüroth's theorem, if $X \to C$ is a conic bundle over a smooth, integral, projective curve C, a necessary condition for the unirationality of X is that $C \simeq \mathbf{P}^1_k$, which lets us, from now on, restrict our attention to that case. The following question was raised by Iskovskikh in [Isk67, §4.5] (see also [Yan92, Problem]).

Question 1.2. If k is a field and $X \to \mathbf{P}^1_k$ is a conic bundle such that $X(k) \neq \emptyset$, is X unirational?

Although this question is still open, it is expected to have a negative answer in general. Using the terminology of [Sko96, Definition 0.1], we say that a conic is *non-split* if it is singular and irreducible. For a conic bundle $f : X \to \mathbf{P}^1_k$, we denote by $\delta \in \mathbf{Z}_{\geq 0}$ the degree of $\{t \in \mathbf{P}^1_k : X_t \text{ is non-split}\}$, which may also be understood as the number of geometric

---

Date: November 19, 2024

singular fibres of a minimal model of the generic fibre of f. When k has characteristic different from 2, conic bundles with a k-point are k-unirational if $\delta \leq 7$, by works of Segre, Iskovskikh, Manin, Kollár and Mella (see section 1 of [KM17] for a summary). Iskovskikh [Isk67] gave a positive answer to Question 1.2 for $k = \mathbf{R}$, or more generally when k is a real closed field. Later, Yanchevskiĭ [Yan85] proved the case of Henselian fields, Voronovich that of pseudo-algebraically closed fields [Vor86] and Yanchevskiĭ generalised this result to pseudo-closed fields, see [Yan92, Theorem 1]. Over finite fields, Yanchevskiĭ raised the following question (see the discussion preceding Problem 3 in [Yan90]).

Question 1.3 (Yanchevskiĭ)**.** Let $\mathbf{F}$ be a finite field. For all conic bundles $f : X \to \mathbf{P}^1_{\mathbf{F}}$, is X unirational?

The only result in this direction was given by Mestre in [Mes96] who proved that X is unirational when $\mathbf{F}$ is finite of characteristic different from 2 and $|\mathbf{F}| \geq \delta^2 \times 2^{\delta-3}$. Our first result is a partial answer to Question 1.3 over the larger family of 2-*quasi-finite* fields that we first define, jointly with the classical notion of quasi-finite fields. We recall that for a prime number p, the ring of p-adic integers is denoted by $\mathbf{Z}_p$.

Definition 1.4. Let k be a field. We say that k is 2-*quasi-finite* if it is perfect and if there exists a set of prime numbers S such that $2 \in S$ and the absolute Galois group of k is isomorphic to $\prod_{p \in S} \mathbf{Z}_p$. Furthermore, k is called *quasi-finite* if S is the set of all prime numbers.

By definition, quasi-finite fields are 2-quasi-finite Three classical examples of quasi-finite fields are that of finite fields, Laurent series over an algebraically closed field of characteristic zero and non-principal ultraproducts of finite fields (see e.g. [Ax68, §7, Proposition 3]). Our first result may then be stated as follows:

Theorem 1.5. Let k be a 2-quasi-finite field of characteristic different from 2 and $f : X \to \mathbf{P}^1_k$ a regular conic bundle. Denote by B the reduced divisor of $\mathbf{P}^1_k$ made of those points whose fibre by f is non-split. Consider the following assertion:

($\star$) *The set B is a union of rational points, one point of degree at most 2 and one point whose degree is odd.*

If condition ($\star$) is verified, then X is unirational.

The case where $B = \mathbf{P}^1(k)$ is not encapsulated in the aforementioned result of Mestre, whose bound is very restrictive for large $|k|$. The proof relies on a criterion of Enriques for unirationality of conic bundles which consists in building a rational curve on X intersecting a general fibre of f (see Proposition 3.2). For fields k with $\mathrm{cd}(k) \leq 1$, we prove that this amounts to constructing a finite morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that for each $t \in \varphi^{-1}(B)$, we have $2 \mid e(t/\varphi(t)) \times [\kappa(t) : \kappa(\varphi(t))]$, where $e(t/\varphi(t))$ denotes the ramification index of φ at t (see Theorem 3.1). More precisely, we show that the unirationality of X is equivalent to the existence of such a cover. When k is 2-quasi-finite, the latter is achieved under assumption ($\star$) by taking φ as a tower of well chosen degree 2 covers (see Section 4.2). Eventually, this gives a further insight into the limits of the method used by Mestre, whose construction of φ is made of precisely *one* degree 2 cover of $\mathbf{P}^1_k$, and a degree 2 cover satisfying the above property does not exist if B contains $\mathbf{P}^1(k)$.

When studying rational points on varieties, another topic of interest is that of *R-equivalence* introduced by Manin in [Man86, Chapter II, §14] and defined as follows.

Definition 1.6. Let k be a field and X a k-scheme. Two points x and y in $X(k)$ are called *directly R-equivalent* if there exists a rational map $g : \mathbf{P}^1_k \dashrightarrow X$ such that x and y both belong to $g(\mathbf{P}^1(k))$. The equivalence relation spanned by this relation is called *R-equivalence*. We further say that R-equivalence is *trivial* if $X(k)/R$ has cardinality 1.

The R-equivalence relation on $X(k)$ turns out to be interesting when, geometrically, X contains many rational curves. This is the case in the setting of geometrically rational surfaces and more generally of separably rationally connected varieties [Kol96, Chapter IV, Definition 3.2] which are defined as follows.

Definition 1.7. A variety X over a field k is *separably rationally connected* if for any algebraic closure $\overline{k}$ of k, there exists an integral $\overline{k}$-variety T and a rational map $e : \mathbf{P}^1_{\overline{k}} \times_{\overline{k}} T \dashrightarrow X_{\overline{k}}$ such that the map $\mathbf{P}^1_{\overline{k}} \times \mathbf{P}^1_{\overline{k}} \times T \dashrightarrow X_{\overline{k}} \times X_{\overline{k}}$ defined by $(z, z', t) \mapsto (e(z, t), e(z', t))$ is dominant and smooth at the generic point.

Various authors have studied triviality and finiteness of $X(k)/R$ for several classes of separably rationally connected varieties. For instance, it is already known that regular conic bundles over $\mathbf{P}^1_k$ with at most 5 reducible geometric fibres have only one R-equivalence class if k is an infinite, perfect, C_1 field of characteristic different from 2 [CTS87][CT15]. We refer the reader to [Kol99], [Pir12] and [CT15] for further results on different families of varieties. Our second result in this paper is on the triviality of R-equivalence for conic bundles $X \to \mathbf{P}^1_k$ over a 2-quasi-finite field k.

Theorem 1.8. Let us use the same notations as in Theorem 1.5 and consider the following assertion:

($\star\star$) *The set B is a union of rational points and one point which has either degree 2 or odd degree.*

If condition ($\star\star$) is verified, then $X(k)/R$ is trivial.

Eventually, in Corollary 5.3, we prove that for a finite field $\mathbf{F}$ and a conic bundle $X \to \mathbf{P}^1_{\mathbf{F}}$, the variety X is unirational and R-equivalence is trivial on it, if one assumes that rational points of a smooth, geometrically integral and separably rationally connected projective surface over $\mathbf{F}(t)$ are dense in its Brauer-Manin set (see Conjecture 5.1). This is an analogue of a conjecture of Colliot-Thélène and Sansuc over number fields [CTS80] that has been studied by various authors. Combined to Theorems 1.5 and 1.8, this leads to the following generalisation of Question 1.3 that we formulate over quasi-finite fields.

Question 1.9. Let k be a quasi-finite field and $X \to \mathbf{P}^1_k$ a conic bundle. Is X unirational and R-equivalence trivial on X?

Outline of the paper

In Section 2, we start by setting notations and definitions in §2.1. In §2.2, we state properties of 2-quasi-finite fields that are used all along the article. The content of §2.3 is known to the experts. We explain how one can detect split fibres of a regular conic bundle through the residues of its generic fibre. Eventually, in §2.4 we recall the definitions of weak approximation and the Brauer-Manin set over the function field of a curve over a finite field. We also recall the dictionary between rational points of a variety and sections of its models.

Section 3 is dedicated to a unirationality criterion for conic bundles over fields of cohomological dimension at moste one, stated in Theorem 3.1. In §3.1, we recall a criterion of Enriques on unirationality of conic bundles and we restate it over fields of cohomological dimension at most one, in Corollary 3.4. Eventually, a proof of Theorem 3.1 is given in §3.2. As a corollary, we also get in §3.3 a criterion for the triviality of R-equivalence.

In Section 4, we apply the criteria given in Section 3 to prove the unirationality and R-equivalence results for conic bundles over a 2-quasi-finite field stated in Theorems 1.5 and 1.8. This amounts to constructing well-chosen ramified covers $\mathbf{P}^1 \to \mathbf{P}^1$, for which we give general recipes in §4.1. We prove Theorem 1.5 in §4.2 and Theorem 1.8 in §4.3.

Eventually, in Section 5, we prove in Corollary 5.3 that the unirationality of conic bundle surfaces over finite fields is implied by Conjecture 5.1, which predicts the closure of rational points in the adelic set of a geometrically integral, smooth and separably rationally connected projective surface over $\mathbf{F}(t)$, for any finite field $\mathbf{F}$. The proof goes through Theorem 5.2, where we show that if rational points are dense in the Brauer-Manin set of a conic bundle surface, then it has weak weak approximation (resp. weak approximation if $|\mathbf{F}|$ is odd) and we conclude by the dictionary of §2.4.

Appendix A is dedicated to the study of Brauer groups of separably rationally connected surfaces. In §A.1 we supply, in Proposition A.1, a proof of the finiteness of the algebraic Brauer group, up to constant classes, of a separably rationally connected variety. In §A.2, we show, in Proposition A.3, that the Brauer group of a conic bundle surface, up to constant classes, is a 2-torsion group.

Acknowledgement

I am very grateful to Olivier Wittenberg who introduced me to this question, for many useful discussions and the time that he generously gave me. I would like to express my gratitude to Antoine Chambert-Loir and Alexei Skorobogatov for their suggestions and detailed comments on a previous version of this article. I also want to thank Jean-Louis Colliot-Thélène for discussions and suggestions on Section 5. The author was supported by a «Contrat doctoral spécifique normalien» from the École normale supérieure de Paris and by UKRI Future Leaders Fellowship `MR/V021362/1`.

2 Preliminaries

2.1 Notations

If A is an abelian group and $n \in \mathbf{Z}_{\geq 0}$, we denote by $A[n]$ the n-torsion part of A, by $A\{p\}$ the p-primary torsion part of A and $A\{p'\} := \bigoplus_{l \text{ prime},\, l \neq p} A\{l\}$.

If X is a scheme, we denote by $\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ the *Brauer group* of X and if R is a commutative ring we set $\mathrm{Br}(R) := \mathrm{Br}(\mathrm{Spec}(R))$. If k is a field, $\mathrm{Br}(k)$ is also the set of central simple algebras over k up to Brauer equivalence, see [GS06, §2.4].

When R is a discrete valuation ring, with function field K and residue field κ whose characteristic is different from 2, then we have Serre's *residue* map [CTS21, Definition 1.4.3.(ii)] $\partial : \mathrm{Br}(K)\{2\} \longrightarrow \mathrm{H}^1(\kappa, \mathbf{Q}_2/\mathbf{Z}_2)$ whose restriction to the 2-torsion of $\mathrm{Br}(K)$ gives rise to a map

$$r : \mathrm{Br}(K)[2] \longrightarrow \mathrm{H}^1(\kappa, \mathbf{Z}/2\mathbf{Z}). \tag{1}$$

If k is a field, we denote by $\mathrm{cd}(k)$ its *cohomological dimension* (see e.g. [Ser94, I.§3.1]). If G is the absolute Galois group of k and A an abelian group on which G acts trivially, we denote by $\mathrm{H}^1(k, A)$ the first Galois cohomology group of G with coefficients in A (see [Ser94, Chapitre I, §2]) which is canonically isomorphic to $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(k), A)$.

When $Y \to X$ is a morphism of schemes and A is an abelian group, we denote by $\mathrm{res}_{Y/X} : \mathrm{H}^1_{\text{ét}}(X, A) \to \mathrm{H}^1_{\text{ét}}(Y, A)$ the *restriction* morphism, see e.g. [CTS21, §2.2.4.(2.7)]. If F/E is an extension of fields, the restriction map $\mathrm{res}_{\mathrm{Spec}(F)/\mathrm{Spec}(E)}$ is also denoted $\mathrm{res}_{F/E}$ and we view it as a morphism $\mathrm{H}^1(E, A) \to \mathrm{H}^1(F, A)$ where the absolute Galois groups of E and F act trivially on A. If F/E is finite and separable, we denote by $\mathrm{cores}_{F/E} : \mathrm{H}^1(F, A) \to \mathrm{H}^1(E, A)$ the *corestriction* map, see e.g. [Ser94, Chapitre I, §2.4]).

A k-variety is a separated k-scheme of finite type and a *nice curve* over k is a proper, smooth and geometrically integral k-variety of dimension one. When X is a scheme, its set of codimension one points is denoted by $X^{(1)}$. Following [Sko96, Definition 0.1], we say that a k-variety X is *split* if it contains a geometrically integral open subscheme. If L is a field extension of k, then we say that L is *a splitting field* of V if $V \otimes_k L$ is split. A splitting field of V is *minimal* if it does not contain any proper subfield that splits V.

2.2 Generalities on 2-quasi-finite fields

Let us start by giving general properties of 2-quasi-finite fields.

Proposition 2.1. Let k be a 2-quasi-finite field and l/k a finite extension of k.

(a) The cohomological dimension of k is one.

(b) We have $l^\times/(l^\times)^2 \simeq \mathbf{Z}/2\mathbf{Z}$.

(c) The extension l/k contains a unique extension of degree 2 of k.

Proof. Denote by G the absolute Galois group of k. First notice that assertions (b) and (c) are immediate combinations of Galois correspondence and the fact that the 2-Sylow of G is $\mathbf{Z}_2$. For (a), using that G is a closed subgroup of $\widehat{\mathbf{Z}}$, [Ser94, Chapitre I, §3.3, Proposition 14] ensures that $\mathrm{cd}(G) \leq \mathrm{cd}(\widehat{\mathbf{Z}})$. This proves that $\mathrm{cd}(G) \leq 1$ by [Ser94, Chapitre I, §3.2, Exemple 1)], which is an equality since $G \neq 1$. □

The following proposition supplies a description of corestriction maps for finite extensions of a 2-quasi-finite field.

Proposition 2.2. If the characteristic of k is different from 2 and $E/F/k$ are finite field extensions of k, then we have canonical isomorphisms $\mathrm{H}^1(F, \mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$ and $\mathrm{H}^1(E, \mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$ under which the morphism $\mathrm{cores}_{E/F} : \mathrm{H}^1(E, \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^1(F, \mathbf{Z}/2\mathbf{Z})$ is the identity morphism of $\mathbf{Z}/2\mathbf{Z}$.

Proof. For the first part of the statement, since k has characteristic different from 2, Kummer's exact sequence ensures that $\mathrm{H}^1(E, \mathbf{Z}/2\mathbf{Z}) \simeq E^\times/(E^\times)^2$ and $\mathrm{H}^1(F, \mathbf{Z}/2\mathbf{Z}) \simeq F^\times/(F^\times)^2$, and they are isomorphic to $\mathbf{Z}/2\mathbf{Z}$ by (b) of Proposition 2.1. For the second part of the statement, since k has cohomological dimension one, the map $\mathrm{cores}_{E/F} : \mathrm{H}^1(E, \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^1(F, \mathbf{Z}/2\mathbf{Z})$ is an isomorphism by [NSW08, Proposition 3.3.11]. The only automorphism of $\mathbf{Z}/2\mathbf{Z}$ being the identity map, this proves the statement. □

2.3 Residues and splitness

Definition 2.3. Let S be a scheme. Two conic bundles $f : X \to S$ and $g : Y \to S$ are said to be *equivalent* if there exists a dense open subset U of S such that $f^{-1}(U)$ and $g^{-1}(U)$ are isomorphic over U.

We recall the following proposition, which is useful in the rest of the article.

Proposition 2.4 ([CTS21, Lemma 11.3.2])**.** Let C be a smooth and geometrically integral curve. If $f : X \to C$ is a conic bundle, then there exists a regular conic bundle $g : Y \to C$ equivalent to f such that all fibres of g are integral.

The following proposition describes a minimal splitting field of the special fibre of a regular conic bundle over a DVR, via the residue of the generic fibre.

Proposition 2.5. Let R be a discrete valuation ring, K its fraction field, κ its residue field. Assume further that 2 is invertible in R and denote by r the associated residue map defined as in (1). Let C be a smooth conic over K and $\mathscr{X}$ an integral R-proper scheme such that $\mathscr{X}$ is regular, with generic fibre $\mathscr{X}_K \simeq C$. Let $\alpha \in \kappa^\times$ be a representative of the class $r([C]) \in \mathrm{H}^1(\kappa, \mathbf{Z}/2\mathbf{Z}) = \kappa^\times/(\kappa^\times)^2$. Then $\kappa(\sqrt{\alpha})$ is a minimal splitting field of $\mathscr{X}_\kappa$.

Before proving Proposition 2.5, we recall that geometrically, a conic is either isomorphic to a projective line, or a union of two projective lines, or a double line, so that a conic is split if and only if it is a union of two projective lines, or it is geometrically irreducible.

Proof. We first construct an explicit integral R-scheme $\mathscr{C}$ such that $\mathscr{C}$ is regular, with generic fibre $\mathscr{C}_K \simeq C$ and we prove that the statement holds for $\mathscr{C}$. Then, for $\mathscr{X}$ as in the statement, since $\mathscr{X}_K \simeq \mathscr{C}_K$ the conclusion follows from [Sko15, Corollary 2.3], whose statement is true without assuming that κ is perfect, the proof being given in the Remark following it.

To construct $\mathscr{C}$, denote by v the valuation on R, pick ϖ a uniformiser of R and choose $q \in K[x, y, z]$ a quadratic form such that $C = V(q) \subset \mathbf{P}^2_K$. Let us prove that q may be chosen of the form $ax^2 + by^2 - z^2$ where $a \in R$ and $b \in R^\times$. Since κ is of characteristic different from 2, we may assume that q is diagonal and write it $q = ax^2 + by^2 + cz^2$ where a, b, c further lie in $K^\times$ as C is smooth. After multiplying q by a constant, we may also assume that $a, b, c \in R$. Furthemore, after the change of variables $x' = \varpi^{\lfloor v(a)/2 \rfloor} x$, $y' = \varpi^{\lfloor v(b)/2 \rfloor} y$ and $z' = \varpi^{\lfloor v(c)/2 \rfloor} z$ we may assume that $v(a), v(b), v(c) \in \{0, 1\}$. Since two among the integers $v(a)$, $v(b)$ and $v(c)$ have same class in $\mathbf{Z}/2\mathbf{Z}$, we may rearrange the variables of q in such a way that $v(b) = v(c)$. If the latter is equal to 0, we may then divide q by $-c \in R^\times$ so that we may assume that $q = ax^2 + by^2 - z^2$ with $v(b) = 0$. Otherwise, we may divide q by c and make the change of variable $x' = \varpi^\varepsilon x$ where $\varepsilon = 0$ if $v(a) = 1$, and $\varepsilon = -1$ is $v(a) = 0$, so that we may again assume that $q = ax^2 + by^2 - z^2$ with $v(b) = 0$.

Let us now verify the statement for $\mathscr{C} = V(q) \subset \mathbf{P}^2_R$, which is regular, and for which we clearly have that $\mathscr{C}_K \simeq C$. By [CTS21, Equation (1.18)], we have that $r([C]) = [b^{v(a)}]$. If $v(a) = 1$, then $a = 0 \in \kappa$. Thus, $\mathscr{C}_\kappa$ is the zero locus of the quadric $by^2 - z^2$ which may be rewritten in $\kappa(\sqrt{b})$ as $(y\sqrt{b} - z)(y\sqrt{b} + z)$, so that $\kappa(\sqrt{b})$ splits $\mathscr{C}_\kappa$. Moreover, if b is a square in κ then $\kappa(\sqrt{b})$ is clearly minimal. Otherwise, $by^2 - z^2$ is irreducible, so that $\kappa(\sqrt{b})$ is again minimal. Now, if $v(a) = 0$, then $\mathscr{C}_\kappa$ is the zero locus of a non-degenerate, hence geometrically irreducible conic, so that $\mathscr{C}_\kappa$ is split. This proves the statement for $\mathscr{C}$. □

The following proposition is also used thereafter.

Proposition 2.6 ([Sko15, Corollary 2.3 and Remark])**.** Let C be a smooth geometrically integral curve over a field k of characteristic different from 2. If $f : X \to C$ and $g : Y \to C$ are two equivalent regular conic bundles, then the non-split fibres of f and g lie over the same points of C.

2.4 Approximation over function fields

Let C be a nice curve over a finite field $\mathbf{F}$, and K its function field, we denote by Ω_K the set of places of K, that is, closed points of C. If X is a proper K-variety and $S \subset \Omega_K$ is finite, we set $X(K_\Omega^S) = \prod_{v \in \Omega_K \setminus S} X(K_v)$ which is endowed with the product topology. When $S = \emptyset$, this set is also denoted by $X(K_\Omega)$.

Definition 2.7. Let X be a proper K-variety. We say that X has *weak weak approximation* if there exists a finite subset $S \subset \Omega_K$ such that the diagonal embedding $X(K) \hookrightarrow X(K_\Omega^S)$ has a dense image. We further say that X has *weak approximation* if we can take $S = \emptyset$.

The *Brauer-Manin pairing* has been introduced by Manin [Man71] to study the defect of weak approximation in the setting of number fields (see also [Har07, §1.3] and [CTS21, §13.3.1]), although the definition is the same over function fields of curves over a finite field. If X is a proper K-variety, it is defined as:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{BM} : \quad X(K_\Omega) \times \mathrm{Br}(X) &\longrightarrow \mathbf{Q}/\mathbf{Z} \\ ((x_v), \alpha) &\longmapsto \textstyle\sum_{v \in \Omega_K} \mathrm{inv}_v \left(x_v^*(\alpha) \right), \end{aligned}$$

where $\mathrm{inv}_v : \mathrm{Br}(K_v) \to \mathbf{Q}/\mathbf{Z}$ is the local invariant and x_v^* stands for the specialisation morphism of Brauer groups $\mathrm{Br}(x_v) : \mathrm{Br}(X) \to \mathrm{Br}(K_v)$. Elements of $X(K_\Omega)$ orthogonal to $\mathrm{Br}(X)$ form a closed subset $X(K_\Omega)^{\mathrm{Br}(X)}$ of $X(K_\Omega)$ containing $X(K)$ and which is called the Brauer-Manin set of X.

Definition 2.8. Let X be a proper K-variety. We say that *the Brauer-Manin obstruction to weak approximation is the only one on* X if $X(K)$ is dense in $X(K_\Omega)^{\mathrm{Br}(X)}$. We abbreviate this by saying that X verifies (BM).

We need to recall how approximation of local points can be translated over function fields and we refer the reader to [Has10, Section 1] for further details.

We fix a proper K-variety X and a *model* of X, that is, a flat proper morphism $\rho : \mathscr{X} \to C$ whose generic fibre is isomorphic to X. Then, if $v \in \Omega_K$ and $P_v \in X(K_v)$, the valuative criterion of properness ensures that P_v extends to a unique $\widehat{\mathscr{O}_v}$-morphism $\widehat{P_v} : \mathrm{Spec}(\widehat{\mathscr{O}_v}) \to \mathscr{X} \times_C \mathrm{Spec}(\widehat{\mathscr{O}_v})$. If N is a positive integer, we set

$$J_{P_v,N} \coloneqq \{Q \in X(K_v) : \text{ the restrictions of } \widehat{Q} \text{ and } \widehat{P_v} \text{ to } \mathrm{Spec}(\widehat{\mathscr{O}_v}/\mathfrak{m}_v^N) \text{ are the same}\}.$$

Reminder 2.9 ([Has10, Section 1])**.** If $S \subset \Omega_K$ is finite and $(P_v)_{v \in \Omega_K \setminus S} \in X(K_\Omega^S)$, then a fundamental system of neighbourhoods of (P_v) is given by the sets

$$W_{T,N} \coloneqq \prod_{v \in T} J_{P_v,N} \times \prod_{v \in \Omega_K \setminus (S \cup T)} X(K_v)$$

where T ranges over finite subsets of $\Omega_K \setminus S$ and N ranges over positive integers. Furthermore, the mapping $P \mapsto \widehat{P}$ is a bijection between $X(K) \cap W_{T,N}$ and sections $: C \to \mathscr{X}$ of ρ such that for all $v \in T$, the pullbacks of j and $\widehat{P_v}$ to $\mathrm{Spec}(\widehat{\mathscr{O}_v}/\mathfrak{m}_v^N)$ coincide.

3 A unirationality criterion

This section is dedicated to the proof of the following unirationality criterion. To state it, let us recall that if $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ is a dominant morphism and $s, t \in \mathbf{P}^1_k$ are such that $\varphi(t) = s$, then we denote by $e(t/s)$ the *ramification index* of φ at t.

Theorem 3.1. Let k be a field of characteristic different from 2 with $\mathrm{cd}(k) \leq 1$, and $f : X \to \mathbf{P}^1_k$ a regular conic bundle . Denote by B the set of points in $\mathbf{P}^1_k$ over which the fibre of f is non-split. Then the following assertions are equivalent:

1) The variety X is k-unirational.

2) There exists a dominant morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that for any $s \in B$ and any $t \in \varphi^{-1}(s)$ one has $2 \mid e(t/s) \times [\kappa(t) : \kappa(s)]$.

3.1 A criterion of Enriques

Before proving Theorem 3.1, let us state Enriques criterion for the unirationality of conic bundles [PS99, Proposition 10.1.1].

Proposition 3.2. Let k be a field and $f : X \to S$ a conic bundle between k-varieties. Then the following assertions are equivalent:

(i) The variety X is unirational.

(ii) There exists a rational map $g : \mathbf{P}^{\dim S}_k \dashrightarrow X$ such that $f \circ g$ is dominant.

(iii) There exists a dominant map $h : \mathbf{P}^{\dim S}_k \dashrightarrow S$ such that the base change of f by h has a rational section.

Proof of Proposition 3.2. The proof of (ii)⇒(iii)⇒(i) is given in [PS99, Proof of Proposition 10.1.1].

Since the proof of (i)⇒(ii) given in *ibidem* works implicitly for an infinite field, let us give a general proof. Let us assume (i), that is X is unirational, and let us prove (ii). First choose a rational dominant map $\psi : \mathbf{P}^r \dashrightarrow X$. Note that the proof of [Kol02, Lemma 2.3] ensures that, given a rational map $\varphi : \mathbf{P}^d_k \dashrightarrow X$ with $f \circ \varphi$ dominant, if $d > \dim S$ then there exists a rational hypersurface $\iota : Z \hookrightarrow \mathbf{P}^d_k$ such that $f \circ \varphi \circ \iota$ is dominant, that is, there exists a rational map $\theta : \mathbf{P}^{d-1}_k \dashrightarrow \mathbf{P}^d_k$ such that $f \circ \varphi \circ \theta$ is dominant. When starting with ψ, we may now apply this procedure iteratively, which proves (ii). ☐

Remark 3.3. Note that if S is a projective curve, then the rational maps in (ii) and (iii) of Proposition 3.2 may be assumed to be morphisms by the valuative criterion of properness.

When the base field has cohomological dimension at most one, we infer the following unirationality criterion for conic bundles.

Corollary 3.4. Let k be a field of characteristic different from 2 such that $\mathrm{cd}(k) \leq 1$. If $f : X \to \mathbf{P}^1_k$ is a regular conic bundle, then X is unirational if and only if there exists a dominant morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that the closed fibres of all regular conic bundles equivalent to the base change of f by φ are split.

The proof of Corollary 3.4 requires the following characterisation of regular conic bundles over $\mathbf{P}^1_k$ having a section.

Lemma 3.5. Let k be a field of characteristic different from 2, with $\mathrm{Br}(k)\,[2] = 0$. If $f : X \to \mathbf{P}^1_k$ is a regular conic bundle, then f has a section if and only if all fibres of f over a closed point of $\mathbf{P}^1_k$ are split.

Proof. If we write the short exact of [GS06, Theorem 6.9.1] with $i = m = 2$ and $j = 1$, we get the following short exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathbf{P}^1_k)[2] \longrightarrow \mathrm{Br}(k(\mathbf{P}^1_k))[2] \xrightarrow{\oplus r_P} \bigoplus_{P \in (\mathbf{P}^1_k)^{(1)}} \mathrm{H}^1(\kappa(P), \mathbf{Z}/2\mathbf{Z}) \qquad (2)$$

where for each $P \in (\mathbf{P}^1_k)^{(1)}$, the map r_P denotes the residue map (1) associated to the discrete valuation ring $\mathscr{O}_{\mathbf{P}^1_k, P}$. Besides, $\mathrm{Br}(\mathbf{P}^1_k)\,[2] = \mathrm{Br}(k)\,[2] = 0$ where the first equality is derived from [Sko01, Corollary 2.3.9] and the second one from our assumption on k. Thus, if X_η denotes the generic fibre of f, then f has a section if and only if the smooth conic X_η has a rational point, that is, if and only if $[X_\eta] = 0 \in \mathrm{Br}(k(\mathbf{P}^1_k))[2]$. By the short exact sequence (2), $[X_\eta] = 0$ if and only if for any closed point P of $\mathbf{P}^1_k$, $r_P([X_\eta]) = 0$. Furthermore, Proposition 2.5 applied to the regular integral scheme $X \times_{\mathbf{P}^1_k} \mathrm{Spec}(\mathscr{O}_{\mathbf{P}^1_k, P})$ ensures that $r_P([X_\eta]) = 0$ if and only if X_P splits. This proves that f has a section if and only if all its closed fibres are split. ☐

Remark 3.6. Let k be a field of characteristic different from 2. Then $\mathrm{Br}(k)\,[2] = 0$ if k verifies one of the following conditions:

(a) the cohomological dimension of k is at most one;

(b) the absolute Galois group of k is a p-group for some prime number $p \neq 2$.

Indeed, if k satisfies (a), this comes from the isomorphism $\mathrm{Br}(k)\,[2] = \mathrm{H}^2(k, \mathbf{Z}/2\mathbf{Z})$ induced by Kummer's exact sequence. If k satisfies (b), all central simple algebras A over k are split by a finite separable extension of k, whose degree is by assumption a power of the odd prime p, that is, $[A] \notin \mathrm{Br}(k)\{2\} \setminus \{0\}$.

We may now give a proof of Corollary 3.4:

Proof of Corollary 3.4. By Remark 3.3 applied to (iii) of Proposition 3.2, X is unirational if and only if there exists a dominant map $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that the conic bundle $g : X \times_{\mathbf{P}^1_k, \varphi} \mathbf{P}^1_k \to \mathbf{P}^1_k$, defined as the base change of f by φ, has a section. By Proposition 2.4, all conic bundles are equivalent to a regular conic bundle. In particular, g has a section if and only if all regular conic bundles over $\mathbf{P}^1_k$ equivalent to g have a section. Now, using Lemma 3.5 and (a) of Remark 3.6, the latter is equivalent to saying that the closed fibres of all regular conic bundle over $\mathbf{P}^1_k$ equivalent to g are split, which proves the statement. ☐

3.2 Proof of Theorem 3.1

Before we give a proof of Theorem 3.1, we recall that when $R \subset S$ is an inclusion of discrete valuation rings, with respective fraction fields $K \subset L$ and residue fields $\kappa \subset \lambda$ of characteristic different from 2, we denote by $\mathrm{Res}_{K/L} : \mathrm{Br}(K)[2] \to \mathrm{Br}(L)[2]$ the restriction morphism defined contravariantly from the morphisms $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$. If R contains a field and e is the ramification index of S/R, the following diagram is commutative [CTS21, Proposition 1.4.7]:

$$\begin{tikzcd} \mathrm{Br}(K)[2] \arrow[r, "r_K"] \arrow[d, "\mathrm{Res}_{K/L}"] & \mathrm{H}^1(\kappa, \mathbf{Z}/2\mathbf{Z}) \arrow[d, "e \times \mathrm{res}_{\kappa/\lambda}"] \\ \mathrm{Br}(L)[2] \arrow[r, "r_L"] & \mathrm{H}^1(\lambda, \mathbf{Z}/2\mathbf{Z}) \end{tikzcd} \qquad (3)$$

where the horizontal maps are the residue maps (1) associated to R and S.

Proof of Theorem 3.1. Before starting the proof, using Propositions 2.4 and 2.6, we may assume that all non-split fibres of f are integral. Thus, non-split fibres of f coincide with its singular fibres.

Let us now make a remark that is useful all along the proof. If $f : X \to \mathbf{P}^1_k$ is a conic bundle and $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ is dominant, denote by $f' : X' \to \mathbf{P}^1_k$ the base change of f by φ (so that X' may not be regular). For any closed point t of $\mathbf{P}^1_k$, if we set $s = \varphi(t)$ and if we denote by r_t (resp. r_s) the residue map (1) at t (resp. at s) and X_η (resp. X'_η) the generic fibre of f (resp. f') then if we apply the commutative diagram (3) to the inclusion of discrete valuation rings $\mathscr{O}_{\mathbf{P}^1_k,s} \subset \mathscr{O}_{\mathbf{P}^1_k,t}$ induced by f, we get

$$r_t(X'_\eta) = e(t/s) \times \mathrm{res}_{\kappa(s)/\kappa(t)}(r_s(X_\eta)). \tag{4}$$

As $\mathrm{cd}(k) \leq 1$, the map $\mathrm{cores}_{\kappa(t)/\kappa(s)}$ is an isomorphism by [NSW08, Proposition 3.3.11]. Moreover, $\mathrm{cores}_{\kappa(t)/\kappa(s)} \circ \mathrm{res}_{\kappa(s)/\kappa(t)} = [\kappa(t) : \kappa(s)]$ [GS06, Proposition 4.2.10], so that (4) may be rewritten as

$$r_t(X'_\eta) = e(t/s) \times [\kappa(t) : \kappa(s)] \times \mathrm{cores}^{-1}_{\kappa(t)/\kappa(s)}(r_s(X_\eta)) \in \mathrm{H}^1(\kappa(t), \mathbf{Z}/2\mathbf{Z}). \tag{5}$$

Now, if $f'' : X'' \to \mathbf{P}^1_k$ is a regular conic bundle equivalent to f', since its generic fibre X''_η is isomorphic to X'_η, we may rewrite (5) as

$$r_t(X''_\eta) = e(t/s) \times [\kappa(t) : \kappa(s)] \times \mathrm{cores}^{-1}_{\kappa(t)/\kappa(s)}(r_s(X_\eta)) \in \mathrm{H}^1(\kappa(t), \mathbf{Z}/2\mathbf{Z}). \tag{6}$$

Let us now assume 1), that is X is k-unirational. By Corollary 3.4, there exists a dominant map $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that if $f'' : X'' \to \mathbf{P}^1_k$ is a regular conic bundle equivalent to the base change $f' : X' \to \mathbf{P}^1_k$ of f by φ, then all closed fibres of f'' are split. For $s \in B$ and $t \in \varphi^{-1}(s)$, since X''_t splits and X_s is non-split, Proposition 2.5 ensures that $r_t(X''_\eta) = 0$ and $r_s(X_\eta) \neq 0$. As $\mathrm{H}^1(\kappa(t), \mathbf{Z}/2\mathbf{Z}) = \kappa(t)^\times/(\kappa(t)^\times)^2$ is a group of exponent 2, by (6) we deduce that $2 \mid e(t/s) \times [\kappa(t) : \kappa(s)]$, which proves 2).

Suppose that 2) holds true and let $f' : X' \to \mathbf{P}^1_k$ be the base change of f by φ and $f'' : X'' \to \mathbf{P}^1_k$ a regular conic bundle such that f' and f'' are isomorphic over $\mathbf{P}^1_k \setminus B$. In the beginning of the proof, X has been chosen such that f is smooth above $\mathbf{P}^1_k \setminus B$, so that the morphism f' is also smooth above $\mathbf{P}^1_k \setminus \varphi^{-1}(B)$. In particular, the fibres of f'' above U are smooth, hence split. Furthermore, for $s \in B$ and $t \in \varphi^{-1}(s)$, the assumption on φ and equation (6) ensure that $r_t(X''_\eta) = 0$, as $\mathrm{H}^1(\kappa(t), \mathbf{Z}/2\mathbf{Z})$ is a group of exponent 2. By Proposition 2.5, the fibre X''_t is then split. This shows that any closed fibre of f'' is split, so that X is unirational by Corollary 3.4, which proves 1). □

Remark 3.7. By Lemma 3.5, the last paragraph of the proof ensures that if φ verifies the condition of 2), then the base change of f by φ has a section.

3.3 A criterion for triviality of R-equivalence

Corollary 3.8. Let k be a field of characteristic different from 2 with $\mathrm{cd}(k) \leq 1$, and $f : X \to \mathbf{P}^1_k$ a regular conic bundle. Denote by B the set of points of $\mathbf{P}^1_k$ over which the fibre of f is non-split. Assume that for all $s_0, s_1 \in \mathbf{P}^1(k)$ there exists a dominant morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ verifying:

(a) for any $s \in B$ and any $t \in \varphi^{-1}(s)$ one has $2 \mid e(t/s) \times [\kappa(t) : \kappa(s)]$;

(b) the fibres $\varphi^{-1}(s_0)$ and $\varphi^{-1}(s_1)$ have a rational point.

Then X is k-unirational and $X(k)/R$ is trivial.

Let us recall the following elementary observation on conics.

Reminder 3.9. Let F be a field of characteristic different from 2 and C a conic over F.

(i) If $C(F) \neq \emptyset$, then R-equivalence is trivial on C.

(ii) If $\mathrm{cd}(F) \leq 1$, then $C(F) \neq \emptyset$.

Proof. For (i), if $C(F) \neq \emptyset$, then C is either a line, or a union of two distinct lines in $\mathbf{P}^2_F$ or a double line, so that R-equivalence is trivial on it. For (ii), if C is smooth, it has a class in $\mathrm{Br}(k)\,[2]$. But $\mathrm{Br}(k)\,[2] = \mathrm{H}^2(k, \mathbf{Z}/2\mathbf{Z})$, and the latter is zero since $\mathrm{cd}(F) \leq 1$. We thus have that $C \simeq \mathbf{P}^1_F$ from which the statement follows. If C is singular, all singular points are rational. ☐

We may now give a proof of Corollary 3.8.

Proof of Corollary 3.8. By Theorem 3.1, X is k-unirational. Let us prove that $X(k)/R$ has cardinality one. Since the conic X_0 has a rational point, by (ii) of Reminder 3.9, $X(k) \neq \emptyset$ so that it remains to prove that for all $x_0, x_1 \in X(k)$, the points x_0 and x_1 are R-equivalent.

Set $s_0 = f(x_0)$, $s_1 = f(x_1)$ which are rational points of $\mathbf{P}^1_k$. Choose $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ as in the statement and denote by $f' : X' \to \mathbf{P}^1_k$ the base change of f by φ and by $g : X' \to X$ the base change of φ by f. Using (b), let us choose t_0 (resp. t_1) a rational point of $\varphi^{-1}(s_0)$ (resp. $\varphi^{-1}(s_1)$). Since φ verifies (a), Remark 3.7 ensures that f' has a section $h : \mathbf{P}^1_k \to X'$. Set $x'_0 = h(t_0)$ and $x'_1 = h(t_1)$, so that $x'_0, x'_1 \in X'(k)$. As the rational points $g(x'_0)$ and $g(x'_1)$ of X lie in $(g \circ h)(\mathbf{P}^1(k))$, they are R-equivalent. Moreover, x_0 and $g(x'_0)$ (resp. x_1 and $g(x'_1)$) lie on the same fibre of f, which is a conic, hence they are R-equivalent by (i) of Reminder 3.9. This proves that x_0 and x_1 are R-equivalent. ☐

4 Proof of the main results

In this section, we assume that k is a 2-quasi-finite field and we prove Theorems 1.5 and 1.8. We respectively make use of Theorem 3.1 and Corollary 3.8.

4.1 Some ramified covers of $\mathbf{P}^1_k$

This subsection encapsulates the construction of particular covers of the projective line that are thoroughly used in the proofs of Theorems 1.5 and 1.8.

Lemma 4.1. Let m be a closed point of $\mathbf{P}^1_k$ such that $\deg(m) = 2d$, with $d \in \mathbf{Z}_{>0}$. Then there exists a degree d morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that $\deg(\varphi(m)) = 2$.

Proof of Lemma 4.1. Let l/k be a degree 2 extension. Note that l is unique up to isomorphism and $\kappa(m)/k$ is Galois, since k is 2-quasi-finite. All along the proof, we denote by σ the nontrivial element of $\mathrm{Gal}(l/k)$, we fix $\alpha \in k^\times \setminus (k^\times)^2$ and set $P \in \mathbf{A}^1_k$ corresponding to the polynomial $x^2 - \alpha$, so that $\deg(P) = 2$. For $h \in l(\mathbf{P}^1)$, we denote by ${}^\sigma h$ the image of h by the left action of σ on $l(\mathbf{P}^1)$.

Since $\kappa(m)/k$ is Galois and l is unique up to isomorphism, the extension l/k sits in $\kappa(m)/k$. Thus, the fibre of m under the morphism $\mathbf{P}^1_l \to \mathbf{P}^1_k$, which is the fibre product of $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ with $\mathbf{P}^1_k$, is made of two degree $2d$ points $m_1, m_2 \in \mathbf{P}^1_l$, as $\kappa(m) \otimes_k l = \kappa(m) \otimes_l (l \otimes_k l) = \kappa(m) \times \kappa(m)$. Since $m_1 - m_2$ has degree zero, there exists $f \in l(\mathbf{P}^1)$ such that $\mathrm{div}(f) = m_1 - m_2$, so that the map $f : \mathbf{P}^1_l \to \mathbf{P}^1_l$ it induces verifies $m_1 = f^{-1}(0)$, $m_2 =$

$f^{-1}(\infty)$. Let g be an automorphism of $\mathbf{P}^1_l$ such that $g(0) = \sqrt{\alpha}$ and $g(\infty) = -\sqrt{\alpha}$ and $({}^{\sigma}g)(x) = g(1/\sigma(x))$. We are going to prove that there exists $u \in l^\times$ such that $g \circ (uf)$ is σ-invariant as an element of $l(\mathbf{P}^1)$. By Galois descent, $g \circ (uf)$ will then be the base change of a morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$. Since $(g \circ (uf))^{-1}\{\sqrt{\alpha}, -\sqrt{\alpha}\} = \{m_1, m_2\}$, this means that $\varphi(m) = P$ and φ is of degree d, which will prove the statement.

Let us first note that $\mathrm{div}({}^{\sigma}f) = m_2 - m_1$ so that $\mathrm{div}(f \times {}^{\sigma}f) = 0$, that is, there exists $v \in l^\times$ such that $f \times {}^{\sigma}f = v$. Furthermore, as v is σ-invariant, we have $v \in k^\times$. Since the cohomological dimension of k is one, the corestriction map $\mathrm{cores}_{l/k} : \mathrm{H}^1(l, \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})$ is surjective by [NSW08, Proposition 3.3.11]. Hence, the norm map $N_{l/k} : l^\times \to k^\times$ is surjective. In particular, there exists $u \in l^\times$ such that $v = u \times \sigma(u)$. After replacing f by f/u, we may then assume that $f \times {}^{\sigma}f = 1$. Thus:

$$ {}^{\sigma}(g \circ f) = g \circ (1/({}^{\sigma}f)) = g \circ f $$

from which we deduce that $g \circ f$ is σ-invariant. □

Corollary 4.2. If P is a closed point of $\mathbf{P}^1_k$ such that $\deg(P) = 2$, then there exists a finite morphism $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ of degree 2 such that $\varphi^{-1}(P)$ is a closed point of degree 4.

Proof. Let m be a point of $\mathbf{P}^1_k$ of degree 4. By Lemma 4.1, there exists a finite morphism $\theta : \mathbf{P}^1_k \to \mathbf{P}^1_k$ of degree 2 such that $\theta(m)$ is a point of degree 2. Since $\theta(m)$ and P are both points of degree 2, there exists an automorphism ψ of $\mathbf{P}^1_k$ mapping $\theta(m)$ to P. Thus, the map $\varphi := \psi \circ \theta$ is the sought cover. □

Lemma 4.3. Let $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ be a degree 2 cover, $U \subset \mathbf{P}^1_k$ be the complement of the branch locus of φ. Then there exists a degree 2 cover $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ with branch locus $\mathbf{P}^1_k \backslash U$ such that for any closed point $m \in U$:

(i) if $\deg(m)$ is odd and $\varphi^{-1}(m)$ is a point of degree 2 over m, then $\psi^{-1}(m)$ is made of two rational points over m;

(ii) if $\deg(m)$ is odd and $\varphi^{-1}(m)$ is made of two rational points over m, then $\psi^{-1}(m)$ is a point of degree 2 over m;

(iii) if $\deg(m)$ is even, then $\varphi^{-1}(m)$ and $\psi^{-1}(m)$ are isomorphic over m.

Proof of Lemma 4.3. Let us first note that if $u : U \to \mathrm{Spec}(k)$ is the structural morphism of U, then the following diagram is commutative

$$\begin{tikzcd}
\mathbf{Z}/2\mathbf{Z} \arrow[rrr, "\times[\kappa(m):k]"] \arrow[d, equal] & & & \mathbf{Z}/2\mathbf{Z} \arrow[d, equal] \\
\mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z}) \arrow[r, "u^*"] \arrow[rr, bend right, "\mathrm{res}_{\kappa(m)/k}"'] & \mathrm{H}^1(U, \mathbf{Z}/2\mathbf{Z}) \arrow[r, "m^*"] & \mathrm{H}^1(\kappa(m), \mathbf{Z}/2\mathbf{Z}) \arrow[r, "\mathrm{cores}_{\kappa(m)/k}"] & \mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})
\end{tikzcd}$$

where the vertical equalities come from Proposition 2.2, by 2-quasi-finiteness of k, and the commutativity of the whole diagram is the restriction-corestriction formula [GS06, Proposition 4.2.10].

Denote by α the nonzero class in $\mathrm{H}^1(k, \mathbf{Z}/2\mathbf{Z})$ and $\tau : \varphi^{-1}(U) \to U$ the restriction of φ above U, which defines a class in $\mathrm{H}^1(U, \mathbf{Z}/2\mathbf{Z})$. Let $\tau' : V \to U$ be a $\mathbf{Z}/2\mathbf{Z}$-torsor whose class in $\mathrm{H}^1(U, \mathbf{Z}/2\mathbf{Z})$ is $u^*(\alpha) + [\tau]$, that is, a twist of τ by any element of $k^\times \setminus (k^\times)^2$. Denote by $\psi : Y \to \mathbf{P}^1_k$ the normalisation of $\mathbf{P}^1_k$ in $\mathrm{Spec}(k(V))$, so that Y is a smooth projective curve. Since $Y \otimes_k \overline{k}$ is the normalisation of $\mathbf{P}^1_{\overline{k}}$ in $\overline{k}(V) = \overline{k}(\varphi^{-1}(U))$, this means that $Y \otimes_k \overline{k} \simeq \mathbf{P}^1_{\overline{k}}$. In particular, Y is a smooth geometrically connected projective curve of genus 0, that is, Y is isomorphic to a smooth conic. As $\mathrm{cd}(k) = 1$, we have $Y(k) \neq \emptyset$ by Reminder 3.9, so that $Y \simeq \mathbf{P}^1_k$ and $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$.

Let us now prove that ψ is the sought cover. Indeed, in $\mathrm{H}^1(\kappa(m), \mathbf{Z}/2\mathbf{Z})$, we have:

$$[\psi^{-1}(m)] = m^*([\tau']) = m^*(u^*(\alpha) + [\tau]) = \mathrm{res}_{\kappa(m)/k}(\alpha) + [\varphi^{-1}(m)] \in \mathbf{Z}/2\mathbf{Z}. \tag{7}$$

Moreover, $\mathrm{cores}_{\kappa(m)/k}$ is an isomorphism by Proposition 2.2. Besides, from the identity

$$\mathrm{cores}_{\kappa(m)/k} \circ \mathrm{res}_{\kappa(m)/k} = [\kappa(m) : \kappa]$$

we may rewrite equation (7) as

$$[\psi^{-1}(m)] = [\kappa(m) : k] + [\varphi^{-1}(m)] \in \mathrm{H}^1(\kappa(m), \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$$

which is the statement we wanted to prove. □

4.2 Proof of Theorem 1.5

Let us prove Theorem 1.5 using Theorem 3.1. For this purpose, we show that for any B as in ($\star$), there exists a cover $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that for all $s \in B$ and $t \in \varphi^{-1}(s)$ we have

$$2 \mid e(t/s) \times [\kappa(t) : \kappa(s)]. \tag{8}$$

In the following lemma, we start by tackling the case where $B \subset \mathbf{P}^1(k)$.

Lemma 4.4. Let $B \subset \mathbf{P}^1(k)$. Then, for any two points $P, Q \in B$, there exists a dominant map $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ whose degree is a power of 2, satisfying condition (8), with the further assumption that φ is totally ramified above P and Q.

Proof. After adding rational points to B, we may assume that $|B|$ is even. Setting $|B| = 2n$, we prove the statement by induction on n. The case where $n = 0$ being trivial, we may assume that $n > 0$ and such a φ exists for strictly lower n. Fix distinct points P and Q in B and, after choosing an automorphism of $\mathbf{P}^1_k$, assume that $P = 0$ and $Q = \infty$. Then, define $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ by $t \mapsto t^2$, so that ψ is totally ramified above P and Q. We let B_{in} be the set of those $m \in B \setminus \{P, Q\}$ such that $\psi^{-1}(m)$ is a point of degree 2 over m, and B_{ts} those such that $\psi^{-1}(m)$ is made of two rational points, so that $|B_{\mathrm{in}}| + |B_{\mathrm{ts}}| = 2n - 2$. By (i) and (ii) of Lemma 4.3, we may assume that $|B_{\mathrm{ts}}| \leq n - 1$. Then $\psi^{-1}(B_{\mathrm{ts}})$ is made of at most $2n - 2$ rational points of $\mathbf{P}^1_k$. By induction, there exists $\theta : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that θ is totally ramified above $\psi^{-1}(P)$, $\psi^{-1}(Q)$ and condition (8) is verified for θ and all $s \in \psi^{-1}(B_{\mathrm{ts}})$ and $t \in \theta^{-1}(s)$. Thus, if $\varphi := \psi \circ \theta$, the cover φ is totally ramified above P and Q. Moreover, for all $s \in B_{\mathrm{ts}}$ and $t \in \varphi^{-1}(s)$, condition (8) is verified, and for all $s \in B_{\mathrm{in}}$ and $t \in \varphi^{-1}(s)$ we have $2 \mid [\kappa(t) : \kappa(s)]$, so that φ is the sought cover. □

We now tackle the general case of Theorem 1.5.

Proof of Theorem 1.5. Let us now choose B as in ($\star$) and, up to enlarging B, we may assume that it is made of rational points, one point P of degree 2 and one point Q of odd degree. Corollary 4.2 supplies a 2-cover $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that $\psi^{-1}(P)$ is a point of degree 4. Using Lemma 4.3, we may further assume that $\psi^{-1}(Q)$ is a point of degree 2 over Q. Then, if B_{ts} denotes those rational points m of $B \setminus \{P, Q\}$ such that $\psi^{-1}(m)$ is made of two rational points, by B_{in} those for which $\psi^{-1}(m)$ is a degree two point over m, condition (8) is satisfied for $s \in \{P, Q\} \cup B_{\text{in}}$ and $t \in \psi^{-1}(s)$. Furthermore, since $\psi^{-1}(B_{\text{ts}})$ is made of rational points of $\mathbf{P}^1_k$, Lemma 4.4 supplies a morphism $\theta : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that condition (8) is verified for θ and all $s \in \psi^{-1}(B_{\text{ts}})$ and $t \in \theta^{-1}(s)$. By construction, the morphism $\varphi := \psi \circ \theta$ then satisfies condition (8) for all $s \in B$ and $t \in \varphi^{-1}(s)$, so that φ is the sought cover. □

4.3 Proof of Theorem 1.8

Let us now prove Theorem 1.8 using Corollary 3.8. If we fix distinct $P, Q \in \mathbf{P}^1(k)$, we then need to find $\varphi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that

$$\varphi^{-1}(P)(k) \neq \emptyset \text{ and } \varphi^{-1}(Q)(k) \neq \emptyset \tag{9}$$

and for any $s \in B$ and $t \in \varphi^{-1}(s)$, condition (8) holds.

We start proving the first case of ($\star\star$), that is, we assume that B is a union of rational points and one point m of degree 2. Up to enlarging B, we may assume that $P, Q \in B$. Using Corollary 4.2, there exists a degree 2 cover $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ such that $\psi^{-1}(m)$ is a point of degree 4. After composing ψ with an automorphism of $\mathbf{P}^1_k$, we may also assume that ψ is totally ramified above Q, and we let $\alpha := \psi^{-1}(Q)$. Furthermore, we may assume that $\psi^{-1}(P)$ contains a rational point. Indeed, if ψ is totally ramified above P, then $\psi^{-1}(P)$ is a rational point, and otherwise, by Lemma 4.3, we may assume that $\psi^{-1}(P)$ is made of two rational points. We thus denote by β a rational point of $\psi^{-1}(P)$. Then, Lemma 4.4 supplies a cover $\theta : \mathbf{P}^1_k \to \mathbf{P}^1_k$ that is totally ramified above α and β and such that condition (8) is satisfied for all $s \in \mathbf{P}^1(k) \cap \psi^{-1}(B \setminus \{P, Q, m\})$ and $t \in \theta^{-1}(s)$. If we set $\varphi := \psi \circ \theta$, by construction, it verifies condition (8), and it is totally ramified above P and Q, so that it also satisfies (9).

We now prove the second case of ($\star\star$) where B is assumed to be a union of rational points and one point m of odd degree. Again, after enlarging B and using an automorphism of $\mathbf{P}^1_k$, we may assume that B contains P and Q, that $P = 0$ and $Q = \infty$. We set $\psi : \mathbf{P}^1_k \to \mathbf{P}^1_k$ the degree 2 cover defined by $t \mapsto t^2$, which is totally ramified above P and Q. In the case where $m \notin \{P, Q\}$, using Lemma 4.3 we make the additional assumption that $\psi^{-1}(m)$ is a point of degree 2 over m. Now, Lemma 4.4 supplies a cover $\theta : \mathbf{P}^1_k \to \mathbf{P}^1_k$ that is totally ramified above $\psi^{-1}(P)$ and $\psi^{-1}(Q)$, and such that condition (8) is satisfied for all $s \in \mathbf{P}^1(k) \cap \psi^{-1}(B \setminus \{P, Q, m\})$ and $t \in \theta^{-1}(s)$. Then, $\psi \circ \theta$ satisfies condition (8) and it is totally ramified above P and Q, hence it satisfies condition (9).

5 Sufficiency of the Brauer-Manin obstruction

In this section, when $\mathbf{F}$ is a finite field, we prove that the unirationality of conic bundles over $\mathbf{P}^1_{\mathbf{F}}$ is implied by an analogue of a conjecture of Colliot-Thélène and Sansuc in positive characteristic.

The following conjecture is an analogue of a conjecture of Colliot-Thélène and Sansuc, stated as an open question over number fields in [CTS80] (see also Colliot-Thélène's conjecture in [CTS21, Conjecture 14.1.2]).

Conjecture 5.1. Let C be a nice curve a finite field $\mathbf{F}$ and K its function field. If X is a proper, smooth, geometrically integral and separably rationally connected surface over K, then X verifies (BM).

This section is dedicated to the following theorem on the existence of curves passing through a given set of rational points, for a conic bundle over $\mathbf{P}^1_{\mathbf{F}}$.

Theorem 5.2. Let $\mathbf{F}$ be a finite field, $f : X \to \mathbf{P}^1_{\mathbf{F}}$ a regular conic bundle. Let C be a nice curve over $\mathbf{F}$ with function field K and assume that X_K verifies (BM). Then the following assertions hold.

(1) There exists a morphism $g : C \to X$ such that $X(\mathbf{F}) \subset g(C)$.

(2) If the characteristic of $\mathbf{F}$ is odd, then for all $A \subset X(\mathbf{F})$ with $|A| \leq |C(\mathbf{F})|$, there exists $g : C \to X$ such that $A \subset g(C(\mathbf{F}))$.

We split the proof of Theorem 5.2 into two parts. In §5.1 we give a proof of (1) of Theorem 5.2 and in §5.2, we show (2) of Theorem 5.2. Let us now deduce the following immediate corollary on unirationality and triviality of R-equivalence for conic bundles.

Corollary 5.3. Let $\mathbf{F}$ be a finite field and $f : X \to \mathbf{P}^1_{\mathbf{F}}$ a regular conic bundle. Assuming that Conjecture 5.1 is true, the following assertions hold.

(a) The variety X is unirational.

(b) If the characteristic of $\mathbf{F}$ is odd, then all two points of $X(\mathbf{F})$ are directly R-equivalent.

Proof. Let $C := \mathbf{P}^1_{\mathbf{F}}$ and $K := (F)(C)$. Since Conjecture 5.1 is true, the variety X_K verifies (BM).

Let us first prove (a). Since $\mathbf{F}$ is finite, it has cohomological dimension one, so that $X_0(\mathbf{F})$ and $X_1(\mathbf{F})$ are nonempty by Reminder 3.9. Choose $x \in X_0(\mathbf{F})$ and $y \in X_1(\mathbf{F})$. We then apply (1) of Theorem 5.2 with $C := \mathbf{P}^1_{\mathbf{F}}$ to f. This supplies a morphism $g : \mathbf{P}^1_F \to X$ such that $x, y \in g(\mathbf{P}^1_{\mathbf{F}})$. Since x and y lie in distinct fibres of f, the morphism $f \circ g$ is dominant, which implies that X is unirational by Proposition 3.2.

Let us now prove (b). If $x, y \in X(\mathbf{F})$ are distinct, then (2) of Theorem 5.2 applied to f and $C := \mathbf{P}^1_{\mathbf{F}}$ supplies $g : \mathbf{P}^1_{\mathbf{F}} \to X$ such that $x, y \in g(\mathbf{P}^1(\mathbf{F}))$. In other words, x and y are directly R-equivalent. □

5.1 Unirationality

Proposition 5.4. Let $\mathbf{F}$ be a finite field, C a nice curve over $\mathbf{F}$ and K its function field. Let also $X \to \mathbf{P}^1_{\mathbf{F}}$ be a regular conic bundle. If X_K verifies (BM), then X_K has weak weak approximation.

Although the proof of Proposition 5.4 relies on classical arguments, it makes use of the finiteness of $\operatorname{Br}(X_K)/\operatorname{Br}(K)$, for which we supply a proof in Appendix A.1.

Proof. First note that $X(\mathbf{F}) \neq \emptyset$ since all fibres over a point of $\mathbf{P}^1(\mathbf{F})$ has a rational point by (ii) of Reminder 3.9, so that $X(K) \neq \emptyset$. By Corollary A.2, the group $\operatorname{Br}(X_K)/\operatorname{Br}(K)$ is finite, and we set $B \subset \operatorname{Br}(X_K)$ a finite set of representatives. Then $X(K_\Omega)^{\operatorname{Br}(X)} = \bigcap_{b \in B} X(K_\Omega)^b$. Since each $X(K_\Omega)^b$ is open in $X(K_\Omega)$ by [Poo17, Corollary 8.2.11.(b)], so is $X(K_\Omega)^{\operatorname{Br}(X)}$. As $X(K_\Omega)^{\operatorname{Br}(X)} \neq \emptyset$, this means that there exists a finite $S \subset \Omega_K$ such that the projection map $X(K_\Omega)^{\operatorname{Br}(X)} \to \prod_{v \in \Omega_K \setminus S} X(K_v)$ is surjective. Since $X(K)$ is dense in $X(K_\Omega)^{\operatorname{Br}(X)}$, it is also dense in $\prod_{v \in \Omega_K \setminus S} X(K_v)$. □

Let us now supply a proof of (1) of Theorem 5.2.

Proof of (1) of Theorem 5.2. Let us use the notations of the statement. We denote by $\rho : X \times C \to C$ the projection morphism and we fix $x \in X(\mathbf{F})$. By Proposition 5.4, the variety X_K has weak weak approximation, so that we can fix a finite set $S \subset \Omega_K$ such that $X(K)$ is dense in $X(K^S_\Omega)$. Let us now choose $T \subset \Omega_K \setminus S$ satisfying $|T| = |X(\mathbf{F})|$, and we fix a bijection $T \to X(\mathbf{F})$ written as $v \mapsto x_v$. We set $(P_v) \in X(K_\Omega^S)$ defined as $P_v := x_v$ for $v \in T$ and $P_v := x$ for $v \in \Omega_K \setminus (S \cup T)$. Using the notations of Reminder 2.9, we then consider the nonempty open subset $W_{T,1}$ of $X(K_\Omega^S)$ associated to the tuple (P_v) and the model ρ of X. Since $X(K)$ is dense in $X(K_\Omega^S)$, we may pick $P \in X(K) \cap W_{T,1}$. Then, using notations of §2.4, the K-point P extends to a section $j : C \to X \times C$ of ρ that coincides with $\widehat{P_v}$ on $\operatorname{Spec}(\kappa(v))$ for $v \in T$. In particular, if $g : C \to X$ is the first coordinate of j, then for each $v \in T$ we have an equality of set theoretical points $g(v) = x_v$, that is, the sets $g(T)$ and $X(\mathbf{F})$ are the same. □

5.2 Triviality of R-equivalence

Proposition 5.5. Let $\mathbf{F}$ be a finite field of odd characteristic, C a nice curve over $\mathbf{F}$ and K its function field. If $f : X \to \mathbf{P}^1_{\mathbf{F}}$ is a regular conic bundle, then the Brauer-Manin pairing $\langle \cdot, \cdot \rangle_{BM}$ on X_K is identically zero. In particular, if X_K verifies (BM), then X_K has weak approximation.

Before we give a proof, let us show how assertion (2) of Theorem 5.2 is inferred from Proposition 5.5.

Proof of (2) of Theorem 5.2. Let us use the notations of the statement. We denote by $\rho : X \times C \to C$ the projection morphism and we fix $x \in X(\mathbf{F})$ and $A \subset X(\mathbf{F})$ with $|A| \leq |C(\mathbf{F})|$. By Proposition 5.5, the variety X_K has weak approximation, so that $X(K)$ is dense in $X(K_\Omega)$. Since $|A| \leq |C(\mathbf{F})|$, let us choose $T \subset C(\mathbf{F})$ satisfying $|T| = |A|$, and we fix a bijection $T \to A$ written as $v \mapsto x_v$. We set $(P_v) \in X(K_\Omega)$ defined as $P_v := x_v$ for $v \in T$ and $P_v := x$ for $v \in \Omega_K \setminus T$. Using the notations of Reminder 2.9, we then consider the nonempty open subset $W_{T,1}$ of $X(K_\Omega^S)$ associated to the tuple (P_v) and the model ρ of X. Since $X(K)$ is dense in $X(K_\Omega)$, we may pick $P \in X(K) \cap W_{T,1}$. Then, using notations of §2.4, the K-point P extends to a section $j : C \to X \times C$ of ρ that coincides with $\widehat{P_v}$ on $\operatorname{Spec}(\kappa(v)) = \operatorname{Spec}(\mathbf{F})$ for $v \in T$. In particular, if $g : C \to X$ is the first coordinate of j, then for each $v \in T$, the restriction of g to $\operatorname{Spec}(\kappa(v)) = \operatorname{Spec}(\mathbf{F})$ is x_v. Since $A = \{x_v : v \in T\}$, this shows that $A = g(T) \subset g(C(\mathbf{F}))$. □

The proof of Proposition 5.5 relies on the following lemma.

Lemma 5.6. Let F be a field of characteristic different from 2 and C a nice curve over F with function field K. Let $f : X \to \mathbf{P}^1_F$ be a regular conic bundle with $X(F) \neq \emptyset$. Then the morphism $\operatorname{Br}(X \times_F C) \to \operatorname{Br}(X_K)/\operatorname{Br}(K)$ is surjective.

Proof. To prove the statement, by Lemma A.3 we need to prove the surjectivity of the map $\operatorname{Br}(X \times_F C)[2] \to (\operatorname{Br}(X_K)/\operatorname{Br}(K))[2]$. Since $X(K) \neq \emptyset$ and since the choice of any element of $X(K)$ gives rise to an isomorphism $\operatorname{Br}(X_K) \simeq \operatorname{Br}(K) \oplus \operatorname{Br}(X)/\operatorname{Br}(K)$, we have $(\operatorname{Br}(X_K)/\operatorname{Br}(K))[2] \simeq \operatorname{Br}(X)[2] / \operatorname{Br}(F)[2]$. It is thus enough to show that the map $\operatorname{Br}(X \times_F C)[2] \to \operatorname{Br}(X)[2] / \operatorname{Br}(F)[2]$ is surjective.

We denote by $\pi : X \times C \to C$ the projection morphism, by η the generic point of C and π_η the base change of π by η. We then have the following commutative diagram whose rows are exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(X \times C)\,[2] \arrow[r] & \mathrm{Br}(X_K)\,[2] \arrow[r] & \bigoplus_{P \in C^{(1)}} \mathrm{H}^1_{\text{ét}}(X_{\kappa(P)}, \mathbf{Z}/2\mathbf{Z}) \arrow[r] & \mathrm{H}^3_{\text{ét}}(X \times C, \mathbf{Z}/2\mathbf{Z}) \\
0 \arrow[r] & \mathrm{Br}(C)\,[2] \arrow[r] \arrow[u, "\pi^*"] & \mathrm{Br}(K)\,[2] \arrow[r] \arrow[u, "\pi^*_\eta"] & \bigoplus_{P \in C^{(1)}} \mathrm{H}^1(\kappa(P), \mathbf{Z}/2\mathbf{Z}) \arrow[r] \arrow[u, "\oplus \operatorname{res}_{X_{\kappa(P)}/\kappa(P)}"'] & \mathrm{H}^3_{\text{ét}}(C, \mathbf{Z}/2\mathbf{Z}) \arrow[u, "\pi^*"]
\end{tikzcd} \tag{10}$$

where the bottom row (resp. top row) is the limit, as U ranges over nonempty open subsets of C, of the exact sequence [CTS21, (3.17)] with $Z := C \setminus U$ (resp. $Z := X \setminus \pi^{-1}(U)$), $l = 2$ and $n = 1$. The commutativity of the left square is due to functoriality of Brauer groups, that of the central square is due to functoriality of residues (see e.g. [CTS21, Theorem 3.7.5]) and to the irreducibility of $X_{\kappa(P)}$, and the commutativity of the right square is given by the functoriality of Gysin's spectral sequence [CTS21, Lemma 2.3.6].

Let us prove that $\pi^* : \mathrm{H}^3_{\text{ét}}(C, \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^3_{\text{ét}}(X \times C, \mathbf{Z}/2\mathbf{Z})$ is injective and that $\oplus \operatorname{res}_{X_{\kappa(P)}/\kappa(P)}$ is an isomorphism. By diagram chasing on (10), this will prove that $\mathrm{Br}(X \times_F C)\,[2] \to \mathrm{Br}(X)\,[2] \,/\, \mathrm{Br}(F)\,[2]$ is surjective. Since $X(F) \neq \emptyset$, the morphism π has a section, which proves that π^* is injective. Furthermore, for each $P \in C^{(1)}$, we fix $\overline{\kappa(P)}$ a separable closure of $\kappa(P)$, a geometric point $\overline{x}$ of $X_{\overline{\kappa(P)}}$ and we still denote by $\overline{x}$ the image of $\overline{x}$ in $X_{\kappa(P)}$ and $\mathrm{Spec}(\kappa(P))$. Then, [Fu11, Proposition 5.7.20] supplies canonical isomorphisms $\mathrm{H}^1_{\text{ét}}(X_{\kappa(P)}, \mathbf{Z}/2\mathbf{Z}) \simeq \mathrm{Hom}_{\text{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}^1(\kappa(P), \mathbf{Z}/2\mathbf{Z}) \simeq \mathrm{Hom}_{\text{cont}}(\pi_1(\mathrm{Spec}(\kappa(P)), \overline{x}), \mathbf{Z}/2\mathbf{Z})$ under which $\operatorname{res}_{X_{\kappa(P)}/\kappa(P)}$ is identified to the pullback $\alpha^* : \mathrm{Hom}_{\text{cont}}(\pi_1(\mathrm{Spec}(\kappa(P)), \overline{x}) \to \mathrm{Hom}_{\text{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}/2\mathbf{Z})$ of the continuous homomorphism $\alpha : \pi_1(X, \overline{x}) \to \pi_1(\mathrm{Spec}(\kappa(P)), \overline{x})$, itself induced by the morphism $X_{\mathrm{Spec}(\kappa(P))} \to \mathrm{Spec}(\kappa(P))$. Moreover, α is surjective with kernel $\pi_1(X_{\overline{\kappa(P)}}, \overline{x})$, see e.g. [Fu11, Proposition 3.3.7]. But Corollary A.2 ensures that $X_{\overline{\kappa(P)}}$ is rational, which, by purity of the étale fundamental group [MR203, X.§3, Corollaire 3.3], implies that $\pi_1(X_{\overline{\kappa(P)}}, \overline{x}) = 1$. In other words, α is an isomorphism, hence so is α^*, that is, $\operatorname{res}_{X_{\kappa(P)}/\kappa(P)}$ is an isomorphism. □

Proof of Proposition 5.5. Denote by $h : X \times \mathrm{Spec}(K) \to X \times C$ the product of id_X with the generic point of C. Let us fix $v \in \Omega_K$ and $x_v \in X(K_v)$. We write $x_v^* : \mathrm{Br}(X_{K_v}) \to \mathrm{Br}(K_v)$ (resp. $h^* : \mathrm{Br}(X \times C) \to \mathrm{Br}(X \times \mathrm{Spec}(K)))$ for the pullback of x_v (resp. h). Let us verify that $x_v^* \circ h^* = 0$. For this purpose, from now on, we use the notation of §2.4. Since the projection morphism $X \times C \to C$ is a model of X, by the valuative criterion of properness, the K_v-point x_v extends to a unique $\widehat{\mathscr{O}_v}$-morphism $\widehat{x_v} : \mathrm{Spec}(\widehat{\mathscr{O}_v}) \to (X \times C) \times_C \mathrm{Spec}(\widehat{\mathscr{O}_v})$. If we still denote by $\widehat{x_v} : \mathrm{Spec}(\widehat{\mathscr{O}_v}) \to X \times C$ its projection to $X \times C$, we thus have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K_v) \arrow[r, "x_v"] \arrow[d] & X_K \arrow[d, "h"] \\
\mathrm{Spec}(\mathscr{O}_v) \arrow[r, "\widehat{x_v}"] & X \times C
\end{tikzcd}$$

so that $x_v^* \circ h^*$ factors through $\widehat{x_v}^* : \mathrm{Br}(X \times C) \to \mathrm{Spec}(\mathscr{O}_v)$. But $\mathrm{Br}(\mathscr{O}_v) = 0$ (see e.g. [Poo17, Corollary 6.9.3]) which proves that $x_v^* \circ h^* = 0$.

Now, for $\alpha \in \mathrm{Br}(X_K)$, Lemma 5.6 supplies $\beta \in \mathrm{Im}(h^*)$ and $\gamma \in \mathrm{Br}(K)$ such that $\alpha = \beta + \gamma$. If $(x_v) \in X(K_\Omega)$, we thus have $\langle (x_v), \alpha \rangle_{BM} = \langle (x_v), \beta \rangle_{BM} + \langle (x_v), \gamma \rangle_{BM}$. But $\langle (x_v), \beta \rangle_{BM} = \sum_{v \in \Omega_K} \mathrm{inv}_v(x_v^*(\beta)) = 0$ since $\mathrm{Im}(h^*) \subset \ker(x_v^*)$ for all $v \in \Omega_K$, and

$\langle (x_v), \gamma \rangle_{BM} = 0$ by Albert-Brauer-Hasse-Noether exact sequence. Thus, $\langle (x_v), \alpha \rangle_{BM} = 0$, so that the Brauer-Manin pairing on X_K is identically zero. □

Appendix A Brauer groups of surfaces in positive characteristic

A.1 Geometrically separably rationally connected surfaces

Needed in our proof of (1) of Theorem 5.2 is the following statement on Brauer groups of separably rationally connected varieties. We recall that if X is a scheme over a field F and F^{s} is a separable closure of F, we denote the *algebraic Brauer group* of X by $\mathrm{Br}_1(X) := \ker \left[\mathrm{Br}(X) \xrightarrow{\mathrm{pr}^*} \mathrm{Br}(X \otimes_F F^{\mathrm{s}}) \right]$, where $\mathrm{pr} : X \otimes_F F^{\mathrm{s}} \to X$ is the first projection morphism.

Proposition A.1. Let F be a field and F^{s} a separable closure of F. Let also X be a smooth, proper and geometrically integral F-variety and set $X^{\mathrm{s}} := X \otimes_F F^{\mathrm{s}}$. If X is separably rationally connected, then $\mathrm{Br}_1(X)/\mathrm{Br}(F)$ is finite. In particular, if X^{s} is rational, then $\mathrm{Br}(X)/\mathrm{Br}(F)$ is finite.

Proof. The statement is well known if the characteristic of F is zero (see e.g. [CTS21, assertion (6) in p.347]), so we may assume that F has positive characteristic p. The second part of the statement is inferred from the first part. Indeed, if X^{s} is rational, then $\mathrm{Br}(X^{\mathrm{s}}) \simeq \mathrm{Br}(\mathbf{P}^{\dim(X)}_{F^{\mathrm{s}}})$ by purity of the Brauer group [Č19], so that $\mathrm{Br}(X^{\mathrm{s}}) = \mathrm{Br}(F^{\mathrm{s}}) = 0$, hence $\mathrm{Br}(X) = \mathrm{Br}_1(X)$. Let us then assume that X is separably rationally connected and prove the first part of the statement. We denote by $\overline{F}$ an algebraic closure of F and we set $\overline{X} := X \otimes_F \overline{F}$.

We extract from exact sequence (2.23) of [Sko01, Corollary 2.3.9] a short exact sequence:

$$\mathrm{Br}(F) \xrightarrow{\pi^*} \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(F, \mathrm{Pic}(X^s))$$

where $\pi : X \to \mathrm{Spec}(F)$ is the structural morphism of X. It is thus enough to prove that $\mathrm{Pic}(X^{\mathrm{s}})$ is finitely generated and torsion-free, from which we infer the finiteness of $\mathrm{H}^1(F, \mathrm{Pic}(X^{\mathrm{s}}))$, hence that of $\mathrm{Br}_1(X)/\mathrm{Br}(F)$. First, separably rational connectedness of X ensures that $\mathrm{H}^1(\overline{X}, \mathscr{O}_{\overline{X}}) = 0$ by (see [BDS13] and [Gou14]). This implies that $\mathrm{Pic}(X^{\mathrm{s}}) = \mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$, the Néron-Severi group of $\overline{X}$, which is finitely generated (see e.g. [CTS21, Corollary 5.1.3.(i)]). Now, using Kummer's exact sequence, for each prime number $\ell \neq p$ and $m \geq 0$ we have $\mathrm{Pic}(\overline{X})[\ell^m] = \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbf{Z}/\ell^m\mathbf{Z})$. But since $\overline{X}$ is separably rationally connected, it is simply connected (see [Bis09, Theorem 2.1]), so that $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbf{Z}/\ell^m\mathbf{Z}) = \mathrm{Pic}(\overline{X})[\ell^m] = 0$. In particular, $\mathrm{Pic}(\overline{X})\{p'\} = 0$. Furthermore, $\mathrm{Pic}(\overline{X})\{p\} = 0$ by [GJ18, Theorem 1.4]. From this we deduce that $\mathrm{Pic}(\overline{X})$ is torsion-free. □

Corollary A.2. Let F be a field and $f : X \to \mathbf{P}^1_F$ a conic bundle. Then, for all field extension K of F and separable closure K^{s} of K, the variety $X_{K^{\mathrm{s}}}$ is rational. In particular, the group $\mathrm{Br}(X_K)/\mathrm{Br}(K)$ is finite.

Proof. Let K be a field extension of F and η the generic point of $\mathbf{P}^1_F$. We let K^{s} (resp. F^{s}) be a separable closure of K (resp. the separable closure of F contained in K^{s}) and $\overline{F}$ an algebraic closure of F containing F^{s}.

Let us first prove that $X_{\overline{F}}$ is rational. Indeed, since X_η is geometrically integral we have an isomorphism $\overline{F}(X) \simeq \overline{F}(\mathbf{P}^1)(X_\eta)$ over $\overline{F}$. But $\mathrm{Br}(\overline{F}(\mathbf{P}^1)) = 0$ by Tsen's theorem so that the class of the smooth conic $X_\eta \otimes_{F(\mathbf{P}^1)} \overline{F}(\mathbf{P}^1)$ in $\mathrm{Br}(\overline{F}(\mathbf{P}^1))$ is trivial, that is, there exists an isomorphism $X_\eta \otimes_{F(\mathbf{P}^1)} \overline{F}(\mathbf{P}^1) \simeq \mathbf{P}^1_{\overline{F}(\mathbf{P}^1)}$ of $\overline{F}(\mathbf{P}^1)$-varieties. This proves that the field $\overline{F}(\mathbf{P}^1)(X_\eta)$ is purely transcendental over $\overline{F}(\mathbf{P}^1)$, hence $\overline{F}(X) \simeq \overline{F}(\mathbf{P}^1)(X_\eta)$ is purely transcendental over $\overline{F}$. In other words, $X_{\overline{F}}$ is rational.

By [Coo88, Theorem 1], this implies that $X_{F^{\mathrm{s}}}$ is also rational, so that the K^{s}-variety $X_{K^{\mathrm{s}}} = X_{F^{\mathrm{s}}} \otimes_{F^{\mathrm{s}}} K^{\mathrm{s}}$ is rational. The last part of the statement is a consequence of the last assertion of Proposition A.1. ☐

A.2 Conic bundles over a curve

In characteristic zero, it is well known that up to constant classes, the Brauer group of a conic bundle over $\mathbf{P}^1$ is a 2-torsion group, see e.g. [CTS21, Corollary 11.3.5]. In the following proposition, we show that it still holds in odd characteristic.

Proposition A.3. Let F be a field of characteristic different from 2 and $X \to \mathbf{P}^1_F$ a regular conic bundle. If $X(F) \neq \emptyset$, then $\mathrm{Br}(X)/\mathrm{Br}(F)$ is a 2-torsion group.

Proof of Lemma A.3. Let us first prove that $(\mathrm{Br}(X)/\mathrm{Br}(F))\{\ell\} = 0$ for all prime numbers $\ell \neq 2$. For this purpose, we fix a prime number $\ell \neq 2$, we denote by F^{s} a separable closure of F and we set E the fixed field of an ℓ-Sylow of $\mathrm{Gal}(F^{\mathrm{s}}/F)$, so that $\mathrm{Gal}(F^{\mathrm{s}}/E)$ is a pro-ℓ group. Using the commutativity of Br with limits [CTS21, §2.2.2] and [CTS21, Proposition 3.8.4], it is enough to prove that $(\mathrm{Br}(X_E)/\mathrm{Br}(E))\{\ell\} = 0$. Since $\mathrm{Gal}(K^{\mathrm{s}}/E)$ is a pro-ℓ group, E has no quadratic extension, so that all the closed fibres of the conic bundle $f_E := f \otimes_F E : X_E \to \mathbf{P}^1_E$ are split. By Lemma 3.5, and (b) of Remark 3.6, there exists a section s of f_E. If we denote by η the generic fibre of $\mathbf{P}^1_E$, we thus have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(X_E) \arrow[r] \arrow[d, bend left, "s^*"] & \mathrm{Br}(X_{E,\eta}) \arrow[d, bend left, "s_\eta^*"] \\
0 \arrow[r] & \mathrm{Br}(\mathbf{P}^1_E) \arrow[r] \arrow[u, "f^*"] & \mathrm{Br}(\eta) \arrow[u, "f_\eta^*"]
\end{tikzcd} \tag{11}$$

whose rows are exact since X_E is regular, and where f_η (resp. s_η) is the base change of f by η (resp. is the rational point of $X_{E,\eta}$ corresponding to s). Since the conic $X_{E,\eta}$ has a rational point s_η, it is isomorphic to $\mathbf{P}^1_{E(\mathbf{P}^1)}$, so that the map f_η^* is an isomorphism. As $s_\eta^* \circ f_\eta^* = id$, the map s_η^* is also an isomorphism. In particular, the commutativity of (11) ensures that s^* is injective. But since $s^* \circ f^* = id$, the morphism s^* is also surjective, so that s^* is an isomorphism. This proves that $\mathrm{Br}(X_E) \simeq \mathrm{Br}(\mathbf{P}^1_E)$, that is, $\mathrm{Br}(X_E) = \mathrm{Br}(E)$, so that $(\mathrm{Br}(X_E)/\mathrm{Br}(E))\{\ell\} = 0$.

It remains to show that $(\mathrm{Br}(X)/\mathrm{Br}(F))\{2\} = (\mathrm{Br}(X)/\mathrm{Br}(F))[2]$. The choice of an element in $X(F)$ supplies an isomorphism $\mathrm{Br}(X) \simeq \mathrm{Br}(F) \oplus \mathrm{Br}(X)/\mathrm{Br}(F)$, so that $(\mathrm{Br}(X)/\mathrm{Br}(F))\{2\} \simeq \mathrm{Br}(X)\{2\}/\mathrm{Br}(F)\{2\}$. Furthermore, if we denote by η the generic point of $\mathbf{P}^1_F$ and f_η the pullback of f by η, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(X)\{2\} \arrow[r] & \mathrm{Br}(X_\eta)\{2\} \arrow[r] & \displaystyle\bigoplus_{P \in (\mathbf{P}^1_F)^{(1)}} \bigoplus_{V \subset X_P} \mathrm{H}^1(F(V), \mathbf{Q}_2/\mathbf{Z}_2) \\
0 \arrow[r] & \mathrm{Br}(F)\{2\} \arrow[r] \arrow[u, "f^*"] & \mathrm{Br}(F(\mathbf{P}^1))\{2\} \arrow[r] \arrow[u, "f_\eta^*"] & \displaystyle\bigoplus_{P \in (\mathbf{P}^1_F)^{(1)}} \mathrm{H}^1(\kappa(P), \mathbf{Q}_2/\mathbf{Z}_2) \arrow[u, "\oplus_P \oplus_V \mathrm{res}_{F(V)/\kappa(P)}"']
\end{tikzcd} .$$

where the bottom row is the limit, as U ranges over nonempty subsets of $\mathbf{P}^1_F$, of the exact sequences [CTS21, Theorem 3.7.2.(ii)] with $Z \coloneqq \mathbf{P}^1_F \setminus U$ and $\ell = 2$, combined with the isomorphism $\mathrm{Br}(\mathbf{P}^1_F) \simeq \mathrm{Br}(F)$. As for the top row, $V \subset X_P$ ranges over the irreducible components of X_P which, by flatness of f, are precisely the codimension one subschemes of X. The top row complex is then obtained as the limit, as U ranges over nonempty subsets of $\mathbf{P}^1_F$ of the short exact sequences [CTS21, Theorem 3.7.2.(3.19)] applied to $Z \coloneqq X \setminus f^{-1}(U)$ and $\ell = 2$. The whole diagram is commutative by the functoriality of residues (see e.g. [CTS21, Theorem 3.7.5]). Let us notice that since X_η is a smooth conic, the middle vertical row is surjective by [CTS21, Proposition 7.2.1]. Also, if κ_V denotes the algebraic closure of $\kappa(P)$ in $F(V)$, the kernel of the right vertical map is $\bigoplus_{P\in(\mathbf{P}^1_F)^{(1)}} \bigcap_{V\subset X_P} \mathrm{H}^1(\mathrm{Gal}(\kappa_V/\kappa(P)), \mathbf{Q}_2/\mathbf{Z}_2)$. Since X_P is a conic, this group is 2-torsion as $\kappa_V/\kappa(P)$ is an extension of degree at most 2. By diagram chasing, we deduce that $2\,(\mathrm{Br}(X)\{2\}) \subset \mathrm{Br}(F)\{2\}$, that is, $\mathrm{Br}(X)\{2\}/\mathrm{Br}(F)\{2\}$ is a 2-torsion group, hence $(\mathrm{Br}(X)/\mathrm{Br}(F))\,\{2\}$ is 2-torsion. □

References

[Ax68] J. Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271. MR 229613 2

[BDS13] I. Biswas and J. P. Dos Santos, *Triviality criteria for bundles over rationally connected varieties*, J. Ramanujan Math. Soc. **28** (2013), no. 4, 423–442. MR 3158990 18

[Bis09] I. Biswas, *On the fundamental group-scheme*, Bull. Sci. Math. **133** (2009), no. 5, 477–483. MR 2538008 18

[Coo88] K. R. Coombes, *Every rational surface is separably split*, Comment. Math. Helv. **63** (1988), no. 2, 305–311. MR 948785 19

[CT15] J.-L. Colliot-Thélène, *Surfaces de del Pezzo de degré 4 sur un corps C_1*, Taiwanese J. Math. **19** (2015), no. 6, 1613–1618. MR 3434268 3

[CTS80] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 223–237. MR 605344 3, 14

[CTS87] J.-L. Colliot-Thélène and A. N. Skorobogatov, *R-equivalence on conic bundles of degree* 4, Duke Math. J. **54** (1987), no. 2, 671–677. MR 899411 3

[CTS21] ———, *The Brauer-Grothendieck group*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 71, Springer, Cham, [2021] ©2021. MR 4304038 4, 5, 6, 7, 9, 14, 17, 18, 19, 20

[Fu11] L. Fu, *Etale cohomology theory*, Nankai Tracts in Mathematics, vol. 13, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. MR 2791606 17

[FvL16] D. Festi and R. van Luijk, *Unirationality of del Pezzo surfaces of degree 2 over finite fields*, Bull. Lond. Math. Soc. **48** (2016), no. 1, 135–140. MR 3455757 1

[GJ18] F. Gounelas and A. Javanpeykar, *Invariants of Fano varieties in families*, Mosc. Math. J. **18** (2018), no. 2, 305–319. MR 3831010 18

[Gou14] F. Gounelas, *The first cohomology of separably rationally connected varieties*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 11, 871–873. MR 3268754 18

[GS06] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. MR 2266528 4, 9, 10, 12

[Har07] D. Harari, *Quelques propriétés d'approximation reliées à la cohomologie galoisienne d'un groupe algébrique fini*, Bull. Soc. Math. France **135** (2007), no. 4, 549–564. MR 2439198 7

[Has10] B. Hassett, *Weak approximation and rationally connected varieties over function fields of curves*, Variétés rationnellement connexes: aspects géométriques et arithmétiques, Panor. Synthèses, vol. 31, Soc. Math. France, Paris, 2010, pp. 115–153. MR 2931861 7

[Isk67] V. A. Iskovskih, *Rational surfaces with a pencil of rational curves*, Mat. Sb. (N.S.) **74(116)** (1967), 608–638. MR 220734 1, 2

[Isk79] ______, *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 19–43, 237. MR 525940 1

[KM17] J. Kollár and M. Mella, *Quadratic families of elliptic curves and unirationality of degree 1 conic bundles*, Amer. J. Math. **139** (2017), no. 4, 915–936. MR 3689320 2

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 3

[Kol99] ______, *Rationally connected varieties over local fields*, Ann. of Math. (2) **150** (1999), no. 1, 357–367. MR 1715330 3

[Kol02] ______, *Unirationality of cubic hypersurfaces*, J. Inst. Math. Jussieu **1** (2002), no. 3, 467–476. MR 1956057 8

[Man66] Yu. I. Manin, *Rational surfaces over perfect fields*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 30, 55–113. MR 225780 1

[Man67] ______, *Rational surfaces over perfect fields. II*, Mat. Sb. (N.S.) **72(114)** (1967), 161–192. MR 225781 1

[Man71] ______, *Le groupe de Brauer-Grothendieck en géométrie diophantienne*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars Éditeur, Paris, 1971, pp. 401–411. MR 427322 7

[Man86] ______, *Cubic forms*, second éd., North-Holland Mathematical Library, vol. 4, North-Holland Publishing Co., Amsterdam, 1986, Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel. MR 833513 2

[Mes96] J.-F. Mestre, *Annulation, par changement de variable, d'éléments de* $\mathrm{Br}_2(k(x))$*, où* k *est un corps fini*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 5, 423–426. MR 1381777 2

[MR203] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR 2017446 17

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, second éd., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026 5, 10, 12

[Pir12] A. Pirutka, *R-equivalence on low degree complete intersections*, J. Algebraic Geom. **21** (2012), no. 4, 707–719. MR 2957693 3

[Poo17] B. Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR 3729254 15, 17

[PS99] A. N. Parshin and I. R. Shafarevich (eds.), *Algebraic geometry. V*, Encyclopaedia of Mathematical Sciences, vol. 47, Springer-Verlag, Berlin, 1999, Fano varieties, A translation of *Algebraic geometry. 5* (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow. MR 1668575 8

[Ser94] J.-P. Serre, *Cohomologie galoisienne*, fifth éd., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994. MR 1324577 4, 5

[Sko96] A. N. Skorobogatov, *Descent on fibrations over the projective line*, Amer. J. Math. **118** (1996), no. 5, 905–923. MR 1408492 1, 5

[Sko01] ______, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001. MR 1845760 9, 18

[Sko15] ______, *Descent on toric fibrations*, Arithmetic and geometry, London Math. Soc. Lecture Note Ser., vol. 420, Cambridge Univ. Press, Cambridge, 2015, pp. 422–435. MR 3467133 6

[STVA14] C. Salgado, D. Testa, and A. Várilly-Alvarado, *On the unirationality of del Pezzo surfaces of degree 2*, J. Lond. Math. Soc. (2) **90** (2014), no. 1, 121–139. MR 3245139 1

[Č19] K. Česnavičius, *Purity for the Brauer group*, Duke Math. J. **168** (2019), no. 8, 1461–1486. MR 3959863 18

[Vor86] I. I. Voronovich, *Splitting fields of central simple algebras over a field of rational functions*, Dokl. Akad. Nauk BSSR **30** (1986), no. 9, 773–775, 860. MR 867120 2

[Yan85] V. I. Yanchevskiĭ, *K-unirationality of conic bundles and splitting fields of simple central algebras*, Dokl. Akad. Nauk BSSR **29** (1985), no. 12, 1061–1064, 1148. MR 826395 2

[Yan90] ______, *K-unirationality of conic bundles, the Kneser-Tits conjecture for spinor groups and central simple algebras*, Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, Part 2, PWN, Warsaw, 1990, pp. 483–490. MR 1171295 2

[Yan92] ______, *K-unirationality of conic bundles over large arithmetic fields*, no. 209, 1992, Journées Arithmétiques, 1991 (Geneva), pp. 16, 311–320. MR 1211025 1, 2

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH - CLAVERTON DOWN, BATH, BA2 7AY, UNITED KINGDOM

Email address: `eb2751@bath.ac.uk`