A QUILLEN THEOREM B FOR STRICT ∞-CATEGORIES

by

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Abstract. — We prove a generalization of Quillen's Theorem B to strict ∞-categories. More generally, we show that under similar hypothesis as for Theorem B, the comma construction for strict ∞-categories, that we introduced with Maltsiniotis in a previous paper, is the homotopy pullback with respect to Thomason equivalences. We give several applications of these results, including the construction of new models for certain Eilenberg–Mac Lane spaces.

Introduction

This paper is a part of an ongoing research project with Maltsiniotis and Gagna about the homotopy theory of ∞-$\mathcal{C}at$, the category of strict ∞-categories and strict ∞-functors, including for the moment the papers and preprints [**3, 4, 5, 6, 7, 16**]. By "homotopy theory of ∞-$\mathcal{C}at$", we mean the study of strict ∞-categories through their classifying spaces, defined by means of the so-called Street nerve [**25**], associating a simplicial set to every strict ∞-category. In other words, the homotopy theory of ∞-$\mathcal{C}at$ is the homotopy theory of the pair $(\infty\text{-}\mathcal{C}at, \mathcal{W}_\infty)$, where $\mathcal{W}_\infty$ denotes the class of *Thomason equivalences* of ∞-$\mathcal{C}at$, that is, of strict ∞-functors sent to simplicial weak homotopy equivalences by Street's nerve. Gagna proved in [**16**] that the localization of ∞-$\mathcal{C}at$ by $\mathcal{W}_\infty$ is equivalent to the homotopy category of spaces. Therefore, the study of the homotopy theory of ∞-$\mathcal{C}at$ is the study of the homotopy theory of spaces from an alternative point of view. We refer to the introduction of [**6**] for a detailed exposition of this project.

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But let us step back. This homotopy theory of strict ∞-categories is inspired by the homotopy theory of $\mathcal{C}at$, the category of small categories, as developed by Quillen [**22**], Thomason [**27**] and Grothendieck [**18**] (see also [**20, 15**]). The starting point of this theory is the idea of Quillen to define his higher algebraic K-theory groups in terms of homotopy types of categories. To study the resulting theory, he introduced his famous Theorems A and B, establishing the main properties of *Thomason equivalences* of $\mathcal{C}at$, that is, functors sent to simplicial weak homotopy equivalences by the usual nerve functor.

Before recalling the statement of Theorem B, let us introduce some notation and terminology. If $u : A \to B$ is a functor and b is an object of B, we will denote by $_b\backslash A$ the category of objects of A under b, that is, of pairs $(a, f : b \to u(a))$, where a is an object of A and f is a morphism of B. In other words, the category $_b\backslash A$ is the comma category $b{\downarrow}u$, where b is identified with the functor from the terminal category to B of value b. We will say that a functor $u : A \to B$ is *colocally homotopically constant* if, for every morphism $f : b \to b'$ of B, the functor $_f\backslash A : {}_{b'}\backslash A \to {}_b\backslash A$ induced by f is a Thomason equivalence.

***Theorem B* (Quillen).** — If $u : A \to B$ is a colocally homotopically constant functor, then, for every object b of B, the category $_b\backslash A$ is canonically the homotopy fiber of u at b.

The homotopy fiber of the statement has to be understood in the sense of the Thomason model category structure [**27**] but can also be interpreted by simply saying that $N(_b\backslash A)$ is the simplicial homotopy fiber of Nu at b, where N denotes the nerve functor. This theorem was generalized by Barwick and Kan [**10**] to the following statement about homotopy pullbacks:

***Theorem* (Barwick–Kan).** — *Consider a diagram*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

of $\mathcal{C}at$, where v is colocally homotopically constant. Then the comma construction $u{\downarrow}v$ is canonically the homotopy pullback $A \times^h_C B$.

This result is also a consequence of a version of Quillen's Theorem B due to Cisinski [**15**, Theorem 6.4.15] [1]. Barwick and Kan actually proved a similar result for (weak) (∞, n)-categories. Note that this generalization, for $n > 0$, is orthogonal to our project as the weak equivalences used by Barwick and Kan are the equivalences of (∞, n)-categories and not the Thomason equivalences. When $n = 0$, these two classes of weak equivalences coincide and the above theorem is essentially the case $n = 0$.

1. More precisely, this follows from condition (vi) of the "dual" of Cisinski's theorem applied to the canonical functor $u{\downarrow}C \to C$, which is a Grothendieck cofibration, so that any functor is transverse to it in the sense of Cisinski (and from the fact that the canonical functor $A \to u{\downarrow}C$ is a deformation retract and hence a Thomason equivalence).

The theorems of Quillen and Barwick–Kan for categories were generalized to 2-categories endowed with Thomason equivalences, and even bicategories, by Cegarra *et al.* [**13, 12, 14**].

The purpose of this paper is to generalize these two theorems to strict ∞-categories. We introduced, with Maltsiniotis, in our study of the ∞-categorical Theorem A [**7**], a comma construction for strict ∞-categories. As for categories, a special case of this construction allows to define a slice construction and one can thus define the notion of a colocally homotopically constant strict ∞-functor. Therefore, using Street's nerve to define homotopy pullbacks, the statements of these two theorems still make sense for strict ∞-categories and our main result can be stated as:

Theorem. — *Consider a diagram*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

of ∞-$\mathcal{C}at$, where v is colocally homotopically constant. Then the comma construction $u{\downarrow}v$ is canonically the homotopy pullback $A \times^h_C B$.

This implies the Theorem B for strict ∞-categories:

Theorem. — *If $u : A \to B$ is a colocally homotopically constant strict ∞-functor, then, for every object b of B, the ∞-category ${}_{b}\backslash A$ is canonically the homotopy fiber of u at b.*

The proof is inspired by Cisinski's proof of the original Quillen Theorem B [**15**, Theorem 6.4.15] and is in particular based on a simplicial result due to Rezk that can be thought of as a simplicial version of Quillen's Theorem B. Besides Rezk's result, our proof rely mainly on two tools. First, we use the sesquifunctoriality of the comma construction that we proved with Maltsiniotis [**7**, Appendix B]. Second, we use Steiner's theory of augmented directed complexes [**24**] to produce "contractions" of Street's orientals [**25**].

As an application of our Theorem B, we prove the following statement about loop spaces of strict ∞-categories:

Theorem. — *Let A be a strict ∞-category endowed with an object a. Suppose that for every 1-cell $f : a' \to a''$ the induced ∞-functor $\underline{\mathsf{Hom}}_A(a'', a) \to \underline{\mathsf{Hom}}_A(a', a)$ is a Thomason equivalence. Then $\underline{\mathsf{Hom}}_A(a, a)$ is a model for the loop space of (A, a).*

Using this theorem and a particular case of the Theorem A for strict ∞-categories [**6, 7**], that we deduce from Theorem B, we obtain new models for some Eilenberg–Mac Lane spaces:

Theorem. — *Let π be a commutative ordered group whose underlying poset is directed. Denote by π^+ its monoid of positive elements. Then, for any $n \geqslant 1$, the ∞-category $B^n\pi^+$ is a $K(\pi, n)$.*

In this statement, $B^n\pi^+$ denotes the obvious strict n-category having only one i-cell for $0 \leqslant i < n$ and whose set of n-cells is π^+. In particular, we get that $B^n\mathbb{N}$ is a $K(\mathbb{Z}, n)$.

Our paper is organized as follows. The first section contains simplicial preliminaries and in particular Rezk's simplicial Theorem B. In the second section, we introduce the ∞-categorical notions that we need. We recall the main properties of the Gray tensor product and of oplax transformations. We give a brief overview of the theory of comma ∞-categories that we introduced with Maltsiniotis in [**7**]. We use this theory to define slice ∞-categories and a kind of mapping space factorization for strict ∞-functors. The third section contains the main results of the paper. We introduce the notion of Thomason equivalences, homotopy pullback squares in ∞-$\mathcal{C}at$ and colocally homotopically constant strict ∞-functors. We study the homotopical behavior of these ∞-functors under base change, from which we deduce our main theorems, including the Theorem B for strict ∞-categories. The fourth section contains several applications of our Theorem B. We prove the non-relative case of the Theorem A for strict ∞-categories. We then apply Theorem B to study loop spaces of strict ∞-categories. Using our results, we produce new models for certain Eilenberg–Mac Lane spaces. We end the section with an application to loop spaces of strict ∞-groupoids. Finally, in an appendix, we produce, using Steiner's theory [**24**], the "contractions" of Street's orientals [**25**] needed in the proof of the main result.

1. Simplicial preliminaries

1.1. — We will denote by $\mathbf{\Delta}$ the *simplex category*. Recall that it is the full subcategory of the category of ordered sets whose objects are the

$$\Delta_n = \{0 < \cdots < n\}$$

for $n \geqslant 0$. The category of *simplicial sets*, that is, of presheaves over $\mathbf{\Delta}$, will be denoted by $\widehat{\mathbf{\Delta}}$. The Yoneda embedding $\mathbf{\Delta} \hookrightarrow \widehat{\mathbf{\Delta}}$ will always be considered as an inclusion.

1.2. — By a *weak equivalence* of simplicial sets, we will always mean a weak homotopy equivalence, that is, a weak equivalence of the Kan–Quillen model category structure. Similarly, by a homotopy pullback square of simplicial sets, we will always mean a homotopy pullback square for the Kan–Quillen model category structure.

The following proposition can be considered as a kind of simplicial Theorem B and will be crucial to our proof of the ∞-categorical Theorem B:

Proposition 1.3 **(Rezk)**. — *Let $p : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:*

(a) *every pullback square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y' & \longrightarrow & Y \end{array}$$

is a homotopy pullback square,

(b) *for every diagram of pullback squares*

$$\begin{array}{ccccc} X'' & \xrightarrow{u'} & X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y'' & \xrightarrow[u]{} & Y' & \longrightarrow & Y \end{array} \quad ,$$

if u is a weak equivalence, then so is u',

(c) *for every diagram of pullback squares*

$$\begin{array}{ccccc} X'' & \xrightarrow{u'} & X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta_n & \longrightarrow & \Delta_m & \longrightarrow & Y \end{array} \quad ,$$

the morphism u' is a weak equivalence,

(d) *for every diagram of pullback squares*

$$\begin{array}{ccccc} X'' & \xrightarrow{u'} & X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta_0 & \longrightarrow & \Delta_m & \longrightarrow & Y \end{array} \quad ,$$

the morphism u' is a weak equivalence.

Proof. — The equivalence between conditions (a) and (b) follows from [**23**, Proposition 2.7]. The equivalence between these two conditions and condition (c) is a consequence of [**23**, Theorem 4.1] (see [**23**, Remark 4.2]). Clearly, condition (c) implies condition (d) and it suffices to prove the converse. Consider a diagram of pullback squares as in condition (c) and form the diagram of pullback squares

$$\begin{array}{ccccccc} X''' & \xrightarrow{u''} & X'' & \xrightarrow{u'} & X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta_0 & \xrightarrow[0]{} & \Delta_n & \longrightarrow & \Delta_m & \longrightarrow & Y \end{array} \quad .$$

By condition (d), the morphisms u'' and $u'u''$ are weak equivalences. This implies that u' is a weak equivalence, thereby proving the result. □

We end the section with a probably well-known fact about fiber products of strong deformation retracts.

1.4. — By a *strong left deformation retract*, we will mean a triple

$$(i : A \to B, r : B \to A, h : \Delta_1 \times B \to B)$$

of simplicial maps such that

(*a*) r is a retraction of i (so that $ri = 1_A$),

(*b*) h is a homotopy from ir to 1_B,

(*c*) we have $h(\Delta_1 \times i) = ip_2$, where $p_2 : \Delta_1 \times A \to A$ denotes the second projection.

If the homotopy h goes from 1_B to ir, instead of going from ir to 1_B, we will talk of a *strong right deformation retract.*

Proposition 1.5. — *Let*

$$(i_k : A_k \to B_k, r_k : B_k \to A_k, h_k : \Delta_1 \times B_k \to B_k),$$

for $k = 0, 1, 2$*, be three left* (resp. *right*) *strong deformation retracts. If* f_0, f_1, g_0, g_1 *are morphisms such that the diagrams*

$$\begin{array}{ccccc} A_0 & \xrightarrow{f_0} & A_2 & \xleftarrow{f_1} & A_1 \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ B_0 & \xrightarrow[g_0]{} & B_2 & \xleftarrow[g_1]{} & B_1 \end{array}$$

and

$$\begin{array}{ccccc} \Delta_1 \times B_0 & \xrightarrow{\Delta_1 \times g_0} & \Delta_1 \times B_2 & \xleftarrow{\Delta_1 \times g_1} & \Delta_1 \times B_1 \\ {\scriptstyle h_0}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_1} \\ B_0 & \xrightarrow[g_0]{} & B_2 & \xleftarrow[g_1]{} & B_1 \end{array}$$

commute, then

$$\begin{aligned} &\big(i_0 \times_{i_2} i_1 : A_0 \times_{A_2} A_1 \to B_0 \times_{B_2} B_1, \\ &\ r_0 \times_{r_2} r_1 : B_0 \times_{B_2} B_1 \to A_0 \times_{A_2} A_1, \\ &\ h_0 \times_{h_2} h_1 : \Delta_1 \times (B_0 \times_{B_2} B_1) \to B_0 \times_{B_2} B_1\big) \end{aligned}$$

is a strong left (resp. *right*) *deformation retract.*

Proof. — We only need to check that $r_0 \times_{r_2} r_1$ is well-defined; the fact that the triple of the statement is a strong left (resp. right) deformation retract will then follow by functoriality of the fiber product. Evaluating the second diagram of the statement

at 0 (resp. at 1) in Δ_1, we get that the diagram

$$\begin{array}{ccccc} B_0 & \xrightarrow{g_0} & B_2 & \xleftarrow{g_1} & B_1 \\ \downarrow{\scriptstyle i_0r_0} & & \downarrow{\scriptstyle i_2r_2} & & \downarrow{\scriptstyle i_1r_1} \\ B_0 & \xrightarrow[g_0]{} & B_2 & \xleftarrow[g_1]{} & B_1 \end{array}$$

commutes. The fact that i_2 is a monomorphism and that the first diagram of the statement commutes implies then that the diagram

$$\begin{array}{ccccc} B_0 & \xrightarrow{g_0} & B_2 & \xleftarrow{g_1} & B_1 \\ \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle r_1} \\ A_0 & \xrightarrow[f_0]{} & A_2 & \xleftarrow[f_1]{} & A_1 \end{array}$$

commutes as well, thereby ending the proof. □

2. Preliminaries on oplax transformations and comma ∞-categories

2.1. — We will denote by ∞-$\mathcal{C}at$ the category of strict ∞-categories and strict ∞-functors. All the ∞-categories and ∞-functors considered in this paper will be strict, and we will drop the adjective "strict" from now on.

If C is an ∞-category, we will denote by C° the ∞-category obtained from C by reversing all the i-cells for $i > 0$.

We will denote by D_0 the terminal ∞-category and by D_1 the ∞-category associated to the category defined by the ordered set $\Delta_1 = \{0 < 1\}$. We have two ∞-functors $\sigma, \tau : \mathrm{D}_0 \to \mathrm{D}_1$ corresponding respectively to the objects 0 and 1 of D_1 and a unique ∞-functor $\kappa : \mathrm{D}_1 \to \mathrm{D}_0$.

We begin the section with some preliminaries on the Gray tensor product and oplax transformations.

2.2. — The category ∞-$\mathcal{C}at$ is endowed with a biclosed monoidal category structure given by the so-called *Gray tensor product*, first introduced in [**1**]. This tensor product is a generalization of the tensor product of 2-categories introduced by Gray in [**17**]. We will not need its precise definition and we will only recall the properties we will need. We refer the reader to [**5**, Appendix A] for a comprehensive presentation in the spirit of our paper. If A and B are two ∞-categories, their (Gray) tensor product will be denoted by $A \otimes B$. For instance, one has

$$\mathrm{D}_1 \otimes \mathrm{D}_1 \simeq \begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & \swarrow & \downarrow \\ (1,0) & \longrightarrow & (1,1) \end{array} .$$

The unit of this tensor product is the terminal ∞-category D_0. The right and left internal Hom will be denoted by $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$ respectively, so that we have bijections

$$\mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(A \otimes B, C) \simeq \mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(A, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, C))$$

and

$$\mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(A \otimes B, C) \simeq \mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(B, \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, C)),$$

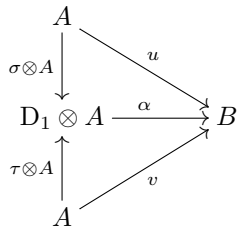
natural in A, B and C in ∞-$\mathcal{C}at$.

Remark 2.3. — The orientation of the non-trivial 2-cell of $\mathrm{D}_1 \otimes \mathrm{D}_1$ in the diagram of the previous paragraph reveals that the Gray tensor product we work with is what we would call the *oplax* Gray tensor product, as opposed to the *lax* Gray tensor product in which this 2-cell would be reversed.

2.4. — Let A and B be two ∞-categories. The objects of the ∞-categories

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \quad \text{and} \quad \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$$

are in canonical bijection with the ∞-functors from A to B. If $u, v : A \to B$ are two such ∞-functors, a 1-cell α of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ from u to v is called an *oplax transformation* from u to v. We will then write $\alpha : u \Rightarrow v$. By definition, an oplax transformation α corresponds to an ∞-functor $\mathrm{D}_1 \to \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ and so, by adjunction, to an ∞-functor $\mathrm{D}_1 \otimes A \to B$. The fact that α has u as source and v as target translates as the commutativity of the diagram



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where we identify A and $\mathrm{D}_0 \otimes A$. Alternatively, again by adjunction, an oplax transformation corresponds to an ∞-functor $A \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B)$. The source and the target of an oplax transformation given by such an ∞-functor are obtained by postcomposing by

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(\sigma, B) : \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B) \simeq B,$$
$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(\tau, B) : \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B) \simeq B,$$

respectively.

Similarly, 1-cells of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$ are called *lax transformations*.

2.5. — Let $u : A \to B$ be an ∞-functor. We define the *identity oplax transformation* $1_u : u \Rightarrow u$ to be the oplax transformation corresponding to the composite

$$\mathrm{D}_1 \otimes A \xrightarrow{\kappa \otimes A} \mathrm{D}_0 \otimes A \xrightarrow{\sim} A \xrightarrow{u} B \quad ,$$

where the middle arrow is the canonical isomorphism.

Let $v : A \to B$ be a second ∞-functor and let $\alpha : u \Rightarrow v$ be an oplax transformation, seen as an ∞-functor $\mathrm{D}_1 \otimes A \to B$. If $w : B \to C$ is an ∞-functor, we get an oplax transformation $w * \alpha : wu \Rightarrow wv$ by composing

$$\mathrm{D}_1 \otimes A \xrightarrow{\alpha} B \xrightarrow{w} C \quad .$$

Similarly, if $w : C \to A$ is an ∞-functor, we get an oplax transformation $\alpha * w : uw \Rightarrow vw$ by composing

$$\mathrm{D}_1 \otimes C \xrightarrow{\mathrm{D}_1 \otimes w} \mathrm{D}_1 \otimes A \xrightarrow{\alpha} B \quad .$$

Finally, let $w : A \to B$ be a third ∞-functor and let $\beta : v \Rightarrow w$ be a second oplax transformation. The composition of 1-cells in the ∞-category $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ gives an oplax transformation that we will denote by $\beta\alpha$. We have $\beta\alpha : u \Rightarrow w$.

2.6. — The ∞-categories, ∞-functors and oplax transformations, with the operations defined in the previous paragraph, form a sesquicategory (see [**26**, Section 2] for a definition) that we will denote by $\infty\text{-}\mathcal{C}\mathsf{at}_{\mathrm{oplax}}$ (but they do not form a 2-category!). Similarly, the ∞-categories, ∞-functors and lax transformations form a sesquicategory that we will denote by $\infty\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$.

2.7. — Let $i : A \to B$ be an ∞-functor. The *structure of a left* (resp. *right*) *oplax transformation retract* on i consists of

(a) a retraction $r : B \to A$ of i (so that we have $ri = 1_A$),

(b) an oplax transformation α from ir to 1_B (resp. from 1_B to ir).

We will say that the structure is *strong* if $\alpha * i = 1_i$ and that it is *above its source* if $r * \alpha = 1_r$.

We will say that (i, r, α), or simply i, is a *left* (resp. *right*) *oplax transformation retract* if (r, α) is a structure of left (resp. right) oplax transformation retract on i. Such a retract will be said to be *strong* or *above its source* according to the properties of the structure (r, α).

All the notions introduced in this paragraph admit lax variants obtained by replacing the oplax transformation α by a lax transformation.

Proposition 2.8. — *Let $i : A \to B$ be a strong left* (resp. *right*) *oplax transformation retract above its source with retraction $r : B \to A$. Then for every diagram of pullback*

squares

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle i'}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ B' & \longrightarrow & B \\ {\scriptstyle r'}\downarrow & \lrcorner & \downarrow{\scriptstyle r} \\ A' & \xrightarrow[u]{} & A \end{array} \quad ,$$

the ∞-functor i' is a strong left (resp. *right*) *oplax transformation retract above its source with retraction r'.*

Proof. — This is a particular case of [**7**, Proposition 5.6]. □

We now recall the basic definitions and some properties of the comma construction for ∞-categories that we introduced with Maltsiniotis in [**7**].

2.9. — Let

$$A \xrightarrow{u} C \xleftarrow{v} B$$

be two ∞-functors. We define the *comma ∞-category* $u{\downarrow_C}\, v$, also simply denoted by $u{\downarrow}v$, to be the iterated fiber product

$$u{\downarrow}v = A \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B,$$

projective limit of the diagram

$$A \xrightarrow{u} C \xleftarrow{\pi_0} \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \xrightarrow{\pi_1} C \xleftarrow{v} B \quad ,$$

where $\pi_0 = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\sigma, C)$ and $\pi_1 = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\tau, C)$.

In the case where $A = C$ and $u = 1_C$, we will denote $u{\downarrow}v$ by $C{\downarrow}v$. Similarly, if $B = C$ and $v = 1_C$, we will denote $u{\downarrow}v$ by $u{\downarrow}C$.

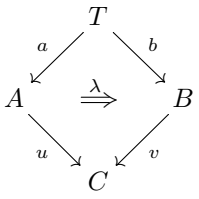
The canonical projections induce ∞-functors

$$A \xleftarrow{p_1} u{\downarrow}v \xrightarrow{p_2} B$$

and, using the description of oplax transformations in terms of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C)$ given in paragraph 2.4, an oplax transformation

$$\begin{array}{ccccc} & & u{\downarrow}v & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ A & & \overset{\kappa}{\Longrightarrow} & & B \\ & {\scriptstyle u}\searrow & & \swarrow{\scriptstyle v} & \\ & & C & & \end{array} \quad .$$

Moreover, the data of an ∞-functor $T \to u\downarrow v$ corresponds to the data of a diagram



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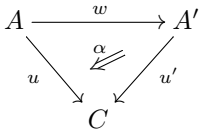
where a and b are ∞-functors and $\lambda : ua \Rightarrow vb$ is an oplax transformation.

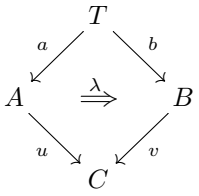
2.10. — Fix $v : B \to C$ an ∞-functor and consider a diagram



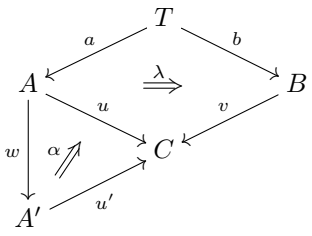
in ∞-$\mathcal{C}at$, where $\alpha : u'w \Rightarrow u$ is an oplax transformation. We define an ∞-functor

$$(w, \alpha)\downarrow v : u\downarrow v \to u'\downarrow v$$

in the following way. Let



be a diagram corresponding to an ∞-functor $T \to u\downarrow v$. By composing the diagram



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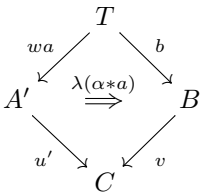
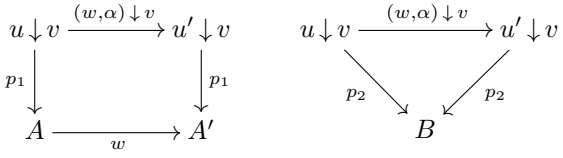
we get a diagram



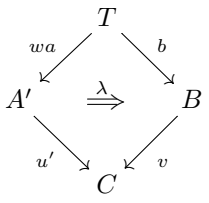
corresponding to an ∞-functor $T \to u'\downarrow v$. This correspondence is natural in T and defines, by the Yoneda lemma, our ∞-functor $(w, \alpha)\downarrow v : u\downarrow v \to u'\downarrow v$.

One checks (see [**7**, Proposition 6.9]) that the square and the triangle



are commutative.

Note that in the case where α is the identity, so that the original triangle is commutative, the diagram corresponding to the ∞-functor $T \to u' \downarrow v$ becomes



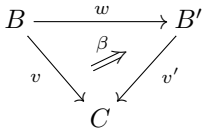
showing that

$$(w, 1_u) \downarrow v : u \downarrow v \to u' \downarrow v$$

is nothing but the ∞-functor

$$w \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B : A \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B \to A' \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B.$$

If now $u : A \to C$ is an ∞-functor and



is a diagram in ∞-$\mathcal{C}at$, where $\beta : v \Rightarrow v'w$ is an oplax transformation, we define similarly an ∞-functor

$$u \downarrow (\beta, w) : u \downarrow v \to u \downarrow v'$$

enjoying analogous properties.

Remark 2.11. — We proved with Maltsiniotis in [**7**, Appendix B] that, if C is an ∞-category, the comma construction actually defines a functor and even a sesquifunctor

$$- \downarrow_C - : \infty\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \infty\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{to}{/}C \to \infty\text{-}\mathcal{C}\mathsf{at}_{\mathrm{oplax}},$$

where ∞-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ and ∞-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{to}{/}C$ are some appropriate sesquicategories. We will only need some consequences of this result which we will recall in this section.

Proposition 2.12. — *Let*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

be two ∞-functors. We have pullback squares

$$\begin{array}{ccc} u\downarrow v & \xrightarrow{(u,1_u)\downarrow v} & C\downarrow v \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p_1} \\ A & \xrightarrow[u]{} & C \end{array} \qquad \begin{array}{ccc} u\downarrow v & \xrightarrow{p_2} & B \\ {\scriptstyle u\downarrow(1_v,v)}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ u\downarrow C & \xrightarrow[p_2]{} & C \end{array} .$$

Proof. — We will only treat the first pullback square, the proof for the second one being similar. We have

$$\begin{aligned} u\downarrow v &= A \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B \\ &\simeq A \times_C \big(C \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B\big) \\ &= A \times_C (C\downarrow v), \end{aligned}$$

where the isomorphism is induced by the ∞-functors p_1 and $u\times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C)\times_C B$. But by paragraph 2.10, the latter ∞-functor is nothing but $(u, 1_u)\downarrow v$, thereby proving the result. □

Proposition 2.13. — *Let*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

be two ∞-functors.

(a) *If $i : A' \to A$ is a strong left oplax transformation retract, then so is*

$$(i, 1_{ui})\downarrow v : (ui)\downarrow v \to u\downarrow v.$$

More precisely, if (r, α) is a structure of strong left oplax transformation retract on i, then there exists a structure of strong left oplax transformation retract on $(i, 1_{ui})\downarrow v$ of the form (r', γ) with γ compatible with α in the sense that $p_1 * \gamma = \alpha * p_1$, where $p_1 : u\downarrow v \to A$.

(b) *If $j : B' \to B$ is a strong right oplax transformation retract, then so is*

$$u\downarrow(1_{vj}, j) : u\downarrow(vj) \to u\downarrow v.$$

More precisely, if (r, β) is a structure of strong right oplax transformation retract on j, then there exists a structure of strong right oplax transformation retract on $u\downarrow(1_{vj}, j)$ of the form (r', γ) with γ compatible with β in the sense that $p_2 * \gamma = \beta * p_2$, where $p_2 : u\downarrow v \to B$.

Proof. — The first assertion of (a) is exactly [**7**, Corollary B.2.8.(a)]. The proof of this corollary actually produces a structure (r', γ) and the fact that this γ is compatible with α follows from [**7**, Proposition B.2.9]. The situation is similar for (b). □

We now introduce slice ∞-categories in terms of comma ∞-categories. For a more concrete description, we refer the reader to [**6**, paragraph 4.1] (see [**7**, Proposition 7.1] for the comparison of the two definitions).

2.14. — If A is an ∞-category and a is an object of A, we will denote by ${}_{a\backslash}A$ the ∞-category

$$ {}_{a\backslash}A = a \downarrow A, $$

where a is seen as an ∞-functor $\mathrm{D}_0 \to A$. The ∞-functor $p_2 : {}_{a\backslash}A \to A$ will be called the *forgetful ∞-functor*.

More generally, if $u : A \to B$ is an ∞-functor and b is an object of B, we set

$$ {}_{b\backslash}A = b \downarrow u, $$

where b is also seen as an ∞-functor $\mathrm{D}_0 \to B$. It follows from Proposition 2.12 that we have

$$ {}_{b\backslash}A = {}_{b\backslash}B \times_B A, $$

where the fiber product involves the forgetful ∞-functor ${}_{b\backslash}B \to B$ and u. In this setting, we also have a *forgetful ∞-functor* ${}_{b\backslash}A \to A$.

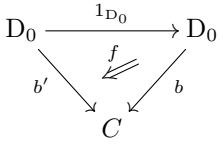
If $f : b \to b'$ is a 1-cell of B, we define the ∞-functor

$$ {}_{f\backslash}A : {}_{b'\backslash}A \to {}_{b\backslash}A $$

to be

$$ (1_{\mathrm{D}_0}, f) \downarrow u : b' \downarrow u \to b \downarrow u, $$

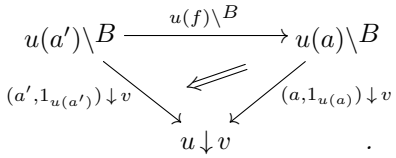
where f is seen as an oplax transformation



Proposition 2.15. — *Let*

$$ A \xrightarrow{u} C \xleftarrow{v} B $$

be two ∞-functors and let $f : a \to a'$ be a 1-cell of A. Then there exists an oplax transformation

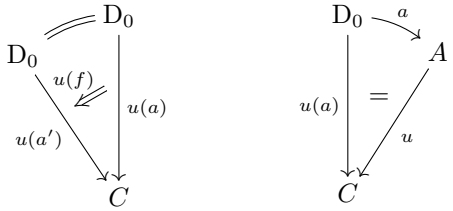


Proof. — We will use the fact that the comma construction $- \downarrow v$ extends to a sesquifunctor from a certain sesquicategory ∞-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ to the sesquicategory ∞-$\mathcal{C}\mathsf{at}_{\mathrm{oplax}}$ (see [**7**, Theorem B.2.6]). All we need to know about the source sesquicategory of this

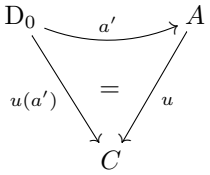
extension (see [**7**, paragraph B.1.18] for a complete description) is that the diagram

,

where the 2-arrows denote oplax transformations and the 3-arrow is formally an identity "oplax 2-transformation" but can be interpreted simply as an equality of oplax transformations, defines a 2-cell from the composite of 1-cells associated to the 2-triangles



to a 1-cell associated to the 2-triangle



.

(In the notation of [**7**, paragraph B.1.18], this 2-cell is denoted by $(f, 1_{u(f)})$ and we have

$$(f, 1_{u(f)}) : (a, 1_{u(a)}) *_0 (1_{\mathrm{D}_0}, u(f)) \Rightarrow (a', 1_{u(a')})$$

in ∞-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$.) By applying the sesquifunctoriality of $- \downarrow v$ to this 2-cell, we get a 2-cell in ∞-$\mathcal{C}\mathsf{at}_{\mathrm{oplax}}$, that is, an oplax transformation, from the composite of the ∞-functors $(a, 1_{u(a)}) \downarrow v$ and $(1_{\mathrm{D}_0}, u(f)) \downarrow v = u(f) \backslash B$ to the ∞-functor $(a', 1_{u(a')}) \downarrow v$, thereby proving the result. □

We end the section with a kind of mapping space factorization for ∞-functors, involving comma ∞-categories, that will be needed in our proof of the ∞-categorical Theorem B.

2.16. — Let $u : A \to B$ be an ∞-functor. We will see that u factors as

$$A \xrightarrow{\ j\ } B \downarrow u \xrightarrow{\ p_1\ } B \quad ,$$

for some ∞-functor j. As the composition

$$\mathrm{D}_0 \xrightarrow{\ \tau\ } \mathrm{D}_1 \xrightarrow{\ \kappa\ } \mathrm{D}_0$$

is the identity, by applying the functor $\underline{\mathsf{Hom}}_{\mathrm{lax}}(-, B)$, we get a factorization

$$B \xrightarrow{\iota} \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \xrightarrow{\pi_1} B$$

of the identity of B. By pulling back this factorization along u, we get a diagram of pullback squares

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle j}\downarrow & \lrcorner & \downarrow{\scriptstyle \iota} \\ B\!\downarrow\! u & \longrightarrow & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \\ {\scriptstyle p_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ A & \xrightarrow[u]{} & B \end{array} \quad ,$$

defining our ∞-functor j. The equality $\pi_0\iota = 1_B$ easily implies that we have $u = p_1 j$, as announced.

Proposition 2.17. — *The ∞-functor $j : A \to B\!\downarrow\! u$ defined in the previous paragraph is a strong right oplax transformation retract above its source with retraction $p_2 : B\!\downarrow\! u \to A$.*

Proof. — Consider the diagram of pullback squares of the previous paragraph. As 1 is a terminal object of the 1-category D_1, the ∞-functor τ is a strong right lax transformation retract above its source with retraction κ. For formal reasons (see [**5**, Example C.23.(f)]), the functor $\underline{\mathsf{Hom}}_{\mathrm{lax}}(-, B)$ extends to a sesquifunctor $(\infty\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{op}} \to \infty\text{-}\mathcal{C}\mathsf{at}_{\mathrm{oplax}}$, where $\mathcal{C}^{\mathrm{op}}$, for $\mathcal{C}$ a sesquicategory, denotes the sesquicategory obtained from $\mathcal{C}$ by reversing the 1-cells. This implies that $\iota = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\kappa, B)$ is a strong right oplax transformation retract above its source with retraction $\pi_1 = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\tau, B)$. The result then follows from Proposition 2.8. □

2.18. — Similarly, any ∞-functor $u : A \to B$ factors as

$$A \xrightarrow{j'} u\!\downarrow\! B \xrightarrow{p_2} B \quad ,$$

where j' is a strong *left* oplax transformation retract above its source with retraction $p_1 : u\!\downarrow\! B \to A$. This can be proven either by adapting the previous proof or by a duality argument involving the automorphism $C \mapsto C^{\circ}$ of $\infty\text{-}\mathcal{C}at$ (see paragraph 2.1).

3. A Quillen Theorem B for ∞-categories

3.1. — We will denote by $N : \infty\text{-}\mathcal{C}at \to \widehat{\mathbf{\Delta}}$ the so-called *Street nerve*, introduced by Street in [**25**]. We will briefly recall in Appendix A (see paragraph A.3) one of its definition using Steiner's theory [**24**]. This definition is not needed in this section and we will recall all the properties we will use.

This nerve functor is induced by a cosimplicial object $\mathcal{O} : \mathbf{\Delta} \to \infty\text{-}\mathcal{C}at$ sending Δ_n to the so-called *n-th oriental* $\mathcal{O}_n$. Here are pictures of orientals in low dimension:

$$\mathcal{O}_0 = \mathrm{D}_0 = \{0\}\,, \qquad \mathcal{O}_1 = \mathrm{D}_1 = 0 \longrightarrow 1\,, \qquad \mathcal{O}_2 = \quad \begin{array}{ccc} & & 2 \\ & \nearrow & \uparrow \\ 0 & \longrightarrow & 1 \end{array}\,,$$

$$\mathcal{O}_3 = \quad \begin{array}{ccc} 0 & \longrightarrow & 3 \\ \downarrow & \Swarrow & \uparrow \\ 1 & \longrightarrow & 2 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} 0 & \longrightarrow & 3 \\ \downarrow & \Swarrow & \uparrow \\ 1 & \longrightarrow & 2 \end{array}\,.$$

By definition, if C is an ∞-category, we have $(NC)_p = \mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(\mathcal{O}_p, C)$. When C is a 1-category, then NC coincides with the classical nerve functor. As Street's nerve is induced by a cosimplicial object, it admits as a left adjoint the Kan extension of this cosimplicial object along the Yoneda embedding. In particular, it preserves limits.

3.2. — We will say that an ∞-functor $u : A \to B$ is a *Thomason equivalence* if its Street's nerve $Nu : NA \to NB$ is a simplicial weak equivalence.

3.3. — Let $u, v : A \to B$ be two ∞-functors and let $\alpha : u \Rightarrow v$ be an oplax transformation. We constructed in [**7**, Appendix A], with Maltsiniotis, a simplicial homotopy $N\alpha$ from Nu to Nv. We will briefly recall the definition of $N\alpha$ in Appendix A (see paragraph A.11) but all we will need about $N\alpha$ in this section is the following proposition.

Proposition 3.4. — *Let $u, v : A \to B$ be two ∞-functors and let $\alpha : u \Rightarrow v$ be an oplax transformation.*

(a) *If $w : B \to C$ is an ∞-functor, then we have*

$$N(w * \alpha) = N(w)N(\alpha).$$

(b) *If $w : C \to A$ is an ∞-functor, then we have*

$$N(\alpha * w) = N(\alpha)(\Delta_1 \times N(w)).$$

Proof. — This is [**7**, Proposition A.14]. □

Proposition 3.5. — *If $(i : A \to B, r, \alpha)$ is a strong left* (resp. *right*) *oplax transformation retract, then $(Ni : NA \to NB, Nr, N\alpha)$ is a strong left* (resp. *right*) *deformation retract and in particular i and r are Thomason equivalences.*

Proof. — This follows from paragraph 3.3 and the previous proposition. □

3.6. — We will say that a commutative square

$$\begin{array}{ccc} A & \xrightarrow{v} & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \xrightarrow[w]{} & B' \end{array}$$

in ∞-$\mathcal{C}at$ is *a homotopy pullback square* if the commutative square

$$\begin{array}{ccc} NA & \xrightarrow{Nv} & NA' \\ {\scriptstyle Nu}\downarrow & & \downarrow{\scriptstyle Nu'} \\ NB & \xrightarrow[Nw]{} & NB' \end{array}$$

of simplicial sets is a homotopy pullback square (as in paragraph 1.2). Homotopy pullback squares in ∞-$\mathcal{C}at$ inherit many properties of homotopy pullback squares in simplicial sets: for instance they compose, and a square as above in which u and u' are both Thomason equivalences is a homotopy pullback square.

Remark 3.7. — One can show that a commutative square in ∞-$\mathcal{C}at$ is a homotopy pullback square in the sense of the previous paragraph if and only if it induces a pullback square in the weak $(\infty, 1)$-category obtained from ∞-$\mathcal{C}at$ by weakly inverting Thomason equivalences. This follows from (a mild generalization) of [**16**, Theorem 5.6].

We now introduce the notion corresponding to the hypothesis of Theorem B.

3.8. — Let $u : A \to B$ be an ∞-functor. We will say that u is *colocally homotopically constant* if, for every 1-cell $f : b \to b'$ of B, the ∞-functor $f\backslash A : b'\backslash A \to b\backslash A$ is a Thomason equivalence.

The following proposition is the crucial step in our proof of the ∞-categorical Theorem B.

Proposition 3.9. — *If $u : A \to B$ is a colocally homotopically constant ∞-functor, then any pullback square*

$$\begin{array}{ccc} C & \longrightarrow & B \downarrow u \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_1} \\ D & \longrightarrow & B \end{array}$$

is a homotopy pullback square.

Proof. — Since the nerve functor preserves fiber products, by Proposition 1.3, it suffices to show that $N(p_1)$ satisfies condition (d) of this proposition. So consider a

diagram of pullback squares of the form

$$\begin{array}{ccccc} Q_i & \xrightarrow{f_i} & P & \longrightarrow & N(B\downarrow u) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle N(p_1)} \\ \Delta_0 & \xrightarrow[i]{} & \Delta_m & \xrightarrow[x]{} & NB \end{array} .$$

By Proposition 2.12, we also have a diagram of pullback squares

$$\begin{array}{ccccc} x(i)\downarrow u & \xrightarrow{i_*} & x\downarrow u & \longrightarrow & B\downarrow u \\ \downarrow{\scriptstyle p_1} & \lrcorner & \downarrow{\scriptstyle p_1} & \lrcorner & \downarrow{\scriptstyle p_1} \\ \mathcal{O}_0 & \xrightarrow[i]{} & \mathcal{O}_m & \xrightarrow[x]{} & B \end{array} ,$$

where $i_* = (i, 1_{x(i)})\downarrow u$. Using again the fact that the nerve functor preserves fiber products, we get a canonical isomorphism between Q_i and $N(x(i)\downarrow u)$. We thus have to show that

$$f_i : N(x(i)\downarrow u) \to P$$

is a weak equivalence.

Denote by $\eta : \Delta_m \to N(\mathcal{O}_m)$ the adjunction morphism. By one of the triangular identities, the composite

$$\Delta_m \xrightarrow{\eta} N(\mathcal{O}_m) \xrightarrow{Nx} NB$$

is $x : \Delta_m \to NB$ and we get a diagram of pullback squares

$$\begin{array}{ccccccc} N(x(i)\downarrow u) & \xrightarrow{f_i} & P & \xrightarrow{g} & N(x\downarrow u) & \longrightarrow & N(B\downarrow u) \\ \downarrow{\scriptstyle N(p_1)} & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle N(p_1)} & \lrcorner & \downarrow{\scriptstyle N(p_1)} \\ \Delta_0 & \xrightarrow[i]{} & \Delta_m & \xrightarrow[\eta]{} & N(\mathcal{O}_m) & \xrightarrow[Nx]{} & NB \end{array} .$$

Note that we have $gf_i = N(i_*)$.

To prove that f_i is a weak equivalence, we proceed in three steps:

(1) We show that f_0 is a weak equivalence. To do so, we will use Proposition 1.5 to prove that

$$f_0 : N(x(0)\downarrow u) \to P$$

is a strong left deformation retract. To begin with, note that f_0 can be identified with the fiber product of the vertical maps of the commutative diagram

$$\begin{array}{ccccc} N(x(0)\downarrow u) & \xrightarrow{N(p_1)} & N(\mathcal{O}_0) & \xleftarrow{\eta} & \Delta_0 \\ \downarrow{\scriptstyle N(0_*)} & & \downarrow{\scriptstyle N(0)} & & \downarrow{\scriptstyle 0} \\ N(x\downarrow u) & \xrightarrow[N(p_1)]{} & N(\mathcal{O}_m) & \xleftarrow[\eta]{} & \Delta_m \end{array} ,$$

the left square being commutative by paragraph 2.10 and $N(\mathcal{O}_0)$ being isomorphic to Δ_0. The morphism $0 : \Delta_0 \to \Delta_m$ is of course a strong left deformation retract, with a unique retraction and a unique simplicial homotopy k. We will prove in Appendix A (see in particular Propositions A.8 and A.12) that there exists a structure of strong left oplax transformation retract (r, α) on $0 : \mathcal{O}_0 \to \mathcal{O}_m$ making the square

$$\begin{array}{ccc} \Delta_1 \times N(\mathcal{O}_m) & \xleftarrow{\Delta_1 \times \eta} & \Delta_1 \times \Delta_m \\ {\scriptstyle N\alpha}\downarrow & & \downarrow{\scriptstyle k} \\ N(\mathcal{O}_m) & \xleftarrow[\eta]{} & \Delta_m \end{array}$$

commute. In particular, by Proposition 3.5, the morphism $N(0) : N(\mathcal{O}_0) \to N(\mathcal{O}_m)$ is a strong left deformation retract with retraction Nr and homotopy $N\alpha$. Proposition 2.13 then implies that there exists a structure of strong left oplax transformation retract (r', γ) on $0_* : x(0)\downarrow u \to x\downarrow u$ satisfying $p_1 * \gamma = \alpha * p_1$. In particular, again by Proposition 3.5, the morphism $N(0_*) : N(x(0)\downarrow u) \to N(x\downarrow u)$ is a strong left deformation retract with retraction Nr' and homotopy $N\gamma$. By Proposition 3.4, applying N to the equality $p_1 * \gamma = \alpha * p_1$ gives the commutativity of the diagram

$$\begin{array}{ccc} \Delta_1 \times N(x\downarrow u) & \xrightarrow{\Delta_1 \times N(p_1)} & \Delta_1 \times N(\mathcal{O}_m) \\ {\scriptstyle N\gamma}\downarrow & & \downarrow{\scriptstyle N\alpha} \\ N(x\downarrow u) & \xrightarrow[N(p_1)]{} & N(\mathcal{O}_m) \end{array} .$$

We are thus in position to apply Proposition 1.5 and we get that f_0 is a strong left deformation retract and hence a weak equivalence.

(2) We show that g is a weak equivalence. We proved in the previous step that 0_* is a strong left oplax transformation retract and $N(0_*)$ is thus a weak equivalence. As $N(0_*) = gf_0$, this implies that g is a weak equivalence.

(3) We show that f_i is a weak equivalence. Let l be any 1-cell from 0 to i in $\mathcal{O}_m$. Using Proposition 2.15, we get an oplax transformation

$$\begin{array}{ccc} x(i)\backslash A & \xrightarrow{x(l)\backslash A} & x(0)\backslash A \\ & {\scriptstyle i_*}\searrow \quad \swarrow{\scriptstyle 0_*} & \\ & x\downarrow u & \end{array} .$$

We already proved that 0_* is a Thomason equivalence and $x(l)\backslash A$ is a Thomason equivalence by hypothesis. Using paragraph 3.3, we get that $N(i_*)$ is homotopic to a weak equivalence and is thus a weak equivalence. The equality $N(i_*) = gf_i$ and the fact proven above that g is a weak equivalence then implies that f_i is a weak equivalence, thereby ending the proof. □

Theorem 3.10. — *Let*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

be two ∞-functors. If v is colocally homotopically constant, then the pullback square

$$\begin{array}{ccc} u\downarrow v & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ u\downarrow C & \xrightarrow[p_2]{} & C \end{array}$$

is a homotopy pullback square.

Proof. — Consider the factorization

$$B \xrightarrow{j} C\downarrow v \xrightarrow{p_1} C$$

of v introduced in paragraph 2.16. The pullback square of the statement factors as a composite of two pullback squares

$$\begin{array}{ccc} u\downarrow v & \longrightarrow & B \\ {\scriptstyle j'}\downarrow & \lrcorner & \downarrow{\scriptstyle j} \\ \cdot & \longrightarrow & C\downarrow v \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_1} \\ u\downarrow C & \xrightarrow[p_2]{} & C \end{array}$$

and it suffices to show that these two squares are homotopy pullback squares. By Proposition 2.17, the ∞-functor j is a strong right oplax transformation retract. On the other hand, by Proposition 2.12, the ∞-functor j' can be identified with the ∞-functor

$$u\downarrow(1_v, j) : u\downarrow v \to u\downarrow p_1,$$

which, by Proposition 2.13, is a strong right oplax transformation retract as well. It follows from Proposition 3.5 that both j and j' are Thomason equivalences, showing that the top square is a homotopy pullback square. As for the bottom square, this follows from the previous proposition. □

Corollary 3.11 **(Theorem B)**. — *If $u : A \to B$ is a colocally homotopically constant ∞-functor and b is an object of B, then the pullback square*

$$\begin{array}{ccc} b\backslash A & \longrightarrow & A \\ {\scriptstyle b\backslash u}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ b\backslash B & \longrightarrow & B \end{array},$$

where the horizontal arrows are the forgetful ∞-functors, is a homotopy pullback square.

Proof. — This is a particular case of the previous theorem. □

Corollary 3.12. — *Let*

$$A \xrightarrow{u} C \xleftarrow{v} B$$

be two ∞-functors. If v is colocally homotopically constant, then the comma construction $u{\downarrow}v$ is canonically the homotopy pullback $A \times^h_C B$.

Proof. — By paragraph 2.18, the ∞-functor u factors as

$$A \xrightarrow{j'} u{\downarrow}C \xrightarrow{p_2} C \quad ,$$

where j' is a strong left oplax transformation retract and hence a Thomason equivalence by Proposition 3.5. We thus get a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{u} & C & \xleftarrow{v} & B \\ {\scriptstyle j'}\downarrow & & \| & & \| \\ u{\downarrow}C & \xrightarrow[p_2]{} & C & \xleftarrow[v]{} & B \end{array} \quad ,$$

where the vertical arrows are Thomason equivalences, and the result follows from the previous theorem. □

Remark 3.13. — More precisely, one can show that, under the same hypothesis as in the previous corollary, the "2-square"

$$\begin{array}{ccc} u{\downarrow}v & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & \overset{\kappa}{\swarrow} & \downarrow{\scriptstyle v} \\ A & \xrightarrow[u]{} & C \end{array}$$

introduced in paragraph 2.9 is a "homotopy pullback 2-square" in some appropriate sense (for instance, its topological realization is a homotopy pullback in the sense of Mather [**21**]).

Corollary 3.14. — *If $u : A \to B$ is a colocally homotopically constant ∞-functor and b is an object of B, then the ∞-category ${}_b\backslash A$ is canonically the homotopy fiber of u at b.*

Proof. — This is a particular case of the previous corollary. □

Remark 3.15. — As the comma construction of two n-functors is an n-category, the four previous statements all restrict to n-categories. In particular, we recover the original Quillen Theorem B and its generalization to 2-categories proven by Cegarra [**13**]. To get direct proofs of these results for n-categories, all one has to do is to change in our proofs all the "∞" to "n" and to replace the m-th oriental $\mathcal{O}_m$ appearing in the proof of Proposition 3.9 by its n-th truncation $\mathcal{O}_m^{\leqslant n}$, obtained from $\mathcal{O}_m$ by keeping only i-cells for $i \leqslant n$ and modding out by $(n+1)$-cells. Of course, some parts of these

proofs get simpler for small n. Most notably, for $n = 1$, the map f_i of the proof of Proposition 3.9 can be identified with the nerve of the functor i_*, so that all one has to prove is that $0_* : x(0)\downarrow u \to x\downarrow u$ is a Thomason equivalence, which can be done by describing an explicit structure of transformation retract on this functor (note that an oplax transformation between 1-functors is nothing but a natural transformation). More generally, for $n = 1$ and $n = 2$, all the intermediate constructions and oplax transformations involved in these proofs can be defined by using explicit formulas.

Remark 3.16. — The four previous results were proven for "under-∞-categories". They remain valid for "over-∞-categories" defined as $A_{/b} = u\downarrow b$, for $u : A \to B$ an ∞-functor and b an object of B. This will follow from the equality $A_{/b} = (b\backslash A^\circ)^\circ$ (see paragraph 2.1 for the notation C°) and the fact, that we will prove with Maltsiniotis in [**8**], that the duality $C \mapsto C^\circ$ sends Thomason equivalences to Thomason equivalences.

If one tries to adapt our proofs to "over-∞-categories", one has to replace the ∞-functor $0 : \mathcal{O}_0 \to \mathcal{O}_m$ appearing in the proof of Proposition 3.9 by the ∞-functor $m : \mathcal{O}_0 \to \mathcal{O}_m$. This ∞-functor is both a right oplax transformation retract and a right lax transformation retract, but only the lax structure is compatible with the structure of right deformation retract of the simplicial map $m : \Delta_0 \to \Delta_m$. Therefore, one has to replace the use of our "oplax" comma construction $u\downarrow v = A \times_C \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \times_C B$, for $u : A \to C$ and $v : B \to C$ two ∞-functors, by its "lax" variant $u\downarrow' v = A \times_C \underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1, C) \times_C B$, which has sesquifunctoriality properties with respect to lax transformations instead of oplax transformations. This leads to a proof of our results for "over-∞-categories" defined as $A\overset{\mathrm{co}}{/}b = u\downarrow' b$ (see [**5**, Remark 6.37] for an explanation of this notation). As $A\overset{\mathrm{co}}{/}b = (b\backslash A^{\mathrm{op}})^{\mathrm{op}}$, where $C \mapsto C^{\mathrm{op}}$ denotes the duality of ∞-$\mathcal{C}at$ consisting in reversing cells in odd dimension, the results for these "over-∞-categories" also follow formally from our results and the fact that the duality $C \mapsto C^{\mathrm{op}}$ sends Thomason equivalences to Thomason equivalences, which is a consequence of the existence of a natural isomorphism between $N(C^{\mathrm{op}})$ and $N(C)^{\mathrm{op}}$ (see [**7**, Proposition 5.2]), where $X \mapsto X^{\mathrm{op}}$ denotes the usual duality of simplicial sets.

Finally, the results for "under-∞-categories" defined as $b\overset{\mathrm{co}}{\backslash}A = b\downarrow' u$ will also follow from the fact that $C \mapsto C^\circ$ sends Thomason equivalences to Thomason equivalences, as $b\overset{\mathrm{co}}{\backslash}A = (A^\circ\overset{\mathrm{co}}{/}b)^\circ$.

4. A few applications

A first consequence of the ∞-categorical Theorem B is the (non-relative) ∞-categorical Theorem A, which is a special case of the main result of [**6**] and [**7**].

4.1. — We will say that an ∞-category A is *aspherical* if the unique ∞-functor from A to the terminal ∞-category is a Thomason equivalence or, in other words, if its nerve NA is weakly contractible.

Theorem 4.2. — *Let $u : A \to B$ be an ∞-functor. If for every object b of B, the ∞-category ${}_{b\backslash}A$ is aspherical, then u is a Thomason equivalence.*

Proof. — The hypothesis implies that u is colocally homotopically constant. We can thus apply Theorem B and more precisely Corollary 3.14. We get that, for every object b of B, the ∞-category ${}_{b\backslash}A$ is the homotopy fiber of u at b. As by hypothesis ${}_{b\backslash}A$ is aspherical, this implies that all the homotopy fibers of Nu are weakly contractible, showing that Nu is a weak equivalence. □

We will now use the ∞-categorical Theorem B to produce models of Eilenberg–Mac Lane spaces. We will need the following lemma:

Lemma 4.3. — *Let A be an ∞-category and let a and a' be two objects of A. There exists a canonical isomorphism*

$$a \downarrow a' \simeq \underline{\mathsf{Hom}}_A(a, a')^\circ,$$

natural in a and a', where a and a' are seen as ∞-functors $\mathrm{D}_0 \to A$ and $C \mapsto C^\circ$ denotes the duality introduced in paragraph 2.1.

Proof. — See [**5**, Proposition B.6.2]. □

Theorem 4.4. — *Let A be an ∞-category endowed with an object a. Suppose that for every 1-cell $f : a' \to a''$ of A the induced ∞-functor $\underline{\mathsf{Hom}}_A(a'', a)^\circ \to \underline{\mathsf{Hom}}_A(a', a)^\circ$ is a Thomason equivalence. Then $\underline{\mathsf{Hom}}_A(a, a)^\circ$ is a model for the loop space of (A, a) in the sense that $N(\underline{\mathsf{Hom}}_A(a, a)^\circ)$ has the homotopy type of the loop space of (NA, a).*

Proof. — By the previous lemma, the hypothesis precisely means that the ∞-functor $a : \mathrm{D}_0 \to A$ is colocally homotopically constant. By Proposition 3.12, we thus get that $a \downarrow a$ is the homotopy pullback of

$$\mathrm{D}_0 \xrightarrow{\ a\ } A \xleftarrow{\ a\ } \mathrm{D}_0 \quad ,$$

that is, that $N(a \downarrow a)$ is the homotopy pullback of

$$\Delta_0 \xrightarrow{\ a\ } NA \xleftarrow{\ a\ } \Delta_0 \quad ,$$

thereby proving the result. □

Remark 4.5. — As mentioned before, we will prove with Maltsiniotis in [**8**] that the duality $C \mapsto C^\circ$ sends Thomason equivalences to Thomason equivalences. Therefore the previous theorem remains valid if all the dualities appearing in its statement are removed.

4.6. — Let $(M, +, e)$ be a commutative monoid. For any $n \geqslant 1$, we define an n-category $B^n M$ in the following way. Its cells are

$$(B^n M)_k = \begin{cases} \{*\} & \text{if } 0 \leqslant k < n, \\ M & \text{if } k = n; \end{cases}$$

the unit of the unique $(n-1)$-cell is the unit e of the monoid; and if x and y are n-cells, then for any $0 \leqslant j < n$, we set $x *_j y = x + y$.

Theorem 4.7. — *For any abelian group π and any $n \geqslant 1$, the ∞-category $B^n\pi$ is a $K(\pi, n)$ in the sense that $N(B^n\pi)$ is a $K(\pi, n)$.*

Proof. — The result is well known for $n = 1$. If $n \geqslant 2$, then all the 1-cells of $B^n\pi$ are identities so that the hypothesis of Theorem 4.4 is satisfied. We thus get that the loop space of $B^n\pi$ is $\underline{\mathsf{Hom}}_{B^n\pi}(*, *)^\circ$, which is isomorphic to $B^{n-1}\pi$. The result thus follows by induction using the fact that (the nerve of) $B^n\pi$ is connected. □

Remark 4.8. — In [**11**], Berger proves that the topological realization of the so-called cellular nerve of $B^n\pi$ is a $K(\pi, n)$, showing that $B^n\pi$ is a $K(\pi, n)$ in a, *a priori*, different sense from the previous theorem (see his Corollary 4.3 and his Section 4.10). It will follow from the comparison of Street's nerve and the cellular nerve, that we will study with Maltsiniotis in [**8**], that these two meanings of "being a $K(\pi, n)$" coincide.

Theorem 4.9. — *Let π be a commutative ordered group whose underlying poset is aspherical (as a category). Denote by π^+ its monoid of positive elements. Then, for any $n \geqslant 1$, the ∞-category $B^n\pi^+$ is a $K(\pi, n)$.*

Proof. — The inclusion $\pi^+ \subset \pi$ induces an ∞-functor $B^n\pi^+ \to B^n\pi$. By the previous theorem, it suffices to prove that this ∞-functor is a Thomason equivalence. We will apply Theorem A (Theorem 4.2). We have to prove that the ∞-category ${}_*\backslash(B^n\pi^+)$ is aspherical. The concrete description of the slice ∞-categories given in [**6**, paragraph 4.1] shows that this ∞-category can be described in the following way: it is an n-category whose underlying $(n-1)$-category is $B^{n-1}\pi$ (where $B^0\pi$ means π, as a set) and whose n-cells are given by the order on π. In particular, for $n = 1$, we get the poset π seen as a 1-category. This poset being aspherical by hypothesis, this ends the proof of the case $n = 1$. If $n > 1$, then we have isomorphisms

$$\underline{\mathsf{Hom}}_{{}_*\backslash(B^n\pi^+)}(*, *)^\circ \simeq \big({}_*\backslash(B^{n-1}\pi^+)\big)^\circ \simeq {}_*\backslash(B^{n-1}((\pi^\circ)^+)),$$

where π° denotes the group π equipped with the opposite order. As π° is aspherical as a poset (since π is), we can assume by induction that the ∞-category $\underline{\mathsf{Hom}}_{{}_*\backslash(B^n\pi^+)}(*, *)^\circ$ is aspherical. We can thus apply Theorem 4.4 and we get that the loop space of ${}_*\backslash(B^n\pi^+)$ is aspherical. This shows that ${}_*\backslash(B^n\pi^+)$ is aspherical, as it is obviously connected, thereby ending the proof. □

Example 4.10. — The previous theorem applies to commutative ordered groups whose underlying poset is directed. In particular, the ∞-category $B^n\mathbb{N}$ is a $K(\mathbb{Z}, n)$.

We end the section with an application to loop spaces of ∞-groupoids.

4.11. — Recall that a (strict) ∞-*groupoid* is an ∞-category in which every i-cell for $i > 0$ is strictly invertible (for the composition in codimension 1), and that an ∞-functor $f : G \to H$ between ∞-groupoids is an *equivalence of* ∞-*groupoids* if

— for every object y of H, there exists an object x of G and a 1-cell $f(x) \to y$ in H,

— for every $i \geqslant 0$, every pair of parallel i-cells u, v in G (two 0-cells being always considered as parallel) and every $(i+1)$-cell $\beta : f(u) \to f(v)$ in H, there exists an $(i+1)$-cell $\alpha : u \to v$ in G and an $(i+2)$-cell $f(\alpha) \to \beta$ in H.

Proposition 4.12. — *An equivalence of ∞-groupoids is a Thomason equivalence.*

Proof. — The equivalences of ∞-groupoids are precisely the weak equivalences between ∞-groupoids of the so-called folk model category structure on ∞-$\mathcal{C}at$ [**19**] (see also [**9**]). To prove the result, it thus suffices, using Ken Brown's lemma, to show that the trivial fibrations of the folk model category structure are Thomason equivalences. We will see that the nerve of such a trivial fibration is actually a trivial Kan fibration. By adjunction, to prove this, it suffices to show that the left adjoint $c : \widehat{\mathbf{\Delta}} \to \infty$-$\mathcal{C}at$ of the nerve functor sends the inclusions $\partial\Delta_n \hookrightarrow \Delta_n$, where $n \geqslant 0$ and $\partial\Delta_n$ denotes the boundary of Δ_n, to a cofibration of the folk model category structure. The explicit description of $c(K)$, where K is a simplicial complex, given in [**4**, Section 9], shows that $c(\partial\Delta_n)$ is the underlying $(n-1)$-category of the n-category $c(\Delta_n) = \mathcal{O}_n$. In other words, the ∞-functor $c(\partial\Delta_n) \to c(\Delta_n)$ corresponds to the free addition of the unique non-trivial n-cell of $\mathcal{O}_n$, and is hence, by definition, a folk cofibration, thereby proving the result. □

Remark 4.13. — One can actually show that an ∞-functor between ∞-groupoids is an equivalence of ∞-groupoids if and only if it is a Thomason equivalence.

Theorem 4.14. — *Let G be a strict ∞-groupoid endowed with an object x. Then the loop space of (G, x) is a product of Eilenberg–Mac Lane spaces (including the discrete space $K(E, 0)$, for E a set, as an Eilenberg–Mac Lane space).*

Proof. — As every 1-cell of G is invertible, the hypothesis of Theorem 4.4 is satisfied and we get that $\underline{\mathsf{Hom}}_G(x, x)^\circ$, which is isomorphic to $\underline{\mathsf{Hom}}_G(x, x)$ as G is an ∞-groupoid, is the loop space of (G, x). As all the connected components of a loop space are weakly equivalent, to prove the result, it suffices to show that the connected component of any object of $\underline{\mathsf{Hom}}_G(x, x)$ is a product of Eilenberg–Mac Lane spaces. Consider the object $1_x : x \to x$. Its connected component is equivalent to the full sub-∞-groupoid of $\underline{\mathsf{Hom}}_G(x, x)$ whose only object is 1_x. This ∞-groupoid is obtained

by "shifting down" a sub-∞-groupoid G' of G having only one object (namely x) and one 1-cell (namely 1_x). Such an ∞-groupoid G' is known to be equivalent to a product of the form $\prod_{n\geqslant 2} B^n\pi_n$ (see for instance [**2**, Theorem 4.17]) and the connected component of 1_x is thus equivalent to $\prod_{n\geqslant 1} B^n\pi_{n+1}$. The result thus follows from Theorem 4.7. □

Appendix A
A contraction of the oriental

The purpose of this appendix is to construct the oplax transformation retract needed in the proof of Proposition 3.9.

A.1. — The appendix relies on Steiner's theory of augmented directed complexes as developed in [**24**]. We will recall the minimal amount of information needed to follow our arguments and we refer the reader to [**5**, Section 2] for a comprehensive introduction to this theory in the spirit of our paper.

We will denote by $\mathcal{C}_{\mathrm{ad}}$ the category of augmented directed complexes. Recall that an *augmented directed complex* is an augmented complex K (of abelian groups in nonnegative degrees) endowed, for every $p \geqslant 0$, with a submonoid K_p^* of K_p of *positive p-chains*, and that a *morphism of augmented directed complexes* is a morphism of augmented complexes sending positive p-chains to positive p-chains. Similarly, a *homotopy* between two such morphisms is a homotopy, in the classical sense, sending positive p-chains to positive $(p+1)$-chains.

To any augmented directed complex, Steiner associates an ∞-category thus defining a functor $\nu : \mathcal{C}_{\mathrm{ad}} \to \infty\text{-}\mathcal{C}at$. We will not need the precise definition of this functor and we will recall all the properties of ν we will use.

A.2. — We will denote by $\mathsf{c} : \widehat{\mathbf{\Delta}} \to \mathcal{C}_{\mathrm{ad}}$ the *normalized complex functor*. If X is a simplicial set, the underlying augmented complex of $\mathsf{c}X$ is the classical normalized complex (its p-chains are freely generated by nondegenerate p-simplices of X) and $(\mathsf{c}X)_p^*$, for $p \geqslant 0$, consists of p-chains with nonnegative coefficients. In particular, if $m \geqslant 0$ and $p \geqslant 0$, we have $(\mathsf{c}\Delta_m)_p \simeq \mathbb{Z}^{(B_{m,p})}$ where

$$B_{m,p} = \{(i_0, \dots, i_p) \mid 0 \leqslant i_0 < \cdots < i_p \leqslant m\}.$$

We will call the graded set $\coprod_p B_{m,p}$ the *base* of $\mathsf{c}\Delta_m$. (It is the unique base in some precise sense that we will not need.)

A.3. — By composing the functors

$$\mathbf{\Delta} \xrightarrow{\ y\ } \widehat{\mathbf{\Delta}} \xrightarrow{\ \mathsf{c}\ } \mathcal{C}_{\mathrm{ad}} \xrightarrow{\ \nu\ } \infty\text{-}\mathcal{C}at \quad ,$$

where y denotes the Yoneda embedding, we get a cosimplicial object

$$\mathcal{O} : \mathbf{\Delta} \to \infty\text{-}\mathcal{C}at.$$

This is Steiner's definition of Street's *cosimplicial object of orientals*. For $n \geqslant 0$, the ∞-category $\mathcal{O}_n$ is the *n-th oriental*. The cosimplicial object $\mathcal{O}$ induces the so-called *Street nerve*

$$N : \infty\text{-}\mathcal{C}at \to \widehat{\mathbf{\Delta}},$$

sending an ∞-category C to the simplicial set $NC : \Delta_p \mapsto \mathsf{Hom}_{\infty\text{-}\mathcal{C}at}(\mathcal{O}_p, C)$. This nerve functor admits as a left adjoint the left Kan extension of $\mathcal{O} : \mathbf{\Delta} \to \infty\text{-}\mathcal{C}at$ along the Yoneda embedding.

From now on, we fix $m \geqslant 0$.

We will start by showing that $\mathsf{c}\Delta_m$ retracts by deformation on $\mathsf{c}\Delta_0$ in some appropriate sense.

A.4. — Consider the morphism $\mathsf{c}(0) : \mathsf{c}\Delta_0 \to \mathsf{c}\Delta_m$ induced by the simplicial morphism $0 : \Delta_0 \to \Delta_m$ corresponding to the 0-simplex 0 of Δ_m, and the morphism $\mathsf{c}(r) : \mathsf{c}\Delta_m \to \mathsf{c}\Delta_0$ induced by the unique morphism $r : \Delta_m \to \Delta_0$. By functoriality, we have $\mathsf{c}(r)\mathsf{c}(0) = 1_{\mathsf{c}\Delta_0}$. We will see that $\mathsf{c}(0)\mathsf{c}(r)$ is homotopic to $1_{\mathsf{c}\Delta_m}$. For $p \geqslant 0$, we define

$$h_p : (\mathsf{c}\Delta_m)_p \to (\mathsf{c}\Delta_m)_{p+1}$$

by

$$h_p(i_0, \dots, i_p) = \begin{cases} (0, i_0, \dots, i_p) & \text{if } i_0 > 0, \\ 0 & \text{if } i_0 = 0. \end{cases}$$

Adopting the convention that, for $0 \leqslant j_0 \leqslant \cdots \leqslant j_q \leqslant m$, if the sequence of the j_k is not strictly increasing then $(j_0, \dots, j_q) = 0$ in $(\mathsf{c}\Delta_m)_q$, we can simply write

$$h_p(i_0, \dots, i_p) = (0, i_0, \dots, i_p).$$

Proposition A.5. — *The morphisms h_p introduced in the previous paragraph define a homotopy h from $\mathsf{c}(0)\mathsf{c}(r)$ to $1_{\mathsf{c}\Delta_m}$.*

Proof. — Let $(i_0, \dots, i_p)$ be an element of the base of $\mathsf{c}\Delta_m$. Note first that we have

$$\mathsf{c}(0)\mathsf{c}(r)(i_0, \dots, i_p) = \begin{cases} (0) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

To prove that h is a homotopy, we distinguish two cases:
— If $p = 0$, then we have

$$dh(i_0) = d(0, i_0) = (i_0) - (0) = (i_0) - \mathsf{c}(0)\mathsf{c}(r)(i_0).$$

— If $p \geqslant 1$, then we have

$$\begin{aligned}
&dh(i_0,\dots,i_p) + hd(i_0,\dots,i_p)\\
&\quad = d(0,i_0,\dots,i_p) + \sum_{k=0}^{p}(-1)^k h(i_0,\dots,\hat{i}_k,\dots,i_p)\\
&\quad = (i_0,\dots,i_p) + \sum_{k=0}^{p}(-1)^{k+1}(0,i_0,\dots,\hat{i}_k,\dots,i_p)\\
&\qquad\qquad + \sum_{k=0}^{p}(-1)^k(0,i_0,\dots,\hat{i}_k,\dots,i_p)\\
&\quad = (i_0,\dots,i_p) - 0\\
&\quad = (i_0,\dots,i_p) - \mathsf{c}(0)\mathsf{c}(r)(i_0,\dots,i_p).
\end{aligned}$$

□

We now recall how such a homotopy induces an oplax transformation.

A.6. — If K and L are two augmented directed complexes, we define their tensor product $K \otimes L$ in the following way: the underlying augmented complex is the classical tensor product of the underlying augmented complexes, and the positive chains are generated by tensor products of positive chains. By [**5**, Proposition A.19], there exists a canonical ∞-functor

$$\nu(K) \otimes \nu(L) \to \nu(K \otimes L),$$

where the tensor product on the left is the Gray tensor product.

In particular, if K is an augmented directed complex, we get an augmented directed complex $\mathsf{c}\Delta_1 \otimes K$. Moreover, the ∞-category $\nu(\mathsf{c}\Delta_1)$ is canonically isomorphic to D_1 and we thus get an ∞-functor $\mathrm{D}_1 \otimes \nu(K) \to \nu(\mathsf{c}\Delta_1 \otimes L)$. One checks that if L is a second augmented directed complex, then morphisms $\mathsf{c}\Delta_1 \otimes K \to L$ correspond to homotopies between morphisms from K to L. This implies that if h is a homotopy between morphisms from K to L, we get an oplax transformation $\nu(h)$ by composing

$$\mathrm{D}_1 \otimes \nu(K) \longrightarrow \nu(\mathsf{c}\Delta_1 \otimes K) \xrightarrow{\nu(h)} \nu L \quad .$$

Moreover, if h is a homotopy from f to g, then $\nu(h)$ is an oplax transformation from $\nu(f)$ to $\nu(g)$.

In our case of interest, that is, the case where $K = \mathsf{c}\Delta_m$, the canonical morphism $\mathrm{D}_1 \otimes \mathcal{O}_m \to \nu(\mathsf{c}\Delta_1 \otimes \mathsf{c}\Delta_m)$ is an isomorphism (see for instance [**5**, Proposition 7.5 and Theorem A.15]), which we will consider as an equality.

We finally produce the announced structure of oplax transformation retract.

A.7. — We will denote by

$$0 : \mathcal{O}_0 \to \mathcal{O}_m \quad \text{and} \quad r : \mathcal{O}_m \to \mathcal{O}_0$$

the ∞-functors induced by the simplicial maps $0 : \Delta_0 \to \Delta_m$ and $r : \Delta_m \to \Delta_0$. Recall that $\mathcal{O}_0$ is the terminal ∞-category D_0 and that the ∞-functor 0 corresponds to the object 0 of $\mathcal{O}_m$. The ∞-functor r is obviously a retraction of 0.

By applying the considerations of the previous paragraph to the homotopy h of Proposition A.5, we obtain an oplax transformation α from the composite $\mathcal{O}_m \xrightarrow{r} \mathcal{O}_0 \xrightarrow{0} \mathcal{O}_m$ to the identity of $\mathcal{O}_m$. By definition, this oplax transformation is obtained by applying the functor ν to the morphism $\mathsf{c}\Delta_1 \otimes \mathsf{c}\Delta_m \to \mathsf{c}\Delta_m$, that we will still denote by h, given by

$$\begin{aligned} h((0) \otimes (i_0, \dots, i_p)) &= \begin{cases} (0) & \text{if } p = 0, \\ 0 & \text{if } p > 0, \end{cases} \\ h((1) \otimes (i_0, \dots, i_p)) &= (i_0, \dots, i_p), \\ h((0,1) \otimes (i_0, \dots, i_p)) &= (0, i_0, \dots, i_p), \end{aligned}$$

where $(i_0, \dots, i_p)$ is in the base of $\mathsf{c}\Delta_m$.

Proposition A.8. — *The ∞-functor $0 : \mathcal{O}_0 \to \mathcal{O}_m$ is a strong left oplax transformation retract. More precisely, the pair (r, α), introduced in the previous paragraph, is a strong left oplax transformation retract structure on $0 : \mathcal{O}_0 \to \mathcal{O}_m$.*

Proof. — This follows from the previous paragraph. (The condition of strongness is automatic as the identity of 0 is the only 1-cell from 0 to 0 in $\mathcal{O}_m$.) □

We end the appendix with a compatibility result, needed in the proof of Proposition 3.9, between the oplax transformation α and a classical simplicial homotopy.

A.9. — We will denote by

$$k : \Delta_1 \times \Delta_m \to \Delta_m$$

the unique simplicial homotopy from the constant endofunctor of Δ_m of value 0 to the identity of Δ_m. Recall that k sends a p-simplex $(\varphi, \psi) : \Delta_p \to \Delta_1 \times \Delta_m$ of $\Delta_1 \times \Delta_m$ to the p-simplex $(0, \dots, 0, \psi(r), \dots, \psi(p))$ of Δ_m, where r denotes the number of 0 in the sequence $\varphi(0), \dots, \varphi(p)$.

A.10. — We will denote by $\eta : \Delta_m \to N(\mathcal{O}_m)$ the adjunction morphism. This morphism sends a p-simplex $\psi : \Delta_p \to \Delta_m$ of Δ_m to the p-simplex $\mathcal{O}(\psi) : \mathcal{O}_p \to \mathcal{O}_m$ of $N(\mathcal{O}_m)$. By definition of $\mathcal{O}$, we have $\mathcal{O}(\psi) = \nu\mathsf{c}(\psi)$, with $\mathsf{c}(\psi) : \mathsf{c}\Delta_p \to \mathsf{c}\Delta_m$ defined on the base of $\mathsf{c}\Delta_p$ by

$$\mathsf{c}(\psi)(i_0, \dots, i_q) = (\psi(i_0), \dots, \psi(i_q)),$$

where, following the convention introduced in paragraph A.4, the right member is zero if the sequence $\psi(i_0), \dots, \psi(i_q)$ is not strictly increasing.

A.11. — Let $u, v : A \to B$ be two ∞-functors and let $\beta : u \Rightarrow v$ be an oplax transformation. Following [**7**, Appendix A], we define a simplicial homotopy

$$N\beta : \Delta_1 \times NA \to NB$$

from Nu to Nv in the following way. Let $(\varphi, x) : \Delta_p \to \Delta_1 \times NA$ be a p-simplex of $\Delta_1 \times NA$. By definition, the homotopy $N\beta$ sends (φ, x) to the p-simplex

$$\mathcal{O}_p \xrightarrow{\nu(g_\varphi)} \mathrm{D}_1 \otimes \mathcal{O}_p \xrightarrow{\mathrm{D}_1 \otimes x} \mathrm{D}_1 \otimes A \xrightarrow{\beta} B$$

of NB, where

$$g_\varphi : \mathsf{c}\Delta_p \to \mathsf{c}\Delta_1 \otimes \mathsf{c}\Delta_p$$

is the morphism defined as follows. Let $(i_0, \dots, i_q)$ be an element of the base of $\mathsf{c}\Delta_p$ and denote by r the number of 0 in the sequence $\varphi(i_0), \dots, \varphi(i_q)$. The morphism g_φ is defined by

$$g_\varphi(i_0, \dots, i_q) = \begin{cases} (1) \otimes (i_0, \dots, i_q) & \text{if } r = 0, \\ (0) \otimes (i_0, \dots, i_q) + (0,1) \otimes (i_1, \dots, i_q) & \text{if } r = 1, \\ (0) \otimes (i_0, \dots, i_q) & \text{if } r \geqslant 2, \end{cases}$$

where $(i_1, \dots, i_q) = 0$ for $q = 0$.

Proposition A.12. — *The square*

$$\begin{array}{ccc} \Delta_1 \times \Delta_m & \xrightarrow{\Delta_1 \times \eta} & \Delta_1 \times N(\mathcal{O}_m) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle N\alpha} \\ \Delta_m & \xrightarrow[\eta]{} & N(\mathcal{O}_m) \end{array}$$

commutes.

Proof. — Let $p \geqslant 0$ and fix $(\varphi, \psi) : \Delta_p \to \Delta_1 \times \Delta_m$ a p-simplex of $\Delta_1 \times \Delta_m$. We want to compare the two p-simplices $\mathcal{O}_p \to \mathcal{O}_m$ of $N(\mathcal{O}_m)$ associated to (φ, ψ) by the square of the statement. Each of these p-simplices will be induced by a morphism $\mathsf{c}\Delta_p \to \mathsf{c}\Delta_m$ and we will prove that these two morphisms are equal. We thus fix $(i_0, \dots, i_q)$ an element of the base of $\mathsf{c}\Delta_p$ and we denote by r the number of 0 in the sequence $\varphi(i_0), \dots, \varphi(i_q)$.

By paragraphs A.9 and A.10, the morphism ηk sends the p-simplex (φ, ψ) to the p-simplex $\nu(f) : \mathcal{O}_p \to \mathcal{O}_m$ of $N(\mathcal{O}_m)$, where the morphism $f : \mathsf{c}\Delta_p \to \mathsf{c}\Delta_m$ satisfies

$$f(i_0, \dots, i_q) = (0, \dots, 0, \psi(i_r), \dots, \psi(i_q)).$$

In other words, we have

$$f(i_0, \dots, i_q) = \begin{cases} (\psi(i_0), \dots, \psi(i_q)) & \text{if } r = 0, \\ (0, \psi(i_1), \dots, \psi(i_q)) & \text{if } r = 1, \\ 0 & \text{if } r \geqslant 2. \end{cases}$$

Similarly, by paragraphs A.10 and A.11, the morphism $(N\alpha)(\Delta_1 \times \eta)$ sends (φ, ψ) to the p-simplex $\nu(g) : \mathcal{O}_p \to \mathcal{O}_m$ of $N(\mathcal{O}_m)$, where the morphism $g : \mathsf{c}\Delta_p \to \mathsf{c}\Delta_m$ is the composite

$$\mathsf{c}\Delta_p \xrightarrow{g_\varphi} \mathsf{c}\Delta_1 \otimes \mathsf{c}\Delta_p \xrightarrow{\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi)} \mathsf{c}\Delta_1 \otimes \mathsf{c}\Delta_m \xrightarrow{h} \mathsf{c}\Delta_m \quad .$$

To compute this composite, we distinguish three cases:

— If $r = 0$, then we have

$$\begin{aligned} h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))g_\varphi(i_0, \dots, i_q) &= h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))((1) \otimes (i_0, \dots, i_q)) \\ &= h((1) \otimes (\psi(i_0), \dots, \psi(i_q))) \\ &= (\psi(i_0), \dots, \psi(i_q)). \end{aligned}$$

— If $r = 1$, then we have

$$\begin{aligned} &h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))g_\varphi(i_0, \dots, i_q) \\ &\quad = h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))((0) \otimes (i_0, \dots, i_q) + (0,1) \otimes (i_1, \dots, i_q)) \\ &\quad = h((0) \otimes (\psi(i_0), \dots, \psi(i_q)) + (0,1) \otimes (\psi(i_1), \dots, \psi(i_q))) \\ &\quad = (0, \psi(i_1), \dots, \psi(i_q)), \end{aligned}$$

where the last equality has to be checked separately in the cases $q = 0$ and $q \neq 0$.

— Finally, if $r \geqslant 2$, then we have

$$\begin{aligned} h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))g_\varphi(i_0, \dots, i_q) &= h(\mathsf{c}\Delta_1 \otimes \mathsf{c}(\psi))((0) \otimes (i_0, \dots, i_q)) \\ &= h((0) \otimes (\psi(i_0), \dots, \psi(i_q))) \\ &= 0. \end{aligned}$$

This shows that $f = g$, thereby ending the proof. □

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